MATHEMATICS

THE PROPERTY P_1 FOR SEMISIMPLE $\mathfrak{p}$-ADIC GROUPS

BY

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SUMMARY

We discuss a property, called P_1, for a special class of locally compact groups. The following theorem is proved. Let G be the group of rational points of a connected, semisimple, linear algebraic group, defined over a local field [2]). Then G does not have the property P_1 unless G is compact.

Let G be a locally compact group. Denote by $\mathscr{K}(G)$ the space of complex-valued continuous functions on G with compact support. Fix a left Haar measure dx on G and let $L^1(G)$ be the space of complex-valued integrable functions on G with norm $\|f\|_1=\int_G |f(x)|dx$ $(f \in L^1(G))$. For any function g on G an $y \in G$, put $(L_y f)(x)=f(y^{-1}x)$ $(x \in G)$. G is said to have the property P_1 if it satisfies the following condition: given any compact set $C \subset G$, there is for every $\varepsilon>0$ a positive function $s(=s_{C,\varepsilon})$ in $L^1(G)$ such that $\|s\|_1=1$ and $\|L_y s-s\|_1<\varepsilon$ for all $y \in C$ (cf. [4], Ch. 8,3). It is known (cf. [4], Ch. 8,6) that the property P_1 is equivalent to the, perhaps more familiar, property $(\mathscr{M})$, which assures the existence of a left invariant mean for the space of bounded complex-valued continuous functions on G. The purpose of this note is to extend the results of ([4], Ch. 8, 7.5) to semisimple $\mathfrak{p}$-adic groups (see below). First we prove a generalization of the lemma in ([4], Ch. 8, 7.3).

LEMMA 1. *Let G be a locally compact group satisfying the following conditions.*

(a) *G contains a compact subgroup K and a closed subgroup H such that $G=KH$.*
(b) *The map $KH \to G$ given by $(k, h) \mapsto kh$ $(k \in K,\ h \in H)$ is open.* [3])
(c) *The group G is unimodular, but the subgroup H is not.*

Then there is a constant $c_0>0$ and an element $h_0 \in G$ with the following

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[2]) For archimedian local fields, the theorem is equivalent to a result of O. Takenouchi, mentioned in ([4], Ch 8, 7.5).

[3]) In case G is separable, (b) is automatically fulfilled (category argument).

property: for all functions $f \in \mathscr{K}(G)$ *such that* $\int_G f(x)dx = 1$ *and* $f(kx) = f(x)$ *for all* $k \in K$ *and* $x \in G$, *the inequality*

$$\|L_{h_0} f - f\|_1 \geqslant c_0$$

holds.

Proof. Let dk, dh denote left Haar measures on K, H respectively. There exists a G-invariant measure $d\dot{x}$ on G/K such that

$$\int_G f(x)dx = \int_{G/K} \{\int_K f(xk)\,dk\}\,d\dot{x} \quad (f \in \mathscr{K}(G)).$$

By (b), the canonical map $H/H \cap K \to G/K$ is a homeomorphism. So, after transport, the G-invariant measure on G/K yields a measure μ on $H/H \cap K$ which is at least H-invariant. Normalize the Haar measure on $H \cap K$ such that the volume of $H \cap K$ is one. Then we have for every function $g \in \mathscr{K}(H)$ such that $g(hk) = g(h)$ $(h \in H,\ k \in K \cap H)$,

$$\int_H g(h)\,dh = \int_{H/H \cap K} g(\dot{h})\,d\mu(\dot{h})$$

provided dh is suitably normalized. If we take

$$g(h) = \int_K f(hk)\,dk \quad (h \in H),$$

we therefore obtain the formula:

$$\int_G f(x)\,dx = \int_H \{\int_K f(hk)\,ak\}\,dh \quad (f \in \mathscr{K}(G)).$$

By passing to the opposite group, we also have (since G and K are unimodular):

$$\int_G f(x)\,dx = \int_K \{\int_H f(kh)\,\Delta_H(h^{-1})\,dh\}\,dk \quad (f \in \mathscr{K}(G)),$$

where Δ_H denotes the Haar modulus of H.

Choose any $h_0 \in H$ such that $\Delta_H(h_0^{-1}) \neq 1$. Given $k \in K$, let $h_1 \in H$ be such that $k_1 h_1 = h_0^{-1} k$ $(k_1 \in K)$. Then $h_1 = h_1(k)$ is uniquely determined modulo $H \cap K$. Clearly $\Delta_H(h) = 1$ for $h \in H \cap K$.

Consider any $f \in \mathscr{K}(G)$ such that $f(k_1 x) = f(x)$ for all $k_1 \in K$, $x \in G$. Then

$$\|L_{h_0} f - f\|_1 \geqslant \int_K |\int_H \{f(h_0^{-1} kh) - f(kh)\}\,\Delta_H(h^{-1})\,dh|\,dk$$

and

$$\int_H \{f(h_0^{-1} kh) - f(kh)\}\,\Delta_H(h^{-1})\,dh = \{\Delta_H(h_1(k)) - 1\} \int_H f(h)\,\Delta_H(h^{-1})\,dh.$$

The left-hand-side of the latter equality is obviously a continuous function of $k \in K$. Hence, by choosing f suitably, the same is true for $\Delta_H(h_1(k)) - 1$. Clearly $|\Delta_H(h_0^{-1}) - 1| > 0$; moreover

$$\int_H f(h)\,\Delta_H(h^{-1})\,dh = \int_G f(x)\,dx$$

(we may assume $\int_K dk = 1$). Thus the lemma holds with h_0 as above and with

$$c_0 = \int_K |\Delta_H(h_1(k)) - 1| dk > 0.$$

By ([4], Ch. 8, 7.4) we may state: if a locally compact group G satisfies the conditions (a, b, c) of Lemma 1, then G does not have the property P_1.

Next we recall some standard facts about algebraic groups (cf. [1]). Let Ω be a field. By an Ω-group we mean a linear algebraic group defined over Ω. Let $\mathbf{G}$ be a connected, semisimple Ω-group. By a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ we mean an algebraic subgroup which contains a Borel subgroup of $\mathbf{G}$. We say that $\mathbf{P}$ is Ω-parabolic if it is parabolic and defined over Ω. Let $\mathbf{N}$ denote the unipotent radical of $\mathbf{P}$. Then $\mathbf{N}$ is an Ω-subgroup and $\mathbf{P}$ is the normalizer of $\mathbf{N}$ in $\mathbf{G}$. By a Levi Ω-subgroup $\mathbf{M}$ of $\mathbf{P}$ we mean a reductive Ω-group such that the mapping $(m, n) \mapsto mn$ $(m \in \mathbf{M},\ n \in \mathbf{N})$ defines an Ω-isomorphism of the algebraic varieties $\mathbf{M} \times \mathbf{N}$ and $\mathbf{P}$. Such a subgroup $\mathbf{M}$ always exists and is connected. Fix $\mathbf{M}$ and let $\mathbf{A}$ be a maximal Ω-split torus lying in the centre of $\mathbf{M}$. Then $\mathbf{A}$ is unique and $\mathbf{M}$ is the centralizer of $\mathbf{A}$ in $\mathbf{G}$. We call $\mathbf{A}$ a split component of $\mathbf{P}$. Let N be the group of Ω-rational points of $\mathbf{N}$. For any split component $\mathbf{A}'$ of $\mathbf{P}$ there exists a unique element $n \in N$ such that $\mathbf{A}' = \mathbf{A}^n = n\mathbf{A}n^{-1}$. Let us denote by $\underline{\mathfrak{g}}$, $\underline{\mathfrak{p}}$, $\underline{\mathfrak{m}}$, $\underline{\mathfrak{n}}$, $\underline{\mathfrak{a}}$ the Lie algebras of $\mathbf{G}$, $\mathbf{P}$, $\mathbf{M}$, $\mathbf{N}$, $\mathbf{A}$ respectively. They are defined over Ω.

Now let Ω be a $\mathfrak{p}$-adic field, i.e. a locally compact field with a non-trivial discrete valuation. We shall assume that the absolute value is given by $d(ay) = |a| dy$ $(a \in \Omega)$, $|0| = 0$, where dy denotes a Haar measure on the additive group of Ω. Let G denote the group of all Ω-rational points of $\mathbf{G}$. By a parabolic subgroup P of G, we mean a subgroup of the form $P = G \cap \mathbf{P}$, where $\mathbf{P}$ is an Ω-parabolic subgroup of $\mathbf{G}$. P determines $\mathbf{P}$ completely. By a split component A of P, we mean a subgroup of the form $A = G \cap \mathbf{A}$, where $\mathbf{A}$ is a split component of $\mathbf{P}$. $\mathbf{A}$ is completely determined by A, since Ω is an infinite field. We call (P, A) a parabolic or cuspidal pair in G. Once A is fixed, we have the corresponding Levi Ω-decompositions $\mathbf{P} = \mathbf{MN}$ and $P = MN$ where $M = \mathbf{M} \cap G$. We shall call N the unipotent radical of P. Finally, let $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{a}$ denote the Lie algebras of the Ω-rational points of $\underline{\mathfrak{g}}$, $\underline{\mathfrak{p}}$, $\underline{\mathfrak{m}}$, $\underline{\mathfrak{n}}$, $\underline{\mathfrak{a}}$ respectively.

Let G be as above. Then $G \subset GL(n, \Omega)$ for a suitable $n \geqslant 0$. $GL(n, \Omega)$, being an open subset of a vector space over Ω of dimension n^2, is a locally compact group. Since G is closed in $GL(n, \Omega)$, it is also locally compact. Moreover G is separable. Given two parabolic pairs (P_i, A_i) $(i = 1, 2)$ in G, we write $(P_1, A_1) < (P_2, A_2)$ if $P_1 \subset P_2$ and $A_1 \supset A_2$. A parabolic pair is called mincuspidal if it is minimal with respect to this partial order.

Let (P, A) be a parabolic pair with Levi decomposition $P = MN$. By a root α of P (or (P, A)) we mean an element α in $\mathfrak{a}^*$, the dual of $\mathfrak{a}$, with the following property. Let $\mathfrak{n}_\alpha$ denote the set of all $X \in \mathfrak{n}$ such that $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}$. Then $\mathfrak{n}_\alpha \neq (0)$. Fix a mincuspidal pair (P, A)

in G with corresponding Levi decomposition $P=MN$. For any root α of (P, A), let ξ_α be the corresponding character of A. Let A^+ be the set of all points $a \in A$ where $|\xi_\alpha(a)| \geqslant 1$ for every root α of (P, A). Then, by Bruhat-Tits (cf. [2], [3] Theorem 5) we can choose an open and compact subgroup K of G with the properties: (i) $G=KP$, (ii) $G=KA^+\omega_M K$, where ω_M is a finite subset of M.

Let Δ_G denote the Haar modulus of G. Let us denote by Ad the adjoint representation of $\mathbf{G}$ on $\underline{\mathfrak{g}}$. Then

$$\Delta_G(x)=|\det \mathrm{Ad}_{\mathfrak{g}}(x)| \quad (x \in G).$$

Similarly $\Delta_P(x)=|\det \mathrm{Ad}_{\mathfrak{p}}(p)|$ $(p \in P)$, where Δ_P is the Haar modulus of P. Put $p=mn$ $(m \in M,\ n \in N)$. Then

$$\begin{aligned}\Delta_P(p)=\Delta_P(m) &= |\det \mathrm{Ad}_{\mathfrak{m}}(m)|\ |\det \mathrm{Ad}_{\mathfrak{n}}(m)| \\ &= \Delta_M(m)\cdot|\det \mathrm{Ad}_{\mathfrak{n}}(m)|,\end{aligned}$$

where Δ_M stands for the Haar modulus of M.

LEMMA 2. *Let* $\mathbf{G}$ *be any connected* Ω*-group. Denote by* G *its subgroup of* Ω*-rational points. Suppose* $\mathbf{G}$ *is reductive. Then* G *is unimodular.*

PROOF. Let $\mathbf{T}$ be the identity component of the centre of $\mathbf{G}$. $\mathbf{T}$ is defined over Ω and $\mathbf{G}=\mathbf{T}\cdot\mathbf{G}_1$, where $\mathbf{G}_1$ is a semisimple Ω-group. $\mathbf{G}_1$ is equal tot its derived group. Hence det Ad $(x)=1$ if $x \in \mathbf{G}_1$ or $x \in \mathbf{T}$. Consequently $\Delta_G(x)=|\det \mathrm{Ad}_{\mathfrak{g}}(x)|=1$ $(x \in G)$. This proves the lemma.

So

$$\Delta_P(p)=|\det \mathrm{Ad}_{\mathfrak{n}}(m)| \quad (p \in P,\ p=mn).$$

If $\Delta_P=1$, then $|\xi_\alpha(a)|=1$ for all roots α of (P, A). In particular A^+ is compact, hence $G=KA^+\omega_M K$ is compact.

In view of Lemma 1 and its consequences, we have our main result:

THEOREM. *Let* Ω *be a* $\mathfrak{p}$*-adic field. Let* G *be the group of* Ω*-rational points of a connected, semisimple, linear algebraic group, defined over* Ω*. Then* G *does not have the property* P_1*, unless* G *is compact.*

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