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STOCHASTIC NONLINEAR DIFFERENTIAL EQUATIONS. I

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Synopsis

A solution method is developed for nonlinear differential equations having the following two properties. Their coefficients are stochastic through their dependence on a Markov process. The magnitude of the fluctuations, multiplied with their auto-correlation time, is a small quantity. Under these conditions, the solution is also approximately a Markov process. Its probability distribution obeys a master equation, whose kernel is found as an expansion in that small quantity. The general formula is derived. Applications will be given in the second part of this work.

1. *Introduction.* In three recent papers[1,2]), stochastic *linear* differential equations have been studied, that is, equations of the type

$$\frac{\mathrm{d}u\,(t)}{\mathrm{d}t} = A(t)\,u(t),$$

where u is a vector and $A(t)$ a random matrix whose stochastic properties are supposed known. It turned out that, in the case that the fluctuations in A are small and rapid, a linear differential equation for the average of u exists

$$\frac{\mathrm{d}\,\langle u(t)\rangle}{\mathrm{d}t} = K(t)\,\langle u(t)\rangle.$$

Here $K(t)$ is a non-stochastic matrix, which could be found in successive powers of $\alpha\tau_c$, where τ_c is the autocorrelation time of the fluctuations in A and α a measure for their size.

In the present article a similar approach is applied to *nonlinear* equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = F(u, t), \tag{1}$$

where F is a stochastic function of its arguments u and t. The method used here, however, is applicable only if this stochastic nature of F is of the following special kind. F is stochastic through its dependence on an additional variable ξ, which itself is a stationary Markov process. This type of stochasticity is particularly simple and has been used by previous authors for certain linear models[3]). In addition we have to assume again that the fluctuations in F are small and rapid, so that an expansion in $\alpha\tau_c$ is valid.

The result will be a differential equation for the probability density $p(u, t)$ of u,

$$\frac{\partial p(u, t)}{\partial t} = K(t)\, p(u, t), \tag{2}$$

where $K(t)$ is a sure operator acting on the u-dependence of $p(u, t)$. It is essential that $K(t)$ does *not* act as operator on the time dependence of $p(u, t)$, that is, $K(t)$ does not involve derivatives with respect to time, let alone integrations over past or future times. Hence (2) has the form of a master equation, which tells that in this approximation u can be regarded as a Markov process. This approximate Markov character is of course due to the assumption of a short τ_c. The operator $K(t)$ will be found in successive powers of $\alpha\tau_c$.

2. *The Markov process* ξ. As ξ is a stationary Markov process its probability density $\Pi(\xi, t)$ obeys a master equation

$$\frac{\partial \Pi(\xi, t)}{\partial t} = \int \{W(\xi \mid \xi')\, \Pi(\xi', t) - W(\xi' \mid \xi)\, \Pi(\xi, t)\}\, \mathrm{d}\xi' = W\Pi(\xi, t). \tag{3}$$

The linear operator W has eigenvalues $-\lambda_j$ with right and left eigenfunctions $\phi_j(\xi)$ and $\psi_j(\xi)$,

$$W\phi_j = -\lambda_j \phi_j, \qquad \int \psi_i \phi_j\, \mathrm{d}\xi = \delta_{ij}.$$

We assume that the eigenvalue $\lambda_0 = 0$ is non-degenerate and isolated. A measure for the auto-correlation time is given by the distance to the next level,

$$\tau_c \sim 1/\lambda_1.$$

The consequence of the special type of stochasticity assumed for F is that the *joint probability density* $P(u, \xi, t)$ of u and ξ obeys a master equation

$$\frac{\partial P(u, \xi, t)}{\partial t} + \frac{\partial}{\partial u} F(u, t; \xi) P(u, \xi, t) = WP(u, \xi, t). \tag{4}$$

Notice the similarity to the Boltzmann equation with a flow term and a collision term, but of course the present equation is linear. If this equation could be solved explicitly, the probability density $p(u, t)$ would be found by integrating $P(u,\xi,t)$ over all values of ξ. Alternatively, one may write

$$p(u, t) = \int \psi_0(\xi) P(u, \xi, t) \, \mathrm{d}\xi, \tag{5}$$

since $\psi_0(\xi) = 1$. Of course, one also has to prescribe an initial value for P. For convenience we take

$$P(u, \xi, 0) = p(u, 0) \phi_0(\xi). \tag{6}$$

This means that at $t = 0$ there is no correlation between u and ξ, and ξ is in equilibrium. That is a fair description of the situation that obtains when u is kept fixed until $t = 0$.

We assume that the eigenfunctions ϕ_j are a complete set and expand P,

$$P(u, \xi, t) = p(u, t) \phi_0(\xi) + \sum_{j=1}^{\infty} p_j(u, t) \phi_j(\xi). \tag{7}$$

On substituting this expansion in (4) one obtains a sequence of coupled equations for the p_j. It is convenient to use the following abbreviation for the "matrix elements".

$$\int \psi_i(\xi) F(u, t; \xi) \phi_j(\xi) \, \mathrm{d}\xi = F_{ij}(u, t).$$

If, moreover, we write the equation for p separately from the equation for the p_i $(i = 1, 2, \ldots)$ the sequence takes the form

$$\frac{\partial}{\partial t} p(u, t) + \frac{\partial}{\partial u} F_{00}(u, t) p(u, t) = -\sum_{j=1}^{\infty} \frac{\partial}{\partial u} F_{0j}(u, t) p_j(u, t), \tag{8a}$$

$$\left(\frac{\partial}{\partial t} + \lambda_i\right) p_i(u, t) = -\frac{\partial}{\partial u} F_{i0}(u, t) p_0(u, t) - \sum_{j=1}^{\infty} \frac{\partial}{\partial u} F_{ij}(u, t) p_j(u, t). \tag{8b}$$

In order to introduce the expansion parameter we make the usual decomposition of F in a sure part and a random part with zero mean,

$$F(u, t; \xi) = F_{00}(u, t) + \alpha f(u, t; \xi), \qquad \int f(u, t; \xi) \phi_0(\xi) \, \partial\xi = 0. \tag{9}$$

Equation (8a) may then be written

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial u} F_{00}\right\} p = -\alpha \frac{\partial}{\partial u} \boldsymbol{g} \cdot \boldsymbol{p}, \tag{10a}$$

where $\boldsymbol{p}$ is the column vector with components $p_1, p_2, \ldots$ and $\boldsymbol{g}(u, t)$ is the row vector with components $f_{01}, f_{02}, \ldots$ In fact, we also define a column vector $\boldsymbol{f}$ and a matrix $\boldsymbol{F}$ by setting

$$(F_{ij}) = \begin{pmatrix} F_{00} & \boldsymbol{g} \\ \boldsymbol{f} & \boldsymbol{F} \end{pmatrix}.$$

Equation (8b) may then be written

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial u} F_{00} + \boldsymbol{\Lambda}\right\} \boldsymbol{p} = -\alpha \frac{\partial}{\partial u} \boldsymbol{f} p - \alpha \frac{\partial}{\partial u} \boldsymbol{F}\boldsymbol{p}, \tag{10b}$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix with elements $\lambda_1, \lambda_2, \ldots$ and F_{00} originates from the diagonal elements of F_{ij}.

3. *The integral equation for p.* It is seen from the equations (10) that p and $\boldsymbol{p}$ are coupled through terms of order α. To eliminate the terms not involving α, we transform to the interaction representation. Let the operator $V(u, t \mid u', t')$ be defined as the solution of the equation

$$\frac{\partial}{\partial t} V(u, t \mid u', t') = -\frac{\partial}{\partial u} F_{00}(u, t)\, V(u, t \mid u', t'),$$

for $t \geq t'$, together with

$$V(t \mid t) = 1 \qquad \text{and} \qquad V(t' \mid t) = V(t \mid t')^{-1}.$$

It is true that the definition of $V(t' \mid t)$ for $t' < t$ through the reciprocal may sometimes give rise to difficulties, but we shall not discuss them here. Using this operator we define the interaction representation of p, indicated by a prime

$$p(u, t) = \int V(u, t \mid u', 0)\, p'(u', t)\, \mathrm{d}u',$$

or more succinctly

$$p(t) = V(t \mid 0)\, p'(t).$$

The interaction representation of $\boldsymbol{p}$ is defined by

$$\boldsymbol{p} = \mathrm{e}^{-t\boldsymbol{\Lambda}}\, V(t \mid 0)\, \boldsymbol{p}'. \tag{11}$$

The interaction representation of the operator $\partial g/\partial u$ appearing in (10a) will be indicated (somewhat inconsistently) by $\boldsymbol{g}'$

$$\boldsymbol{g}'(t) = V(0 \mid t) \frac{\partial}{\partial u} g(u, t) V(t \mid 0) \mathrm{e}^{-tA}. \tag{12a}$$

More explicitly, its kernel in the interaction representation is

$$g'(t, u \mid u') = \int V(u, 0 \mid u'', t) \frac{\partial}{\partial u''} g(u'', t) V(u'', t \mid u', 0) \, \mathrm{d}u'' \, \mathrm{e}^{-tA}.$$

Similarly we set

$$\boldsymbol{f}'(t) = \mathrm{e}^{tA} V(0 \mid t) \frac{\partial}{\partial u} f(u, t) V(t \mid 0), \tag{12b}$$

$$\boldsymbol{F}'(t) = \mathrm{e}^{tA} V(0 \mid t) \frac{\partial}{\partial u} F(u, t) V(t \mid 0) \mathrm{e}^{-tA}. \tag{12c}$$

In this new representation the equations (10) become

$$\frac{\partial p'(u, t)}{\partial t} = -\alpha \boldsymbol{g}'(t) \boldsymbol{p}'(u, t), \tag{13a}$$

$$\frac{\partial p'(u, t)}{\partial t} = -\alpha \boldsymbol{F}'(t) p'(u, t) - \alpha \boldsymbol{f}'(t) p'(u, t). \tag{13b}$$

The aim is to obtain an equation for p' alone. We therefore solve the second line formally by defining an operator $\boldsymbol{T}'(t \mid t')$ for $t > t'$ as the solution of

$$\frac{\partial}{\partial t} \boldsymbol{T}'(t \mid t') = -\alpha \boldsymbol{F}'(t) \boldsymbol{T}'(t \mid t') \qquad (t > t'), \tag{14a}$$

$$\boldsymbol{T}'(t' \mid t') = 1. \tag{14b}$$

Then

$$\boldsymbol{p}'(t) = \boldsymbol{T}'(t \mid 0) \boldsymbol{p}'(0) - \alpha \int_0^t \boldsymbol{T}'(t \mid t') \boldsymbol{f}(t') p'(t') \, \mathrm{d}t'. \tag{15}$$

The first term vanishes owing to our choice of initial state (6). Substitution of (15) into (13a) yields an integro-differential equation for p' alone

$$\frac{\partial p'(t)}{\partial t} = \alpha^2 \int_0^t \boldsymbol{g}'(t) \boldsymbol{T}'(t \mid t') \boldsymbol{f}'(t') p'(t') \, \mathrm{d}t' = \alpha^2 \int_0^t \Gamma'(t \mid t') p'(t') \, \mathrm{d}t'. \tag{16}$$

In the original representation this equation reads

$$\frac{\partial p\,(u,t)}{\partial t} + \frac{\partial}{\partial u} F_{00}\,(u,t)\,p\,(u,t) = \alpha^2 \int_0^t \Gamma\,(t \mid t')\,p\,(u,t')\,\mathrm{d}t'. \tag{17}$$

With the kernel $\Gamma\,(t \mid t')$, which is an operator on u,

$$\Gamma\,(t \mid t') = \frac{\partial}{\partial u} g\,(u,t)\,V\,(t \mid 0)\,\mathrm{e}^{-tA}\,\boldsymbol{T}'\,(t \mid t')\,\mathrm{e}^{t'A}\,V\,(0 \mid t')\,\frac{\partial}{\partial u} f(u,t'). \tag{18}$$

4. *Expansion in* α. The operator $\boldsymbol{T}'\,(t \mid t')$ defined by (14) is easily found as a power series in

$$\boldsymbol{T}'\,(t \mid t') = 1 + \alpha \int_{t'}^{t} \mathrm{d}t_1\,\boldsymbol{F}'(t_1) + \alpha^2 \int_{t'}^{t} \mathrm{d}t_1 \int_{t'}^{t} \mathrm{d}t_2\,\boldsymbol{F}'(t_1)\,\boldsymbol{F}'(t_2) + \cdots. \tag{19}$$

This leads to an expansion of $\Gamma'\,(t \mid t')$

$$\Gamma'\,(t \mid t') = \sum_{n=0}^{\infty} \alpha^n \Gamma_n'\,(t \mid t'), \tag{20a}$$

$$\Gamma_0'\,(t \mid t') = \boldsymbol{g}'(t)\,\boldsymbol{f}'(t'), \tag{20b}$$

$$\Gamma_n'\,(t \mid t') = \int_{t'}^{t} \mathrm{d}t_1 \int_{t'}^{t_1} \mathrm{d}t_2 \cdots \int_{t'}^{t_{n-1}} \mathrm{d}t_n\,\boldsymbol{g}'(t)\,\boldsymbol{F}'(t_1)\,\boldsymbol{F}'(t_2) \cdots \boldsymbol{F}'(t_n)\,\boldsymbol{f}'(t'). \tag{20c}$$

In the original representation one has for instance

$$\Gamma_1\,(t \mid t') = \int_{t'}^{t} \mathrm{d}t_1\,\frac{\partial}{\partial u}\,\boldsymbol{g}\,(u,t)\,V\,(t \mid 0)\,\mathrm{e}^{-tA}\,\boldsymbol{F}(t_1)\,\mathrm{e}^{t'A}\,V\,(0 \mid t')\,\frac{\partial}{\partial u} f(u,t').$$

However, this is only the trival part of the expansion in α.

The other part of the expansion has the effect of turning the integral equation (16) into a differential equation. The idea is that, according to the equation itself, p' only varies slowly with time and that therefore the factor $p'(t')$ in the integrand may be replaced with $p'(t)$, provided that $\Gamma'\,(t \mid t')$ vanishes as soon as $t - t'$ exceeds a certain value. Assuming this for the present (see the following section for further details), we give the formal expansion of the integral equation following Terwiel's iteration method.[2])

First write (16) in the form

$$\frac{\partial p'(t)}{\partial t} = \alpha^2 \int_0^t \Gamma'(t \mid t') \left\{ p'(t) - \int_{t'}^t \mathrm{d}t_1 \frac{\partial p'(t_1)}{\partial t_1} \right\} \mathrm{d}t'$$

$$= \alpha^2 \int_0^t \Gamma'(t \mid t')\, \mathrm{d}t'\, p'(t) - \alpha^2 \int_0^t \mathrm{d}t' \int_{t'}^t \mathrm{d}t_1 \Gamma'(t \mid t') \frac{\partial p'(t_1)}{\partial t_1}.$$

Then substitute for the derivative at the end again the same equation, and so on. The result is

$$\frac{\partial p'(t)}{\partial t} = \alpha^2 K'(t)\, p'(t), \tag{21}$$

where

$$K'(t) = \sum_{n=0}^{\infty} \alpha^{2n} K_n'(t), \tag{22a}$$

$$K_n'(t) = (-1)^{n-1} \int_0^t \mathrm{d}t' \int_{t'}^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_1' \cdots$$

$$\cdots \int_{t'_{n-1}}^t \mathrm{d}t_n \int_0^{t_n} \mathrm{d}t_n' \Gamma'(t \mid t')\, \Gamma'(t_1 \mid t_1') \cdots \Gamma'(t_n \mid t_n'). \tag{22b}$$

This is a differential equation for $p'(t)$ of the general form of a master equation. Of course $K(t)$ is an operator acting on the u-dependence of p'.

On substituting in this result the expansion (20) of Γ' and rearranging terms according their order in α one finally finds

$$K'(t) = \sum_{n=0}^{\infty} \alpha^n k_n'(t). \tag{23a}$$

The first three terms are

$$k_0'(t) = \int_0^t \mathrm{d}t' \boldsymbol{g}'(t)\, \boldsymbol{f}'(t'), \tag{23b}$$

$$k_1'(t) = \int_0^t \mathrm{d}t' \int_{t'}^t \mathrm{d}t_1 \boldsymbol{g}'(t)\, \boldsymbol{F}'(t_1)\, \boldsymbol{f}'(t'), \tag{23c}$$

$$K_2'(t) = \int_0^t \mathrm{d}t' \int_{t'}^t \mathrm{d}t_1 \int_{t'}^{t_1} \mathrm{d}t_2 \boldsymbol{g}'(t)\, \boldsymbol{F}'(t_1)\, \boldsymbol{F}'(t_2)\, \boldsymbol{f}'(t')$$

$$- \int_0^t \mathrm{d}t' \int_{t'}^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_1' \boldsymbol{g}'(t)\, \boldsymbol{f}'(t')\, \boldsymbol{g}'(t_1)\, \boldsymbol{f}'(t_1'). \tag{23d}$$

The general term k_n' can be constructed by the following rules.

(i) Write g' on the left and f' on the right, and insert any sequence of n factors F', g', and f' between them such that g' and f' only occur in the continuation $f'g'$.

(ii) Supply the first factor g' with the argument t, and the remaining factors with $t_1, t_2, \ldots, t_{n+1}$ in any permutation, subject to the conditions that in any sequence

$$g'F'F' \cdots F'f'$$

the indices increase, while they decrease in each pair $f'g'$.

(iii) Integrate each term over the domain

$$t > t_1 > t_2 \cdots t_{n+1} > 0.$$

(iv) Multiply each term by one factor (-1) for each pair $f'g'$ in it and add the terms.

5. *Justification of the expansion.* In this section we give general considerations on the conditions under which the derivation given in the preceding sections are valid. In the second part of this work we treat specific forms of $F_{00}(u, t)$; in these cases we can give more precise results on the nature of the convergence of the series in eq. (23a).

First we have to convince ourselves that Γ' actually does have a finite decay time τ_c such that $\Gamma(t \mid t') \approx 0$ for $|t - t'| > \tau_c$. Equation (18) may be written

$$\Gamma(t \mid t') = \frac{\partial}{\partial u} g(u, t) \, T(t \mid t') \frac{\partial}{\partial u} f(u, t'), \tag{24}$$

where T in the operator T' transformed back to the original representation

$$T'(t \mid t') = V(0 \mid t) \, e^{tA} \, T(t \mid t') \, e^{-t'A} \, V(t' \mid 0). \tag{25}$$

$T(t \mid t')$ reduces to the unit operator at $t = t'$ and obeys the equation

$$\frac{\partial}{\partial t} T(t \mid t') = -\frac{\partial}{\partial u} \{F_{00}(u, t) + \alpha F(u, t)\} \, T(t \mid t') - A T(t \mid t'), \tag{26}$$

which is simply the homogeneous part of (10b).

The last term causes an exponential decay at least as fast as $e^{-\lambda_1 t} \sim \exp(-t/\tau_c)$. It is essential to assume that this is fast compared to the variation caused by the flow term. That is, if the averaged (or unperturbed) process

$$\frac{du}{dt} = F_{00}(u, t),$$

has a characteristic correlation time τ_m ($F_{00}(u, t) \sim 1/\tau_m$), we assume

$$\tau_c \ll \tau_m.$$

Then one may conclude from (26) to the estimate

$$T(t \mid t') \sim e^{-(t-t')\Lambda}. \tag{27}$$

With the aid of this estimate, and again, ignoring all changes on the slow time scale τ_m, one obtains, using successively (16), (25) and (27),

$$\Gamma'(t \mid t') = g'(t)\, T'(t \mid t')\, f'(t') \sim g\, e^{-t\Lambda}\, e^{t\Lambda}\, e^{-(t-t')\Lambda}\, e^{-t'\Lambda}\, e^{t'\Lambda} f$$

$$\sim \sum_{j=1}^{\infty} g_j f_j \exp(-\lambda_j (t - t')).$$

This shows that Γ' does decay in a time of order τ_c, which justifies the assumptions made when going from the integral equation (16) to the differential equation (21). It also follows that each of the integrations in (23) virtually extends only over a range of length τ_c, so that $k'_n \sim \tau_c^{2n}$. That means that (22) actually amounts to an expansion in the dimensionless parameter $\alpha\tau_c$.

The final formula (24), however, involves the expansion (20) of Γ' as well. In order to estimate the terms (20c) we again ignore the slower variation on the time scale τ_m and compute $\Gamma'_1(t \mid t')$ using (12),

$$\Gamma'_1(t \mid t') = \int_{t'}^{t} dt_1\, g'(t)\, F'(t_1)\, f'(t') \sim \int_{t'}^{t} dt_1\, g\, e^{-(t-t_1)\Lambda}\, F\, e^{-(t_1-t')\Lambda} f$$

$$\sim \sum_{i \neq j} g_i F_{ij} f_j \frac{\exp(-\lambda_j (t - t')) - \exp(-\lambda_i (t - t'))}{\lambda_i - \lambda_j}$$

$$+ \sum_{j} g_j F_{jj} f_j (t - t') \exp(-\lambda_j (t - t'))$$

$$\sim \sum_{i,j} g_i F_{ij} f_j \exp(-(t - t')/\tau_c).$$

Similarly one finds that the successive Γ'_n are of order $\tau_c^n\, e^{-(t-t')/\tau_c}$, so that (20) is likewise an expansion in $\alpha\tau_c$. Consequently (21) is a series of terms of orders $\alpha^2 (\alpha\tau_c)^n$ with $n = 0, 1, 2, \ldots$. The term proportional to α is missing owing to the fact that we have absorbed the average of the fluctuations in F_{00}.

It has here been tacitly assumed that the spectrum λ_j of W is discrete, but only minor adjustments are required for continuous spectra. It is essential, however, that λ_0 is isolated in order that there is a finite τ_c.

6. *The final result*. One more step is needed to arrive at the final formula. It appears from (23) that the operator $K'(t)$ still depends on the initial time $t = 0$, that is the time at which it was supposed that u and ξ are uncorrelated. This dependence would make the formula of very limited interest in physics. Fortunately it can be shown that as soon as $t \gg \tau_c$, the influence of the initial condition on the equation is negligible.

To show this we note that, owing to the rapid decay of Γ', no serious error is made by extending the range of integration of (23), that is, replacing (23) with

$$\bar{K}_n'(t) = (-1)^{n-1} \int_{-\infty}^{t} dt' \int_{t'}^{t} dt_1 \int_{-\infty}^{t_1} dt_1' \cdots \int_{t'_{n-1}}^{t} dt_n \int_{-\infty}^{t_n} dt_n' \, \Gamma'(t \mid t') \cdots \Gamma'(t_n \mid t_n').$$

This also has the effect of replacing in (22b) the lower limit of integration $t = 0$, wherever it occurs, with $t = -\infty$. The resulting equation

$$\frac{\partial p'(u,t)}{\partial t} = \alpha^2 \bar{K}'(t)\, p'(t)$$

is valid for each stochastic process, u, obeying (1), regardless of the initial condition which has served to specify it, provided that an initial transient period of order τ_c has had time to elapse.

For completeness we rewrite this equation in the original representation

$$\frac{\partial p(u,t)}{\partial t} = -\frac{\partial}{\partial u} F_{00}(u,t)\, p(u,t) + \alpha^2 V(t \mid 0)\, \bar{K}'(t)\, V(0 \mid t)\, p(t).$$

This is the equation (2). In most applications the second order in α is sufficient; this can be written more explicitly

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial u} F_{00}(u,t)\right\} p(u,t)$$

$$= \alpha^2 \frac{\partial}{\partial u} g(u,t) \int_{-\infty}^{t} dt' V(t \mid t')\, e^{-(t-t')\Lambda} \frac{\partial}{\partial u} f(u,t')\, V(t' \mid t)\, p(u,t)$$

$$= \alpha^2 \frac{\partial}{\partial u} \sum_{j=1}^{\infty} F_{0j}(u,t) \int_{-\infty}^{t} dt' V(t \mid t') \exp(-(t-t')\lambda_j)$$

$$\times \frac{\partial}{\partial u} F_{j0}(u,t')\, V(t' \mid t)\, p(u,t).$$

Note that this equation has the form of a continuity equation for $p\,(u, t)$

$$\frac{\partial p\,(u,t)}{\partial t} + \frac{\partial}{\partial u}\,\Phi p\,(u, t) = 0,$$

with a "renormalized" force Φ, which is actually an operator acting on the u-dependence. To be sure, in order to find Φ one has to determine $V(t \mid t')$ and hence to solve the unrenormalized equation.

In many cases F does not depend explicitly on time, apart from its implicit dependence through ξ. Then $V(t \mid t') = V(t - t')$ and the equation becomes

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\,F_{00}(u)\right\} p\,(u, t) = \alpha^2\,\frac{\partial}{\partial u}\sum_{j=1}^{\infty} F_{0j}(u)\int_0^{\infty} \mathrm{d}\tau\; V(\tau)\exp\,(-\lambda_j\tau)\,\frac{\partial}{\partial u}\,F_{j0}(u)\;V(-\tau)\,p\,(u, t).$$

In the special case that $F(u, \xi)$ has the form

$$F(u, \xi) = F_{00}(u) + b(\xi)\,h(u),$$

the equation reduces to

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial u}\,F_{00}(u)\right\} p\,(u, t) = \alpha^2\sum_{j=1}^{\infty} b_{0j}b_{j0}h\,(u)\int_0^{\infty} \mathrm{d}\tau\; V(\tau)\exp\,(-\lambda_j\tau)\,\frac{\partial}{\partial u}\,h(u)\;V(-\tau)\,p\,(u, t).$$

In the second part of this work we shall apply these equations to several examples.

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