The linear decomposition of λ^2-models

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Abstract

In this paper it is shown that the category of sets and relations gives rise to a model of weak second-order linear type theory. To describe the structure of this model, categorical notions based on the concept of *semi-functor* [3] are used. It is easy to transform the model into a λ^2-*algebra* [10], i.e. a categorical model of the non-extensional λ^2-calculus. A slight extension of this λ^2-algebra yields a *non-stable* version of the *coherence space model* [1] of the λ^2-calculus.

1 Introduction

In [6] we have shown that the category Rel of sets and relations is a model of Linear Type Theory (LTT). To describe the linear structure on Rel we used *semi*-notions rather than ordinary categorical notions. These semi-notions are based on the concept of *semi-functor*, which was introduced in [3]. A semi-functor is a "functor" that does not preserve identities.

In this paper we shall extend Rel with second-order quantifiers. This new model built out of Rel is a model of second-order Linear Type Theory (LTT^2). Again the structure of the model is described by semi-notions. For example, the second-order quantifiers are semi-functors, and they form semi-adjunctions with appropriate substitution functors.

In the Rel-based model of LTT^2 several equalities hold. For example, the product is equal to the coproduct, and the existential quantifier is equal to the universal quantifier. By equipping the sets involved with unary or binary predicates, we find models of LTT^2 in which less equalities hold. In particular, the model built out of the category of sets with binary predicates may be considered as a *non-stable* version of the *coherence space model* [1] of LTT^2.

We describe two operations on models of LTT^2. Firstly, any model of LTT^2 can be transformed into a λ^2-algebra, i.e. a model of the non-extensional λ^2-calculus. Secondly, we show how the models of this paper can be "extensionalised" by equipping the sets involved with a pre-order.

2 Semi-functors

In this section the categorical notions of *adjunction* and *comonad* are generalised to *semi*-functors.

2.1 Semi-adjunctions

Let C, D be categories. A *semi-functor* $F : \mathsf{C} \to \mathsf{D}$ is defined just as a functor, except that it need not preserve identities [3]. Hence, every functor is a semi-functor, but not vice-versa.

Example 1 *Let* Rel *be the category of sets and relations. Define the semi-functor* $! : \mathsf{Rel} \to \mathsf{Rel}$ *on objects* A *as* $!A = \{X \subseteq A | X \text{ finite }\}$*, and on arrows* $R : A \to B$ *as* $X!(R)Y \Leftrightarrow \forall b \in Y \exists a \in X(aRb)$.

The natural transformation F(id) with components $F(id_A) : FA \to FA$ plays an important role. We write $\mathsf{D}(FA, B)_s$ for the set of arrows $f \in \mathsf{D}(FA, B)$ that satisfy

$$f \circ F(id_A) = f$$

If F happens to be a functor, then $\mathsf{D}(FA, B)_s = \mathsf{D}(FA, B)$. Analogously, the set $\mathsf{D}(B, FA)_s$ is defined.
Various category-theoretic definitions which involve functors can be generalised to semi-functors. For example, the notion of *semi*-adjunction is defined as follows.

Definition 2 *Let* C, D *be categories. A* semi-adjunction *from* C *to* D *is a tuple* $\langle F, G, \alpha, \beta \rangle$ *where* $F : \mathsf{C} \to \mathsf{D}$ *and* $G : \mathsf{D} \to \mathsf{C}$ *are semi-functors, and* α *and* β *are families of functions*

$$\mathsf{D}(FA, B) \underset{\beta_{A,B}}{\overset{\alpha_{A,B}}{\rightleftarrows}} \mathsf{C}(A, GB)$$

natural in A, B*, which cut down to isomorphisms*

$$\mathsf{D}(FA, B)_s \cong \mathsf{C}(A, GB)_s$$

This definition of semi-adjunction is equivalent to the original definition of [3] (see [7])[1]. We write $F \dashv_s G$ iff F, G are components of a semi-adjunction.
In this paper we shall in particularly be interested in semi-adjunctions of which the left-adjoint F is a functor. If both adjoints happen to be functors, the definitions of adjunction and semi-adjunction coincide.

[1] We might simplify our definition of semi-adjunction by just requiring a natural isomorphism between the restricted Hom-sets [7]

2.2 Semi-comonads

A further notion which may be generalised is that of a *comonad* [11].

Definition 3 *Let* C *be a category. A* semi-comonad *on* C *is a tuple* $\langle T, \eta, \mu \rangle$ *where* $T : \mathsf{C} \to \mathsf{C}$ *is a semi-functor, and* $\eta : T \to Id_{\mathsf{C}}$ *and* $\mu : T \to TT$ *are natural transformations, satisfying*

1. $\eta_{TA} \circ \mu_A = T(\eta_A) \circ \mu_A = T(id_A)$
2. $\mu_{TA} \circ \mu_A = T(\mu_A) \circ \mu_A$
3. $\mu_A \circ T(id_A) = \mu_A$

If T happens to be a functor, then the definitions of comonad and semi-comonad coincide.

Example 4 *The semi-functor* ! *of example 1 is part of a semi-comonad structure on* Rel. *Define* $X \eta_A a \Leftrightarrow a \in X$ *and* $X \mu_A \chi \Leftrightarrow \bigcup \chi \subseteq X$.

We also define an appropriate notion of morphism between semi-comonads.

Definition 5 *A* semi-comonad morphism $\langle F, m \rangle$ *between semi-comonads* $\langle T : \mathsf{C} \to \mathsf{C}, \eta, \mu \rangle$ *and* $\langle T' : \mathsf{D} \to \mathsf{D}, \eta', \mu' \rangle$ *consists of a semi-functor* $F : \mathsf{C} \to \mathsf{D}$ *and a natural transformation* $m : T'F \to FT$ *such that*

1. $F(\eta_A) \circ m_A = \eta'_{FA} \circ T'F(id_A)$
2. $F(\mu_A) \circ m_A = m_{TA} \circ T'(m_A) \circ \mu'_{FA}$
3. $m_A \circ T'F(id_A) = m_A$

Let CoMnd_s be the category with as objects semi-comonads and as arrows semi-comonad morphisms. The identity on $\langle T, \eta, \mu \rangle$ is $\langle Id_{\mathsf{C}}, Tid \rangle$, and if $\langle F, m \rangle : \langle T, \eta, \mu \rangle \to \langle T', \eta', \mu' \rangle$ and $\langle G, n \rangle : \langle T', \eta', \mu' \rangle \to \langle T'', \eta'', \mu'' \rangle$ are semi-comonad morphisms, then their composition is the arrow $\langle GF, Gm \cdot nF \rangle : \langle T, \eta, \mu \rangle \to \langle T'', \eta'', \mu'' \rangle$.
In analogy with the relation between comonads and adjunctions [11], there is a relation between semi-comonads and semi-adjunctions. Part of this relation is the construction of *(semi-)Kleisli categories.*

Definition 6 *Let* $\langle T, \eta, \mu \rangle$ *be a semi-comonad. The* semi-Kleisli category $Kl(T)$ *of* T *is the category with as objects the objects of* C, *and as arrows* $f : A \to B$ *arrows* $f \in \mathsf{C}(TA, B)_s$. *The identity* id_A *on an object* A *in* $Kl(T)$ *is the arrow* $\eta_A : TA \to A$, *and the composition* $g * f$ *of arrows* $f : A \to B$ *and* $g : B \to C$ *in* $Kl(T)$ *is defined by* $g * f = g \circ T(f) \circ \mu_A$.

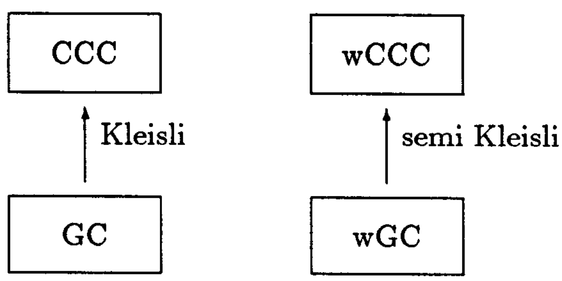
If T is clear from the context, then we write $Kl(\mathsf{C})$ for $Kl(T)$. The operation Kl can be extended to a functor on CoMnd_s.

Definition 7 *Let* $Kl : \mathsf{CoMnd}_s \to \mathsf{Cat}_s$ *be the functor defined on objects by* $Kl(\langle T : \mathsf{C} \to \mathsf{C}, \eta, \mu\rangle = Kl(T)$. *If* $\langle F, m\rangle : \langle T, \eta, \mu\rangle \to \langle T', \eta', \mu'\rangle$ *is a semi-comonad morphism, then* $Kl(\langle F, m\rangle) : Kl(T) \to Kl(T')$ *is the semi-functor defined on objects* C *by* $Kl(\langle F, m\rangle)(C) = F(C)$ *and on arrows* $f : C \to C'$ *by* $Kl(\langle F, m\rangle)(f) = F(f) \circ m_C$.

In this definition Cat_s is the category of categories and semi-functors. If m is clear from the context, then we write $Kl(F)$ for $Kl(\langle F, m\rangle)$. Note that if F in $\langle F, m\rangle$ is a functor, then $Kl(\langle F, m\rangle)$ is a functor.

3 Linear Decomposition of Typed Lambda Calculus Models

First order Linear Type Theory (LTT) [2] may be viewed as a decomposition of the usual type theory belonging to the typed lambda calculus. The main feature of this decomposition is the decomposition of the exponential type $\Rightarrow$ into two new type constructors $\multimap$ (linear implication) and ! (of-course). The type $A \Rightarrow B$ may then be written as $!(A) \multimap B$.
Further linear type constructors are the binary type operators $\times$ (direct product) and $\otimes$ (tensor product), and the constant types 1, I, $\perp$.
The syntactical decomposition of type theory into LTT has a semantical counterpart. It is well-known that Cartesian closed categories (CCC's) are models of typed lambda calculi. In [15] *Girard categories* (GC's) have been defined as models for LTT. Among other things, a GC C has finite products (1, $\times$), it is monoidal closed (where $\otimes, I$ is the monoidal structure on C, and $\multimap$ makes it closed), and there is a comonad $! : \mathsf{C} \to \mathsf{C}$. Corresponding to the decomposition of type theory into LTT, each GC C is a decomposition of a CCC, which may be regained by taking the Kleisli category $Kl(\mathsf{C})$ of C.
It is also possible to decompose the type theory of *non-extensional* typed lambda



calculi, i.e. of typed lambda calculi which do not satisfy the η-rule:

$$\lambda x : \sigma.(tx) = t$$

Models of these non-extensional calculi are *weak* Cartesian closed categories (wCCC's) [3, 6, 10]. A wCCC is defined as a CCC, except that the functionspace constructor is a semi-functor rather than a functor, and hence it forms a semi-adjunction rather than an adjunction with the product-functor (see the appendix for an algebraic description of a wCCC). In [6] *weak* Girard categories (wGC's) have been defined. Roughly, the difference between GC's and wGC's is that ! need only be a *semi*-comonad on a wGC (see appendix for exact definition). Each wGC C is a decomposition of a wCCC, which may be regained by taking the semi-Kleisli category $Kl(\mathsf{C})$.

4 Linear Decomposition of λ^2-Models

Linear Type Theory with second-order quantifiers (LTT^2) may be viewed as a decomposition of the type theory belonging to the second-order lambda calculus [1, 2]. In addition to the type structure of LTT, in LTT^2 there are linear type variables $\alpha, \beta, \ldots$ and if σ is a linear type, we may abstract over α and form the type $\Pi\alpha.\sigma$. The possibility to abstract over types carries over to the type theory constructed out of LTT^2, which is therefore the type theory belonging to λ^2-calculus.
In this section we will describe the categorical structure needed to interpret the non-extensional second-order type calculi, and we will show how λ^2-models may be constructed out of LTT^2-models by means of an analogue to the (semi-) Kleisli construction.

4.1 λ^2-models

In [14] *PL-categories* are defined which provide semantics for the λ^2-calculus. In [10] (the second-order version of) PL-category is generalised to *λ^2-algebra.* In λ^2-algebras we can interpret the *non-extensional* λ^2-calculus, i.e. the λ^2-calculus without the two η-rules:

$$\lambda x : \sigma.(tx) = t$$

$$\Pi\alpha.(t\alpha) = t$$

Definition 8 [2] *A* λ^2-algebra H *consists of the following data:*

- *A category* B*, called the* base *category of* H*, with finite products and with a distinguished object* Ω.
- *A functor* $H : \mathsf{B}^{op} \rightarrow \mathsf{WCCC}$ *into the category of wCCC's and (up to equality) structure preserving functors.*
- *For each* $N \in \mathsf{B}$ *a semi-functor* $\Pi_N : H(N \times \Omega) \rightarrow HN$.

satisfying the following requirements:

[2] Note that this definition slightly differs from [10] as we require the fibers to be wCCC's instead of semi-CCC's.

1. *For each $N \in \mathsf{B}$ we have $\mathsf{B}(N, \Omega) \cong Obj(HN)$, and for each $u : M \to N$ in B the functor $u^* = H(u) : HN \to HM$ acts on objects by composition.*

2. *For each $N \in \mathsf{B}$ we have $\pi^*_{N,\Omega} \dashv_s \Pi_N$, where $\pi_{N,\Omega} : N \times \Omega \to N$ is the projection in B.*

3. Beck-Chevalley condition *(omitting indices):*

   (a) $\Pi \circ (u \times id)^* = u^* \circ \Pi$

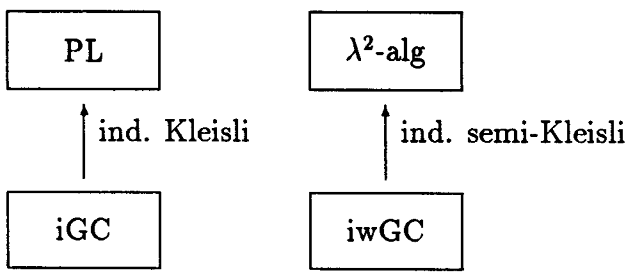
   (b) $u^* \alpha = \alpha (u \times id)^*$

   (c) $(u \times id)^* \beta = \beta u^*$

   where α, β are the natural transformations belonging to the semi-adjunction $\pi^* \dashv_s \Pi$.

4.2 LTT^2-models

In [15] the notion of *indexed GC (iGC)*, the categorical structure belonging to LTT^2, is roughly described. It is shown that an analogue of the Kleisli construction may be applied to an iGC to obtain (the second order version of) a PL-category. In this



section we define the corresponding decomposition of a λ^2-algebra. An *indexed wGC (iwGC)* is defined exactly as an λ^2-algebra except that the fibers H(N) are wGC's instead of GC's (and that the functors u^* preserve the wGC-structure rather than the wCCC-structure).

Definition 9 *An* indexed weak Girard category (iwGC) *H consists of the following data:*

- *A base category B with finite products and with a distinguished object Ω.*

- *A functor $H : \mathsf{B}^{op} \to \mathsf{WGC}$ into the category of wGC's and structure preserving functors.*

- *For each $N \in \mathsf{B}$ a semi-functor $\Pi_N : H(N \times \Omega) \to HN$.*

satisfying the following requirements:

1. *For each* $N \in \mathsf{B}$ *we have* $\mathsf{B}(N,\Omega) \cong Obj(HN)$, *and for each* $u : M \rightarrow N$ *in* B *the functor* $u^* = H(u) : HN \rightarrow HM$ *acts on objects by composition.*

2. *For each* $N \in \mathsf{B}$ *we have* $\pi^*_{N,\Omega} \dashv_s \Pi_N$, *where* $\pi_{N,\Omega} : N \times \Omega \rightarrow N$ *is the projection in* B.

3. Beck-Chevalley condition *(omitting indices):*

   (a) $\Pi \circ (u \times id)^* = u^* \circ \Pi$

   (b) $u^*\alpha = \alpha(u \times id)^*$

   (c) $(u \times id)^*\beta = \beta u^*$

   where α, β *are the natural transformations belonging to the semi-adjunction* $\pi^* \dashv_s \Pi$.

4.3 The Indexed Kleisli Construction

Just like the Kleisli category can be taken of ordinary semi-comonads, the *indexed* Kleisli category construction may be applied to *indexed* semi-comonads.

Definition 10 *An* indexed semi-comonad *is a functor* $H : \mathsf{B}^{op} \rightarrow \mathsf{CoMnd}_s$ *such that if* $H(f) = \langle F, m \rangle$, *then* F *is a functor.*

Definition 11 *The* indexed Kleisli category $iKL(H)$ *of an indexed semi-comonad* H *is the indexed category* $Kl \circ H : \mathsf{B} \rightarrow \mathsf{Cat}$.

Let $I : \mathsf{WGC} \rightarrow \mathsf{CoMnd}_s$ be the inclusion functor defined by $I(\mathsf{C}) = !_{\mathsf{C}}$ on objects, and if $F \in \mathsf{WGC}(\mathsf{C}, \mathsf{D})$, then $I(F) = \langle F, !_{\mathsf{D}} Fid \rangle$. An iwGC $H : \mathsf{B}^{op} \rightarrow \mathsf{WGC}$ may be considered as an indexed semi-comonad $I \circ H : \mathsf{B}^{op} \rightarrow \mathsf{CoMnd}_s$, hence the indexed Kleisli category construction may be applied to indexed weak Girard categories. We write $iKl(H)$ for $iKl(I \circ H)$. For each iwGC H $iKl(H)$ is a λ^2-algebra.

Lemma 12 *Let* $\langle T : \mathsf{C} \rightarrow \mathsf{C}, \eta, \mu \rangle$ *and* $\langle T' : \mathsf{D} \rightarrow \mathsf{D}, \eta', \mu' \rangle$ *be semi-comonads, and let* $F : \mathsf{C} \rightarrow \mathsf{D}$ *be a functor such that* $FT = T'F$, $F\eta = \eta' F$ *and* $F\mu = \mu' F$. *If* $F \dashv_s G$, *then there is a natural transformation* $n : TG \rightarrow GT'$ *such that*

- $\langle G, n \rangle : T' \rightarrow T$ *is a semi-comonad morphism.*
- $Kl(\langle F, T'Fid \rangle) \dashv_s Kl(\langle G, n \rangle)$

Proof: Suppose α, β are the natural transformations belonging to the semi-adjunction $F \dashv_s G$. Take $n = \alpha(T'(\beta(G(id))))$, then $\langle Kl(\langle F, T'Fid \rangle), Kl(\langle G, n \rangle), \alpha, \beta \rangle$ is a semi-adjunction. ∎

Theorem 13 *If H is an iwGC, then $iKl(H)$ is a λ^2-algebra.*

Proof: We already know that each $iKl(H)(N) = Kl(HN)$ is a wCCC. It is easy to check that the functors $iKl(H)(u)$ preserve the wCCC-structure. By the previous lemma, the functors $iKl(H)(\pi) = Kl(\pi^*)$ have semi-rightadjoints $Kl(\Pi)$. It is easy to check that $iKl(H)$ satisfies the Beck-Chevalley conditions. ∎

5 Rel as iwGC

In [6] we have shown that the category Rel of sets and relations is a wGC. In this section we will construct an iwGC out of Rel. Taking the indexed Kleisli category of this iwGC gives a simple example of a λ^2-algebra.

5.1 Inj

Define Inj as the category with as objects sets and as arrows injective functions. This category has a number of (well-known) useful properties, which are similar to the properties of algebraic dcpo's. Firstly, it is *directed complete.*

Definition 14 *Let C be a category. A* directed diagram *in C is a functor $D : I \to$ C, where I is a directed poset (i.e. every two elements have an upperbound in I) considered as a category.*

Theorem 15 Inj *is* directed complete, *i.e. each directed diagram in* Inj *has a colimit.*

Furthermore, finite sets are "compact" in Inj.

Theorem 16 *If $(\rho_i : A_i \to A | i \in I)$ is a directed colimit in* Inj, *and $f \in$ Inj(X, A) where X is finite, then there exists $k \in I$ and $f' \in$ Inj(X, A_k) such that $\rho_k \circ f' = f$.*

Let **S** be a set of finite sets such that each finite set is isomorphic to an element of **S**, then **S** forms a *basis* for Inj in the sense that each object of Inj is the colimit of a directed diagram of sets in **S**.

Theorem 17 *Let $A \in$ Inj, and let I be the set $\{\langle X, f\rangle | X \in \mathbf{S}, f \in$ Inj$(X, A)\}$. Order I by $\langle X, f\rangle \leq \langle X', f'\rangle$ iff there exist $g \in$ Inj(X, X') such that $f' = f \circ g$. Note that g is unique if it exists, and that I is directed.*
Define $D : I \to$ Inj by $D(\langle X, f\rangle) = X$ and $D(\langle X, f\rangle \leq \langle X', f'\rangle) = g : X \to X'$, then A is a colimit of D.

Note that the n-fold products Inj^n inherit these properties of Inj.

5.2 The functor H

The base category B has as objects the categories Inj^n. We take $\Omega = \mathsf{Inj}$. The arrows of B are *continuous* functors.

Definition 18 *Let* C, D *be directed complete categories. A functor* $F : \mathsf{C} \to \mathsf{D}$ *is* continuous *iff it preserves directed colimits, i.e. if* $(\rho_i : A_i \to A | i \in I)$ *is a directed colimit in* C*, then* $(F(\rho_i) : F(A_i) \to F(A) | i \in I)$ *is a directed colimit in* D.

The fibre $H_n = H(\Omega^n)$ has as objects continuous functors $\Omega^n \to \Omega$. The arrows R in H^n between objects $F, G : \Omega^n \to \Omega$ are *continuous families of relations*, i.e. $R = (R_A \subseteq F(A) \times G(A) | A \in \mathsf{Inj})$, and if $(\rho_i : A_i \to A | i \in I)$ is a directed colimit in Inj^n, then

$$R_A = \bigcup_{i \in I} \{(F(\rho_i)(a), G(\rho_i)(b)) | a R_{A_i} b\}$$

Note that a continuous family of relations is *monotone*: for $f : A \to B$ one has

$$a R_A b \Rightarrow F(f)(a) R_B G(f)(b)$$

The identities in H_n are the families of identities, and composition is defined componentwise: $(S \circ R)_A = S_A \circ R_A$. Given a continuous functor $U : \Omega^m \to \Omega^n$ we define $U^* : H_n \to H_m$ on objects F as $U^*(F) = F \circ U$, and on arrows R as $U^*(R)_A = R_{U(A)}$.

5.3 The functors Π_n

First we define the *trace* of a functor.

Definition 19 *Let* $F : \mathsf{Inj} \to \mathsf{Inj}$ *be a continuous functor. The* trace $Tr(F)$ *of* F *is the set* $\{\langle X, a \rangle | X \in \mathbf{S}, a \in F(X)\}$.

For each natural number n we define a semi-functor $\Pi_n : H_{n+1} = H(\Omega^n \times \Omega) \to H_n$. Let $F : \Omega^{n+1} \to \Omega$ be an object of H_{n+1}, then $\Pi_n(F)$ is an object of H_n, i.e. a continuous functor $\Omega^n \to \Omega$. On objects $A \in \Omega^n$ we define

$$\Pi_n(F)(A) = Tr(F(A, -))$$

and on arrows $f : A \to B$ in Ω^n we define $\Pi_n(F)(f) : Tr(F(A, -)) \to Tr(F(B, -))$ by

$$\Pi_n(F)(f)(X, a) = \langle X, F(f, id_X)(a) \rangle$$

Let $R : F \to G$ be an arrow in H_{n+1}, then $\Pi_n(R) : \Pi_n(F) \to \Pi_n(G)$ is an arrow in H_n defined by

$$\langle X, a \rangle (\Pi_n(R))_A \langle Y, b \rangle \Leftrightarrow \exists f : X \to Y (F(id_A, f)(a) R_{A,Y} b)$$

It is easy to see that in general Π_n is only a *semi*-functor:

$$\langle X, a \rangle (\Pi_n(id))_A \langle Y, b \rangle$$

$$\Leftrightarrow$$
$$\exists f : X \to Y(F(id_A, f)(a) = b)$$
$$\not\Leftrightarrow$$
$$\langle X, a\rangle = \langle Y, b\rangle$$

Let $P_n : \Omega^{n+1} \to \Omega^n$ be the projection in B.

Theorem 20 $P_n^* \dashv_s \Pi_n$

Proof: The functor P_n^* is defined on objects $F \in \Omega^n$ as $P_n^*(F) = F \circ \Pi_n$. We define natural transformations $\alpha^n_{F,G} : H_{n+1}(F \circ P_n, G) \to H_n(F, \Pi_n(G))$ and $\beta^n_{F,G} : H_n(F, \Pi_n(G)) \to H_{n+1}(F \circ P_n, G)$ as follows. Let $R : F \circ P_n \to G$ then

$$a\alpha(R)_A\langle Y, b\rangle \Leftrightarrow aR_{A,Y}b$$

Let $R : F \to \Pi_n(G)$, then

$$a\beta(R)_{A,A'}b \Leftrightarrow \exists Y, f : Y \to A', b' \in G(A,Y)(aR_A(Y,b')\&G(id_A, f)(b') = b)$$

It can be checked that $\langle P_n^*, \Pi_n, \alpha^n, \beta^n\rangle$ is a semi-adjunction. ■

5.4 The structure on the fibers

The wGC-structure on the fibers H_n is similar to the structure on Rel [6] We shall give the definitions of the linear operators in H_n on objects F, G. Let A be an object of Ω^n.

- $1_n(A) = \emptyset$
- $I_n(A) = \{*\}$
- $\bot_n(A) = \{*\}$
- $(!_n F)(A) = \mathcal{P}_f(F(A))$
- $(F \times_n G)(A) = F(A) \uplus G(A)$
- $(F \otimes_n G)(A) = F(A) \diamond G(A)$
- $(F \multimap_n G)(A) = F(A) \diamond G(A)$

where $A \uplus B$, $A \diamond B$ are resp. the disjoint union and the cartesian product of two sets. It is easy to define the linear operators on arrows, and to show that the functors U^* preserve the structure on the fibers.

6 Some Related iwGC's

In this section we describe three iwGC's which are related to the model of the previous section.

6.1 PRel as iwGC

The objects A of the category PRel are pairs $\langle Dom_A, p_A\rangle$, where Dom_A is a set, and p_A is a *predicate* on Dom_A, i.e. p_A is a subset of Dom_A. We write $p_A(a)$ iff $a \in p_A$. The arrows $R : A \to B$ of PRel are relations $R \subseteq Dom_A \times Dom_B$ which *preserve truth*, i.e. $p_A(a)\&aRb$ implies $p_B(b)$. It is clear that Rel is a full subcategory of PRel. In [6] we have shown that PRel is, like Rel, a wGC. For example, a semi-comonad structure on PRel may be defined[3] by $Dom_{!A} = \{X \subseteq Dom_A | X \text{ finite } \}$ and $p_{!A}(X) \Leftrightarrow \forall a \in X(p_A(a))$
Analogous to the construction of an iwGC out of Rel we can build an iwGC out of PRel. In fact, there are only three differences. Firstly, instead of Inj we use the category PInj. This category has as objects sets and as arrows $f : A \to B$ injective functions $f : Dom_A \to Dom_B$ satisfying

$$p_A(a) \Leftrightarrow p_B(f(b))$$

The category PInj has properties similar to Inj. The second difference is that the arrows in the categories H_n are continuous families of *truth preserving* relations. Finally, on the trace of a continuous functor $F : \mathsf{PInj}^n \to \mathsf{PInj}$ a predicate $p_{Tr(F)}$ is defined by

$$p_{Tr(F)}(\langle X, a\rangle) \Leftrightarrow p_{F(X)}(a)$$

6.2 WCohl as iwGC

The category WCohl has as objects pairs $A = \langle Dom_A, q_A\rangle$, where Dom_A is a set and q_A is a *binary*, rather than an *unary*, predicate on Dom_A. In fact, we shall require that these predicates are symmetric. Arrows $R : A \to B$ are relations $R \subseteq Dom_A \times Dom_B$ which preserve the predicates (i.e. $q_A(a, a')\&arb\&a'Rb'$ implies $q_B(b, b')$).
It has been shown in [6] that WCohl can be equipped with the structure of a wGC. By now it should be clear how to build an iwGC out of WCohl. We shall not give details. The indexed semi-Kleisli category of this iwGC turns out to be the same as the λ^2-algebra that we get by dropping everywhere the word "stable" in the description of the *coherence space model* of the λ^2-calculus [1].

[3] Alternatively, we might take $Dom_{!A} = \{X \subseteq p_A | X \text{ finite}\}$.

6.3 NRel as iwGC

A very simple iwGC can be defined by using only sets of natural numbers. Firstly, we define the category NRel as the full subcategory of Rel with as objects elements of $\mathcal{P}\omega$. Because there are encodings of finite subsets $(-)^+ : \mathcal{F}(\omega) \cong \omega$ and of pairs $\langle \omega \times \omega \cong \omega$ of natural numbers as natural numbers, the category NRel can be equipped with essentially the same wGC structure as Rel. For example, we may define $!x = \{X^+ | X \subseteq x, X \text{ finite }\}$ for $x \subseteq \omega$.
It is well-known that $\mathcal{P}\omega$ ordered by inclusion is an algebraic lattice. The compact elements of $\mathcal{P}\omega$ are the finite sets. To convert NRel into an iwGC, we take base category B with as objects $(\mathcal{P}\omega)^n$ for $n \in \omega$ and as arrows continuous functions (i.e. functions that preserve directed joins). Furthermore, we take $\Omega = \mathcal{P}\omega$.
The objects in the category H_n are continuous functions $\Omega^n \to \Omega$, and the arrows $R : f \to g$ are families $(R_x \subseteq f(x) \times g(x) | x \in (\mathcal{P}\omega)^n)$ such that if $U \subseteq \mathcal{P}\omega$ is directed, then

$$R_{\bigcup U} = \bigcup_{x \in U} R_x$$

For every continuous function $f : \Omega^m \to \Omega^n$ a functor $f^* : H_n \to H_m$ is defined by $f^*(g) = g \circ f$ and $(f^*(R))_x = R_{f(x)}$, for $x \in \mathcal{P}\omega$.
If $f : \mathcal{P}\omega \to \mathcal{P}\omega$ is a continuous function, then $Tr(f) = \{\langle X^+, n\rangle | X \subseteq \omega, X \text{ finite }, n \in f(X)\}$. Functors $\Pi_n : H_{n+1} \to H_n$ are defined on objects by $\Pi_n(f)(x) = Tr(f(x,-))$, and on arrows by $\langle X^+, n\rangle \Pi_n(R)_x \langle Y^+, m\rangle \Leftrightarrow X \subseteq Y \& n R_{x,Y} m$.
In general, the Π_n are only *semi*-functors:

$$(X,n)(\Pi_n(id))_x(X',n')$$

$$\Leftrightarrow$$

$$X \subseteq X' \& n(id)_{x,X'} n'$$

$$\Leftrightarrow$$

$$X \subseteq X' \& n = n'$$

$$\not\Leftrightarrow$$

$$(X,n) = (X',n')$$

Note that this iwGC is in fact a simplification of the iwGC constructed out of Rel by requiring the arrows in Ω to be inclusions rather than, more general, injections.

7 Preordered Sets

The *Karoubi envelope* construction may be used to transform various semi notions to the corresponding ordinary notions [3, 6]. For example, the Karoubi envelope $K(\mathsf{C})$ of a category C is a CCC, resp. a GC if C is a wCCC, resp. a wGC. We do not

know how to extend the Karoubi envelope construction to indexed categories such that the (extended) Karoubi envelope of an iwGC is an iGC. However, in special cases we can transform iwGC's into iGC's in a manner which is reminiscent of some sort of Karoubi envelope construction. In this section we shall transform the iwGC constructed out of NRel into a iGC. In a similar way the other iwGC's defined in this paper can be transformed into iGC's.
In [6] we have seen that taking the Karoubi envelope of Rel is more or less the same as equipping the objects A with a preorder [4] $\leq_A$ and requiring that the arrows $R : A \to B$ satisfy

$$a' \leq_A aRb \leq_B b' \Rightarrow a'Rb'$$

Hence, to transform the iwGC constructed out of NRel into a iGC, we shall equip all sets with preorders.

Definition 21 *$\mathcal{PR}\omega$ is the set of pairs $u = (x_u, \leq_u)$, where $x_u \in \mathcal{P}\omega$ and $\leq_u$ is a preorder on x_u.*

The set $\mathcal{PR}\omega$ can be ordered by

$$u \sqsubseteq v \Leftrightarrow (x_u \subseteq x_v \& \leq_u \subseteq \leq_v)$$

It is easy to check that $(\mathcal{PR}\omega, \sqsubseteq)$ is an algebraic lattice and that the compact elements are the finite preorders. The base category B has as objects $(\mathcal{PR}\omega)^n$ for $n \in \omega$, and as arrows continuous functions. The categories H_n are defined just as in the NRel-model except that we require the families of relations to satisfy the additional requirement

$$a' \leq_{f(u)} aR_u b \leq_{g(u)} b' \Rightarrow a'R_u b'$$

Note that the identity id_f on an object f in H_n is given by the family $(\leq_{f(u)} | u \in (\mathcal{PR}\omega)^n)$.
The GC-structure on the categories H_n is similar to the structure in the NRel-model. For example, for each n a functor $!_n : H_n \to H_n$ is defined on objects by $x_{!f(u)} = \{X^+ | X \subseteq x_{f(u)}, X \text{ finite}\}$, $X^+ \leq_{!f(u)} Y^+ \Leftrightarrow \forall b \in Y \exists a \in X (a \leq_{f(u)} b)$, and on arrows by $X^+ !R_u Y^+ \Leftrightarrow \forall b \in Y \exists a \subset X(aR_u b)$. Furthermore, there are natural transformations $\eta^n :! \to Id_{H_n}$, $\mu^n :! \to !!$ defined by $X^+(\eta^n_f)_u a \Leftrightarrow \exists a' \in X(a' \leq_{f(u)} a$ and $X^+(\mu^n_f)_u \chi^+ \Leftrightarrow \forall a' \in \bigcup \chi \exists a \in X(a \leq_{f(u)} a')$. It is easy to see that $(!_n, \eta^n, \mu^n)$ is a comonad on H_n.
The functors f^* are also as in the NRel-model.
Let $f : \mathcal{PR}\omega \to \mathcal{PR}\omega$ be a continuous function. The preorder $Tr(f)$ has domain $\{U^\pm | U \in \mathcal{PR}\omega, U \text{ finite}, n \in x_{f(U)}\}$ (where $(-)^\pm : \mathcal{FR}\omega \cong \omega$) and $\langle U^\pm, n \rangle \leq_{Tr(f)}$

[4] In fact, the objects are equipped with transitive relations $<$ having the interpolation property $a < c \Rightarrow \exists b(a < b < c)$.

$\langle V^{\pm}, m\rangle \Leftrightarrow U \sqsubseteq V \& n \leq_{f(V)} m$. Next the functors Π_n are defined as in the NRel-model, but using this preordered trace.
It is easy to check that Π_n preserves identities:

$$\langle U^{\pm}, n\rangle(\Pi_n(id_f))_u\langle V^{\pm}, m\rangle$$
$$\Leftrightarrow$$
$$U \sqsubseteq V \& n(id_f)_{u,V} m$$
$$\Leftrightarrow$$
$$U \sqsubseteq V \& n \leq_{f(u,V)} m$$
$$\Leftrightarrow$$
$$\langle U^{\pm}, n\rangle \leq_{Tr(f(u,-))} \langle V^{\pm}, m\rangle$$
$$\Leftrightarrow$$
$$\langle U^{\pm}, n\rangle \leq_{\Pi_n(f)(u)} \langle V^{\pm}, m\rangle$$
$$\Leftrightarrow$$
$$\langle U^{\pm}, n\rangle(id_{\Pi_n(f)})_u\langle V^{\pm}, m\rangle$$

As the categories H_n are GC's rather than wGC's we have in fact built an iGC. Applying the indexed Kleisli category gives a (second-order version of a) PL-category.

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Appendix

A wCCC's

Definition 22 *A* weak Cartesian closed category (wCCC) C *[3, 6, 10, 12] is a category* C *with a terminal object* 1 *and binary products* $A \times B$*, and with the following data:*

- *For each pair of objects* $A, B \in \mathsf{C}$ *an object* $A \Rightarrow B \in \mathsf{C}$*, and an arrow* $e_{A,B} \in \mathsf{C}((A \Rightarrow B) \times A, B)$*. Furthermore, for each arrow* $f \in \mathsf{C}(D \times A, B)$ *an arrow* $\Lambda(f) \in \mathsf{C}(D, A \Rightarrow B)$*.*

satisfying the following equations (omitting subscripts):

1. $e \circ (\Lambda(f) \times id) = f$
2. $\Lambda(f \circ (g \times id)) = \Lambda(f) \circ g$

B wGC's

Definition 23 *A* linear category *[6, 13, 15]* $\langle \mathsf{C}, \times, 1, \otimes, I, \multimap, \bot \rangle$ *is a category* C *such that:*

1. *The functor* $\times : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ *is a chosen product in* C*, and* 1 *is a terminal object in* C*.*

2. $\langle \mathsf{C}, \otimes, I \rangle$ *is a symmetric monoidal category, i.e.* $\otimes$ *is a functor* $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$*,* I *is an object in* C*, and there are natural isomorphisms* $\rho_{A,B} : A \otimes B \cong B \otimes A$*,* $\lambda_A : A \otimes I \cong A$ *and* $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ *satisfying certain commutative diagrams, the MacLane-Kelly coherence conditions (for more details see [11]).*

3. $\langle \mathsf{C}, \otimes, I, \multimap \rangle$ *is a symmetric monoidal closed category, i.e.* $\multimap$ *is a functor* $\mathsf{C}^{op} \times \mathsf{C} \to \mathsf{C}$ *and* $(-) \otimes B$ *is a left adjoint of* $B \multimap (-)$ *(i.e. there is a natural isomorphism* $\mathsf{C}(A \otimes B, C) \cong \mathsf{C}(A, B \multimap C)$*).*

4. $\bot$ *is a* dualising object *in* C*, i.e. for each object* A *the arrow* τ_A *given by the next derivation is an isomorphism:*

$$\dfrac{\dfrac{\dfrac{(A \multimap \bot) \stackrel{id}{\to} (A \multimap \bot)}{(A \multimap \bot) \otimes A \to \bot}}{A \otimes (A \multimap \bot) \to \bot}}{A \stackrel{\tau_A}{\to} (A \multimap \bot) \multimap \bot}$$

Definition 24 *A* wGC *[6]* $\langle \mathsf{C}, \times, 1, \otimes, I, \multimap, \bot, !, \eta, \mu, i, \sim \rangle$ *is a linear category* $\langle \mathsf{C}, \times, 1, \otimes, I, \multimap, \bot$ *such that*

1. $\langle \eta, \mu \rangle$ *is a* semi comonad *on* C*.*

2. $i : !1 \cong I$ *is an isomorphism, such that:*
   - $i \circ !(id_1) = i$

3. $\sim_{A,B} : !A \otimes !B \cong !(A \times B)$ *is a isomorphism natural in* A, B*, such that:*
   - $!(\langle pi_{A,B}), !(\pi'_{A,B}) \rangle) \circ \mu_{A \times B} = \sim_{!A,!B} \circ (\mu_A \otimes \mu_B) \circ \sim^{-1}_{A,B}$

   where $\pi_{A,B} : A \times B \to A$ *and* $\pi'_{A,B} : A \times B \to B$ *are projections, and* $\langle f, g \rangle : C \to A \times B$ *is the unique arrow such that* $\pi \circ \langle f, g \rangle = f$ *and* $\pi' \circ \langle f, g \rangle = g$*.*

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The Linear Decomposition of λ^2-Models

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Abstract

In this paper it is shown that the category of sets and relations gives rise to a model of weak second-order linear type theory. To describe the structure of this model, categorical notions based on the concept of *semi-functor* [3] are used. It is easy to transform the model into a λ^2-*algebra* [10], i.e. a categorical model of the non-extensional λ^2-calculus. A slight extension of this λ^2-algebra yields a *non-stable* version of the *coherence space model* [1] of the λ^2-calculus.

1 Introduction

In [6] we have shown that the category Rel of sets and relations is a model of Linear Type Theory (LTT). To describe the linear structure on Rel we used *semi*-notions rather than ordinary categorical notions. These semi-notions are based on the concept of *semi-functor*, which was introduced in [3]. A semi-functor is a "functor" that does not preserve identities.

In this paper we shall extend Rel with second-order quantifiers. This new model built out of Rel is a model of second-order Linear Type Theory (LTT^2). Again the structure of the model is described by semi-notions. For example, the second-order quantifiers are semi-functors, and they form semi-adjunctions with appropriate substitution functors.

In the Rel-based model of LTT^2 several equalities hold. For example, the product is equal to the coproduct, and the existential quantifier is equal to the universal quantifier. By equipping the sets involved with unary or binary predicates, we find models of LTT^2 in which less equalities hold. In particular, the model built out of the category of sets with binary predicates may be considered as a *non-stable* version of the *coherence space model* [1] of LTT^2.

We describe two operations on models of LTT^2. Firstly, any model of LTT^2 can be transformed into a λ^2-algebra, i.e. a model of the non-extensional λ^2-calculus. Secondly, we show how the models of this paper can be "extensionalised" by equipping the sets involved with a pre-order.

2 Semi-functors

In this section the categorical notions of *adjunction* and *comonad* are generalised to *semi*-functors.

2.1 Semi-adjunctions

Let C, D be categories. A *semi-functor* $F : \mathsf{C} \to \mathsf{D}$ is defined just as a functor, except that it need not preserve identities [3]. Hence, every functor is a semi-functor, but not vice-versa.

Example 1 *Let* Rel *be the category of sets and relations. Define the semi-functor* $! : \mathsf{Rel} \to \mathsf{Rel}$ *on objects* A *as* $!A = \{X \subseteq A | X \text{ finite } \}$*, and on arrows* $R : A \to B$ *as* $X!(R)Y \Leftrightarrow \forall b \in Y \exists a \in X(aRb)$*.*

The natural transformation F(id) with components $F(id_A) : FA \to FA$ plays an important role. We write $\mathsf{D}(FA, B)_s$ for the set of arrows $f \in \mathsf{D}(FA, B)$ that satisfy

$$f \circ F(id_A) = f$$

If F happens to be a functor, then $\mathsf{D}(FA, B)_s = \mathsf{D}(FA, B)$. Analogously, the set $\mathsf{D}(B, FA)_s$ is defined.
Various category-theoretic definitions which involve functors can be generalised to semi-functors. For example, the notion of *semi*-adjunction is defined as follows.

Definition 2 *Let* C, D *be categories. A* semi-adjunction *from* C *to* D *is a tuple* $\langle F, G, \alpha, \beta \rangle$ *where* $F : \mathsf{C} \to \mathsf{D}$ *and* $G : \mathsf{D} \to \mathsf{C}$ *are semi-functors, and* α *and* β *are families of functions*

$$\mathsf{D}(FA, B) \underset{\beta_{A,B}}{\overset{\alpha_{A,B}}{\rightleftarrows}} \mathsf{C}(A, GB)$$

natural in A, B*, which cut down to isomorphisms*

$$\mathsf{D}(FA, B)_s \cong \mathsf{C}(A, GB)_s$$

This definition of semi-adjunction is equivalent to the original definition of [3] (see [7])[1]. We write $F \dashv_s G$ iff F, G are components of a semi-adjunction.
In this paper we shall in particularly be interested in semi-adjunctions of which the left-adjoint F is a functor. If both adjoints happen to be functors, the definitions of adjunction and semi-adjunction coincide.

[1] We might simplify our definition of semi-adjunction by just requiring a natural isomorphism between the restricted Hom-sets [7]

2.2 Semi-comonads

A further notion which may be generalised is that of a *comonad* [11].

Definition 3 *Let* C *be a category. A* semi-comonad *on* C *is a tuple* $\langle T, \eta, \mu\rangle$ *where* $T : \mathsf{C} \to \mathsf{C}$ *is a semi-functor, and* $\eta : T \to Id_{\mathsf{C}}$ *and* $\mu : T \to TT$ *are natural transformations, satisfying*

1. $\eta_{TA} \circ \mu_A = T(\eta_A) \circ \mu_A = T(id_A)$
2. $\mu_{TA} \circ \mu_A = T(\mu_A) \circ \mu_A$
3. $\mu_A \circ T(id_A) = \mu_A$

If T happens to be a functor, then the definitions of comonad and semi-comonad coincide.

Example 4 *The semi-functor* ! *of example 1 is part of a semi-comonad structure on* Rel. *Define* $X\eta_A a \Leftrightarrow a \in X$ *and* $X\mu_A\chi \Leftrightarrow \bigcup\chi \subseteq X$.

We also define an appropriate notion of morphism between semi-comonads.

Definition 5 *A* semi-comonad morphism $\langle F, m\rangle$ *between semi-comonads* $\langle T : \mathsf{C} \to \mathsf{C}, \eta, \mu\rangle$ *and* $\langle T' : \mathsf{D} \to \mathsf{D}, \eta', \mu'\rangle$ *consists of a semi-functor* $F : \mathsf{C} \to \mathsf{D}$ *and a natural transformation* $m : T'F \to FT$ *such that*

1. $F(\eta_A) \circ m_A = \eta'_{FA} \circ T'F(id_A)$
2. $F(\mu_A) \circ m_A = m_{TA} \circ T'(m_A) \circ \mu'_{FA}$
3. $m_A \circ T'F(id_A) = m_A$

Let CoMnd_s be the category with as objects semi-comonads and as arrows semi-comonad morphisms. The identity on $\langle T, \eta, \mu\rangle$ is $\langle Id_{\mathsf{C}}, Tid\rangle$, and if $\langle F, m\rangle : \langle T, \eta, \mu\rangle \to \langle T', \eta', \mu'\rangle$ and $\langle G, n\rangle : \langle T', \eta', \mu'\rangle \to \langle T'', \eta'', \mu''\rangle$ are semi-comonad morphisms, then their composition is the arrow $\langle GF, Gm \cdot nF\rangle : \langle T, \eta, \mu\rangle \to \langle T'', \eta'', \mu''\rangle$.
In analogy with the relation between comonads and adjunctions [11], there is a relation between semi-comonads and semi-adjunctions. Part of this relation is the construction of *(semi-)Kleisli categories.*

Definition 6 *Let* $\langle T, \eta, \mu\rangle$ *be a semi-comonad. The* semi-Kleisli category $Kl(T)$ *of* T *is the category with as objects the objects of* C, *and as arrows* $f : A \to B$ *arrows* $f \in \mathsf{C}(TA, B)_s$. *The identity* id_A *on an object* A *in* $Kl(T)$ *is the arrow* $\eta_A : TA \to A$, *and the composition* $g * f$ *of arrows* $f : A \to B$ *and* $g : B \to C$ *in* $Kl(T)$ *is defined by* $g * f = g \circ T(f) \circ \mu_A$.

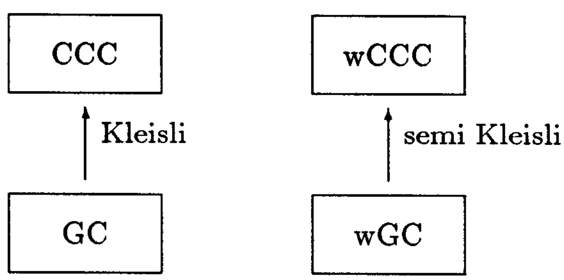
If T is clear from the context, then we write $Kl(\mathsf{C})$ for $Kl(T)$. The operation Kl can be extended to a functor on CoMnd_s.

Definition 7 *Let $Kl : \mathsf{CoMnd}_s \to \mathsf{Cat}_s$ be the functor defined on objects by $Kl(\langle T : \mathsf{C} \to \mathsf{C}, \eta, \mu\rangle = Kl(T)$. If $\langle F, m\rangle : \langle T, \eta, \mu\rangle \to \langle T', \eta', \mu'\rangle$ is a semi-comonad morphism, then $Kl(\langle F, m\rangle) : Kl(T) \to Kl(T')$ is the semi-functor defined on objects C by $Kl(\langle F, m\rangle)(C) = F(C)$ and on arrows $f : C \to C'$ by $Kl(\langle F, m\rangle)(f) = F(f) \circ m_C$.*

In this definition Cat_s is the category of categories and semi-functors. If m is clear from the context, then we write $Kl(F)$ for $Kl(\langle F, m\rangle)$. Note that if F in $\langle F, m\rangle$ is a functor, then $Kl(\langle F, m\rangle)$ is a functor.

3 Linear Decomposition of Typed Lambda Calculus Models

First order Linear Type Theory (LTT) [2] may be viewed as a decomposition of the usual type theory belonging to the typed lambda calculus. The main feature of this decomposition is the decomposition of the exponential type $\Rightarrow$ into two new type constructors $\multimap$ (linear implication) and ! (of-course). The type $A \Rightarrow B$ may then be written as $!(A) \multimap B$.
Further linear type constructors are the binary type operators $\times$ (direct product) and $\otimes$ (tensor product), and the constant types 1, I, $\bot$.
The syntactical decomposition of type theory into LTT has a semantical counterpart. It is well-known that Cartesian closed categories (CCC's) are models of typed lambda calculi. In [15] *Girard categories* (GC's) have been defined as models for LTT. Among other things, a GC C has finite products $(1, \times)$, it is monoidal closed (where $\otimes, I$ is the monoidal structure on C, and $\multimap$ makes it closed), and there is a comonad $! : \mathsf{C} \to \mathsf{C}$. Corresponding to the decomposition of type theory into LTT, each GC C is a decomposition of a CCC, which may be regained by taking the Kleisli category $Kl(\mathsf{C})$ of C.
It is also possible to decompose the type theory of *non-extensional* typed lambda



calculi, i.e. of typed lambda calculi which do not satisfy the η-rule:

$$\lambda x : \sigma.(tx) = t$$

Models of these non-extensional calculi are *weak* Cartesian closed categories (wCCC's) [3, 6, 10]. A wCCC is defined as a CCC, except that the functionspace constructor is a semi-functor rather than a functor, and hence it forms a semi-adjunction rather than an adjunction with the product-functor (see the appendix for an algebraic description of a wCCC). In [6] *weak* Girard categories (wGC's) have been defined. Roughly, the difference between GC's and wGC's is that ! need only be a *semi*-comonad on a wGC (see appendix for exact definition). Each wGC C is a decomposition of a wCCC, which may be regained by taking the semi-Kleisli category $Kl(\mathsf{C})$.

4 Linear Decomposition of λ^2-Models

Linear Type Theory with second-order quantifiers (LTT^2) may be viewed as a decomposition of the type theory belonging to the second-order lambda calculus [1, 2]. In addition to the type structure of LTT, in LTT^2 there are linear type variables $\alpha, \beta, \ldots$ and if σ is a linear type, we may abstract over α and form the type $\Pi\alpha.\sigma$. The possibility to abstract over types carries over to the type theory constructed out of LTT^2, which is therefore the type theory belonging to λ^2-calculus.

In this section we will describe the categorical structure needed to interpret the non-extensional second-order type calculi, and we will show how λ^2-models may be constructed out of LTT^2-models by means of an analogue to the (semi-) Kleisli construction.

4.1 λ^2-models

In [14] *PL-categories* are defined which provide semantics for the λ^2-calculus. In [10] (the second-order version of) PL-category is generalised to *λ^2-algebra*. In λ^2-algebras we can interpret the *non-extensional* λ^2-calculus, i.e. the λ^2-calculus without the two η-rules:

$$\lambda x : \sigma.(tx) = t$$

$$\Pi\alpha.(t\alpha) = t$$

Definition 8 [2] *A* λ^2-algebra *H consists of the following data:*

- *A category* B*, called the* base *category of H, with finite products and with a distinguished object* Ω*.*
- *A functor* $H : \mathsf{B}^{op} \to \mathsf{WCCC}$ *into the category of wCCC's and (up to equality) structure preserving functors.*
- *For each* $N \in \mathsf{B}$ *a semi-functor* $\Pi_N : H(N \times \Omega) \to HN$*.*

satisfying the following requirements:

[2] Note that this definition slightly differs from [10] as we require the fibers to be wCCC's instead of semi-CCC's.

1. *For each $N \in \mathsf{B}$ we have $\mathsf{B}(N, \Omega) \cong Obj(HN)$, and for each $u : M \to N$ in B the functor $u^* = H(u) : HN \to HM$ acts on objects by composition.*

2. *For each $N \in \mathsf{B}$ we have $\pi^*_{N,\Omega} \dashv_s \Pi_N$, where $\pi_{N,\Omega} : N \times \Omega \to N$ is the projection in B.*

3. Beck-Chevalley condition *(omitting indices):*

   (a) $\Pi \circ (u \times id)^* = u^* \circ \Pi$

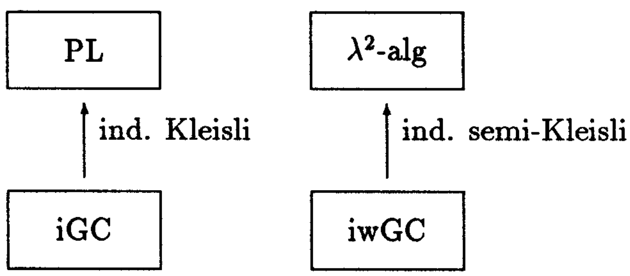
   (b) $u^* \alpha = \alpha(u \times id)^*$

   (c) $(u \times id)^* \beta = \beta u^*$

   where α, β are the natural transformations belonging to the semi-adjunction $\pi^* \dashv_s \Pi$.

4.2 LTT^2-models

In [15] the notion of *indexed GC (iGC)*, the categorical structure belonging to LTT^2, is roughly described. It is shown that an analogue of the Kleisli construction may be applied to an iGC to obtain (the second order version of) a PL-category. In this



section we define the corresponding decomposition of a λ^2-algebra. An *indexed wGC (iwGC)* is defined exactly as an λ^2-algebra except that the fibers H(N) are wGC's instead of GC's (and that the functors u^* preserve the wGC-structure rather than the wCCC-structure).

Definition 9 *An* indexed weak Girard category (iwGC) *H consists of the following data:*

- *A base category B with finite products and with a distinguished object Ω.*
- *A functor $H : \mathsf{B}^{op} \to \mathsf{WGC}$ into the category of wGC's and structure preserving functors.*
- *For each $N \in \mathsf{B}$ a semi-functor $\Pi_N : H(N \times \Omega) \to HN$.*

satisfying the following requirements:

1. *For each $N \in \mathsf{B}$ we have $\mathsf{B}(N, \Omega) \cong Obj(HN)$, and for each $u : M \to N$ in B the functor $u^* = H(u) : HN \to HM$ acts on objects by composition.*

2. *For each $N \in \mathsf{B}$ we have $\pi^*_{N,\Omega} \dashv_s \Pi_N$, where $\pi_{N,\Omega} : N \times \Omega \to N$ is the projection in B.*

3. Beck-Chevalley condition *(omitting indices):*

   (a) $\Pi \circ (u \times id)^* = u^* \circ \Pi$

   (b) $u^* \alpha = \alpha(u \times id)^*$

   (c) $(u \times id)^* \beta = \beta u^*$

   where α, β are the natural transformations belonging to the semi-adjunction $\pi^* \dashv_s \Pi$.

4.3 The Indexed Kleisli Construction

Just like the Kleisli category can be taken of ordinary semi-comonads, the *indexed* Kleisli category construction may be applied to *indexed* semi-comonads.

Definition 10 *An* indexed semi-comonad *is a functor $H : \mathsf{B}^{op} \to \mathsf{CoMnd}_s$ such that if $H(f) = \langle F, m \rangle$, then F is a functor.*

Definition 11 *The* indexed Kleisli category $iKL(H)$ *of an indexed semi-comonad H is the indexed category $Kl \circ H : \mathsf{B} \to \mathsf{Cat}$.*

Let $I : \mathsf{WGC} \to \mathsf{CoMnd}_s$ be the inclusion functor defined by $I(\mathsf{C}) = !_{\mathsf{C}}$ on objects, and if $F \in \mathsf{WGC}(\mathsf{C}, \mathsf{D})$, then $I(F) = \langle F, !_{\mathsf{D}} Fid \rangle$. An iwGC $H : \mathsf{B}^{op} \to \mathsf{WGC}$ may be considered as an indexed semi-comonad $I \circ H : \mathsf{B}^{op} \to \mathsf{CoMnd}_s$, hence the indexed Kleisli category construction may be applied to indexed weak Girard categories. We write $iKl(H)$ for $iKl(I \circ H)$. For each iwGC H $iKl(H)$ is a λ^2-algebra.

Lemma 12 *Let $\langle T : \mathsf{C} \to \mathsf{C}, \eta, \mu \rangle$ and $\langle T' : \mathsf{D} \to \mathsf{D}, \eta', \mu' \rangle$ be semi-comonads, and let $F : \mathsf{C} \to \mathsf{D}$ be a functor such that $FT = T'F$, $F\eta = \eta' F$ and $F\mu = \mu' F$. If $F \dashv_s G$, then there is a natural transformation $n : TG \to GT'$ such that*

- *$\langle G, n \rangle : T' \to T$ is a semi-comonad morphism.*
- *$Kl(\langle F, T'Fid \rangle) \dashv_s Kl(\langle G, n \rangle)$*

Proof: Suppose α, β are the natural transformations belonging to the semi-adjunction $F \dashv_s G$. Take $n = \alpha(T'(\beta(G(id))))$, then $\langle Kl(\langle F, T'Fid \rangle), Kl(\langle G, n \rangle), \alpha, \beta \rangle$ is a semi-adjunction. ■

Theorem 13 *If H is an iwGC, then $iKl(H)$ is a λ^2-algebra.*

Proof: We already know that each $iKl(H)(N) = Kl(HN)$ is a wCCC. It is easy to check that the functors $iKl(H)(u)$ preserve the wCCC-structure. By the previous lemma, the functors $iKl(H)(\pi) = Kl(\pi^*)$ have semi-rightadjoints $Kl(\Pi)$. It is easy to check that $iKl(H)$ satisfies the Beck-Chevalley conditions. ■

5 Rel as iwGC

In [6] we have shown that the category Rel of sets and relations is a wGC. In this section we will construct an iwGC out of Rel. Taking the indexed Kleisli category of this iwGC gives a simple example of a λ^2-algebra.

5.1 Inj

Define Inj as the category with as objects sets and as arrows injective functions. This category has a number of (well-known) useful properties, which are similar to the properties of algebraic dcpo's. Firstly, it is *directed complete.*

Definition 14 *Let* C *be a category. A* directed diagram *in* C *is a functor $D : I \to$ C, where I is a directed poset (i.e. every two elements have an upperbound in I) considered as a category.*

Theorem 15 Inj *is* directed complete, *i.e. each directed diagram in* Inj *has a colimit.*

Furthermore, finite sets are "compact" in Inj.

Theorem 16 *If $(\rho_i : A_i \to A | i \in I)$ is a directed colimit in* Inj, *and $f \in$ Inj(X, A) where X is finite, then there exists $k \in I$ and $f' \in$ Inj(X, A_k) such that $\rho_k \circ f' = f$.*

Let **S** be a set of finite sets such that each finite set is isomorphic to an element of **S**, then **S** forms a *basis* for Inj in the sense that each object of Inj is the colimit of a directed diagram of sets in **S**.

Theorem 17 *Let $A \in$ Inj, and let I be the set $\{\langle X, f\rangle | X \in \mathbf{S}, f \in$ Inj$(X, A)\}$. Order I by $\langle X, f\rangle \leq \langle X', f'\rangle$ iff there exist $g \in$ Inj(X, X') such that $f' = f \circ g$. Note that g is unique if it exists, and that I is directed.*
Define $D : I \to$ Inj by $D(\langle X, f\rangle) = X$ and $D(\langle X, f\rangle \leq \langle X', f'\rangle) = g : X \to X'$, then A is a colimit of D.

Note that the n-fold products Inj^n inherit these properties of Inj.

5.2 The functor H

The base category B has as objects the categories Inj^n. We take $\Omega = \mathsf{Inj}$. The arrows of B are *continuous* functors.

Definition 18 *Let* C, D *be directed complete categories. A functor* $F : \mathsf{C} \to \mathsf{D}$ *is* continuous *iff it preserves directed colimits, i.e. if* $(\rho_i : A_i \to A | i \in I)$ *is a directed colimit in* C*, then* $(F(\rho_i) : F(A_i) \to F(A) | i \in I)$ *is a directed colimit in* D.

The fibre $H_n = H(\Omega^n)$ has as objects continuous functors $\Omega^n \to \Omega$. The arrows R in H^n between objects $F, G : \Omega^n \to \Omega$ are *continuous families of relations*, i.e. $R = (R_A \subseteq F(A) \times G(A) | A \in \mathsf{Inj})$, and if $(\rho_i : A_i \to A | i \in I)$ is a directed colimit in Inj^n, then

$$R_A = \bigcup_{i \in I} \{(F(\rho_i)(a), G(\rho_i)(b)) | a R_{A_i} b\}$$

Note that a continuous family of relations is *monotone*: for $f : A \to B$ one has

$$a R_A b \Rightarrow F(f)(a) R_B G(f)(b)$$

The identities in H_n are the families of identities, and composition is defined componentwise: $(S \circ R)_A = S_A \circ R_A$. Given a continuous functor $U : \Omega^m \to \Omega^n$ we define $U^* : H_n \to H_m$ on objects F as $U^*(F) = F \circ U$, and on arrows R as $U^*(R)_A = R_{U(A)}$.

5.3 The functors Π_n

First we define the *trace* of a functor.

Definition 19 *Let* $F : \mathsf{Inj} \to \mathsf{Inj}$ *be a continuous functor. The* trace $Tr(F)$ *of* F *is the set* $\{\langle X, a\rangle | X \in \mathbf{S}, a \in F(X)\}$.

For each natural number n we define a semi-functor $\Pi_n : H_{n+1} = H(\Omega^n \times \Omega) \to H_n$. Let $F : \Omega^{n+1} \to \Omega$ be an object of H_{n+1}, then $\Pi_n(F)$ is an object of H_n, i.e. a continuous functor $\Omega^n \to \Omega$. On objects $A \in \Omega^n$ we define

$$\Pi_n(F)(A) = Tr(F(A, -))$$

and on arrows $f : A \to B$ in Ω^n we define $\Pi_n(F)(f) : Tr(F(A, -)) \to Tr(F(B, -))$ by

$$\Pi_n(F)(f)(X, a) = \langle X, F(f, id_X)(a)\rangle$$

Let $R : F \to G$ be an arrow in H_{n+1}, then $\Pi_n(R) : \Pi_n(F) \to \Pi_n(G)$ is an arrow in H_n defined by

$$\langle X, a\rangle(\Pi_n(R))_A\langle Y, b\rangle \Leftrightarrow \exists f : X \to Y(F(id_A, f)(a) R_{A,Y} b)$$

It is easy to see that in general Π_n is only a *semi*-functor:

$$\langle X, a\rangle(\Pi_n(id))_A\langle Y, b\rangle$$

$$\Leftrightarrow$$

$$\exists f : X \to Y(F(id_A, f)(a) = b)$$

$$\not\Leftrightarrow$$

$$\langle X, a\rangle = \langle Y, b\rangle$$

Let $P_n : \Omega^{n+1} \to \Omega^n$ be the projection in B.

Theorem 20 $P_n^* \dashv_s \Pi_n$

Proof: The functor P_n^* is defined on objects $F \in \Omega^n$ as $P_n^*(F) = F \circ \Pi_n$. We define natural transformations $\alpha^n_{F,G} : H_{n+1}(F \circ P_n, G) \to H_n(F, \Pi_n(G))$ and $\beta^n_{F,G} : H_n(F, \Pi_n(G)) \to H_{n+1}(F \circ P_n, G)$ as follows. Let $R : F \circ P_n \to G$ then

$$a\alpha(R)_A\langle Y, b\rangle \Leftrightarrow aR_{A,Y}b$$

Let $R : F \to \Pi_n(G)$, then

$$a\beta(R)_{A,A'}b \Leftrightarrow \exists Y, f : Y \to A', b' \in G(A, Y)(aR_A(Y, b')\&G(id_A, f)(b') = b)$$

It can be checked that $\langle P_n^*, \Pi_n, \alpha^n, \beta^n\rangle$ is a semi-adjunction. ∎

5.4 The structure on the fibers

The wGC-structure on the fibers H_n is similar to the structure on Rel [6] We shall give the definitions of the linear operators in H_n on objects F, G. Let A be an object of Ω^n.

- $1_n(A) = \emptyset$
- $I_n(A) = \{*\}$
- $\perp_n(A) = \{*\}$
- $(!_n F)(A) = \mathcal{P}_f(F(A))$
- $(F \times_n G)(A) = F(A) \uplus G(A)$
- $(F \otimes_n G)(A) = F(A) \diamond G(A)$
- $(F \multimap_n G)(A) = F(A) \diamond G(A)$

where $A \uplus B$, $A \diamond B$ are resp. the disjoint union and the cartesian product of two sets. It is easy to define the linear operators on arrows, and to show that the functors U^* preserve the structure on the fibers.

6 Some Related iwGC's

In this section we describe three iwGC's which are related to the model of the previous section.

6.1 PRel as iwGC

The objects A of the category PRel are pairs $\langle Dom_A, p_A\rangle$, where Dom_A is a set, and p_A is a *predicate* on Dom_A, i.e. p_A is a subset of Dom_A. We write $p_A(a)$ iff $a \in p_A$. The arrows $R : A \to B$ of PRel are relations $R \subseteq Dom_A \times Dom_B$ which *preserve truth*, i.e. $p_A(a)\&aRb$ implies $p_B(b)$. It is clear that Rel is a full subcategory of PRel. In [6] we have shown that PRel is, like Rel, a wGC. For example, a semi-comonad structure on PRel may be defined[3] by $Dom_{!A} = \{X \subseteq Dom_A | X \text{ finite } \}$ and $p_{!A}(X) \Leftrightarrow \forall a \in X(p_A(a))$
Analogous to the construction of an iwGC out of Rel we can build an iwGC out of PRel. In fact, there are only three differences. Firstly, instead of Inj we use the category PInj. This category has as objects sets and as arrows $f : A \to B$ injective functions $f : Dom_A \to Dom_B$ satisfying

$$p_A(a) \Leftrightarrow p_B(f(b))$$

The category PInj has properties similar to Inj. The second difference is that the arrows in the categories H_n are continuous families of *truth preserving* relations. Finally, on the trace of a continuous functor $F : \mathsf{PInj}^n \to \mathsf{PInj}$ a predicate $p_{Tr(F)}$ is defined by

$$p_{Tr(F)}(\langle X, a\rangle) \Leftrightarrow p_{F(X)}(a)$$

6.2 WCohl as iwGC

The category WCohl has as objects pairs $A = \langle Dom_A, q_A\rangle$, where Dom_A is a set and q_A is a *binary*, rather than an *unary*, predicate on Dom_A. In fact, we shall require that these predicates are symmetric. Arrows $R : A \to B$ are relations $R \subseteq Dom_A \times Dom_B$ which preserve the predicates (i.e. $q_A(a, a')\&arb\&a'Rb'$ implies $q_B(b, b')$).
It has been shown in [6] that WCohl can be equipped with the structure of a wGC. By now it should be clear how to build an iwGC out of WCohl. We shall not give details. The indexed semi-Kleisli category of this iwGC turns out to be the same as the λ^2-algebra that we get by dropping everywhere the word "stable" in the description of the *coherence space model* of the λ^2-calculus [1].

[3] Alternatively, we might take $Dom_{!A} = \{X \subseteq p_A | X \text{ finite}\}$.

6.3 NRel as iwGC

A very simple iwGC can be defined by using only sets of natural numbers. Firstly, we define the category NRel as the full subcategory of Rel with as objects elements of $\mathcal{P}\omega$. Because there are encodings of finite subsets $(-)^+ : \mathcal{F}(\omega) \cong \omega$ and of pairs $\langle \omega \times \omega \cong \omega$ of natural numbers as natural numbers, the category NRel can be equipped with essentially the same wGC structure as Rel. For example, we may define $!x = \{X^+ | X \subseteq x, X \text{ finite } \}$ for $x \subseteq \omega$.
It is well-known that $\mathcal{P}\omega$ ordered by inclusion is an algebraic lattice. The compact elements of $\mathcal{P}\omega$ are the finite sets. To convert NRel into an iwGC, we take base category B with as objects $(\mathcal{P}\omega)^n$ for $n \in \omega$ and as arrows continuous functions (i.e. functions that preserve directed joins). Furthermore, we take $\Omega = \mathcal{P}\omega$.
The objects in the category H_n are continuous functions $\Omega^n \to \Omega$, and the arrows $R : f \to g$ are families $(R_x \subseteq f(x) \times g(x) | x \in (\mathcal{P}\omega)^n)$ such that if $U \subseteq \mathcal{P}\omega$ is directed, then

$$R_{\bigcup U} = \bigcup_{x \in U} R_x$$

For every continuous function $f : \Omega^m \to \Omega^n$ a functor $f^* : H_n \to H_m$ is defined by $f^*(g) = g \circ f$ and $(f^*(R))_x = R_{f(x)}$, for $x \in \mathcal{P}\omega$.
If $f : \mathcal{P}\omega \to \mathcal{P}\omega$ is a continuous function, then $Tr(f) = \{\langle X^+, n\rangle | X \subseteq \omega, X \text{ finite } , n \in f(X)\}$. Functors $\Pi_n : H_{n+1} \to H_n$ are defined on objects by $\Pi_n(f)(x) = Tr(f(x, -))$, and on arrows by $\langle X^+, n\rangle \Pi_n(R)_x \langle Y^+, m\rangle \Leftrightarrow X \subseteq Y \& n R_{x,Y} m$.
In general, the Π_n are only *semi*-functors:

$$(X, n)(\Pi_n(id))_x(X', n')$$

$$\Leftrightarrow$$

$$X \subseteq X' \& n(id)_{x,X'} n'$$

$$\Leftrightarrow$$

$$X \subseteq X' \& n = n'$$

$$\not\Leftrightarrow$$

$$(X, n) = (X', n')$$

Note that this iwGC is in fact a simplification of the iwGC constructed out of Rel by requiring the arrows in Ω to be inclusions rather than, more general, injections.

7 Preordered Sets

The *Karoubi envelope* construction may be used to transform various semi notions to the corresponding ordinary notions [3, 6]. For example, the Karoubi envelope $K(\mathsf{C})$ of a category C is a CCC, resp. a GC if C is a wCCC, resp. a wGC. We do not know how to extend the Karoubi envelope construction to indexed categories such that the (extended) Karoubi envelope of an iwGC is an iGC. However, in special cases we can transform iwGC's into iGC's in a manner which is reminiscent of some sort of Karoubi envelope construction. In this section we shall transform the iwGC constructed out of NRel into a iGC. In a similar way the other iwGC's defined in this paper can be transformed into iGC's.

In [6] we have seen that taking the Karoubi envelope of Rel is more or less the same as equipping the objects A with a preorder [4] $\leq_A$ and requiring that the arrows $R : A \to B$ satisfy

$$a' \leq_A aRb \leq_B b' \Rightarrow a'Rb'$$

Hence, to transform the iwGC constructed out of NRel into a iGC, we shall equip all sets with preorders.

Definition 21 *$\mathcal{PR}\omega$ is the set of pairs $u = (x_u, \leq_u)$, where $x_u \in \mathcal{P}\omega$ and $\leq_u$ is a preorder on x_u.*

The set $\mathcal{PR}\omega$ can be ordered by

$$u \sqsubseteq v \Leftrightarrow (x_u \subseteq x_v \& \leq_u \subseteq \leq_v)$$

It is easy to check that $(\mathcal{PR}\omega, \sqsubseteq)$ is an algebraic lattice and that the compact elements are the finite preorders. The base category B has as objects $(\mathcal{PR}\omega)^n$ for $n \in \omega$, and as arrows continuous functions. The categories H_n are defined just as in the NRel-model except that we require the families of relations to satisfy the additional requirement

$$a' \leq_{f(u)} aR_u b \leq_{g(u)} b' \Rightarrow a'R_u b'$$

Note that the identity id_f on an object f in H_n is given by the family $(\leq_{f(u)} | u \in (\mathcal{PR}\omega)^n)$.

The GC-structure on the categories H_n is similar to the structure in the NRel-model. For example, for each n a functor $!_n : H_n \to H_n$ is defined on objects by $x_{!f(u)} = \{X^+ | X \subseteq x_{f(u)}, X \text{ finite}\}$, $X^+ \leq_{!f(u)} Y^+ \Leftrightarrow \forall b \in Y \exists a \in X(a \leq_{f(u)} b)$, and on arrows by $X^+ !R_u Y^+ \Leftrightarrow \forall b \in Y \exists a \subset X(aR_u b)$. Furthermore, there are natural transformations $\eta^n :! \to Id_{H_n}$, $\mu^n :! \to !!$ defined by $X^+(\eta^n_f)_u a \Leftrightarrow \exists a' \in X(a' \leq_{f(u)} a$ and $X^+(\mu^n_f)_u \chi^+ \Leftrightarrow \forall a' \in \bigcup \chi \exists a \in X(a \leq_{f(u)} a')$. It is easy to see that $(!_n, \eta^n, \mu^n)$ is a comonad on H_n.

The functors f^* are also as in the NRel-model.

Let $f : \mathcal{PR}\omega \to \mathcal{PR}\omega$ be a continuous function. The preorder $Tr(f)$ has domain $\{U^\pm | U \in \mathcal{PR}\omega, U \text{ finite}, n \in x_{f(U)}\}$ (where $(-)^\pm : \mathcal{FR}\omega \cong \omega$) and $\langle U^\pm, n\rangle \leq_{Tr(f)}$

[4] In fact, the objects are equipped with transitive relations $<$ having the interpolation property $a < c \Rightarrow \exists b(a < b < c)$.

$\langle V^{\pm}, m\rangle \Leftrightarrow U \sqsubseteq V \& n \leq_{f(V)} m$. Next the functors Π_n are defined as in the NRel-model, but using this preordered trace.
It is easy to check that Π_n preserves identities:

$$\langle U^{\pm}, n\rangle(\Pi_n(id_f))_u\langle V^{\pm}, m\rangle$$

$$\Leftrightarrow$$

$$U \sqsubseteq V \& n(id_f)_{u,V} m$$

$$\Leftrightarrow$$

$$U \sqsubseteq V \& n \leq_{f(u,V)} m$$

$$\Leftrightarrow$$

$$\langle U^{\pm}, n\rangle \leq_{Tr(f(u,-))} \langle V^{\pm}, m\rangle$$

$$\Leftrightarrow$$

$$\langle U^{\pm}, n\rangle \leq_{\Pi_n(f)(u)} \langle V^{\pm}, m\rangle$$

$$\Leftrightarrow$$

$$\langle U^{\pm}, n\rangle(id_{\Pi_n(f)})_u\langle V^{\pm}, m\rangle$$

As the categories H_n are GC's rather than wGC's we have in fact built an iGC. Applying the indexed Kleisli category gives a (second-order version of a) PL-category.

Acknowledgement

I would like to thank Bart Jacobs and Jan van Leeuwen for reading draft versions of this paper and for giving useful suggestions.

Appendix

A wCCC's

Definition 22 *A* weak Cartesian closed category (wCCC) C *[3, 6, 10, 12] is a category* C *with a terminal object* 1 *and binary products* $A \times B$, *and with the following data:*

- *For each pair of objects* $A, B \in$ C *an object* $A \Rightarrow B \in$ C, *and an arrow* $e_{A,B} \in$ C$((A \Rightarrow B) \times A, B)$. *Furthermore, for each arrow* $f \in$ C$(D \times A, B)$ *an arrow* $\Lambda(f) \in$ C$(D, A \Rightarrow B)$.

satisfying the following equations (omitting subscripts):

1. $e \circ (\Lambda(f) \times id) = f$
2. $\Lambda(f \circ (g \times id)) = \Lambda(f) \circ g$

B wGC's

Definition 23 *A* linear category *[6, 13, 15]* $\langle \mathsf{C}, \times, 1, \otimes, I, -\!\circ, \bot \rangle$ *is a category* C *such that:*

1. *The functor* $\times : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ *is a chosen product in* C, *and* 1 *is a terminal object in* C.

2. $\langle \mathsf{C}, \otimes, I \rangle$ *is a symmetric monoidal category, i.e.* $\otimes$ *is a functor* $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$, I *is an object in* C, *and there are natural isomorphisms* $\rho_{A,B} : A \otimes B \cong B \otimes A$, $\lambda_A : A \otimes I \cong A$ *and* $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ *satisfying certain commutative diagrams, the MacLane-Kelly coherence conditions (for more details see [11]).*

3. $\langle \mathsf{C}, \otimes, I, -\!\circ \rangle$ *is a symmetric monoidal closed category, i.e.* $-\!\circ$ *is a functor* $\mathsf{C}^{op} \times \mathsf{C} \to \mathsf{C}$ *and* $(-) \otimes B$ *is a left adjoint of* $B -\!\circ (-)$ *(i.e. there is a natural isomorphism* $\mathsf{C}(A \otimes B, C) \cong \mathsf{C}(A, B -\!\circ C)$*).*

4. $\bot$ *is a* dualising object *in* C, *i.e. for each object* A *the arrow* τ_A *given by the next derivation is an isomorphism:*
$$\dfrac{\dfrac{\dfrac{(A -\!\circ \bot) \stackrel{id}{\to} (A -\!\circ \bot)}{(A -\!\circ \bot) \otimes A \to \bot}}{A \otimes (A -\!\circ \bot) \to \bot}}{A \stackrel{\tau_A}{\to} (A -\!\circ \bot) -\!\circ \bot}$$

Definition 24 *A* wGC *[6]* $\langle \mathsf{C}, \times, 1, \otimes, I, -\!\circ, \bot, !, \eta, \mu, i, \sim \rangle$ *is a linear category* $\langle \mathsf{C}, \times, 1, \otimes, I, -\!\circ, \bot$ *such that*

1. $\langle \eta, \mu \rangle$ *is a* semi comonad *on* C.

2. $i : !1 \cong I$ *is an isomorphism, such that:*
   - $i \circ !(id_1) = i$

3. $\sim_{A,B} : !A \otimes !B \cong !(A \times B)$ *is a isomorphism natural in* A, B, *such that:*
   - $!(\langle pi_{A,B}), !(\pi'_{A,B}) \rangle) \circ \mu_{A \times B} = \sim_{!A,!B} \circ (\mu_A \otimes \mu_B) \circ \sim^{-1}_{A,B}$

   where $\pi_{A,B} : A \times B \to A$ *and* $\pi'_{A,B} : A \times B \to B$ *are projections, and* $\langle f, g \rangle : C \to A \times B$ *is the unique arrow such that* $\pi \circ \langle f, g \rangle = f$ *and* $\pi' \circ \langle f, g \rangle = g$.

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