MATHEMATICS

A REMARK ON THE MILNOR RING

BY

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1. Let F be a field. We denote its multiplicative group by $F^\times$. The Milnor ring of F is the graded associative ring with unit K_*F which is generated by symbols $l(a)$ ($a \in F^\times$) of degree 1, with relations $l(ab)=l(a)+$ $+l(b)$, $l(a)l(1-a)=0$. We have $K_0F \simeq \mathbb{Z}$, $K_1F \simeq F^\times$ (see [3] for details).

If F has a discrete valuation, with residue field $\bar{F}$, then Milnor has defined an additive homomorphism $K_*F \to K_*\bar{F}$, which can be used to obtain information about K_*F (loc. cit., section 2). In the present note we shall show that if F has an arbitrary valuation, one can construct a homomorphism of K_*F onto a ring of a simple nature and we shall give some applications of that homomorphism.

2. Let v be a valuation of F with value group Γ. Then Γ is a totally ordered abelian group and v is a surjective homomorphism of $F^\times$ onto Γ such that $v(a+b) \geqslant \mathrm{Min}\,(v(a), v(b))$. One knows that equality holds here if $v(a) \neq v(b)$. The rank of v is the dimension over $\mathbb{Q}$ of the vector space $\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $E=F(t_\alpha)_{\alpha \in I}$ be a purely transcendental extension of F, the t_α being algebraically independent over F. We fix a well-ordering of the index set I. Let A be the free abelian group with base I. The well-ordering of I defines a total lexicographical order on A.

If $a=\sum\limits_{\alpha \in I} n_\alpha \cdot \alpha \in A$, $n_\alpha \geqslant 0$, let $m_a=\prod\limits_{\alpha \in I} t_\alpha^{n_\alpha}$. This is a monomial in $F[t_\alpha]_{\alpha \in I}$. The elements of $F[t_\alpha]_{\alpha \in I}$ are the finite linear combinations $\sum\limits_{a \in A} c_a m_a$ (with $c_a \in F$). We now have the following well-known result, the proof of which is omitted. (See [2, p. 160–161] for the case of transcendence degree 1).

Lemma 1. *There exists a unique valuation w of E with the following properties*

(a) *its value group is $A \oplus \Gamma$, ordered lexicographically,*

(b) *if $f=\sum\limits_{a \in A} c_a m_a \in F[t_\alpha]$ and if $b \in A$ is the maximal element of A with $c_b \neq 0$, then $w(f)=(b, v(c_b))$.*

3. We denote by $\Lambda^*\Gamma$ the exterior algebra of Γ (over $\mathbb{Z}$), then $\Lambda^0\Gamma \simeq \mathbb{Z}$, $\Lambda^1\Gamma \simeq \Gamma$.

PROPOSITION 1. *Let v be a valuation of F with value group Γ. There exists a unique surjective homomorphism of graded algebras $\alpha\colon K_*F \to \Lambda^*\Gamma$ such that $\alpha(l(a)) = v(a)$.*

The uniqueness of α is clear, since the $l(a)$ generate K_*F. The existence will follow if we prove that $v(a) \wedge v(1-a) = 0$. Now if $v(a) \geqslant 0$, we have $v(a) = 0$ or $v(1-a) = 0$, while if $v(a) < 0$ we have $v(1-a) = v(a)$. In either case $v(a) \wedge v(1-a) = 0$.

COROLLARY 1. *Assume that the rank of v is at least $r \geqslant 2$. Then K_rF is not a torsion group.*

α induces a surjective homomorphism of $K_rF \otimes \mathbf{Q}$ onto $\Lambda^r\Gamma \otimes \mathbf{Q} \simeq \Lambda^r(\Gamma \otimes \mathbf{Q})$ which is a nonzero vector space over $\mathbf{Q}$, whence the assertion.

COROLLARY 2. *Assume that v has uncountable rank. Then K_rF is uncountable for $r \geqslant 2$.*

This follows by a similar argument.

In the applications we use the integer $\delta(F)$ which is defined in the following way. Let F_0 be the prime field of F. Then $\delta(F)$ is the transcendence degree of F/F_0 if char $F = p > 0$ and this transcendence degree plus 1 if char $F = 0$.

LEMMA 2. *If $\delta(F) \geqslant r$ there exists a valuation of F with rank $\geqslant r$.*

Let $E = F_0(t_1, \ldots, t_s)$ be a purely transcendental extension of F_0 contained in F, the t_i being algebraically independent. Using Lemma 1 one constructs a valuation of E of rank s (if char $F > 0$) or of rank $s+1$ (if char $F = 0$). Lemma 2 then follows by using the standard result about extending valuations (see [2, p. 89]).

PROPOSITION 2. *Assume that K_nF is a torsion group for some $n \geqslant 2$. Then $\delta(F) < n$.*

This is a direct consequence of Cor. 1 of Prop. 1 and Lemma 2.

COROLLARY. *Let F be an algebraically closed field such that $K_2F = 0$. Then F is isomorphic to the algebraic closure of one of the three following fields: a finite field, $\mathbf{Q}$, a field of rational functions in 1 indeterminate over a finite field.*

REMARK. By a result of Tate we have that K_nF is a vector space over $\mathbf{Q}$ if F is an algebraically closed field and $n \geqslant 2$.

It would be interesting to know if the converse of Prop. 2 is true, viz.: $\delta(F) < n$ implies that K_nF is a torsion group. If F is a global field this is indeed the case, by recent results of Bass, Garland and Tate (reported on in [1]). But this converse would decidedly be a more profound statement than the fairly trivial Prop. 2.

Proposition 3. *Let F be an uncountable field. Then K_nF is uncountable for all $n \geqslant 2$.*

Choose an uncountable transcendence basis $(t_\alpha)_{\alpha\in I}$ of F over F_0. By Lemma 1 there exists a valuation of $F_0(t_\alpha)$ of uncountable rank, which can be extended to a valuation of F with the same property. The assertion is now a consequence of Cor. 2 of Prop. 1.

Prop. 3 for $n=2$ was proved in [4, Theorem 11.9] by another method.

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REFERENCES

1. Bass, H., K_2 des corps globaux, Sém. Bourbaki, 1970/71, no. 394 (Juin 1971).
2. Bourbaki, N., Eléments de mathématique, Fasc. **XXX**, Algèbre Commutative, Chap. 5, 6, Paris, Hermann 1964.
3. Milnor, J., Algebraic K-Theory and Quadratic Forms, Inv. Math. **9**, 318–344, (1970).
4. ———, Notes on Algebraic K-theory, to appear.