MATHEMATICS

ON SPACES OF OPERATORS BETWEEN LOCALLY K-CONVEX SPACES

BY

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I. INTRODUCTION

In this paper several spaces of operators between locally K-convex spaces are studied. We are especially interested in the relations between them as well as in the characterization of the locally K-convex spaces for which some of those spaces of operators coincide.

In the final section we show how certain spaces of operators can be represented as sequence spaces.

For the theory of locally K-convex spaces we refer to [7] and [12].

II. DEFINITIONS

Let X and Y be complete locally K-convex Hausdorff spaces over a c-compact, non-archimedian (n.a.) valued field K *).

Then we define the following spaces of operators from X to Y:

$C(X, Y)$ = The compact operators.

I.e. the operators $f: X \to Y$ such that there exists a K-convex zero-neighbourhood U in X whose image $f(U)$ is bounded and relatively c-compact in Y.

$CF(X, Y)$ = The compactifying operators.

I.e. the operators $f: X \to Y$ such that for every K-convex and bounded subset B of X the image $f(B)$ is relatively c-compact in Y.

$CC(X, Y)$ = The completely continuous operators.

I.e. the closure of the space of the operators with finite rank in the space $L_\beta(X, Y)$ (the space of all the operators from X to Y with the topology of uniform convergence on the K-convex, bounded subsets of X).

$N(X, Y)$ = The nuclear operators.

I.e. the operators $f: X \to Y$ such that there exist n.a. Banach spaces X_1 and Y_1, operators $g: X \to X_1$, $h: Y_1 \to Y$ and a nuclear operator $\bar{f}: X_1 \to Y_1$ with $f = h \circ \bar{f} \circ g$.

*) The letters X and Y, unless stated otherwise, will always have this particular meaning.

In the case X and Y are n.a. Banach spaces it was proved by SERRE [9] and GRUSON [5] that

$$C(X, Y) = CF(X, Y) = CC(X, Y) = N(X, Y) = X' \hat{\otimes} Y.$$

This property is not valid anymore for arbitrary locally K-convex spaces.

III. THE C-APPROXIMATION PROPERTY

Just as in the archimedian theory of compact operators we have to assume that some of the spaces involved have an "approximation property" in the sense of Grothendieck.

We state this property here in a form adapted to our theory.

The space X is said to have the c-approximation property if the identity operator on X can be approximated by operators with finite rank, uniformly on every K-convex, bounded and c-compact subset of X.

At this moment we don't answer the question whether or not every X has the c-approximation property. (The answer seems to be positive.)

We state however the following results:

PROPOSITION 1. Every n.a. perfect sequence space with the natural topology (see [4] § 4) has the c-approximation property.

PROOF. Every element $\alpha = (\alpha_i)$ of such a sequence space Λ can be written as

$$\alpha = \sum_{i=1}^{\infty} \alpha_i e_i,$$

where e_i is the sequence with 1 on the i-th place and zero's elsewhere. ([4] prop. 19)

We now consider operators g_k on Λ with finite rank, defined by

$$g_k(\alpha) = \sum_{i=1}^{k} \alpha_i e_i, \qquad k = 1, 2, \ldots.$$

For every element β of the dual space Λ^* and the corresponding n.a. semi-norm p_β on Λ, we have:

$$\begin{aligned} \forall k: p_\beta(g_k(\alpha)) &= \max_{i=1,\ldots,k} |\beta_i \alpha_i| \\ &\leqslant \max_i |\beta_i \alpha_i| = p_\beta(\alpha). \end{aligned}$$

Hence the sequence (g_k) is equicontinuous.

On the other hand we have:

$$\forall \alpha \in \Lambda: \lim_{k\to\infty} g_k(\alpha) = \alpha.$$

Therefore $\lim_{k\to\infty} g_k(\alpha) = \alpha$ uniformly on every K-convex, bounded and c-compact subset of Λ. ([4] prop. 13)

EXAMPLE. The space c_0 of the sequences in K converging to zero, with the n.a. norm

$$\|\alpha\| = \max_i |\alpha_i|$$

has the c-approximation property.

COROLLARY. Every n.a. Banach space of countable type has the c-approximation property.

Such a space is indeed isomorphic to the space c_0 ([8] Th. 2.1).

IV. GENERAL PROPERTIES

PROPOSITION 2. Let S, T also be complete locally K-convex spaces and suppose $g \in L(S, X)$, $h \in L(Y, T)$, then

If $f \in C(X, Y)$ then $f \circ g \in C(S, Y)$ and $h \circ f \in C(X, T)$,

If $f \in CF(X, Y)$ then $f \circ g \in CF(S, Y)$ and $h \circ f \in CF(X, T)$,

If $f \in CC(X, Y)$ then $f \circ g \in CC(S, Y)$ and $h \circ f \in CC(X, T)$,

If $f \in N(X, Y)$ then $f \circ g \in N(S, Y)$ and $h \circ f \in N(X, T)$.

PROOF. Immediate consequence of the definitions and the general properties of bounded and c-compact sets. ([10] and [12])

PROPOSITION 3. Every element of $CF(X, Y)$ transforms weakly convergent sequences into convergent sequences.

PROOF. Let $f \in CF(X, Y)$ and suppose that $\lim_{n\to\infty} x_n = x$, weakly in X. Since f is also continuous for the weak topologies on X and Y we have that

$$\lim_{n\to\infty} f(x_n) = f(x)$$

weakly in Y.

On the other hand, the K-convex hull $A = \mathscr{C}\ \{x, x_1, x_2, \ldots, x_n, \ldots\}$ of the set $\{x, x_1, x_2, \ldots, x_n, \ldots\}$ is bounded in X.

Hence $f(A)$ is bounded and relatively c-compact in Y, which implies that on $f(A)$ the weak topology of Y coincides with the original topology of Y. ([5] § 5 prop. 4)

Hence

$$\lim_{n\to\infty} f(x_n) = f(x) \text{ in } Y.$$

PROPOSITION 4.

$$CC(X, Y) \subset CF(X, Y).$$

PROOF. If the operator $f\colon X \to Y$ has finite rank then $\operatorname{Im} f$ is relatively c-compact since K is c-compact ([10] prop. 1.17) and thus is $f \in CF(X, Y)$.

We consider now a net $\{f^\nu\}$ of operators with finite rank converging in $L_\beta(X, Y)$ to the operator f.

We then have to prove that $f \in CF(X, Y)$.

Let D be a K-convex, bounded subset of X and let V be a zero-neighbourhood in Y. *)

Then

$$\exists \nu_0 \text{ such that } (f-f^{\nu})(D) \subset V \text{ for all } \nu \geqslant \nu_0.$$

The set $f^{\nu_0}(D)$ is K-convex, bounded and relatively c-compact in Y and thus there exist elements $y_1, \ldots, y_n$ in Y such that

$$f^{\nu_0}(D) \subset V + \mathscr{C}\{y_1, \ldots, y_n\} \quad ([5]\ \S\ 5 \text{ prop. } 3).$$

Hence

$$f(D) = (f-f^{\nu_0})(D) + f^{\nu_0}(D)$$
$$\subset V + V + \mathscr{C}\{y_1, \ldots, y_n\} = V + \mathscr{C}\{y_1, \ldots, y_n\},$$

which means that the set $f(D)$ is relatively c-compact in Y. (Y is complete).

PROPOSITION 5.

$$CC(X, Y) = CF(X, Y),$$

whenever Y has the c-approximation property.

PROOF. Suppose $f \in CF(X, Y)$, let $B \subset X$ be a K-convex, bounded set and let V be a zero-neighbourhood in Y.

Then $f(B)$ is K-convex, bounded and relatively c-compact in Y. Hence there is an operator $h \colon Y \to Y$ with finite rank such that

$$(\mathrm{Id}_Y - h)(f(B)) \subset V.$$

We put $g = h \circ f$.

Then the operator $g \colon X \to Y$ has finite rank and

$$(f-g)(B) = (\mathrm{Id}_Y \circ f - h \circ f)(B) = (\mathrm{Id}_Y - h)(f(B)) \subset V,$$

which means that $f \in CC(X, Y)$.

The proof is then complete by proposition 4.

PROPOSITION 6. If the space $L_\beta(X, Y)$ is complete, then

$$CC(X, Y) = X'_\beta \hat{\otimes} Y.$$

(For the definition of the tensor product topology we refer to [11]).

PROOF. We only have to prove that the topology induced by $L_\beta(X, Y)$ on $X'_\beta \otimes Y$ (which is nothing else than the space of operators with finite rank from X to Y) is exactly the tensor product topology.

Let $B \subset X$ be bounded and K-convex and let V be a zero-neighbourhood in Y. We denote by $W_{B,V}$ the corresponding zero-neighbourhood in $L_\beta(X, Y)$.

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*) By a zero-neighbourhood we always mean a K-convex, open and closed zero-neighbourhood.

For $u \in X'_\beta \otimes Y : u = \sum_{i=1}^k a_i \otimes y_i$ and the corresponding operator

$$f_u : X \to Y : f_u(x) = \sum_{i=1}^k \langle x, a_i \rangle \, y_i$$

we now obtain:

$$\begin{aligned} W_{B,V} &= \{f_u | f_u(B) \subset V\} \\ &= \{f_u | \sup_{x\in B} \sup_{a \in V^0} |\langle f_u(x), a\rangle| \leqslant 1\} \\ &= \{f_u | \sup_{x\in B} \sup_{a \in V^0} | \textstyle\sum_{i=1}^k \langle x, a_i\rangle \langle y_i, a\rangle| \leqslant 1\} \\ &= \{f_u | \varepsilon_{B,V}(u) \leqslant 1\} \ \ ([11]). \end{aligned}$$

PROPOSITION 7.

$$C(X, Y) \subset CF(X, Y).$$

PROOF. Suppose $f \in C(X, Y)$ and let $B \subset X$ be bounded and K-convex. There is in X a zero-neighbourhood U such that $f(U)$ is bounded and relatively c-compact in Y.

There is also an element $\varrho \in K$ such that $B \subset \varrho U$. Hence $f(B)$ is relatively c-compact in Y.

PROPOSITION 8.

$$N(X, Y) \subset C(X, Y).$$

PROOF. Suppose $f \in N(X, Y)$, then from the definition we have the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ X_1 \arrow[r, "\bar{f}"'] & Y_1 \arrow[u, "h"'] \end{tikzcd}$$

where $\bar{f}$ is nuclear and thus compact. ([5] §5 prop. 2). Hence f is compact.

Counterexamples

We have the following relations between the spaces of operators that are considered here:

$$\begin{array}{c} N(X, Y) \subset C(X, Y) \subset CF(X, Y) \subset L(X, Y), \\ \phantom{N(X, Y) \subset C(X, Y) \subset {}} \cup \phantom{{} \subset L(X, Y),} \\ \phantom{N(X, Y) \subset C(X, Y) \subset {}} CC(X, Y) \phantom{{} \subset L(X, Y),} \end{array}$$

while the equality $CC(X, Y) = CF(X, Y)$ depends on the c-approximation property.

The following examples prove that in the general case we cannot ask for more.

EXAMPLE 1. $C \neq N$.

Consider the product K^I where I is any infinite set and take for f the identy operator on K^I. Then f is compact.

Assume $f = g \circ h$ where $h\colon K^I \to Y_1$ and $g\colon Y_1 \to K^I$ are operators and Y_1 is a n.a. Banach space.

It is now easy to see that under these assumptions the space K^I would have a bounded zero-neighbourhood, which is impossible. (Kolmogoroff: [12] Th. 3.1).

Hence f cannot be nuclear.

EXAMPLE 2. $C \neq CF$.

Let X be reflexive and infinite dimensional and take for f the identity on X.

Then $f \in CF(X, X)$ ([3] prop. 3).

However $f \notin C(X, X)$ since X is not normable ([8] Th. 3.1).

EXAMPLE 3. $CF \neq L$.

Let X be an infinite dimensional n.a. Banach space and take for f the identity on X.

Then $f \in L(X, X)$ but $f \notin CF(X, X)$ since X is not locally c-compact ([10] prop. 3.2).

V. SPECIAL CASES

PROPOSITION 9.

a. $\mathrm{Id}_X \in CF(X, X)$ if and only if X is semi-reflexive,

b. $\mathrm{Id}_X \in CC(X, X)$ if and only if X is semi-reflexive and has the c-approximation property,

c. $\mathrm{Id}_X \in C(X, X)$ if and only if X is finite dimensional.

PROOF.

a. $\mathrm{Id}_X \in CF(X, X)$ if and only if every closed, K-convex and bounded subset of X is c-compact and this is the case if and only if X is semi-reflexive ([3] prop. 3).

b. If $\mathrm{Id}_X \in CC(X, X)$ then X is already semi-reflexive by a. and proposition 4.

Hence $L_\beta(X, X) = L_c(X, X)$, where the c stands for the topology of uniform convergence on the K-convex bounded and c-compact subsets of X.

This implies that the identity on X can be approximated in $L_c(X, X)$ by operators with finite rank.

The other half of the proof follows from a. and proposition 5.

c. If $\mathrm{Id}_X \in C(X, X)$ then X has a bounded and c-compact zero-neighbourhood. Hence X is normable and locally c-compact, which implies that X is finite dimensional.

The other part of the proof is trivial.

REMARK 1. More generally we can write for c.: X (resp. Y) is finite dimensional if and only if $C(X, Y) = L(X, Y)$ for all Y (resp. all X).

The generalizations of a. and b. are more interesting. They are discussed later.

Remark 2. Comparing the results of prop. 3 and prop. 9.a. with a result of Monna ([6] p. 472) we can conclude that the property stated in prop. 3 is not sufficient for an operator to be compactifying.

Proposition 10. If X is a n.a. Banach space then

a. $CF(X, Y) = C(X, Y)$ and
b. $C(Y, X) = N(Y, X)$

for all Y.

Proof.

a. Immediate consequence of the definitions.

b. We only have to prove that $C(Y, X) \subset N(Y, X)$ (prop. 8). Suppose $f \in C(Y, X)$ and let B be the unit ball in X.

In Y there exists a zero-neighbourhood U such that $f(U)$ is bounded and relatively c-compact in X.

We can assume without loss of generality that $f(U) \subset B$.

We consider now the normed linear space Y_U associated with Y and the zero-neighbourhood U *) and its completion $\hat{Y}_U$. It is then easy to see that we have the following factorization:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & X \\ \hat{Y}_U \arrow[ur, "h"'] & \end{tikzcd}$$

where g and h are operators.

The set $\overline{g(U)}$ is a zero-neighbourhood in the completion $\hat{Y}_U$ and

$$h(\overline{g(U)}) \subset \overline{h(g(U))} = \overline{f(U)}.$$

Therefore the set $h(\overline{g(U)})$ is relatively c-compact in X, which implies that h is a compact and thus also a nuclear operator. ([5] § 5 prop. 2). Hence f is nuclear.

Definition. The space X is said to be quasi-normable if for every K-convex, equicontinuous subset A of X' there exists in X' a K-convex, equicontinuous set $D \supset A$ such that on A the topologies induced by the strong dual X'_β and by the n.a. normed linear space X'_D *) coincide.

Every n.a. Banach space is quasi-normable.

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*) Same definition as in the archimedean theory.

PROPOSITION 11.

a. If X is quasi-normable, then

$$C(X, Y) = CF(X, Y)$$

for all n.a. Banach spaces Y.

b. If X is quasi-normable and bornological, then

$$N(Y, X'_\beta) = C(Y, X'_\beta)$$

for all n.a. Banach spaces Y. *)

PROOF. We only have to prove that $CF(X, Y) \subset C(X, Y)$ (prop. 7). Let B be the unit ball in Y.

If $f \in C(X, Y)$ and α is any element in K with $|\alpha| > 1$, then there is in X a zero-neighbourhood U such that $f(\alpha U) \subset B$. We can assume that U is of the form $U = A^0$, where A is an equicontinuous, K-convex subset of X'.

Since X is quasi-normable there exists a K-convex equicontinuous set $D \subset X'$ with the property: $\forall \lambda \in K$, $\exists W$ (zero-neighbourhood in X'_β) such that

$$A \cap W \subset \lambda D.$$

Taking the polars we have successively:

$$\begin{aligned} 1/\lambda D^0 &\subset (A \cap W)^0 \subset (A^{00} \cap W^{00})^0 = (A^0 \cup W^0)^{00} \\ &= (\mathscr{C}(A^0 \cup W^0))^{00} \subset \alpha \cdot \mathscr{C}(A^0 \cup W^0) \quad ([12] \text{ Th. } 4.15) \\ &= \alpha(A^0 + W^0) = \alpha U + \alpha W^0, \end{aligned}$$

where $V = D^0$ is a zero-neighbourhood in X and $M = \alpha W^0$ is K-convex and bounded in Y.

Since $f \in CF(X, Y)$ the set $f(M)$ is relatively c-compact in Y, which means that there exist elements $y_1, \ldots, y_n$ in Y with

$$f(M) \subset B + \mathscr{C}\{y_1, \ldots, y_n\}$$

and thus

$$f(V) \subset \lambda B + \lambda \mathscr{C}\{y_1, \ldots, y_n\}.$$

Hence $f(V)$ is bounded and relatively c-compact in Y, which means that f is a compact operator.

b. We only have to prove that

$$C(Y, X'_\beta) \subset N(Y, X'_\beta) \text{ (prop. 8).}$$

Let B be the unit ball in Y and take $f \in C(Y, X'_\beta)$.

If $\overline{f(B)}$ stands for the closure of $f(B)$ in X'_β, then $\overline{f(B)}$ is bounded and c-compact in X'_β.

*) X'_β is complete since X is bornological.

Hence $\overline{f(B)}$ is equicontinuous in X'.

Since X is quasi-normable there is a K-convex equicontinuous subset $D \supset \overline{f(B)}$ in X' such that $f(B)$ is still c-compact in X'_D.

Hence $f(B)$ is relatively c-compact in X'_D.

Now it is easy to see that we have a factorization

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "g"'] & X'_\beta \\ & X'_D \arrow[u, "h"'] \end{tikzcd}$$

where h and g are operators (h is the canonical injection).

The mapping h can be extended continuously to the completion $\hat{X}'_D$. Let $\hat{h}$ be that extended mapping.

Then we still have that $f = \hat{h} \circ g$, where $g(B) = f(B)$ is relatively c-compact in $\hat{X}'_D$.

The mapping g is thus a compact mapping between the n.a. Banach spaces Y and $\hat{X}'_D$. Hence g is nuclear ([5] § 5 prop. 2) and so is f.

LEMMA. Suppose X is metrizable. Then for every K-convex bounded subset A of X there exists a K-convex bounded subset $B \supset A$ such that on A the topologies induced by X and by X_B coincide.

PROOF. The proof is the same as in the archimedian case. We only give a sketch of it.

Let V_n be a fundamental sequence of zero-neighbourhoods in X. Then there exists a sequence (λ_i) in K such that $A \subset \bigcup_i \lambda_i V_i$.

Consider then in K a sequence (μ_i) such that $\lim_i \lambda_i/\mu_i = 0$ and take $B = \bigcap_i \mu_i V_i$.

Then for every $\lambda \in K$ there is an index i_λ such that

$$A \subset \lambda \mu_i V_i \qquad \text{for } i > i_\lambda.$$

If now n is chosen in such a way that $V_n \subset \bigcap_{i=1,\ldots,i_\lambda} \lambda_i \mu_i V_i$, then $A \cap V_n \subset \lambda B$ which proves the lemma.

PROPOSITION 12. If X is metrizable, then

a. $N(Y, X) = C(Y, X)$ and
b. $C(X'_\beta, Y) = CF(X'_\beta, Y)$

for all n.a. Banach spaces Y.

PROOF. Applying the lemma, this proof is analogous to that of proposition 11 and therefore is omitted.

PROPOSITION 13. The following statements are equivalent:

a. X is semi-reflexive
b. $CF(X, Y)=L(X, Y)$ for all Y
c. $CF(Y, X)=L(Y, X)$ for all Y
d. $CF(X, Y)=L(X, Y)$ for all n.a. Banach spaces Y
e. $CF(Y, X)=L(Y, X)$ for all n.a. Banach spaces Y
f. $C(Y, X)=L(Y, X)$ for all n.a. Banach spaces Y.

PROOF.

a $\Rightarrow$ b: Suppose $f \in L(X, Y)$ and let $B \subset X$ be bounded and K-convex. Then B is relatively c-compact. ([3] prop. 3) and therefore the image $f(B)$ is relatively c-compact in Y ([10] prop. 1.14).

b $\Rightarrow$ a: Take $Y=X$ and apply proposition 9.a.

a $\Rightarrow$ c: Suppose $f \in L(Y, X)$ and let $B \subset Y$ be bounded and K-convex. Then $f(B)$ is bounded and K-convex in X and thus relatively c-compact in X by a ([3] prop. 3).

Hence $f \in CF(Y, X)$.

c $\Rightarrow$ a: Take $X=Y$ and apply proposition 9.a.

b $\Rightarrow$ d: Trivial.

c $\Rightarrow$ e: Trivial.

d $\Rightarrow$ a: It is easy to see that every complete locally K-convex space X is a closed linear subspace of the product

$$\Pi_{U \in \mathscr{U}} \hat{X}_U$$

where $\mathscr{U}$ stands for a fundamental system of zero-neighbourhoods in X. By d. every projection $\varphi_U: X \to \hat{X}_U$ is an element of $CF(X, \hat{X}_U)$.

Let now D be a K-convex and bounded subset of X.

Then $\overline{\varphi_U(D)}$ (where the closure is taken in $\hat{X}_U$) is c-compact in $\hat{X}_U$.

The product $\Pi_{U \in \mathscr{U}} \overline{\varphi_U(D)}$ is still c-compact ([10] prop. 1.17).

Hence the set D, as a K-convex subset of this product is still relatively c-compact.

This implies that X is semi-reflexive.

e $\Rightarrow$ a: Let $B \subset X$ be closed, K-convex and bounded in X and consider the space X_B (X_B is complete).

Let f be the canonical injection $f: X_B \to X$.

Then $f \in CF(X_B, X)$ by e.

Hence the set $B=f(B)$ is c-compact in X. This means that X is semi-reflexive.

e $\Leftrightarrow$ f: Proposition 10.a.

PROPOSITION 14. The following statements are equivalent:

a. X is semi-reflexive and quasi-normable
b. $L(X, Y)=C(X, Y)$ for all n.a. Banach spaces Y
c. $L(X, Y)=N(X, Y)$ for all n.a. Banach spaces Y
d. For every zero-neighbourhood U in X there exists a zero-neighbourhood $V \subset U$ in X such that the canonical mapping $\hat{X}_V \to \hat{X}_U$ is compact.

PROOF.

a $\Rightarrow$ b: Propositions 11.a. and 13.d.

b $\Leftrightarrow$ c: Proposition 10.b.

b $\Rightarrow$ d: The proof is the same as in the archimedian theory and is therefore omitted.

d $\Rightarrow$ a: Suppose $f \in L(X, Y)$ where Y is any n.a. Banach space with unit ball B.

In X there exists a zero-neighbourhood U such that $f(U) \subset B$. We then consider the zero-neighbourhood $V \subset U$ such that the canonical mapping $\varphi: \hat{X}_V \to \hat{X}_U$ is compact.

We now obtain a factorization

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ \hat{X}_V \arrow[r, "\varphi"'] & \hat{X}_U \arrow[u, "h"'] \end{tikzcd}$$

where g and h are operators.

Let now D be a K-convex and bounded subset of X, then

$$f(B) = h \circ \varphi \circ g(D),$$

where $\varphi(g(D))$ is relatively c-compact in $\hat{X}_U$.

Hence $f(D)$ is relatively c-compact in Y which means that $f \in CF(X, Y)$.

The space X is then semi-reflexive by proposition 13.d. In order to prove that X is quasi-normable we take a K-convex equicontinuous subset A of X' (A can be supposed to have the form $A = U^0$ where U is a zero-neighbourhood in X).

Then we choose V such that the canonical mapping $\varphi: \hat{X}_V \to \hat{X}_U$ is compact.

The transposed mapping ${}^\tau\varphi: X'_A \to X'_{V^0}$ is still compact ([9] prop. 14).

This means that $A = {}^\tau\varphi(A)$ is K-convex, bounded and c-compact in X'_{V^0}.

As V^0 is an equicontinuous subset of X', the topology induced on A by X'_β is weaker than that induced by X'_{V^0}. Hence on A both topologies coincide ([5] § 5 prop. 4).

DEFINITION. A space X satisfying the equivalent conditions of proposition 14 will be called a Schwartz space.

Immediately from proposition 14 it can be deduced that there are no infinite dimensional n.a. Banach spaces which are Schwartz spaces.

Stability properties of Schwartz spaces can be deduced from proposition 14.c. just as in the archimedian theory.

PROPOSITION 15. Consider the following statements:

a. X is metrizable and semi-reflexive

b. $L(Y, X) = N(Y, X)$ for all n.a. Banach spaces Y

c. For every K-convex, closed and bounded subset A of Y there exists a K-convex, bounded subset $B \subset A$ such that Y_B is complete and that the canonical mapping φ: $Y_A \to Y_B$ is compact.

d. X'_β is a Schwartz space.

Then we have

i) $a \Rightarrow b$
ii) $b \Leftrightarrow c$
iii) $a + b \Rightarrow d$.

PROOF.

$a \Rightarrow b$: Propositions 12.a. and 13.f.

$b \Rightarrow c$: If A is choosen as indicated in b., then the injection $I: X_A \to X$ is nuclear by b.

Hence there exist n.a. Banach spaces X_1 and X_2 and a factorization

$$\begin{tikzcd} X_A \arrow[r, "I"] & X \\ X_1 \arrow[u, "\alpha"] \arrow[r, "\beta"'] & X_2 \arrow[u, "\gamma"'] \end{tikzcd}$$

where α, γ are operators an β is nuclear.

Let S be the unit ball in X_2.

We then consider the space $X_{\gamma(S)}$.

The mapping $\gamma: X_2 \to X_{\gamma(S)}$ is continuous and onto. Hence it is open ([12] Th. 3.19) and $X_{\gamma(S)}$ is complete as a Hausdorff quotient space of X_2.

It is now easy to see that we have an other factorization

$$\begin{tikzcd} X & \\ X_2 \arrow[u, "\gamma"] \arrow[r, "\sigma"'] & X_{\gamma(S)} \arrow[ul, "\tau"'] \end{tikzcd}$$

The mapping

$$\varphi = \sigma \circ \beta \circ \alpha : X_A \to X_{\gamma(S)}$$

is injective since I and τ are injective.

Hence φ is the canonical injection and φ is compact since β is.

$c \Rightarrow b$: Suppose $f \in L(Y, X)$ and let S be the unit ball in Y. The set $A = \overline{f(S)}$ is K-convex, closed and bounded in X.

We then take B as given by c.

Since we have now a factorization

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr] & X & \\ & X_{\overline{f(S)}} \arrow[u, dash] \arrow[r, "g"] & X_B \arrow[ul] \end{tikzcd}$$

where all the arrows stand for operators and g is compact (or nuclear), the mapping f is also nuclear.

a+b $\Rightarrow$ d: We remark first that under the condition a. the space X is reflexive as well as its strong dual X'_β.

Then the proof is given by propositions 12.b. and 13.d.

COROLLARY. The strong dual of a metrizable Schwartz space is a Schwartz space.

VI. CONNECTION WITH GENERALIZED SEQUENCE SPACES

Suppose Λ is a n.a. perfect sequence space (For the theory of n.a. perfect sequence spaces we refer to [4]).

On Λ we consider the natural topology, which is determined by the n.a. seminorms:

$$p_\beta(\alpha) = \max_n |\alpha_n \beta_n|, \ \alpha = (\alpha_n) \in \Lambda, \ \beta = (\beta_n) \in \Lambda^*.$$

Let the topology of the space X be determined by the family $\mathscr{P}$ of n.a. semi-norms.

The generalized sequence space $\Lambda(X)$ is then defined by

$$\Lambda(X) = \{(x_n) | x_n \in X \text{ and } \forall \beta \in \Lambda^* : \lim_{n\to\infty} \beta_n x_n = 0 \text{ in } X\}.$$

On $\Lambda(X)$ we define a locally K-convex topology by the family of n.a. semi-norms

$$\lambda_{\beta,q}((x_n)) = \max_n (|\beta_n| \cdot q(x_n)), \ \beta \in \Lambda^*, \ q \in \mathscr{P}.$$

REMARK. In the case $\Lambda = c_0$ and X is a n.a. Banach space then $c_0(X)$ is the space of the sequences in X converging to zero, while the topology on $c_0(X)$ is determined by the n.a. norm

$$\|(x_n)\| = \max_n \|x_n\|.$$

It was proved by SERRE in [9] that in this case

$$c_0(X') = C(X, c_0).$$

Hence, since X and c_0 are n.a. Banach spaces, we also have

$$c_0(X') = C(X, c_0) = CC(X, c_0) = CF(X, c_0) = X' \hat{\otimes} c_0$$

(see [9] and [5]).

In the following propositions some of these properties will be generalized.

PROPOSITION 16. The space $\Lambda(X)$ is complete.

PROOF. Let $\{(x_n^\nu)\}$ be a Cauchy-net in $\Lambda(X)$. I.e.

$$(*) \qquad \forall \varepsilon > 0, \ \forall \beta \in \Lambda^*, \ \forall q \in \mathscr{P}, \ \exists \nu_0$$

such that

$$\max_n |\beta_n| q(x_n^\nu - x_n^\mu) \leqslant \varepsilon \text{ for all } \nu, \mu \geqslant \nu_0.$$

Hence, for every n (n fixed), $\{x_n^\nu\}$ is a Cauchy net in X and thus converges to an element $x_n \in X$.

Then from (*) we obtain:

$$(**) \qquad \max_n |\beta_n| q(x_n^\nu - x_n) \leqslant \varepsilon \text{ for all } \nu \geqslant \nu_0.$$

Further we have that

$$q(\beta_n x_n) \leqslant \max\,(|\beta_n| q(x_n^{\nu_0}),\ |\beta_n| q(x_n^{\nu_0} - x_n)).$$

Hence by (**):

$$\forall n: q(\beta_n x_n) \leqslant \max\,(|\beta_n| q(x_n^{\nu_0}),\ \varepsilon),$$

which means that $(x_n) \in \Lambda(X)$.

Finally (**) proves that $(x_n) = \lim_\nu (x_n^\nu)$ in $\Lambda(X)$.

PROPOSITION 17. Every element of $\Lambda(X)$ is the limit of its sections.

PROOF. If $(x_n) \in \Lambda(E)$, then

$$\lambda_{\beta,q}((0, 0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots)) = \max_{n>k} |\beta_n| q(x_n).$$

But

$$\lim_{n\to\infty} |\beta_n| q(x_n) = 0.$$

Hence

$$\lim_{k\to\infty} \lambda_{\beta,q}((0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots)) = 0,$$

or

$$(x_n) = \lim_{k\to\infty} ((x_1, x_2, \ldots, x_k, 0, 0, \ldots)) \text{ in } \Lambda(X).$$

PROPOSITION 18. $\Lambda(X) \cong \Lambda \hat{\otimes} X$.

PROOF. We consider the linear mapping

$$\varphi: \Lambda \otimes X \to \Lambda(X):$$
$$u = \textstyle\sum_{i=1}^k \alpha^i \otimes x_i \to (\sum_{i=1}^k \alpha_n^i \cdot x_i)_n.$$

As in the archimedian theory ([2] prop. 5.1) it can be proved that φ is an bijection from $\Lambda \otimes X$ onto a dense subspace of $\Lambda(X)$ (Because of prop. 17).

Hence, since $\Lambda(X)$ is complete (prop. 16) it is left to prove that the space $\Lambda(X)$ induces on $\Lambda \otimes X$ exactly the tensor product topology.

This tensor product topology can be defined by n.a. semi-norms $\pi_{\beta,q}$ ($\beta \in \Lambda^*$, $q \in \mathscr{P}$), as follows:

$$\text{If } u = \textstyle\sum_{i=1}^k \alpha^i \otimes x_i \text{ then } \pi_{\beta,q}(u) = \inf\,(\max_i p_\beta(\alpha^i) \cdot q(x_i)).$$

Where the inf is taken over all the representations of u.

We also can define, if U denotes the semi-ball corresponding to the n.a. semi-norm q,

$$\varepsilon_{\beta,q}(u) = \sup_{\gamma\in\{\beta\}^\wedge,\, a\in U^\circ} |\textstyle\sum_{i=1}^k \langle \alpha^i, \gamma\rangle \langle x_i, a\rangle|.$$

It is proved in [11] that $\varepsilon_{\beta,q} = \pi_{\beta,q}$.

In order to prove the proposition we shall prove that for all $u \in \Lambda \otimes X$ we have

$$\varepsilon_{\beta,q}(u) \leqslant \lambda_{\beta,q}(\varphi(u)) \leqslant \pi_{\beta,q}(u).$$

If $u = \sum_{i=1}^{k} \alpha^i \otimes x_i$, then

$$\begin{aligned}\lambda_{\beta,q}(\varphi(u)) &= \max_n |\beta_n| q(\varphi(u)) \\ &= \max_n (|\beta_n| \cdot q(\textstyle\sum_{i=1}^{k} \alpha_n{}^i x_i)) \\ &\leqslant \max_n (|\beta_n| \cdot \max_i (|\alpha_n{}^i| q(x_i))) \\ &= \max_i (q(x_i) \cdot \max_n |\alpha_n{}^i|\, |\beta_n|) \\ &= \max_i q(x_i) p_\beta(\alpha^i).\end{aligned}$$

Hence, since the representation of u was arbitrary,

$$\lambda_{\beta,q}(\varphi(u)) \leqslant \pi_{\beta,q}(u).$$

On the other hand we have:

$$\begin{aligned}\varepsilon_{\beta,q}(u) &= \sup_{\gamma \in \{\beta\}^\wedge,\, a \in U^0} |\textstyle\sum_{i=1}^{k} \langle \alpha^i, \gamma\rangle \langle x_i, a\rangle| \\ &= \sup_{\gamma \in \{\beta\}^\wedge,\, a \in U^0} |\textstyle\sum_{i=1}^{k} (\sum_{n=1}^{\infty} \alpha_n{}^i \gamma_n) \langle x_i, a\rangle| \\ &= \sup_{\gamma \in \{\beta\}^\wedge,\, a \in U^0} |\textstyle\sum_{n=1}^{\infty} \gamma_n \sum_{i=1}^{k} \alpha_n{}^i \langle x_i, a\rangle| \\ &\leqslant \sup_{\gamma \in \{\beta\}^\wedge,\, a \in U^0} \max_n (|\gamma_n|\, |\textstyle\sum_{i=1}^{k} \alpha_n{}^i \langle x_i, a\rangle|) \\ &\leqslant \sup_{a \in U^0} \max_n (|\beta_n|\, |\textstyle\sum_{i=1}^{k} \alpha_n{}^i \langle x_i, a\rangle|) \\ &= \max_n |\beta_n| \cdot \sup_{a \in U^0} |\textstyle\sum_{i=1}^{k} \alpha_n{}^i \langle x_i, a\rangle| \\ &= \max_n |\beta_n| \cdot q(\textstyle\sum_{i=1}^{k} \alpha_n{}^i \langle x_i, a\rangle| \\ &= \lambda_{\beta,q}(\varphi(u)).\end{aligned}$$

PROPOSITION 19. If X is a bornological space, then

$$\Lambda(X'_\beta) = CF(X, \Lambda) = CC(X, \Lambda).$$

PROOF.

a. Suppose $f \in CF(X, \Lambda)$.

Then f is continuous and thus is the mapping

$$X \xrightarrow{f} \Lambda \xrightarrow{P_h} K : x \to f(x) \to (f(x))_h$$

a continuous linear form on X.

Hence there exists an element a_h in X' such that

$$P_h \circ f(x) = \langle x, a_h\rangle$$

and thus f can be written as

$$f(x) = (\langle x, a_n\rangle)_n.$$

We now prove that the sequence (a_n) is an element of $\Lambda(X'_\beta)$.

Let B be a K-convex, bounded subset of X and let p_B denote the corresponding n.a. semi-norm on X'_β. We take further $\delta \in \Lambda^*$ and prove that

$$\lim_{n\to\infty} |\delta_n| p_B(a_n) = 0.$$

Since $f(B)$ is K-convex, bounded and relatively c-compact in Λ, there exists in Λ an element γ such that

$$f(B) \subset \{\gamma\}^\wedge \quad ([4] \text{ prop. } 15).$$

I.e.

$$\forall n: p_B(a_n) = \sup_{x\in B} |\langle x, a_n\rangle| \leqslant |\gamma_n|.$$

But

$$\lim_{n\to\infty} \gamma_n\delta_n = 0, \text{ since } \gamma \in \Lambda, \ \delta \in \Lambda^*.$$

Hence

$$\lim_{n\to\infty} p_B(a_n)|\delta_n| = 0.$$

b. Suppose $f: X \to \Lambda$ has the form

$$f(x) = (\langle x, a_n\rangle)_n \text{ with } (a_n) \in \Lambda(X'_\beta).$$

Since X is bornological it is sufficient to prove that the image $f(B)$ of any bounded, K-convex subset B of X is bounded and relatively c-compact in Λ (f is then automatically continuous).

In order to do so we have to prove the existence of an element $\gamma \in \Lambda$ such that $f(D) \subset \{\gamma\}^\wedge$ ([4] prop. 15).

We know that ([12] p. 251) there exists a real number $\varrho > 1$ such that for every integer n we can find in K an element γ_n with $|\gamma_n| = \varrho^n$.

Now $p_D(a_i)$ is a real number and thus there exists n_i such that

$$\varrho^{n_i-1} \leqslant p_D(a_i) < \varrho^{n_i}.$$

We take $\gamma_i \in K$ such that $|\gamma_i| = \varrho^{n_i}$.

We then obtain for every $\beta \in \Lambda^*$:

$$\begin{aligned}
\lim_{i\to\infty} |\beta_i\gamma_i| &= \lim_{i\to\infty} (|\beta_i|(|\gamma_i| - p_D(a_i)) + |\beta_i| p_D(a_i)) \\
&\leqslant \lim_{i\to\infty} |\beta_i|(\varrho^{n_i} - \varrho^{n_i-1}) + 0 \\
&= (\varrho - 1) \lim_{i\to\infty} |\beta_i|\varrho^{n_i-1} \\
&\leqslant (\varrho - 1) \lim_{i\to\infty} |\beta_i| \cdot p_D(a_i) = 0.
\end{aligned}$$

Hence $\gamma = (\gamma_i) \in \Lambda$ and obviously $f(D) \subset \{\gamma\}^\wedge$.

c. $CF(X, \Lambda) = CC(X, \Lambda)$ by propositions 1 and 5.

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REFERENCES

1. Bourbaki, N., Espaces vectoriels topologiques. Hermann, 1955.
2. De Grande-De Kimpe, N., Generalized Sequence Spaces. To appear in Bull. Soc. Math. Belg.
3. ———, C-compactness in locally K-convex Spaces. Proc. Kon. Ned. Akad. v. Wet., A **74**, 176–180 (1971).
4. ———, Perfect locally K-convex Sequence Spaces. Proc. Kon. Ned. Akad. v. Wet. A **74**, 471–482 (1971).
5. Gruson, L., Théorie de Fredholm p-adique. Bull. Soc. Math. France, **94**, 67–95 (1966).
6. Monna, A. F., Sur les espaces normés non-archimédiens IV. Proc. Kon. Ned. Akad. v. Wet. A **60**, 468–476 (1957).
7. ———, Espaces localement convexes sur un corps valué. Proc. Kon. Ned. Akad. v. Wet., A **62**, 391–405 (1959).
8. ——— and T. A. Springer, Sur la structure des espaces de Banach non-archimédiens. Proc. Kon. Ned. Akad. v. Wet., A **68**, 602–614 (1965).
9. Serre, J-P.,Endomorphismes complètement continus des espaces de Banach p-adiques. Publ. Math. I.H.E.S., **12**, 1962.
10. Springer, T. A., Une notion de compacité dans la théorie des espaces vectoriels topologiques. Proc. Kon. Ned. Akad. v. Wet., A **68**, 182–189 (1965).
11. Put, M. van der and J. van Tiel, Espaces nucléaires non-archimédiens. Proc. Kon. Ned. Akad. v. Wet., A **70**, 556–561 (1965).
12. Tiel, J. van, Espaces localement K-convexes. Proc. Kon. Ned. Akad. v. Wet., A **68**, 249–289 (1965).