Limit Theorems for Random Recurrences and Renewal-Type Processes

Limietstellingen voor Random Recurrente Betrekkingen en Processen van Vernieuwing Type
(met een samenvatting in het Nederlands)

Proefschrift

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CONTENTS

INTRODUCTION

A central theme of this thesis are random sequences $\{X_n : n \in \mathbb{N}_0\}$ which satisfy a linear recurrence relation of the form

$$X_n \stackrel{d}{=} X'_{I_n} + Z_n, \quad X_0 := a \geq 0, \ n \in \mathbb{N}, \tag{1}$$

where I_n is a random index with values in the set $\{0, 1, \ldots, n\}$, the sequence $\{X'_n : n \in \mathbb{N}_0\}$ is a distributional copy of $\{X_n : n \in \mathbb{N}_0\}$, the random vector (I_n, Z_n) is independent of $\{X'_n : n \in \mathbb{N}_0\}$, and a is some constant. A *solution* to (1) is understood as a sequence of marginal probability distributions for $X_0, X_1, \ldots$, thus two random sequences $\{X_n\}, \{Y_n\}$ correspond to the same solution if they satisfy $X_n \stackrel{d}{=} Y_n$ for $n \in \mathbb{N}_0$. With this convention, a unique solution exists under the assumption that $I_n < n$ holds with positive probability for all $n > 0$.

With recurrence relation (1) one naturally associates a nonincreasing discrete-time Markov chain on $\mathbb{N}_0$, which jumps from the generic state $n > 0$ to a random state $I_n \leq n$, and is eventually absorbed at 0. Then X_n is a sum of some random number of random terms that are conditionally independent given the path of the Markov chain starting at n. For instance, when $Z_n \equiv 1$ and $a = 0$, the variable X_n is the *absorption time*, i.e. the number of jumps (possibly of zero size) which the Markov chain needs to approach the absorbing state 0.

A well-known example of the latter kind is related to the cycle structure of random permutations [2]. Consider the random permutation π_n of $\{1, \ldots, n\}$ under the uniform distribution, meaning that all $n!$ possible realizations of π_n are equally likely. Then the number of elements I_n not included in the cycle containing element 1 has uniform distribution on $\{0, \ldots, n-1\}$. Moreover, if it turns that $I_n = m > 0$ then the permutation of the set of m elements is a stochastic copy of π_m, subject to the obvious re-labeling of these elements by $\{1, \ldots, m\}$. Therefore (1) holds for the random variable X_n equal to the number of cycles in the random permutation of $\{1, \dots, n\}$. Exactly the same distributional recurrence holds for the number of records in an i.i.d. sample of size n from a continuous distribution, and for many functionals of random combinatorial structures, notably for the number of nodes in the right-most (or the left-most) branch in a random binary search tree on n nodes [13].

Recurrence relations (1) often appear in connection with analysis of continuous-time Markov processes with complicated state-space. A range of examples is suggested by functionals of exchangeable coalescent processes, in which particles (or, depending on the context, masses, blocks, alleles etc) collide to merge larger particles [115]. In particlular, the absorption time in the coalescent starting with n particles satisfies (1), where Z_n is an exponentially distributed random time elapsed before the first collision event.

Distributional recurrences (1) are common in probability theory and many its applications including random algorithms, insurance models, processes of coalescence and fragmentation, random trees, combinatorial structures and random walks, just to mention a few. The last five years have seen an outbreak of activity around the random recurrence relations. Typically the analysis involves finding large-n asymptotics of moments of the solutions and deriving conditions which ensure weak convergence of X_n (properly normalized and centered) to some nondegenerate probability law. To tackle these problems several methods have been developed by various authors, of which we mention

(I) the method based on the singularity analysis of generating functions (P. Flajolet, M. Drmota);

(II) the contraction method (R. Neininger, L. Rüschendorf, U. Rösler);

(III) the method of moments (L. Devroye, H. Mahmoud).

Neither of these approaches is universal. In particular, (I) requires knowing the distribution of vector (I_n, Z_n) explicitly, while an indispensable ingredient of applicability of (II) and (III) is the availability of at least two-term asymptotic expansions of moments. Therefore, the analysis of each particular recursion with random indices calls for developing some peculiar method.

The present thesis establishes new limit results for several patterns of recurrence relation (1) intrinsically associated with three types of stochastic processes: random walks with a barrier, a sampling scheme which has become known as Bernoulli sieve, and branching random walks. Specifically, we shall study the absorption time and the number of zero increments for random walks with a barrier, and some functionals of the ordered partition resulting from the Bernoulli sieve. For branching random walks we shall study, in fact, a more complicated type of recursion related to the intrinsic martingale.

The random walks with a barrier have been recently introduced by A. Iksanov and M. Möhle, and earlier in somewhat different form by K. Hinderer and H. Walk. It has been found recently that random walks with a barrier offer powerful technical tools to treat the asymptotics in exchangeable partition-valued processes known as beta-coalescents. The Bernoulli sieve is Karlin's occupancy scheme with infinitely many 'boxes' and a random environment, which is given by a sequence of independent and identically distributed random variables $\{\eta_n : n \in \mathbb{N}\}$. The name *Bernoulli sieve* was introduced into probability usage by A. Gnedin [55]. From the viewpoint of applications in species sampling and Bayesian inference, the Bernoulli sieve has a certain appeal, as it provides a tractable model of random partition structures that lead to generalizations of both the Dirichlet process from nonparametric statistics and Ewens' sampling formula from population genetics. A complex of models from the theoretical computer science, like the leader election algorithm and search algorithms (as studied by A. Knopfmacher, G. Louchard, H. Mahmoud, H. Prodinger, J. Fill, P. Hitczenko, W. Szpankowski and many others) receive a natural extension in this new context. Ewens' sampling formula corresponds to the case where η_1 has the beta law with parameters $\theta > 0$ and 1, but also for general η_1 there is a deep connection between the stick-breaking model, random permutations and more general logarithmic combinatorial structures [2]. Branching random walks generalize the classical Galton-Watson processes. J.F.C. Kingman and J. Biggins were the first to investigate these processes in the 1970s. Since then branching random walks have become a popular object of research. Fundamental results which shaped the modern understanding of branching random walks were obtained by G. Alsmeyer, J. Biggins, A. Iksanov, A. Kyprianou, U. Rösler and others. It is known that the intrinsic martingales $\{W_n : n \in \mathbb{N}_0\}$ in branching random walks satisfy certain recurrence relations which reflect the internal recursive structure of these processes. This thesis will reveal the tail behavior of the distribution of $\sup_{n\geq 0} W_n$ and establish the asymptotics of $\mathbb{E}[W_n \log^+ W_n]$ for $\{W_n\}$ which is not uniformly integrable.

On the methodological side, we shall avoid direct assumptions on distribution of the ingredients (Z_n, I_n), rather relate the asymptotics of solutions of the random recursion with some underlying stochastic process. From the general viewpoint, the approach taken amounts to coupling the data Z_n, I_n with some more rich stochastic process. After this general introduction we now describe the contents and key results in more details.

Chapter 1. Let ξ be a random variable with proper distribution

$$p_k := \mathbb{P}\{\xi = k\}, \ k \in \mathbb{N}.$$

Denote by $\{S_n : n \in \mathbb{N}_0\}$ a zero-delayed random walk with the generic step distributed like ξ, i.e.,

$$S_0 := 0, \quad S_n := \xi_1 + \xi_2 + \ldots + \xi_n, \quad n \in \mathbb{N},$$

where $\{\xi_k : k \in \mathbb{N}\}$ are independent copies of ξ.

We define a *random walk with barrier* $n \in \mathbb{N}$ as the random sequence $\{R_k^{(n)} : k \in \mathbb{N}_0\}$ which satisfies the recursion

$$R_0^{(n)} := 0, \quad R_k^{(n)} := R_{k-1}^{(n)} + \xi_k 1_{\{R_{k-1}^{(n)} + \xi_k < n\}}, \quad k \in \mathbb{N}.$$

Informally, this is a standard positive random walk modified by the condition that a jump is canceled each time the random walk overshoots $n-1$.

Throughout it will be assumed that $p_1 > 0$. Introduce the quantities

$$M_n := \#\{k \in \mathbb{N} : R_{k-1}^{(n)} \neq R_k^{(n)}\} = \sum_{l=0}^{\infty} 1_{\{R_l^{(n)} + \xi_{l+1} < n\}};$$

$$T_n := \inf\{k \in \mathbb{N} : R_k^{(n)} = n-1\} = \sum_{l=1}^{\infty} 1_{\{R_l^{(n)} < n-1\}} + 1;$$

$$V_n := T_n - M_n = \#\{i \le T_n : R_{i-1}^{(n)} = R_i^{(n)}\} = \sum_{l=0}^{T_n - 1} 1_{\{R_l^{(n)} + \xi_{l+1} \ge n\}}$$

which correspond, respectively, to the number of positive jumps, the absorption time and the number of zero increments before the absorption for the random walk with barrier n.

The aim of this chapter is deriving weak convergence results for $\{T_n\}$ and $\{V_n\}$ which satisfy the distributional recurrence relations:

$$T_1 = 1, \quad T_n \overset{d}{=} T'_{Y_n} + N_n - 2 \cdot 1_{\{Y_n = 1\}}, \quad n = 2, 3, \ldots, \tag{1.9}$$

$$V_1 = 1, \quad V_n \overset{d}{=} V'_{Y_n} + 1 - 2 \cdot 1_{\{Y_n = 1\}}, \quad n = 2, 3, \ldots, \tag{1.10}$$

where

$$N_n := \inf\{k \in \mathbb{N} : S_k \ge n\}, \quad Y_n := n - S_{N_n - 1}, \quad n \in \mathbb{N}.$$

In both formulas random vector (Y_n, N_n) is independent of both $\{T'_k : k \in \mathbb{N}\}$, which is a copy of $\{T_k : k \in \mathbb{N}\}$, and $\{V'_k : k \in \mathbb{N}\}$, which is a copy of $\{V_k : k \in \mathbb{N}\}$, respectively.

Theorems 4, 5 and 6 provide conditions which ensure that T_n, properly normalized and centered, weakly converges. Explicit formulas for normalizing constants and possible limiting laws are derived. For instance, Theorem 6 states that, under a regular variation assumption, T_n, properly normalized with zero centering, weakly converges to the exponential functional of a subordinator (increasing Lévy process).

Assuming that the law of ξ belongs to the domain of attraction of an α-stable law, $\alpha \in (1, 2)$, Theorem 11 gives the two-term asymptotic expansions of the first two moments of the absorption time T_n. As a consequence, the asymptotics of $\operatorname{Var} T_n$ is obtained.

For the case of finite mean $\mathbb{E}\xi$, Theorem 12 establishes the weak convergence (without normalization) of V_n, the number of zero increments before the absorption. For the case $\mathbb{E}\xi = \infty$, Theorem 13 establishes a weak law of large numbers for V_n and provides the asymptotics of $\mathbb{E}V_n$.

Chapter 2. Denote $\{J_n : n \in \mathbb{N}_0\}$ a zero-delayed random walk with the generic step distributed like $|\log \eta|$, where a random variable η takes values in the open interval $(0,1)$. Let $E_1, \ldots, E_n$ be independent of $\{J_n\}$ i.i.d. sample from the standard exponential distribution. Denote $E_{1,n} \le E_{2,n} \le \ldots \le E_{n,n}$ the corresponding exponential order statistics.

The random walk together with the exponential sample define a random occupancy scheme, called *the Bernoulli sieve*, in which n 'balls' $1, 2, \ldots, n$ occupy an infinite array of 'boxes' indexed by the integers $1, 2, \ldots$, according to the rule: ball i falls in box k iff the exponential point E_i hits the interval (J_{k-1}, J_k). We say that the index of interval (J_{i-1}, J_i) is i, and we call this interval *occupied*, if it contains at least one of n exponential points, and call it *empty*, otherwise. Define the following functionals of the Bernoulli sieve:

$U_n = \inf\{k \in \mathbb{N} : J_k > E_{n,n}\}$, i.e., U_n is the index of the right-most occupied interval;

$K_{n,0} = \#\{1 \le k \le U_n - 1 : (J_{k-1}, J_k) \text{ is empty }\}$, i.e., $K_{n,0}$ is the number of empty intervals with indices not exceeding $U_n - 1$;

$K_n = \#\{k \in \mathbb{N} : (J_{k-1}, J_k) \text{ is occupied}\}$, i.e., K_n is the number of occupied intervals;

$Z_n := \#\{1 \le k \le n : E_{k,n} \in (J_{U_n-1}, J_{U_n})\}$, i.e., Z_n is the number of points in the right-most occupied interval.

It will be shown that the following recurrence relations hold:

$$U_0 := 0, \quad U_n \overset{d}{=} U'_{P_1^{(n)}} + 1, \quad n \in \mathbb{N}, \tag{2.9}$$

$$K_0 = 0, \quad K_n \overset{d}{=} K'_{Q_1^{(n)}} + 1, \quad n \in \mathbb{N}, \tag{2.10}$$

$$K_{0,0} := 0, \quad K_{n,0} \overset{d}{=} K'_{P_1^{(n)},0} + 1_{\{P_1^{(n)}=n\}}, \quad n \in \mathbb{N}, \tag{2.11}$$

where the random variable $P_1^{(n)}$ is independent of $\{U'_j : j \in \mathbb{N}_0\}$ and $\{K'_{j,0} : j \in \mathbb{N}_0\}$, copies of $\{U'_j : j \in \mathbb{N}_0\}$ and $\{K'_{j,0} : j \in \mathbb{N}_0\}$, respectively, and the random variable $Q_1^{(n)}$ is independent of $\{K'_j : j \in \mathbb{N}\}$, a copy of $\{K_j : j \in \mathbb{N}_0\}$.

The main results of this chapter are concerned with the weak convergence of the just introduced functionals acting on the Bernoulli sieve under the additional assumption that the law of $|\log \eta|$ is nonlattice. In particular, Theorem 19 establishes an ultimate criterion for the existence of limiting law for properly normalized and centered U_n. Under a side condition, Theorem 21 proves an analogous result for K_n. Among other things, this condition ensures a more delicate result given in Theorem 20: $K_{n,0}$ weakly converges without normalization. Finally, under natural assumptions on the law of η, Theorem 22 investigates the weak convergence of Z_n.

Chapter 3. Consider a population starting from one progenitor and evolving like a generalized Galton-Watson process, in which individuals may have infinitely many children. It is assumed that the structure of the branching process is enriched by the locations of individuals on the real line, so that the progenitor is located at the origin, and the displacements of children relative to their mother are described by a point process $\mathcal{Z} = \sum_{i=1}^{N} \delta_{X_i}$ on $\mathbb{R}$. Thus $N := \mathcal{Z}(\mathbb{R})$ is the total size of offspring of a particular member of the population, and X_i is the displacement of the ith child. The displacement processes of all population members are supposed to be independent copies of $\mathcal{Z}$. It is further assumed that $\mathcal{Z}(\{-\infty\}) = 0$ and $\mathbb{E}N > 1$ (supercriticality).

For $n \in \mathbb{N}_0$ let $\mathcal{Z}_n$ be the point process that defines the positions on $\mathbb{R}$ of the individuals of the n-th generation, with their total number being given by $\mathcal{Z}_n(\mathbb{R})$. The random process $\{\mathcal{Z}_n : n \in \mathbb{N}_0\}$ is called *the branching random walk* (BRW).

Suppose there exists $\gamma > 0$ such that

$$m(\gamma) \ := \ \mathbb{E} \int_{\mathbb{R}} e^{\gamma x} \, \mathcal{Z}(\mathrm{d}x) \in (0, \infty). \tag{2}$$

For $n \in \mathbb{N}$, define $\mathcal{F}_n$ to be the σ-field containing all information about the first n generations of the population, and let $\mathcal{F}_0$ be the trivial σ-field. Put

$$W_n \ := \ m(\gamma)^{-n} \int_{\mathbb{R}} e^{\gamma x} \, \mathcal{Z}_n(dx). \tag{3}$$

The sequence $\{(W_n, \mathcal{F}_n) : \ n \in \mathbb{N}_0\}$ forms a non-negative martingale with mean one which is called *intrinsic martingale* of the branching random walk.

Theorem 29 proves the power-like tail behavior of the law of $\sup_{n\geq 0} W_n$ for uniformly integrable (regular) martingales $\{W_n : n \in \mathbb{N}_0\}$ under a Kesten-like moment condition. Finally, Theorem 34 investigates the asymptotics of $\mathbb{E}[W_n \log^+ W_n]$, as $n \to \infty$, for non-regular martingales $\{W_n : n \in \mathbb{N}_0\}$.

Chapter 1

Random walks with barrier

1.1 Definition of the random walk with barrier and some applications

Let ξ be a random variable with proper distribution

$$p_k := \mathbb{P}\{\xi = k\},\ \ k \in \mathbb{N}.$$

Denote by $\{S_n : n \in \mathbb{N}_0\}$ a zero-delayed (i.e. starting at 0) random walk with step distributed as ξ:

$$S_0 := 0, \quad S_n := \xi_1 + \xi_2 + \ldots + \xi_n, \quad n \in \mathbb{N},$$

where $\{\xi_k : k \in \mathbb{N}\}$ are independent copies of ξ.

Definition 1. Call *random walk with (fixed) barrier* $n \in \mathbb{N}$ the sequence

$$R_0^{(n)} := 0, \quad R_k^{(n)} := R_{k-1}^{(n)} + \xi_k 1_{\{R_{k-1}^{(n)}+\xi_k<n\}}, \quad k \in \mathbb{N}.$$

Plainly, $\{R_k^{(n)} : k \in \mathbb{N}_0\}$ is a non-decreasing Markov chain which cannot reach the state n. Throughout the rest of this chapter we assume that $p_1 > 0$. Then the random walk with the barrier n will eventually get absorbed in the state $n-1$.

We shall be interested in the quantities

$$M_n := \#\{k \in \mathbb{N} : R_{k-1}^{(n)} \neq R_k^{(n)}\} = \sum_{l=0}^{\infty} 1_{\{R_l^{(n)} + \xi_{l+1} < n\}};$$

$$T_n := \inf\{k \in \mathbb{N} : R_k^{(n)} = n-1\} = \sum_{l=1}^{\infty} 1_{\{R_l^{(n)} < n-1\}} + 1;$$

$$V_n := T_n - M_n = \#\{i \leq T_n : R_{i-1}^{(n)} = R_i^{(n)}\} = \sum_{l=0}^{T_n - 1} 1_{\{R_l^{(n)} + \xi_{l+1} \geq n\}}$$

which are, respectively, the number of jumps, the absorption time and the number of zero increments before the absorption in the random walk with barrier n.

The random walks with barrier appear in various contexts.

Example 1. Meir and Moon in [104] proposed a procedure of isolating the root of random recursive tree with n vertices by using successive deletions of some edges. After deleting (cutting) one edge chosen at random the tree splits into two subtrees. At the next stage, only the subtree containing the root is considered and another edge of this subtree is deleted randomly, etc. This recursive process stops once the root has been isolated, i.e., the subtree containing the root obtained after cutting an edge consists of one vertex. Denote by X_n the (random) number of cuts required to isolate the root. It was shown in [83] that X_n has the same law as the number of jumps M_n in the random walk with barrier n provided the law of a step is

$$p_k = \frac{1}{k(k+1)}, \quad k \in \mathbb{N}.$$

Example 2. Exchangeable coalescent with multiple collisions (also known as Λ-*coalescent*) is a Markov process that starts at $t = 0$ with n blocks and evolves according to the following dynamics. When m blocks are present, each k-tuple of them collides and merges to form a single block at rate

$$\lambda_{mk} = \int_{[0,1]} x^{k-2}(1-x)^{m-k} \Lambda(\mathrm{d}x), \quad 2 \leq k \leq m,$$

where Λ is a finite measure on $[0, 1]$. Denote by X_n the number of collisions in Λ-coalescent which occur until there is just a single block.

It was proved in [84] that when

$$\Lambda(\mathrm{d}x) = \mathrm{const} \cdot x^{a-1} 1_{(0,1)}(x)(\mathrm{d}x) \tag{1.1}$$

for some $a \in (0,2)$, X_n has the same law as M_n provided

$$p_k = \frac{(2-a)\Gamma(a+k-1)}{\Gamma(a)\Gamma(k+2)}, \quad k \in \mathbb{N}. \tag{1.2}$$

Comparing Example 1 with Example 2 for $a = 1$ reveals a known fact: the number of collisions in the Bolthausen-Sznitman coalescent (which is the Λ-coalescent with Λ the Lebesgue measure on [0,1]) has the same law as the number of cuts required to isolate the root of a random recursive tree.

Example 3. One very general appearance of the random walks with barrier is the following. Let $\{Z_k : k \in \mathbb{N}_0\}$ be a non-increasing Markov chain with the state space $\mathbb{N}$ and transition probabilities π_{ij} for $j < i$, and $\pi_{ij} = 0$, otherwise. Define a random variable

$$X_n := \inf\{k \geq 1 : Z_k = 1 \text{ given } Z_0 = n\},$$

which is the absorption time of the Markov chain. It is clear that in case when

$$\pi_{n,n-k} = \frac{\mathbb{P}\{\xi = k\}}{\mathbb{P}\{\xi < n-1\}},$$

the distribution of X_n coincides with that of M_n.

Example 4. A class of one-dimensional online bin-packing problems is naturally associated with random walks with barrier. Suppose there is a bin of capacity $n-1$, in which some number of items of random integer sizes can be packed. The items arrive sequentially. By the 'greedy, on-line' packing policy, every item requiring some volume ξ_j is packed as it arrives, provided there is enough free space in the bin. However, if the remaining space is smaller than ξ_j, the item is rejected. Then M_n is the number of packed items, T_n is the number of packing trials until the bin is filled, and V_n is the number of rejected items that arrived until the bin is filled.

1.2 Connection with linear random recurrences

For fixed $m, i \in \mathbb{N}$ define

$$\widehat{R}_0^{(m)}(i) := 0, \quad \widehat{R}_k^{(m)}(i) := \widehat{R}_{k-1}^{(m)}(i) + \xi_{i+k} 1_{\{\widehat{R}_{k-1}^{(m)}(i)+\xi_{i+k}<m\}}, \ k \in \mathbb{N},$$

and

$$\widehat{T}_m(i) := \sum_{l=1}^{\infty} 1_{\{\widehat{R}_l^{(m)}(i)<m-1\}} + 1, \ \widehat{V}_m(i) := \sum_{l=0}^{\widehat{T}_m(i)-1} 1_{\{\widehat{R}_l^{(m)}(i)+\xi_{i+l+1}\geq m\}}.$$

For fixed $n \in \mathbb{N}$ and any $i \in \mathbb{N}$ the distributions of the sequences $\{R_k^{(n)} : k \in \mathbb{N}_0\}$ and $\{R_k^{(n)}(i) : k \in \mathbb{N}_0\}$ are the same. Hence

$$\widehat{T}_n(i) \stackrel{d}{=} T_n \quad \text{and} \quad \widehat{V}_n(i) \stackrel{d}{=} V_n. \tag{1.3}$$

Set $\widehat{T}_\cdot := \widehat{T}_\cdot(1)$ and recall that the condition $p_1 > 0$ is assumed to hold. The following lemma holds, in fact, for arbitrary nondecreasing Markov chains with a single absorbing state. Still, we present a proof for the special case in order to exemplify the kind argument adopted here.

Lemma 2. *For every fixed $n \in \mathbb{N}$ random variables T_n and V_n are almost surely finite.*

Proof. We prove the a.s. finiteness of T_n by induction on n. Clearly, $T_1 = 1$ a.s. Assuming that $T_k < \infty$ a.s., $k = 1, 2, \dots, n-1$, we will prove that $T_n < \infty$ a.s.

The sequence $\{\widehat{R}_k^{(n)}(1) : k \in \mathbb{N}\}$ and, hence, $\widehat{T}_n$ do not depend on ξ_1. Furthermore, the following equality holds

$$T_n = (1 + \widehat{T}_{n-\xi_1}(1)) 1_{\{\xi_1 \leq n-2\}} + 1_{\{\xi_1 = n-1\}} + (1 + \widehat{T}_n(1)) 1_{\{\xi_1 \geq n\}} \quad \text{a.s.}$$

By the induction assumption, $\widehat{T}_k < \infty$, $k = 1, 2 \dots, n-1$ a.s. Therefore,

$$\widehat{T}_{n-\xi_1} 1_{\{\xi_1 \leq n-2\}} = \sum_{k=1}^{n-2} \widehat{T}_{n-k} 1_{\{\xi_1 = k\}} < \infty \quad \text{a.s.}$$

Consequently,

$$\mathbb{P}\{T_n = \infty\} = \mathbb{P}\{(1+\widehat{T}_n)1_{\{\xi_1 \ge n\}} = \infty\} = \mathbb{P}\{T_n = \infty\}\mathbb{P}\{\xi_1 \ge n\},$$

which implies that

$$\mathbb{P}\{T_n = \infty\} = 0, \tag{1.4}$$

as $\mathbb{P}\{\xi_1 \ge n\} < 1$.

Since $M_1 = 0$ and $M_n \le n-1$ a.s. for $n = 2, 3, \ldots$, the a.s. finiteness of M_n's is obvious. Hence, rvs $V_n = T_n - M_n$ are a.s. finite, as well. The proof is complete. $\square$

We aim next to show that the sequences $\{T_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ satisfy linear random recurrences of the kind describe in the Introduction.

Define

$$N_n := \inf\{k \ge 1 : S_k \ge n\} \quad \text{and} \quad Y_n := n - S_{N_n - 1}, \ n \in \mathbb{N}.$$

Recall that the rv N_n is the time of the first passage through the threshold $n-1$ by a random walk $\{S_k : k \in \mathbb{N}_0\}$, and the rv Y_n is called the undershoot at level n for the random walk.

Proposition 3. *For each $n = 2, 3, \ldots$ the identities hold almost surely:*

$$\begin{aligned} T_n &= (1+\widehat{T}_{Y_n}(N_n))1_{\{Y_n \ne 1\}} + N_n - 1 \\ &= \widehat{T}_{Y_n}(N_n) + N_n - 2\cdot 1_{\{Y_n = 1\}} \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} V_n &= (1+\widehat{V}_{Y_n}(N_n))1_{\{Y_n \ne 1\}} \\ &= \widehat{V}_{Y_n}(N_n) + 1 - 2\cdot 1_{\{Y_n = 1\}}, \end{aligned} \tag{1.6}$$

Proof. We have

$$\begin{aligned} T_n &= T_n(1_{\{Y_n=1\}} + 1_{\{Y_n \ne 1\}}) = (N_n - 1)1_{\{Y_n = 1\}} + T_n 1_{\{Y_n \ne 1\}} \\ &= (N_n-1)1_{\{Y_n=1\}} + \sum_{k=1}^{N_n-1} 1_{\{R_k^{(n)} < n-1\}} 1_{\{Y_n \ne 1\}} + \sum_{k=N_n+1}^{\infty} 1_{\{R_k^{(n)} < n-1\}} 1_{\{Y_n \ne 1\}} \\ &+ 2\cdot 1_{\{Y_n \ne 1\}} = N_n - 1 + 2\cdot 1_{\{Y_n \ne 1\}} + \sum_{k=N_n+1}^{\infty} 1_{\{R_k^{(n)} < n-1\}} 1_{\{Y_n \ne 1\}}. \end{aligned}$$

On the event $\{Y_n = 1, k \geq N_n + 1\}$ the following equality holds $R_k^{(n)} = n - 1$. Hence

$$\sum_{k=N_n+1}^{\infty} 1_{\{R_k^{(n)}<n-1\}} 1_{\{Y_n \neq 1\}} = \sum_{k=N_n+1}^{\infty} 1_{\{R_k^{(n)}<n-1\}}.$$

Therefore, we obtain

$$\begin{aligned}
T_n &= N_n - 1 + 2 \cdot 1_{\{Y_n \neq 1\}} + \sum_{k=N_n+1}^{\infty} 1_{\{R_k^{(n)}<n-1\}} \\
&= N_n - 1 + 2 \cdot 1_{\{Y_n \neq 1\}} + \sum_{k=1}^{\infty} 1_{\{\widehat{R}_k^{(n-S_{N_n-1})}(N_n)<n-S_{N_n-1}-1\}} \\
&= N_n + \widehat{T}_{n-S_{N_n-1}}(N_n) - 2 \cdot 1_{\{Y_n=1\}},
\end{aligned}$$

as required.

Let us turn now to V_n. If $n - R_{N_n-1}^{(n)} = n - S_{N_n-1} = Y_n = 1$, then $V_n = 0$. If $Y_n \neq 1$, than the first zero increment of the random walk with the barrier n equals $R_{N_n}^{(n)} - R_{N_n-1}^{(n)}$, and $T_n \geq N_n + 1$. Hence

$$\begin{aligned}
V_n &= \left(1 + \sum_{l=N_n}^{T_n-1} 1_{\{R_l^{(n)}+\xi_{l+1}\geq n\}}\right) 1_{\{Y_n \neq 1\}} \\
&= \left(1 + \sum_{l=0}^{T_n-N_n-1} 1_{\{R_{N_n+l}^{(n)}+\xi_{N_n+l+1}\geq n\}}\right) 1_{\{Y_n \neq 1\}} \\
&\overset{(1.5)}{=} \left(1 + \sum_{l=0}^{\widehat{T}_{Y_n}(N_n)-1} 1_{\{\widehat{R}_l^{(Y_n)}(N_n)+\xi_{N_n+l+1}\geq Y_n\}}\right) 1_{\{Y_n \neq 1\}} \\
&= (1 + \widehat{V}_{Y_n}(N_n)) 1_{\{Y_n \neq 1\}}.
\end{aligned}$$

The proof is complete. □

Now we intend to show that equalities of distributions (1.3) remain still hold if n is replaced by Y_n and i is replaced by N_n. Indeed, for each fixed

$m \in \mathbb{N}_0$,

$$\begin{aligned}
\mathbb{P}\{\widehat{T}_{Y_n}(N_n) = m\} &= \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathbb{P}\{\widehat{T}_{n-j}(i) = m, N_n = i, S_{N_n-1} = j\} \\
= \ & \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathbb{P}\{\sum_{l=0}^{\infty} 1_{\{\widehat{R}_l^{(n-j)}(i) < n-j-1\}} = m-1, N_n = i, S_{N_n-1} = j\} \\
= \ & \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathbb{P}\{\sum_{l=0}^{\infty} 1_{\{\widehat{R}_l^{(n-j)}(i) < n-j-1\}} = m-1\} \mathbb{P}\{N_n = i, S_{N_n-1} = j\} \\
= \ & \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathbb{P}\{\sum_{l=0}^{\infty} 1_{\{R_l^{(n-j)} < n-j-1\}} = m-1\} \mathbb{P}\{N_n = i, S_{N_n-1} = j\} \\
= \ & \sum_{i=1}^{n} \sum_{j=0}^{n-1} \mathbb{P}\{T_{n-j} = m\} \mathbb{P}\{N_n = i, S_{N_n-1} = j\} \\
= \ & \mathbb{P}\{T_{Y_n} = m\},
\end{aligned}$$

which proves the following distributional equality

$$\widehat{T}_{Y_n}(N_n) \overset{d}{=} T_{Y_n}, \tag{1.7}$$

where on the right-hand side Y_n is independent of $\{T_k : k \in \mathbb{N}\}$.

The distributional equality

$$\widehat{V}_{Y_n}(N_n) \overset{d}{=} V_{Y_n}, \tag{1.8}$$

where on the right-hand side Y_n is independent of $\{V_k : k \in \mathbb{N}\}$, follows along the same lines.

From (1.7) and (1.5) it follows that

$$T_1 = 1, \quad T_n \overset{d}{=} T'_{Y_n} + N_n - 2 \cdot 1_{\{Y_n = 1\}}, \quad n = 2, 3, \ldots, \tag{1.9}$$

where a random vector (Y_n, N_n) is independent of $\{T'_k : k \in \mathbb{N}\}$, which in turn is a distributional copy of $\{T_k : k \in \mathbb{N}\}$.

Analogously, from (1.6) and (1.8) we deduce that

$$V_1 = 1, \quad V_n \overset{d}{=} V'_{Y_n} + 1 - 2 \cdot 1_{\{Y_n = 1\}}, \quad n = 2, 3, \ldots, \tag{1.10}$$

where Y_n is independent of $\{V'_k : k \in \mathbb{N}\}$, a copy of $\{V_k : k \in \mathbb{N}\}$.

1.3 Weak convergence of the absorption time in random walk with barrier

1.3.1 Formulation of the main results and discussion

This subsection is concerned with the weak convergence of the sequence $\{T_n : n \in \mathbb{N}\}$, properly normalized and centered. Below we will prove that, depending on the tail behavior of the distribution of rv ξ (a generic step of the random walk $\{S_k : k \in \mathbb{N}_0\}$) several scalings for T_n and corresponding limiting distributions come into play, among them are the stable distributions (including normal) and the distributions of the so-called exponential functionals of subordinators.

Prior to giving a strict formulation of the results let us discuss the asymptotics informally. The renewal theory suggests that the limiting behaviors of T_n and N_n must be similar, to the extent that the difference $T_n - N_n$ is in a suitable sense small. Specifically, the following will be shown.

(a) If $\mathbb{E}\xi < \infty$, then $T_n - N_n$ weakly converges. Therefore, T_n, properly normalized and centered, possesses a weak limit if and only if the same is true for N_n.

(b) Assuming that $\mathbb{E}\xi = \infty$ we have further cases.

(b1) If $\sum_{k=n}^{\infty} p_k \sim L(n)/n$, where L slowly varies at ∞ and if $(N_n - b_n)/a_n$ weakly converges to some measure μ then $(T_n - N_n)/a_n \xrightarrow{P} 0$, which proves that also $(T_n - b_n)/a_n$ weakly converges to μ. Thus, in this case and case (a) the weak behavior of T_n and N_n is the same.

(b2) If, for some $\alpha \in (0,1)$ and some L slowly varying at ∞, $\sum_{k=n}^{\infty} p_k \sim n^{-\alpha} L(n)$, and N_n/a_n weakly converges to some ν_1 then $(T_n - N_n)/a_n$ weakly converges to some ν_2. Although the argument exploited above does not apply, it will be proved that T_n/a_n weakly converges to some $\nu_3 \neq \nu_1$. Thus, in case (b2) a weak behavior of T_n is not completely determined by that of N_n, rather it is influenced by the weak behavior of both N_n and the undershoot $Y_n = n - S_{N_n-1}$ to, approximately, the same extent. This observation can be explained as follows. The probability of one big jump of $\{S_n : n \in \mathbb{N}_0\}$ in comparison to cases (a) and (b1) is higher and, therefore, the epoch N_n comes more quickly. As a consequence, a contribution to T_n of the sequence $\{R_k^{(n)} : k \in \mathbb{N}_0\}$, while $R_k^{(n)}$ is proceeding from $R_{N_n-1}^{(n)} = S_{N_n-1}$ to $n-1$, becomes significant.

We are ready to formulate our main results. In the sequel, we denote by $\mu_\alpha, \alpha \in [1,2)$ an α-stable law with characteristic function of the form

$$t \mapsto \exp\{-|t|^\alpha \Gamma(1-\alpha)(\cos(\pi\alpha/2) + i \sin(\pi\alpha/2)\operatorname{sgn}(t))\}, \quad t \in \mathbb{R}, \quad \alpha \in (1,2); \tag{1.11}$$

$$t \mapsto \exp\{-|t|(\pi/2 - i \log|t| \operatorname{sgn}(t))\}, \quad t \in \mathbb{R}, \quad \alpha = 1. \tag{1.12}$$

Theorem 4 provides necessary and sufficient conditions which ensure that T_n, properly normalized and centered, weakly converges.

Theorem 4. *If $m := \mathbb{E}\xi < \infty$ then the following assertions are equivalent.*

(i) There exist sequences of numbers $\{a(n), b(n) : n \in \mathbb{N}\}$ with $a(n) > 0$ and $b(n) \in \mathbb{R}$ such that, as $n \to \infty$, $(T_n - b(n))/a(n)$ converges weakly to a nondegenerate and proper probability law.

(ii) For some $\alpha \in [1,2]$ and some L slowly varying at ∞,

$$\sum_{k=1}^{n} k^2 p_k \sim n^{2-\alpha} L(n), \ n \to \infty. \tag{1.13}$$

If $\sigma^2 := \operatorname{Var}\xi < \infty$ then, with $b(n) := n/m$ and $a(n) := (m^{-3}\sigma^2 n)^{1/2}$, the limiting law is standard normal (with mean zero and variance one).

If $\sigma^2 = \infty$ and (1.13) holds with $\alpha = 2$ then, with $b(n) := n/m$ and $a(n) := m^{-3/2}c(n)$, where $\{c(n) : n \in \mathbb{N}\}$ is any positive sequence satisfying $\lim_{n\to\infty} nL(c(n))/c^2(n) = 1$, the limiting law is standard normal.

If (1.13) *holds with* $\alpha \in (1,2)$ *then, with* $b(n) := n/m$ *and* $a(n) := m^{-(\alpha+1)/\alpha}c(n)$*, where* $\{c(n) : n \in \mathbb{N}\}$ *is any positive sequence satisfying* $\lim_{n\to\infty} nL(c(n))/c^\alpha(n) = \frac{2-\alpha}{\alpha}$*, the limiting law is* μ_α*.*

Assume that (1.13) *holds with* $\alpha = 1$*. Let* $c : \mathbb{R}^+ \to \mathbb{R}^+$ *be any function satisfying* $\lim_{t\to\infty} tL(c(t))/c(t) = 1$*, and set* $\psi(t) := t\int_0^{c(t)} \mathbb{P}\{\xi > y\}\mathrm{d}y$*. Let* $b : \mathbb{R}^+ \to \mathbb{R}^+$ *be any function satisfying*

$$b(\psi(t)) \sim \psi(b(t)) \sim t,\ t \to \infty,$$

and set $a(t) := t^{-1}b(t)c(b(t))$*. Then, with the so defined* $b(n)$ *and* $a(n)$*, the limiting law is* μ_1*.*

In case $\alpha = 1$ and m is infinite, Theorem 5 provides conditions under which T_n, properly normalized and centered, weakly converges.

Theorem 5. *Suppose* $\mathbb{E}\xi = \infty$ *and*

$$\sum_{k=n}^{\infty} p_k \ \sim\ L(n)/n,\ n \to \infty, \tag{1.14}$$

for some L *slowly varying at* ∞*. Then, with the same* $b(n)$ *and* $a(n)$ *as in the case* $\alpha = 1$ *of Theorem 4, as* $n \to \infty$*,* $\frac{T_n - b(n)}{a(n)}$ *weakly converges to the* 1*-stable law* μ_1*.*

In the two previous assertions the weak asymptotic behavior of T_n was the same as that of N_n, the time of the first exceedance of the threshold $n-1$ by the random walk. In the next result we will encounter a different situation.

Theorem 6 addresses the case when $\mathbb{E}\xi$ is infinite, and specifies the conditions under which T_n, properly normalized without centering, weakly converges.

Theorem 6. *Assume that for some* $\alpha \in (0,1)$ *and some* L *slowly varying at* ∞*,*

$$\sum_{k=n}^{\infty} p_k \sim n^{-\alpha}L(n),\ n \to \infty. \tag{1.15}$$

Then, as $n \to \infty$,

$$\frac{T_n}{a(n)} \xrightarrow{d} \int_0^\infty e^{-U(t)} \mathrm{d}t,$$

where $a(n) := n^\alpha L^{-1}(n)$, and $\{U(t) : t \geq 0\}$ is a drift-free subordinator with the Lévy measure

$$\theta(\mathrm{d}t) = \frac{e^{-t/\alpha}}{(1 - e^{-t/\alpha})^{\alpha+1}} \mathrm{d}t, \quad t > 0.$$

1.3.2 Auxiliary results.

We start by showing that the sequences $\{\mathbb{E}T_n^k : n \in \mathbb{N}\}$, $k \in \mathbb{N}$, satisfy certain recurrences like (1.5).

Proposition 7. *For each $n \in \mathbb{N}$ and $k \in \mathbb{N}$ it holds that*

$$\mathbb{E}T_n^k = r_n \sum_{l=1}^{n-1} \mathbb{E}T_{n-l}^k p_l + k \cdot r_n \mathbb{E}T_n^{k-1} + D_k(\mathbb{E}T_n, \mathbb{E}T_n^2, \ldots, \mathbb{E}T_n^{k-2}), \quad (1.16)$$

where

$$r_n := \frac{1}{p_1 + p_2 + \cdots + p_{n-1}}, \ D_\cdot(x_1, x_2, \ldots, x_n) := \upsilon_\cdot^{(0)} + \sum_{i=1}^n \upsilon_\cdot^{(i)} x_i,$$

and $\{\upsilon_\cdot^{(i)} : i = 0, 1, 2, \ldots, n\}$ are some real numbers.

In particular, for $k = 1$

$$\mathbb{E}T_n = r_n \mathbb{P}\{\xi \neq n-1\} + r_n \sum_{k=1}^{n-1} \mathbb{E}T_{n-k} p_k \quad (1.17)$$

Proof. We have

$$\begin{aligned}
\mathbb{P}\{T_n = i\} &= \sum_{l=1}^{\infty} \mathbb{P}\{T_n = i, \xi_1 = l\} \\
&= \sum_{l=1}^{n-2} \mathbb{P}\{T_n = i, \xi_1 = l\} \\
&+ \mathbb{P}\{T_n = i, \xi_1 = n-1\} + \mathbb{P}\{T_n = i, \xi_1 \geq n\} \\
&= \sum_{l=1}^{n-1} \mathbb{P}\{T_{n-l} = i-1\}\mathbb{P}\{\xi_1 = l\} \\
&- 1_{\{i=2\}}\mathbb{P}\{\xi_1 = n-1\} + 1_{\{i=1\}}\mathbb{P}\{\xi_1 = n-1\} \\
&+ \mathbb{P}\{T_n = i-1\}\mathbb{P}\{\xi_1 \geq n\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbb{E}T_n^k &= \sum_{i=0}^{\infty} i^k \mathbb{P}\{T_n = i\} \\
&= \sum_{i=0}^{\infty} i^k \Bigg(\sum_{l=1}^{n-1} \mathbb{P}\{T_{n-l} = i-1\}\mathbb{P}\{\xi_1 = l\} - 1_{\{i=2\}}\mathbb{P}\{\xi_1 = n-1\} \\
&+ 1_{\{i=1\}}\mathbb{P}\{\xi_1 = n-1\} + \mathbb{P}\{T_n = i-1\}\mathbb{P}\{\xi_1 \geq n\} \Bigg).
\end{aligned}$$

Since

$$\begin{aligned}
& \sum_{i=0}^{\infty} (i-1+1)^k \sum_{l=1}^{n-1} \mathbb{P}\{T_{n-l} = i-1\}\mathbb{P}\{\xi_1 = l\} \\
= & \sum_{l=1}^{n-1} \mathbb{E}T_{n-l}^k p_l + \sum_{j=1}^{n-1} \binom{k}{j} \mathbb{E}T_{n-l}^j + 1,
\end{aligned}$$

and

$$\sum_{i=0}^{\infty} i^k \mathbb{P}\{T_n = k-1\}\mathbb{P}\{\xi_1 \geq n\} = \left(\mathbb{E}T_n^k + \sum_{j=1}^{n-1} \binom{k}{j} \mathbb{E}T_n^j + 1 \right) \sum_{l=n}^{\infty} p_l,$$

we conclude that

$$\begin{aligned}
\mathbb{E}T_n^k &= r_n \sum_{l=1}^{n-1} \mathbb{E}T_{n-l}^k p_l + k \cdot r_n \cdot \left(\sum_{l=1}^{n-1} \mathbb{E}T_{n-l}^{k-1} p_l + \mathbb{E}T_n^{k-1} \sum_{l=n}^{\infty} p_l \right) \\
&+ D_k(\mathbb{E}T_n, \mathbb{E}T_n^2, \ldots, \mathbb{E}T_n^{k-2}).
\end{aligned}$$

By using the latter relation for $\mathbb{E}T_n^{k-1}$ we arrive at (1.16). □

The next result will be used in the proof of Theorem 6.

Lemma 8. *If condition* (1.15) *holds then*

$$\lim_{n\to\infty} \frac{L(n)}{n^\alpha} \mathbb{E}T_n = \frac{\Gamma(1-\alpha)\Gamma(1+\alpha)}{\Gamma(1-\alpha)\Gamma(1+\alpha) - 1}.$$

Proof. It is known (see, for instance, [84]) that

$$\lim_{n\to\infty} \frac{L(n)}{n^\alpha} \mathbb{E}N_n = \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} =: \beta.$$

We first prove that

$$\limsup_{n\to\infty} \frac{\mathbb{E}T_n}{\mathbb{E}N_n} \leq b := \frac{1}{1-\beta}. \tag{1.18}$$

Assume the contrary. Then, for any $\varepsilon > 0$, inequality

$$\mathbb{E}T_n > (b+\varepsilon)\mathbb{E}N_n$$

holds for infinitely many n. Thus, we can pick ε in such a way that inequality

$$\mathbb{E}T_n > (b+\varepsilon)\mathbb{E}N_n + c$$

holds infinitely often for each fixed $c > 0$. Define $n_c := \inf\{n \geq 1 : \mathbb{E}T_n > (b+\varepsilon)\mathbb{E}N_n + c\}$ and notice that $\lim_{c\to\infty} n_c = \infty$. Then

$$\mathbb{E}T_n \leq (b+\varepsilon)\mathbb{E}N_n + c, \quad n = 1, 2, \ldots, n_c - 1.$$

Therefore,

$$\begin{aligned}
(b+\varepsilon)\mathbb{E}N_{n_c} + c \;&<\; \mathbb{E}T_{n_c} \overset{(1.17)}{=} r_{n_c}\mathbb{P}\{\xi_1 \neq n_c - 1\} + r_{n_c}\sum_{k=1}^{n_c-1}\mathbb{E}T_{n_c-k}p_k \\
&\leq\; r_{n_c}\mathbb{P}\{\xi_1 \neq n_c - 1\} + c + r_{n_c}(b+\varepsilon)\sum_{k=1}^{n_c-1}\mathbb{E}N_{n_c-k}p_k.
\end{aligned}$$

By analyzing what happens with N_n after the first step of random walk one can check that

$$\mathbb{E}N_n = 1 + \sum_{k=1}^{n-1}\mathbb{E}N_{n-k}p_k. \tag{1.19}$$

Hence

$$(b+\varepsilon)\mathbb{E}N_{n_c} + c < r_{n_c}\mathbb{P}\{\xi_1 \neq n_c - 1\} + c + r_{n_c}(b+\varepsilon)(\mathbb{E}N_{n_c} - 1),$$

or, equivalently,

$$0 < r_{n_c}\mathbb{P}\{\xi \neq n_c - 1\} - r_{n_c}(b+\varepsilon) + \mathbb{E}N_{n_c}(r_{n_c} - 1)(b+\varepsilon).$$

Since $r_n - 1 \sim n^{-\alpha}L(n), n \to \infty$ then letting in the last inequality c go to ∞ gives

$$\frac{\varepsilon}{b} < b - 1 + 1 - b = 0,$$

a contradiction. This proves (1.18). The same reasoning allows us to establish the converse inequality for the lower limit. The proof is complete. □

The next result establishes a weak law of large numbers for the absorption time.

Theorem 9. *If*

$$\sum_{l=1}^{n}\sum_{k=l}^{\infty} p_k \sim L(n), \ n \to \infty, \tag{1.20}$$

for some L slowly varying at ∞, then

$$\frac{T_n}{\mathbb{E}T_n} \xrightarrow{P} 1, \quad n \to \infty.$$

Moreover, $\mathbb{E}T_n \sim \dfrac{n}{L(n)}, \ n \to \infty.$

Proof. It suffices to check that, for each $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{\mathbb{E}T_n^k}{\mathbb{E}N_n^k} = 1, \tag{1.21}$$

as, according to [84, Proposition 2.2],

$$\mathbb{E}N_n^k \sim \frac{n^k}{L^k(n)}, \quad n \to \infty. \tag{1.22}$$

We prove (1.21) by induction on k, and proceed along the same path as in the proof of Lemma 8. Assume that (1.21) holds for $k = 1, 2, \ldots, m-1$, yet

$$\limsup_{n\to\infty} \frac{\mathbb{E}T_n^m}{\mathbb{E}N_n^m} > 1.$$

Then, for any $\varepsilon > 0$, inequality

$$\mathbb{E}T_n^m > (\varepsilon + 1)\mathbb{E}N_n^m$$

holds for infinitely many n. Hence, one can pick ε in such a way that inequality

$$\mathbb{E}T_n^m > (\varepsilon + 1)\mathbb{E}N_n^m + c$$

holds infinitely often for each fixed $c > 0$. Define $n_c := \inf\{n \geq 1 : \mathbb{E}T_n^m > (\varepsilon + 1)\mathbb{E}N_n^m + c\}$ and note that $\lim_{c\to\infty} n_c = \infty$. Then

$$\mathbb{E}T_n^m \leq (1+\varepsilon)\mathbb{E}N_n^m + c, \quad n = 1, 2, \ldots, n_c - 1. \tag{1.23}$$

The same argument which has led us to (1.19) allows one to show that

$$\mathbb{E}N_n^m = D_m^*(\mathbb{E}N_n, \dots, \mathbb{E}N_n^{m-2}) + m\mathbb{E}N_n^{m-1} + \sum_{i=1}^{n-1} \mathbb{E}N_{n-i}^m p_i,$$

where $D_\cdot^*(x_1, x_2, \dots, x_n) := v_\cdot^{(0)} + \sum_{i=1}^n w_\cdot^{(i)} x_i$, and $\{w_\cdot^{(i)} : i = 0, 1, 2, \dots, n\}$ are some real numbers.

Recalling the notation of Proposition 7, and writing for shorthand $D_m(\mathbb{E}T_{n_c})$ for $D_m(\mathbb{E}T_{n_c}, \mathbb{E}T_{n_c}^2, \dots, \mathbb{E}T_{n_c}^{m-2})$, and $D_m^*(\mathbb{E}N_{n_c})$ for $D_m^*(\mathbb{E}N_{n_c}, \mathbb{E}N_{n_c}^2, \dots, \mathbb{E}N_{n_c}^{m-2})$ we have

$$\begin{aligned}
(1+\varepsilon)\mathbb{E}N_{n_c}^m + c \quad &< \quad \mathbb{E}T_{n_c}^m \overset{(1.16)}{=} r_n \sum_{l=1}^{n_c-1} \mathbb{E}T_{n_c-l}^m p_l \\
&+ \quad m \cdot r_n \mathbb{E}T_{n_c}^{m-1} + D_m(\mathbb{E}T_{n_c}) \\
&\overset{(1.23)}{\leq} \quad r_n(1+\varepsilon) \sum_{l=1}^{n_c-1} \mathbb{E}N_{n_c-l}^m p_l + m \cdot r_n \mathbb{E}T_{n_c}^{m-1} + \\
&+ \quad D_m(\mathbb{E}T_{n_c}) + c \\
&= \quad (1+\epsilon)(r_{n_c} - 1)(\mathbb{E}N_{n_c}^m - D_m^*(\mathbb{E}N_{n_c}) - m\mathbb{E}N_{n_c}^{m-1}) + \\
&+ \quad (1+\epsilon)\mathbb{E}N_{n_c}^m - (1+\epsilon)(D_m^*(\mathbb{E}N_{n_c}) + m\mathbb{E}N_{n_c}^{m-1}) \\
&+ \quad m \cdot r_n \mathbb{E}T_{n_c}^{m-1} + D_m(\mathbb{E}T_{n_c}) + c,
\end{aligned}$$

Hence

$$\begin{aligned}
0 \quad &< \quad (1+\epsilon)(r_{n_c} - 1)(\mathbb{E}N_{n_c}^m - D_m^*(\mathbb{E}N_{n_c}) - m\mathbb{E}N_{n_c}^{m-1}) \\
&- \quad (1+\epsilon)(D_m^*(\mathbb{E}N_{n_c}) + m\mathbb{E}N_{n_c}^{m-1}) + m \cdot r_n \mathbb{E}T_{n_c}^{m-1} + D_m(\mathbb{E}T_{n_c}).
\end{aligned} \tag{1.24}$$

By [32, Theorem 1.7.2] condition (1.20) implies that $\lim_{n\to\infty} \frac{n}{L(n)}(r_n - 1) = 0$. By the induction assumption,

$$\lim_{c\to\infty} \frac{D_m(\mathbb{E}T_{n_c}, \dots, \mathbb{E}T_{n_c}^{m-2})}{\mathbb{E}N_n^{m-1}} \; = \; 0 \quad \text{and} \quad \lim_{c\to\infty} \frac{\mathbb{E}T_{n_c}^{m-1}}{\mathbb{E}N_{n_c}^{m-1}} \; = \; 1,$$

where for the first relation (1.22) has to be recalled. Dividing (1.24) by $\mathbb{E}N_{n_c}^{m-1}$ and letting c tend to ∞ we arrive at

$$0 < m - (1+\varepsilon)m = -\varepsilon m,$$

a contradiction which proves that

$$\limsup_{n\to\infty} \frac{\mathbb{E}T_n^k}{\mathbb{E}N_n^k} \le 1.$$

A symmetric argument proves the converse inequality for the lower bound. Thence (1.21) holds for every $k \in \mathbb{N}$. The proof is complete. □

Remark 10. Actually, Theorem 9 reveals that condition (1.20) ensures that

$$\frac{T_n}{N_n} \xrightarrow{P} 1, \quad n \to \infty.$$

1.3.3 Proof of Theorem 4.

A classical result of the renewal theory states that if $m = \mathbb{E}\xi < \infty$ then

$$Y_n \xrightarrow{d} Y, \quad n \to \infty,$$

where the random variable Y has distribution

$$\mathbb{P}\{Y = k\} = m^{-1}\mathbb{P}\{\xi \ge k\}, \quad k \in \mathbb{N}.$$

Now (1.3) and (1.5) together imply that

$$T_n - N_n \xrightarrow{d} T'_Y - 2 \cdot 1_{\{Y=1\}}, \quad n \to \infty.$$

Therefore for any sequence $\{d(n) : n \in \mathbb{N}\}$ such that $\lim_{n\to\infty} d(n) = \infty$,

$$\frac{T_n - N_n}{d(n)} \xrightarrow{P} 0, \quad n \to \infty.$$

Assume that the law of ξ does not belong to the domain of attraction of any α-stable law, $\alpha \in [1, 2]$. Then there do not exist sequences $\{x(n), y(n) : n \in \mathbb{N}\}$ with $x(n) \in \mathbb{R}$ and $y(n) > 0$, such that $(S_n - x(n))/y(n)$ weakly converges to a proper and non-degenerate probability law. In view of equality

$$\mathbb{P}\{N_n > m\} = \mathbb{P}\{S_m \le n - 1\},$$

the same is true for N_n.
In the reverse direction, assume that (1.13) holds.

CASE 1: $\alpha \in [1, 2)$. Since

$$\sum_{k=n}^{\infty} p_k \sim \frac{2-\alpha}{\alpha} n^{-\alpha} L(n),$$

then, according to [30, Theorem 3b] and [7, Theorem 2], if $\alpha \in (1, 2)$ and $\alpha = 1$, respectively,

$$\frac{N_n - b(n)}{a(n)} \Rightarrow \mu_\alpha, n \to \infty,$$

with $a(n)$ and $b(n)$ as defined in the formulation of the theorem.

CASE 2: $\alpha = 2$. It is known (see, for instance, [48]) that there exists a sequence $\{a_n : n \in \mathbb{N}\}$ such that $na_n^{-2} L(a_n) \to 1$ and

$$\frac{S_n - nm}{a_n} \Rightarrow \mu_2, \quad n \to \infty,$$

where μ_2 is the standard normal law. Then, by [73, Theorem 2]

$$\frac{N_n - b(n)}{a(n)} \Rightarrow \mu_2.$$

The proof is complete.

1.3.4 Proof of Theorem 5.

Condition (1.14) ensures that

$$m(x) := \int_0^x \mathbb{P}\{\xi > y\} \mathrm{d}y, \quad x > 0$$

belongs de Haan's class Π of regular varing functions [32], i.e.

$$\lim_{t \to \infty} \frac{m(\lambda t) - m(t)}{L(t)} = \log \lambda.$$

Hence $m(x)$ is slowly varying at ∞. Since $\sum_{l=1}^{n} \sum_{k=l}^{\infty} p_k \sim m(n)$ then, according to Remark 10,

$$\frac{T_n}{N_n - 1} \xrightarrow{P} 1, \quad n \to \infty.$$

By Theorem 3(c) and formula on p. 42 in [30] (see also [7, Theorem 2]),

$$\frac{N_n - b(n) - 1}{a(n)} \Rightarrow \mu_1.$$

Therefore,

$$\frac{T_n - b(n)}{a(n)} - \frac{T_n - N_n + 1}{N_n - 1}\frac{b(n)}{a(n)} \Rightarrow \mu_1.$$

Thus, it remains to prove that

$$\begin{aligned}\frac{T_n - N_n + 1}{N_n - 1}\frac{b(n)}{a(n)} &\overset{(1.5)}{=} \frac{\widehat{T}_{Y_n} + 1 - 2 \cdot 1_{\{Y_n=1\}}}{Y_n/m(Y_n)}\frac{m(n)}{m(Y_n)}\frac{b(n)Y_n}{na(n)}\frac{n}{m(n)(N_n - 1)} \\ &=: \prod_{i=1}^{4} K_i(n) \overset{P}{\to} 0.\end{aligned}$$

Theorem 9 implies that

$$\frac{m(n)T_n}{n} \overset{P}{\to} 1 \quad n \to \infty.$$

Hence, by using equality of distributions (1.7) and the fact that

$$Y_n \overset{P}{\to} \infty, \quad n \to \infty,$$

we conclude that

$$K_1(n) \overset{P}{\to} 1.$$

By [46, Theorem 6],

$$K_2(n) \overset{d}{\to} 1/R, \quad n \to \infty,$$

where R is an rv with the uniform law on $[0, 1]$. Finally, by Proposition 2.3 and Corollary 2.1 from [84], respectively,

$$K_3(n) \overset{P}{\to} 0, \quad K_4(n) \overset{P}{\to} 1, \quad n \to \infty.$$

The proof is complete.

1.3.5 Proof of Theorem 6.

The Laplace exponent of $\{U(t) : t \geq 0\}$ takes the form

$$\begin{aligned}\Phi(x) &= \int_0^\infty (1 - e^{-xy})\frac{e^{-y/\alpha}}{(1 - e^{-y/\alpha})^{\alpha+1}}\mathrm{d}y \\ &= \frac{\Gamma(1-\alpha)\Gamma(\alpha x + 1)}{\Gamma(\alpha(x-1)+1)} - 1, \quad x \geq 0.\end{aligned}$$

Then, as is well-known (see, for instance, [17]),

$$\mathbb{E}T^k = \frac{k!}{\Phi(1)\cdots\Phi(k)}, \quad k \in \mathbb{N},$$

where $T := \int_0^\infty e^{-U(t)}\mathrm{d}t$, and the moment sequence $\{\mathbb{E}T^k : k \in \mathbb{N}\}$ uniquely determines the law of T.

Recall that an rv X_β has the Mittag-Leffler law with parameter $\beta \in [0,1)$, if

$$\mathbb{E}X_\beta^k = \frac{k!}{\Gamma^k(1-\beta)\Gamma(1+k\beta)} =: b_k(\beta), \quad k \in \mathbb{N}.$$

By [84, Proposition 2.1],

$$\lim_{n\to\infty} \frac{L^k(n)}{n^{\alpha k}} \mathbb{E}N_n^k = \mathbb{E}X_\alpha^k = b_k(\alpha), \quad k \in \mathbb{N}, \tag{1.25}$$

which, among other things, means that

$$\frac{L(n)}{n^\alpha} N_n \xrightarrow{d} X_\alpha, \quad n \to \infty.$$

In view of Lemma 8,

$$\lim_{n\to\infty} \frac{L(n)}{n^\alpha} \mathbb{E}T_n = \mathbb{E}T.$$

By using the induction and arguments which are similar to those exploited in the proof of Theorem 9, one can check that

$$\lim_{n\to\infty} \frac{\mathbb{E}T_n^k}{\mathbb{E}N_n^k} = \beta_k, \quad k = 2, 3, \ldots,$$

where

$$\beta_k := \frac{1}{b_{k-1} - k^{-1}b_k} \prod_{i=1}^{k-1} \frac{b_{i-1}}{b_{i-1} - i^{-1}b_i}, \quad b_k = b_k(\alpha), \quad k = 2, 3, \ldots$$

Thence, taking into account (1.25), we have

$$\lim_{n\to\infty} \frac{L^k(n)}{n^{\alpha k}} \mathbb{E}T_n^k = \mathbb{E}T^k, \quad k \in \mathbb{N},$$

which completes the proof.

The method of moments used in the proof above does not shed any light on the appearance in the limit of the law of exponential subordinator. Below we provide necessary explanations, only assuming that $\frac{T_n}{a(n)}$ converges in distribution to some rv T, but not assuming that the law of T is known.

By virtue of (1.9), we obtain

$$\frac{T_n}{a(n)} \overset{d}{=} \frac{T'_{Y_n}}{a(Y_n)} \frac{a(Y_n)}{a(n)} + \frac{N_n - 2 \cdot 1_{\{Y_n=1\}}}{a(n)}, \quad n = 2, 3, \dots, \tag{1.26}$$

where a random vector (Y_n, N_n) is independent of $\{T'_k : k \in \mathbb{N}\}$, a copy of $\{T_k : k \in \mathbb{N}\}$. Since

$$Y_n \overset{P}{\to} \infty, \quad n \to \infty, \tag{1.27}$$

then

$$\frac{T'_{Y_n}}{a(Y_n)} \overset{p}{\to} T', \quad n \to \infty, \tag{1.28}$$

where T' has the same law as T. [84, Proposition 2.5] implies that

$$\left(\frac{a(Y_n)}{a(n)}, \frac{N_n}{a(n)} \right) \overset{d}{\to} \left(e^{-U(V)}, \int_0^V e^{-U(t)} \mathrm{d}t \right), \ n \to \infty, \tag{1.29}$$

where V is a rv with the standard exponential law which is independent of $\{U(t) : t \geq 0\}$. Set $M := e^{-U(V)}$, $Q := \int_0^V e^{-U(t)} \mathrm{d}t$ and notice that Q has the Mittag-Leffler law with parameter α. As the left-hand side of (1.26) weakly converges,

$$\rho_n := \left(\frac{T'_{Y_n}}{a(Y_n)}, \frac{a(Y_n)}{a(n)}, \frac{N_n - 2 \cdot 1_{\{Y_n=1\}}}{a(n)} \right),$$

weakly converges, as well. Recalling (1.26) and (1.29)

$$\rho_n \overset{d}{\to} (T', M, Q) \, .$$

Consequently, letting $n \to \infty$ we obtain a distributional identity

$$T \overset{d}{=} MT' + Q, \tag{1.30}$$

where T' is independent of (M, Q).

To establish the stated independence, it suffices to show that

$$\lim_{n\to\infty}\left|\mathbb{E}e^{i(t\frac{T'_{Y_n}}{a(Y_n)}+v\frac{a_{Y_n}}{a(n)}+u\frac{N_n-2\cdot 1_{\{Y_n=1\}}}{a(n)})}-\mathbb{E}e^{itT'}\mathbb{E}e^{i(vM+uQ)}\right|=0,$$

for all real t, u, v.

From (1.28), (1.29) and (1.27) it follows that for any $\varepsilon > 0$ there exist $N_i = N_i(\varepsilon), i = 1, 2, 3$ such that

$$\left|\mathbb{E}e^{it\frac{T'_{Y_n}}{a(Y_n)}}-\mathbb{E}e^{itT'}\right|<\varepsilon,\quad t\in\mathbb{R},\quad n>N_1, \tag{1.31}$$

$$\left|\mathbb{E}e^{i(v\frac{a_{Y_n}}{a(n)}+u\frac{N_n-2\cdot 1_{\{Y_n=1\}}}{a(n)})}-\mathbb{E}e^{i(vM+uQ)}\right|<\varepsilon,\quad u,v\in\mathbb{R},\quad n>N_2, \tag{1.32}$$

$$\mathbb{P}\{Y_n\le N\}<\varepsilon,\quad n>N_3,$$

where $N := \max\{N_1, N_2\}$.

Therefore,

$$\begin{aligned}
&\left|\mathbb{E}e^{i(t\frac{T'_{Y_n}}{a(Y_n)}+v\frac{a_{Y_n}}{a(n)}+u\frac{N_n-2\cdot 1_{\{Y_n=1\}}}{a(n)})}-\mathbb{E}e^{itT'}\mathbb{E}e^{i(vM+uQ)}\right|\\
=&\left|\sum_{s=0}^{n-1}\sum_{l=1}^{s+1}\left(\mathbb{E}e^{i(t\frac{T'_{n-s}}{a(n-s)}+v\frac{a_{n-s}}{a(n)}+u\frac{l-2\cdot 1_{\{n-s=1\}}}{a(n)})}-\mathbb{E}e^{itT'}\mathbb{E}e^{i(vM+uQ)}\right)\right.\\
&\cdot\left.\mathbb{P}\{S_{l-1}=s,N_n=l\}\right|\le\left|\sum_{s=0}^{n-N}\sum_{l=1}^{s+1}\ldots\right|+\left|\sum_{s=n-N+1}^{n-1}\sum_{l=1}^{s+1}\ldots\right|.
\end{aligned}$$

For all $t, u, v \in \mathbb{R}$ and $n > \max(N, N_3)$ the second summand can be estimated as follows:

$$\begin{aligned}
&\left|\sum_{s=n-N+1}^{n-1}\sum_{l=1}^{s+1}\left(\mathbb{E}e^{i(t\frac{T'_{n-s}}{a(n-s)}+v\frac{a_{n-k}}{a(n)}+u\frac{l-2\cdot 1_{\{n-s=1\}}}{a(n)})}-\mathbb{E}e^{itT'}\mathbb{E}e^{i(vM+uQ)}\right)\right.\\
&\cdot\left.\mathbb{P}\{S_{l-1}=s,N_n=l\}\right|\le 2\mathbb{P}\{Y_n\le N\}\le 2\varepsilon.
\end{aligned}$$

As for the first summand, we obtain

$$\begin{aligned}
&\Big|\sum_{s=0}^{n-N}\sum_{l=1}^{s+1}\Big(\mathbb{E}e^{i(t\frac{T'_{n-s}}{a(n-s)}+v\frac{a_{n-s}}{a(n)}+u\frac{l-2\cdot 1_{\{n-s=1\}}}{a(n)})}-\mathbb{E}e^{itT'}\mathbb{E}e^{i(vM+uQ)}\Big)\\
&\cdot\quad \mathbb{P}\{S_{l-1}=s,N_n=l\}\Big|\\
\le\quad&\sum_{s=0}^{n-N}\sum_{l=1}^{s+1}\Big|(\mathbb{E}e^{it\frac{T'_{n-s}}{a(n-s)}}-\mathbb{E}e^{itT'})\mathbb{E}e^{i(vM+uQ)}\Big|\mathbb{P}\{S_{l-1}=s,N_n=l\}\\
+\quad&\sum_{s=0}^{n-N}\sum_{l=1}^{s+1}\Big|\mathbb{E}e^{i\frac{T'_{n-s}}{a(n-s)}}\Big(\mathbb{E}e^{i(v\frac{a_{n-s}}{a(n)}+u\frac{N_n-2\cdot 1_{\{n-s=1\}}}{a(n)})}-\mathbb{E}e^{i(vM+uQ)}\Big)\Big|\\
&\cdot\quad \mathbb{P}\{S_{l-1}=s,N_n=l\}\overset{(1.31)(1.32)}{\le}2\varepsilon.
\end{aligned}$$

According to [36, Lemma 6.2], the law of $\int_0^\infty e^{-U(t)}\,\mathrm{d}t$ is a solution to (1.30). From the results of [126] it follows that this solution is unique.

1.4 Asymptotics of moments of the absorption time

Assuming that the law of ξ belongs to the domain of attraction of an α-stable law, $\alpha\in(1,2)$, below we will find two-term asymptotic expansions of the first two moments of the absorption time. As a consequence, we will derive the asymptotics of the variance.

Theorem 11. *Assume that, for some $\alpha\in(1,2)$ and some L slowly varying at ∞,*

$$\sum_{k=1}^{n}k^2p_k\sim n^{2-\alpha}L(n),\quad n\to\infty. \tag{1.33}$$

Then, as $n\to\infty$,

$$\mathbb{E}T_n=\frac{n}{m}+\frac{1}{(\alpha-1)m^2}n^{2-\alpha}L(n)+o(n^{2-\alpha}L(n)), \tag{1.34}$$

$$\mathbb{E}T_n^2=\frac{n^2}{m^2}+\frac{5-\alpha}{(3-\alpha)(\alpha-1)m^3}n^{3-\alpha}L(n)+o(n^{3-\alpha}L(n)), \tag{1.35}$$

$$\operatorname{Var} T_n \sim \frac{1}{(3-\alpha)m^3} n^{3-\alpha} L(n),$$

where $m = \mathbb{E}\xi < \infty$.

Proof. By a renewal theorem,

$$u_n := \sum_{i=0}^{n} \mathbb{P}\{S_i = n\} \to m^{-1}, \quad n \to \infty.$$

In view of Theorem 9 and its proof we have

$$\mathbb{E}T_n \sim \frac{n}{m}, \quad \mathbb{E}T_n^2 \sim \frac{n^2}{m^2}, \quad n \to \infty. \tag{1.36}$$

Integrating by parts and using [32, Theorem 1.5.11(ii)] allows us to conclude that condition (1.33) implies the following:

$$\mathbb{P}\{\xi > n\} \sim \frac{2-\alpha}{\alpha} n^{-\alpha} L(n), \quad n \to \infty.$$

Hence

$$\mathbb{E}T_n \mathbb{P}\{\xi \geq n\} \sim \frac{2-\alpha}{\alpha m} n^{1-\alpha} L(n), \quad n \to \infty.$$

Thus, as $n \to \infty$,

$$\begin{aligned}
\mathbb{E}T_{Y_n} &= \sum_{k=0}^{n-1} \mathbb{E}T_{n-k} \mathbb{P}\{S_{N_n-1} = k\} \\
&= \sum_{k=0}^{n-1} \mathbb{E}T_{n-k} \sum_{j=0}^{k} \mathbb{P}\{S_j = k,\ N_n = j+1\} \\
&= \sum_{k=0}^{n-1} \mathbb{E}T_{n-k} \sum_{j=0}^{k} \mathbb{P}\{S_j = k,\ \xi_{j+1} \geq n-k\} \\
&= \sum_{k=0}^{n-1} \mathbb{E}T_{n-k} \mathbb{P}\{\xi \geq n-k\} u_k \sim \frac{1}{\alpha m^2} n^{2-\alpha} L(n).
\end{aligned}$$

Analogously

$$\mathbb{E}T^2_{n-S_{N_n-1}} \sim \frac{(2-\alpha)}{\alpha(3-\alpha)m^3} n^{3-\alpha} L(n), \quad n \to \infty.$$

Further

$$\begin{aligned}\sum_{l=1}^{n-1}\sum_{k=2}^{l+1} k\mathbb{P}\{S_{k-1}=l\} &= \sum_{k=1}^{n} k\mathbb{P}\{N_n \ge k\} - 1\\ &= (1/2)(\mathbb{E}N_n^2+\mathbb{E}N_n) - 1 \sim \frac{n^2}{2m^2}, \quad n\to\infty.\end{aligned}$$

Hence, according to [32, Corollary 1.7.3],

$$\sum_{k=2}^{n+1} k\mathbb{P}\{S_{k-1}=l\} \sim \frac{n}{m^2}, \quad n\to\infty.$$

All these relations altogether imply that

$$\begin{aligned}\mathbb{E}T_{n-S_{N_n-1}}N_n &= \mathbb{E}T_n\mathbb{P}\{\xi\ge n\}\\ &+ \sum_{l=1}^{n-1}\mathbb{E}T_{n-l}\mathbb{P}\{\xi\ge n-l\}\sum_{k=2}^{l+1}k\mathbb{P}\{S_{k-1}=l\}\\ &\sim \frac{1}{\alpha(3-\alpha)m^3}n^{3-\alpha}L(n), \quad n\to\infty.\end{aligned}$$

From (1.9) we obtain

$$\mathbb{E}T_n = \mathbb{E}N_n + \mathbb{E}T_{Y_n} - 2\;\mathbb{P}\{Y_n=1\};$$

$$\mathbb{E}T_n^2 = \mathbb{E}T_{Y_n}^2 + \mathbb{E}(N_n - 2\cdot 1_{\{Y_n=1\}})^2 + 2\mathbb{E}T_{Y_n}N_n - 4\mathbb{E}T_{Y_n}1_{\{Y_n=1\}}.$$

By [47, Theorem 10(a)]) (see also Remark 1 on p. 866 and Theorem 2.4 in [105], respectively),

$$\mathbb{E}N_n = \frac{n}{m} + \frac{1}{\alpha(\alpha-1)m^2}n^{2-\alpha}L(n) + o(n^{2-\alpha}L(n))$$

and

$$\mathbb{E}N_n^2 = \frac{n^2}{m^2} + \frac{4}{\alpha(\alpha-1)(3-\alpha)m^3}n^{3-\alpha}L(n) + o(n^{3-\alpha}L(n)).$$

Recalling the asymptotics of $\mathbb{E}T_{Y_n}$ and using the following relation

$$\begin{aligned}\mathbb{P}\{Y_n=1\} &= \sum_{k=0}^{n-1}\mathbb{P}\{S_k=n-1, N_n=k+1\}\\ &= \sum_{k=0}^{n-1}\mathbb{P}\{S_k=n-1, \xi_{k+1}\ge 1\}\\ &= \sum_{k=0}^{n-1}\mathbb{P}\{S_k=n-1\} \to m^{-1}, \; n\to\infty\end{aligned}$$

leads to

$$\begin{aligned}
\mathbb{E}T_n &= \mathbb{E}N_n + \mathbb{E}T_{Y_n} - 2\,\mathbb{P}\{Y_n = 1\} \\
&= \frac{n}{m} + \frac{1}{\alpha(\alpha-1)m^2}n^{2-\alpha}L(n) + \frac{1}{\alpha m^2}n^{2-\alpha}L(n) + o(n^{2-\alpha}L(n)) = \\
&= \frac{n}{m} + \frac{1}{(\alpha-1)m^2}n^{2-\alpha}L(n) + o(n^{2-\alpha}L(n)),
\end{aligned}$$

which proves (1.34). Since

$$\mathbb{E}T_n^2 = \mathbb{E}T_{Y_n}^2 + \mathbb{E}N_n^2 + 2\mathbb{E}T_{Y_n}N_n + o(n^{3-\alpha}L(n))$$

then utilizing the principal terms asymptotics we arrive at (1.35). The proof is complete. □

1.5 Weak convergence and a weak law of large numbers for the number of zero increments

Recall that by the number of zero increments (before the absorption) of a random walk with barrier n is meant a rv

$$V_n := T_n - M_n = \#\{i \le T_n : R_{i-1}^{(n)} = R_i^{(n)}\} = \sum_{l=0}^{T_n-1} 1_{\{R_l^{(n)}+\xi_{l+1}\ge n\}}.$$

Our results given below prove the weak convergence of V_n (without normalization) in case of finite mean $\mathbb{E}\xi$ and establish a weak law of large numbers in case when $\mathbb{E}\xi = \infty$. Although the asymptotic behavior of both M_n and T_n is known, it gives no clue about the asymptotics of V_n in view of strong dependence of these rvs.

Theorem 12. *If $m = \mathbb{E}\xi < \infty$ then, as $n \to \infty$,*

$$V_n \xrightarrow{d} V.$$

A random variable V has the same law as

$$V_Y + 1 - 2 \cdot 1_{\{Y=1\}},$$

where a random variable Y with law

$$\mathbb{P}\{Y = k\} = m^{-1}\mathbb{P}\{\xi \geq k\},\ k \in \mathbb{N}$$

is independent of $\{V_n : n \in \mathbb{N}\}$. In particular, $\mathbb{P}\{V = 0\} = m^{-1}$.

In the next result $\psi(x) := \Gamma'(x)/\Gamma(x)$ denotes the logarithmic derivative of the gamma function. Set also

$$m(x) := \int_0^x \mathbb{P}\{\xi > y\}\mathrm{d}y, \quad x > 0.$$

Theorem 13. *Assume that, for some $\alpha \in (0, 1]$ and some L slowly varying at ∞,*

$$\sum_{k=n}^{\infty} p_k \ \sim\ n^{-\alpha}L(n), \quad n \to \infty, \tag{1.37}$$

and

$$\sup_{n \geq 0} \frac{np_n}{\sum_{k=n+1}^{\infty} p_k} < \infty \quad \textit{for} \ \ \alpha \in [0, 1/2], \tag{1.38}$$

Then, as $n \to \infty$,

$$\frac{V_n}{\mathbb{E}V_n} \xrightarrow{P} 1, \tag{1.39}$$

and $\mathbb{E}V_n \sim (\psi(1) - \psi(1-\alpha))^{-1} \log n$, if $\alpha \in (0, 1)$, and $\mathbb{E}V_n \sim \log m(n)$, if $\alpha = 1$.

Letting in (1.10) $n \to \infty$ and recalling the convergence $Y_n \xrightarrow{d} Y$ immediately proves Theorem 12.

The proof of Theorem 13 relies upon the following auxiliary result.

Lemma 14. *If (1.37) holds for $\alpha \in [0, 1)$ and (1.38) holds for $\alpha \in [0, 1/2]$ then, as $n \to \infty$,*

$$\mathbb{E} \log Y_n = \log n - (\psi(1) - \psi(1-\alpha)) + o(1) \ \textit{and} \tag{1.40}$$

$$\mathbb{E} \log^2 Y_n = \log^2 n - 2(\psi(1) - \psi(1-\alpha)) \log n + o(\log n). \tag{1.41}$$

If (1.37) holds for $\alpha = 1$ then, as $n \to \infty$,

$$\mathbb{E} \log m(Y_n) = \log m(n) - 1 + o(1) \ \textit{and}$$

$$\mathbb{E} \log^2 m(Y_n) = \log^2 m(n) - 2 \log m(n) + o(\log m(n)).$$

Proof. If (1.37) holds for $\alpha \in [0,1)$ then, according, for instance, [32, Theorem 8.6.5]

$$\log n - \log Y_n \xrightarrow{d} (-\log \eta_\alpha),$$

where an rv η_α, $\alpha \in (0,1)$ has the beta law with parameters $1-\alpha$ and α, i.e.,

$$\mathbb{P}\{\eta_\alpha \in \mathrm{d}x\} = \frac{\sin \pi\alpha}{\pi} x^{-\alpha}(1-x)^{\alpha-1} 1_{(0,1)}(x)\mathrm{d}x,$$

and $\mathbb{P}\{\eta_0 = 1\} = 1$.

If we proved that, for each $\delta \in (0, 1-\alpha)$,

$$\sup_{n\ge1} \mathbb{E}(n/Y_n)^\delta = \sup_{n\ge1} \mathbb{E} f_k(\log^k(n/Y_n)) < \infty, \quad k \in \mathbb{N}, \tag{1.42}$$

where $f_k(x) := \exp(\delta x^{1/k})$, then by Vallée-Poussin theorem [104, Theorem T22], for each $k \in \mathbb{N}$, the sequence $\{(\log n - \log Y_n)^k : n \in \mathbb{N}\}$ would be uniformly integrable. If this were so then we would have

$$\lim_{n\to\infty} (\log n - \mathbb{E}\log Y_n) = \mathbb{E}(-\log \eta_\alpha) = \psi(1) - \psi(1-\alpha) < \infty,$$

which would prove (1.40), and

$$\lim_{n\to\infty} \mathbb{E}(\log n - \log Y_n)^2 = \mathbb{E}\log^2 \eta_\alpha = (\psi(1-\alpha)-\psi(1))^2 + \psi'(1-\alpha) - \psi'(1) < \infty,$$

which together with (1.40) would prove (1.41).

Now we have to check (1.42). For $j \in \mathbb{N}_0$, set $u_j := \sum_{k=0}^{j} \mathbb{P}\{S_k = j\}$. Then

$$\mathbb{E}Y_n^{-\delta} = \sum_{k=0}^{n-1} (n-k)^{-\delta} \mathbb{P}\{\xi \ge n-k\} u_k. \tag{1.43}$$

It is known (see Theorem 1.1 [52]), that if $\alpha \in (1/2, 1)$ then

$$u_n \sim \frac{\sin \pi\alpha}{\pi} n^{\alpha-1}/L(n).$$

Under additional condition (1.38) the same holds for $\alpha \in [0, 1/2]$ (see [41]). Thus from (1.43)

$$\mathbb{E}Y_n^{-\delta} \sim n^{-\delta} \frac{\Gamma(1-\alpha-\delta)\Gamma(\alpha)\sin \pi\alpha}{\Gamma(1-\delta)\pi},$$

which proves (1.42).

Assume now that (1.37) holds for $\alpha = 1$. By [46, Theorem 6] and its proof,

$$\log m(n) - \log m(Y_n) \xrightarrow{d} (-\log R),$$

where R is an rv with the uniform law on $[0,1]$. The wanted result will follow if, for example, we can show that the sequence $\{(\log m(n) - \log m(Y_n))^k : n \in \mathbb{N}\}$, $k = 1, 2$, is uniformly integrable. The latter will follow from the relation: for each $\epsilon \in (0,1)$

$$\sup_{n\geq 1} \mathbb{E}\left(\frac{m(n)}{m(Y_n)}\right)^{\epsilon} < \infty. \tag{1.44}$$

The subsequent part of the proof is similar to the previous one. In the case $\alpha = 1$ we have $u_n \sim \frac{1}{m(n)}$ and $\sum_{k=0}^{n} u_k \sim n/m(n)$ (see formula (2,4) and p. 266 [46]). Fix any $\epsilon \in (0,1)$, then

$$\mathbb{E}m(Y_n)^{-\varepsilon} = \sum_{k=0}^{n-1} m(n-k)^{-\varepsilon}\mathbb{P}\{\xi \geq n-k\}u_k \leq \sum_{k=0}^{[\Delta n]} + \sum_{k=[\Delta n]}^{n-1} =: I_1(n) + I_2(n).$$

Next

$$I_1(n) \leq \frac{\mathbb{P}\{\xi \geq n - [\Delta n]\}}{m^{\varepsilon}(n - [\Delta n])} \sum_{k=0}^{[\Delta n]} u_k \sim \frac{\mathbb{P}\{\xi \geq n - [\Delta n]\}}{m^{\varepsilon}(n - [\Delta n])} \frac{[\Delta n]}{m([\Delta n])}.$$

Since

$$\frac{\mathbb{P}\{\xi > x\}x}{m(x)} = \frac{xm'(x)}{m(x)} \to 0, \quad n \to \infty,$$

We have

$$m^{\varepsilon}(n)I_1(n) \to 0 \quad n \to \infty.$$

For $I_2(n)$ (since $m(k)u_k \leq C$ for big k)

$$\begin{aligned} I_2(n) = \quad & \sum_{k=[\Delta n]}^{n-1} m(n-k)^{-\varepsilon}\mathbb{P}\{\xi \geq n-k\}\frac{m(k)u_k}{m(k)} \leq \\ & \frac{C}{m(\Delta n)} \sum_{k=1}^{n-[\Delta n]} m(k)^{-\varepsilon}\mathbb{P}\{\xi \geq k\}. \end{aligned}$$

The function

$$r_\epsilon(x) := \int_0^x m^{-\epsilon}(y)\mathbb{P}\{\xi \geq y\}\mathrm{d}y$$

slowly varies at ∞, and $\sum_{k=1}^n m^{-\epsilon}(k)\mathbb{P}\{\xi \geq k\} \sim r_\epsilon(n)$. By de l' Hôpital's rule, $r_\epsilon(n)/m(n) \sim (1-\epsilon)^{-1}m^{-\epsilon}(n)$. Then

$$\overline{\lim_{n\to\infty}} m^\varepsilon(n)I_2(n) \leq C\overline{\lim_{n\to\infty}} \frac{m(n)}{m(\Delta n)} = C. \tag{1.45}$$

Now (1.44) follows (1.45). $\square$

Proof of Theorem 13. Assume that (1.37) holds for $\alpha \in (0,1)$. Set $a_n := \mathbb{E}V_n$ and $b_n := \mathbb{E}V_n^2$. From (1.10) we have $a_1 = 1$,

$$a_n = \frac{1}{1-\mathbb{P}\{Y_n = n\}}\left(\sum_{k=1}^{n-1} a_k\mathbb{P}\{Y_n = k\} + 1 - 2\mathbb{P}\{Y_n = 1\}\right),\ n = 2,3,\dots, \tag{1.46}$$

and $b_1 = 1$,

$$b_n = \frac{1}{1-\mathbb{P}\{Y_n = n\}}\left(\sum_{k=1}^{n-1} b_k\mathbb{P}\{Y_n = k\} + 2a_n - 1\right). \tag{1.47}$$

We are going to check that

$$b_n \sim a_n^2 \sim k_\alpha^2 \log^2 n, \tag{1.48}$$

where $k_\alpha := (\psi(1) - \psi(1-\alpha))^{-1}$. Assume that the relation

$$\limsup_{n\to\infty} \frac{a_n}{\log n} \leq k_\alpha \tag{1.49}$$

does not hold. Then we can pick $\epsilon > 0$ such that inequality

$$a_n > (k_\alpha + \epsilon)\log n + c$$

holds infinitely often for every fixed positive c. Define

$$n_c := \inf\{n \geq 1 : a_n > (k_\alpha + \epsilon)\log n + c\}.$$

Then

$$a_n \leq (k_\alpha + \epsilon)\log n + c, \quad n = 1, 2, \dots, n_c - 1.$$

It follows from (1.46) that

$$
\begin{aligned}
& (k_\alpha+\epsilon)\log n_c + c < \frac{1}{1-\mathbb{P}\{Y_{n_c}=n_c\}} \\
\cdot & \left(\sum_{i=1}^{n_c-1}((k_\alpha+\epsilon)\log i + c)\mathbb{P}\{Y_{n_c}=i\} + 1 - 2\mathbb{P}\{Y_{n_c}=1\}\right) \\
= \ & c + (k_\alpha+\epsilon)\left(\mathbb{E}\log Y_{n_c} + \frac{\mathbb{P}\{Y_{n_c}=n_c\}}{1-\mathbb{P}\{Y_{n_c}=n_c\}}(\mathbb{E}\log Y_{n_c} - \log n_c)\right) \\
+ \ & \frac{1}{1-\mathbb{P}\{Y_{n_c}=n_c\}}(1-2\mathbb{P}\{Y_{n_c}=1\})
\end{aligned}
$$

(since $n_c \to \infty$ as $c \to \infty$ then by using (1.40), equality $\mathbb{P}\{Y_n = 1\} = u_{n-1}$ and the fact that, by a renewal theorem, $\lim_{n\to\infty} u_n = 0$, we can continue as follows)

$$= c + (k_\alpha+\epsilon)\log n_c - (k_\alpha+\epsilon)k_\alpha^{-1} + 1 + o(1).$$

Sending c to ∞ gives $\epsilon \le 0$, a contradiction. Thus, we have proved (1.49). Analogously, one can establish the converse inequality for the lower bound. Thence

$$a_n \sim k_\alpha \log n.$$

Asymptotic relation (1.48) for b_n can be established similarly with the aid of (1.47). Thus, $\mathbb{E}V_n^2 \sim (\mathbb{E}V_n)^2$. Therefore, by Chebyshev's inequality, for each $\delta > 0$,

$$\mathbb{P}\{|V_n/\mathbb{E}V_n - 1| > \delta\} \le \frac{\operatorname{Var} V_n}{(\delta\mathbb{E}V_n)^2} \to 0,$$

which proves (1.39).

In case when (1.37) holds for $\alpha = 1$ the proof of relation $\mathbb{E}V_n^2 \sim (\mathbb{E}V_n)^2 \sim (\log m(n))^2$ goes along the same route as that of the previous case. It suffices to exploit the second part of Lemma 14. The proof is complete.

1.6 Another linear recurrence relation

1.6.1 Definition and examples. In this Section we investigate another pattern of the linear recurrence (1) of the form:

$$X_n \overset{d}{=} X'_{n-J_n} + Z_n, \quad X_1 := 0, \ n = 2, 3 \ldots, \tag{1.50}$$

where random variables $\{Z_k : k \in \mathbb{N}\}$ and J_n are independent and independent of $\{X_j', j \in \mathbb{N}\}$, and J_n has the same distribution as ξ conditioned on $\xi < n$, that is

$$\mathbb{P}\{J_n = k\} \;=\; \frac{p_k}{p_1 + \cdots + p_{n-1}}, \qquad k, n \in \mathbb{N}, k < n,$$

for some proper and non-degenerate random variable ξ with probability law

$$p_k \;:=\; \mathbb{P}\{\xi = k\}, \quad k \in \mathbb{N},\ p_1 > 0.$$

We start by giving two examples of the recurrences (1.50).

Example. As was observed in [84], the number of jumps M_n in the random walk with barrier n satisfies (1.50) with $Z_k \equiv 1$, $k \in \mathbb{N}$.

Example. Extending Example 2 in Section 1.1, consider the beta $(a, 1)$ coalescent, $a \in (0, 2)$, which is the Λ-coalescent with Λ defined by (1.1). Let $X_n^{(1)}$ be the total branch length of the coalescent, i.e., the sum of the lengths of all branches of the coalescent tree, and $X_n^{(2)}$ be the absorption time of the coalescent. Provided (p_k)'s are given by (1.2), $X_n^{(1)}$ satisfies (1.50) with Z_n having the exponential distribution with mean n/g_n (see [43, formula (2)] and [84, Section 7]), and $X_n^{(2)}$ satisfies (1.50) with Z_n having the exponential distribution with mean $1/g_n$ (see [115, formula (33)] and [84, Section 7]), where

$$g_n := \frac{a}{2-a}\bigg(\frac{\Gamma(a)\Gamma(n+1)}{\Gamma(a+n-1)} - 1\bigg).$$

The results to be presented below extend to some extent those obtained in [84] under the assumption $Z_k \equiv 1$, $k \in \mathbb{N}$.

1.6.2 A coupling.

For fixed $m, i \in \mathbb{N}$, define the events $A_{0,m}(i) = \Omega$ and

$$A_{j,m}(i) := \{\{R_k^{(m)}(i) : k \in \mathbb{N}_0\} \text{ visits } j\},\ j \in \mathbb{N}$$

with $\{R_k^{(m)}(i) : k \in \mathbb{N}_0\}$ as defined in Section 1.2. Notice that

$$1_{A_{j,m}(i)} = \sum_{k=0}^{\infty} 1_{\{R_k^{(m)}(i) \neq j,\, R_{k+1}^{(m)}(i) = j\}}, \quad j \in \mathbb{N}.$$

Write $A_{j,m}$ for $A_{j,m}(0)$.

The following result is a variation on the theme of Lemma 1 [83].

Lemma 15. *Set* $X_1^* := 0$ *and*

$$X_n^* := \sum_{j=0}^{n-2} Z_{n-j} 1_{A_{j,n}}, \quad n = 2, 3, \ldots. \tag{1.51}$$

The sequence of marginal distributions for $\{X_n^* : n \in \mathbb{N}_0\}$ *is a unique solution to* (1.50).

Proof. Let $I := \inf\{k \geq 1 : R_k^{(n)} > 0\}$ and $B_n := X_n^* - Z_n$. For fixed $i \in \mathbb{N}$ define $X_1^*(i) := 0$ and

$$X_n^*(i) := \sum_{j=0}^{n-1} Z_{n-j} 1_{A_{j,n}(i)}, \ n = 2, 3, \ldots$$

Then the sequence $\{X_n^*(i) : n \in \mathbb{N}_0\}$ has the same distribution as $\{X_n^* : n \in \mathbb{N}_0\}$ and is independent of $\{\xi_1, \ldots, \xi_i\}$. For $x \in \mathbb{R}$, we have

$$\begin{aligned} \mathbb{P}\{B_n \leq x\} &= \sum_{i=1}^{\infty} \sum_{k=1}^{n-1} \mathbb{P}\{I = i, R_I^{(n)} = k, B_n \leq x\} \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{n-1} \mathbb{P}\{\xi_1 \geq n, \ldots, \xi_{i-1} \geq n, \xi_i = k, B_n \leq x\}. \end{aligned}$$

Let $D_{i,k,n} := \{\xi_1 \geq n, \ldots, \xi_{i-1} \geq n, \xi_i = k\}$. Then $1_{A_{j,n}} 1_{D_{i,k,n}} = 0$, for $k > j$; $= 1_{D_{i,k,n}}$, for $k = j$, and $= 1_{A_{j-k,n}(i)} 1_{D_{i,k,n}}$, for $k < j$. Therefore,

$$B_n 1_{D_{i,k,n}} = \left(\sum_{j=0}^{n-k-2} Z_{n-k-j} 1_{A_{j,n-k}(i)} \right) 1_{D_{i,k,n}} = X_{n-k}^*(i) 1_{D_{i,k,n}},$$

and

$$\begin{aligned} \mathbb{P}\{B_n \leq x\} &= \sum_{i=1}^{\infty} \sum_{k=1}^{n-1} \mathbb{P}\{\xi_1 \geq n, \ldots, \xi_{i-1} \geq n, \xi_i = k, X_{n-k}^*(i) \leq x\} \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{n-1} \mathbb{P}\{\xi \geq n\} p_k \mathbb{P}\{X_{n-k}^* \leq x\} \\ &= \sum_{k=1}^{n-1} \mathbb{P}\{I_k = n\} \mathbb{P}\{X_{n-k}^* \leq x\}. \end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}\{X_n^* \le x\} &= \int_{\mathbb{R}} \mathbb{P}\{B_n \le x-y\}\mathbb{P}\{Z_n \in \mathrm{d}y\} \\ &= \int_{\mathbb{R}} \sum_{k=1}^{n-1} \mathbb{P}\{I_k = n\}\mathbb{P}\{X_{n-k}^* \le x-y\}\mathbb{P}\{Z_n \in \mathrm{d}y\},\end{aligned}$$

which is the same as (1.50). □

Remark 16. Assume that the marginal distributions of a random sequence $\{Q_n : n \in \mathbb{N}\}$ of scalar random variables are defined recursively as follows:

$$Q_n \stackrel{d}{=} Q_{n-\xi} 1_{\{\xi<n\}} + Z_n, \quad n \in \mathbb{N}, \tag{1.52}$$

where $\{Z_k : k \in \mathbb{N}\}$ are independent (not necessarily identically distributed) random variables, which are independent of ξ.

Recall that $\{S_n : n \in \mathbb{N}_0\}$ is a zero-delayed random walk with a step distributed like ξ, which is independent of $\{Z_k : k \in \mathbb{N}\}$, and that

$$N_n := \inf\{k \in \mathbb{N} : S_k \ge n\}, \quad n \in \mathbb{N}.$$

Define the events

$$A_k := \{\{S_n : n \in \mathbb{N}_0\} \text{ ever enters state } k\}, \quad k \in \mathbb{N}_0.$$

Then $1_{A_0} = 1$ and

$$1_{A_k} = \sum_{i=1}^{k} 1_{\{S_i=k\}}, \quad k \in \mathbb{N}.$$

Set

$$Q_n^* := \sum_{j=0}^{n-1} Z_{n-j} 1_{A_j}, \quad n \in \mathbb{N}. \tag{1.53}$$

In a similar, but simpler way as above, it can be checked that the marginal distributions of $\{Q_n^* : n \in \mathbb{N}\}$ is a unique solution to (1.52).

Using the identity

$$\sum_{k=0}^{n-1} 1_{A_k} = N_n, \quad n \in \mathbb{N},$$

lying in the core of the renewal theory, we conclude that in the case $Z_n \equiv b$, where $b \in \mathbb{R}$ is non-random, $Q_n^* = bN_n$, the observation previously remarked in [84]. If $\{Z_k : k \in \mathbb{N}\}$ are identically distributed then

$$Q_n^* = Z_1 + Z_2 + \ldots + Z_{N_n}.$$

Thus, in the latter case, the weak limiting behavior of Q_n can be obtained by a suitable combining known results on the weak convergence of random walks and first-passage time sequences.

Rearranging terms in (1.53) we obtain a representation

$$Q_n^* := \sum_{j=0}^{n-1} Z_{n-S_j} 1_{\{S_j<n\}} = \sum_{j=0}^{N_n-1} Z_{n-S_j}, \quad n \in \mathbb{N}. \tag{1.54}$$

which is more suitable for asymptotic analysis in the case when Z_k's are not identically distributed.

Recall that a shot-noise process is a random process of the form

$$Q(t) := \sum_{k=0}^{\infty} U(t-\tau_k), \quad t \geq 0,$$

where $\{U(s) : s \in \mathbb{R}\}$ is some random process and $\{\tau_j : j \in \mathbb{N}_0\}$ is an independent zero-delayed random walk with positive steps. If $U_s \equiv 0$ for $s \leq 0$ then

$$Z(t) := \sum_{k=0}^{M(t)} U(t-\tau_k), \quad t \geq 0, \tag{1.55}$$

where $M(t) := \sup\{k \in \mathbb{N}_0 : \tau_k \leq t\}$. Now, by analogy with (1.55), $\{Q_n^* : n \in \mathbb{N}\}$ can be called a shot-noise sequence. Indeed, making in (1.55) the time discrete ($n = 1, 2, \ldots$) and setting $U(n) = Z_n$ for $n \in \mathbb{N}$ and $U(n) = 0$ for non-positive integer n, and replacing $M(n)$ by $M(n-1) = N_n - 1$ we arrive at (1.54).

Now we are ready to point out a basic coupling construction.

Lemma 17. *For X_n^* defined in* (1.51) *we have $X_1^* := 0$ and*

$$X_n^* = \sum_{k=0}^{Y_n-2} Z_{Y_n-k} 1_{A_{k,Y_n}(N_n)} + \sum_{k=0}^{n-2} Z_{n-k} 1_{A_k} - Z_{Y_n} 1_{\{Y_n \geq 2\}}, \quad n = 2, 3, \ldots \quad a.s.,$$

where $Y_n := n - S_{N_n-1}$. *In particular,*

$$X_n^* \overset{d}{=} X_{Y_n}^* + \sum_{k=0}^{n-2} Z_{n-k} 1_{A_k} - Z_{Y_n} 1_{\{Y_n \geq 2\}} \quad n = 2, 3, \ldots,$$

where $\{X_n^* : n \in \mathbb{N}_0\}$ *is independent of* $(N_n, n - S_{N_n-1})$.

Proof. Let us first prove that

$$\sum_{i=1}^{N_n-1} 1_{\{S_i = k\}} = 1_{A_k}, \quad k \in \{1, 2, \ldots, n-2\}. \tag{1.56}$$

Indeed, on the event $\{N_n = 1\}$ the right-hand side of (1.56) equals 0. Also, we have

$$\left(\sum_{i=1}^{N_n-1} 1_{\{S_i = k\}} \right) 1_{\{N_n \geq k+1\}} = 1_{A_k} 1_{\{N_n \geq k+1\}}.$$

Now fix $j \in \{2, 3, \ldots, k\}$. Then $1_{\{S_l = k,\, N_n = j\}} = 0$ for all $l \in \{j, \ldots, k\}$. Therefore,

$$\begin{aligned}
\left(\sum_{i=1}^{N_n-1} 1_{\{S_i = k\}} \right) 1_{\{N_n = j\}} &= \sum_{i=1}^{j-1} 1_{\{S_i = k, N_n = j\}} \\
&= \sum_{i=1}^{k} 1_{\{S_i = k, N_n = j\}} = 1_{A_k} 1_{\{N_n = j\}}.
\end{aligned}$$

Combining these arguments leads to (1.56).

Further it is important that for $i \in \{0, 1, \ldots, N_n - 1\}$

$$R_i^{(n)} = S_i \quad \text{and} \quad R_{N_n}^{(n)} = S_{N_n-1}.$$

The latter implies that, for $k \in \mathbb{N}$,

$$1_{\{R_{N_n-1}^{(n)} \neq k,\, R_{N_n}^{(n)} = k\}} = 1_{\{S_{N_n-1} \neq k,\, S_{N_n} = k\}} = 0 \quad \text{a.s.} \tag{1.57}$$

Also it can be checked that, for $l \in \mathbb{N}_0$ and $k \in \{1, 2, \ldots, n-2\}$,

$$\begin{aligned}
& 1_{\{R_{N_n+l}^{(n)} \neq k, R_{N_n+l+1}^{(n)} = k\}} \\
= \; & 1_{\{R_l^{(n-S_{N_n-1})}(N_n) \neq k - S_{N_n-1}, R_{l+1}^{(n-S_{N_n-1})}(N_n) = k - S_{N_n-1}\}} 1_{\{S_{N_n-1} < k\}}
\end{aligned} \tag{1.58}$$

almost surely. In what follows, it is tacitly assumed that empty sums equal zero. For fixed $k \in \{1, 2, \ldots, n-2\}$, we have

$$\begin{aligned}
1_{A_{k,n}} &= \sum_{i=1}^{\infty} 1_{\{R_{i-1}^{(n)} \neq k, R_i^{(n)} = k\}} \\
&= \sum_{i=1}^{N_n - 1} \cdots + \sum_{i=N_n}^{\infty} \cdots \\
&\overset{(1.57)}{=} \sum_{i=1}^{N_n-1} 1_{\{S_i = k\}} + \sum_{i=N_n+1}^{\infty} 1_{\{R_{i-1}^{(n)} \neq k, R_i^{(n)} = k\}} \\
&\overset{(1.56),(1.58)}{=} 1_{A_k} + 1_{A_{k-S_{N_n-1}, n-S_{N_n-1}}(N_n)} 1_{\{S_{N_n-1} < k\}}.
\end{aligned}$$

By using the obtained representation for $1_{A_{k,n}}$, we can write

$$\begin{aligned}
Q_n^* &= \sum_{k=0}^{n-2} Z_{n-k} 1_{A_k} + \sum_{k=1}^{n-2} Z_{n-k} 1_{A_{k-S_{N_n-1}, n-S_{N_n-1}}(N_n)} 1_{\{S_{N_n-1} < k\}} \\
&= \sum_{k=0}^{n-2} Z_{n-k} 1_{A_k} + \sum_{i=1}^{n-S_{N_n-1}-2} Z_{n-S_{N_n-1}-i} 1_{A_{i,n-S_{N_n-1}}(N_n)} \\
&= \sum_{i=0}^{Y_n-2} Z_{Y_n-i} 1_{A_{i,Y_n}(N_n)} + \sum_{k=0}^{n-2} Z_{n-k} 1_{A_k} - Z_{Y_n} 1_{\{Y_n \geq 2\}}.
\end{aligned}$$

□

1.6.3 A weak convergence result.

Below is given a weak convergence result under a regular variation assumption.

Theorem 18. *Suppose that for some $\alpha \in (0, 1)$, $\beta > -1$ and some functions L_1 and L_2 slowly varying at ∞*

$$\mathbb{P}\{\xi \geq n\} = \sum_{k=n}^{\infty} p_k \sim \frac{L_1(n)}{n^\alpha}, \quad n \to \infty,$$

$Z_n = b_n$ is non-random and

$$b_n \sim n^\beta L_2(n), \quad n \to \infty.$$

Then, as $n \to \infty$,

$$\frac{L_1(n)}{L_2(n)} \frac{X_n}{n^{\alpha+\beta}} \stackrel{d}{\to} \int_0^T e^{-U_t}\, dt,$$

where $\{U_t : t \geq 0\}$ is a drift-free subordinator with the Lévy measure

$$\nu(\mathrm{d}t) \;=\; \frac{\alpha}{\alpha+\beta} \frac{e^{-t/(\alpha+\beta)}}{(1-e^{-t/(\alpha+\beta)})^{\alpha+1}}\, dt, \quad t > 0, \tag{1.59}$$

and T is a random variable with the standard exponential law which is independent of the subordinator.

Proof. The proof has much in common with the proof of [84, Theorem 1.3]. We thus only give a sketch.

For $k, n \in \mathbb{N}$, set $a_k(n) := \mathbb{E}X_n^k$. For $x \geq 0$, define

$$\Phi(x) \;:=\; \frac{\Gamma(1-\alpha)\Gamma((\alpha+\beta)x+1)}{\Gamma((\alpha+\beta)x-\alpha+1)} - 1.$$

Since $\Phi(x) = \int_0^\infty (1-e^{-xy})\nu(\mathrm{d}y)$, where ν is given in (1.59), Φ is the Laplace exponent of a drift-free subordinator with the Lévy measure ν.

It can be checked that

$$\lim_{n\to\infty} \frac{L_1^k(n)}{L_2^k(n)} \frac{a_k(n)}{n^{k(\alpha+\beta)}} \;=\; \frac{k!}{\prod_{i=0}^k (1+\Phi(i))} \;=:\; a_k, \quad k \in \mathbb{N}. \tag{1.60}$$

This implies (see, for example, [84]) that (i) $a_k = \mathbb{E}(\eta^k)$, $k \in \mathbb{N}$, where $\eta \stackrel{d}{=} \int_0^T e^{-U_t} dt$, and that (ii) the moments $a_1, a_2, \ldots$ uniquely determine the law of η, and the result follows. $\square$

1.7 Bibliographic comments

By using a complicated technique based on the singularity analysis of generating functions, it was proved in [43] that the number of cuts X_n needed to isolate the root of a random recursive tree with n vertices with the aid of Meir-Moon's procedure [103], weakly converges, after normalization and centering, to the 1-stable law μ_1. In [83] an alternative, purely probabilistic proof of this result was given, in which *random walks with barrier* have naturally arisen. In particular, it was checked that the X_n has the same law as the number of jumps M_n in the random walk with the barrier n in case

$$\mathbb{P}\{\xi = k\} = \frac{1}{k(k+1)}, \quad k \in \mathbb{N},$$

and proved that

$$\frac{\log^2 n}{n} M_n - \log n - \log\log n \Rightarrow \mu_1, \quad n \to \infty.$$

In this thesis we use the same definition of the *random walk with barrier* as in [83] and [84] (the name was coined in [84]). Notice that earlier similar sequences were investigated in [72], but the problems addressed in the cited work were different from ours.

While Section 2.1 contains several examples of applications of the random walks with barrier, some others can be found in [72, 83, 84].

Λ-coalescents were introduced independently in [115] and [122]. Recently there have appeared a number of papers (see, for instance, [40, 43, 44, 51, 57, 83, 84, 81, 106]) which investigated the weak convergence of several functionals acting on the Λ-coalescents, most notably, the number of collisions, the absorpton time and the total branch length.

Based on the results derived in [107] and [86], in Section 2.2 it is shown that the sequences $\{T_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$, where T_n and V_n are the absorption time and the number of zero increments before the absorption, respectively, in the random walk with barrier n, are the linear random recurrences. General linear random recurrences is a popular object of research in Applied Probability. They arise in random regenerative structures [16, 55, 58, 62, 63], random trees [43, 42, 83, 113, 114], the theory of exchangeable coalescents [44, 57, 64, 81]. Also such recurrences are common in studies of the absorption times of non-increasing Markov chains [38, 88], recursive algorithms [111, 117, 118] and many other fields. A lot of diverse applications of a particular linear recurrence, where $Y_n \equiv 1$ and I_n has the uniform distribution on $\{0, 1, \ldots, n-1\}$, can be found in [13].

Although the asymptotic analysis of random recurrences is a hard analytic problem, some more or less effective methods have been elaborated to date. Evidently the most popular existing approach is the *method of singular analysis of generating functions* [42, 50]. For instance, provided the distributions of all input data of a recurrence are known explicitly this purely analytic method can allow one to obtain high order asymptotic expansions of moments. The *contraction method* [111, 117, 119] is another popular approach which is more probabilistic. It performs especially well whenever a normalized and centered recurrence weakly converges to a probability law which is a fixed point of appropriate transformation. On the other hand, under quite strong moment restrictions this method may lead to establishing a central limit theorem [111]. Finally, we mention a method [118] which is based on the harmonic analysis and potential theory.

In [83] it was remarked that the linear random recurrence generated by the number of jumps in the random walks with barrier can be successfully investigated by using a probabilistic method alone. An essence of that approach is a coupling of a random walk with barrier with the corresponding standard random walk and subsequent use of known methods of the renewal theory. In Section 2.3 we extend this technique to two other linear random recurrences arising in the random walks with barrier.

Section 2.3 is based on the results obtained in [107]. Our Theorems 4, 5 and 6 are analogues of the results given in [84] for the number of jumps in the random walks with barrier. Although our statements on the weak convergence of the absorption times do not follow from the results of [84] on the number of jumps, our proofs rely heavily on the arguments worked out in the cited paper. It is worth mentioning that recently a new, more probabilistic proof of [84, Theorem 1.3] was given in [70]. One may wonder whether that new approach can be used to prove our Theorem 6.

The results of Section 2.4 which are based on [108] are closely related to two-term asymptotic expansions of the renewal functions. Although numerous results of this kind have already been developed [14, 47, 54, 105, 123, 124], the latter topic is relevant and still popular in the renewal theory.

Section 2.5 is based on the results obtained in [86].

Chapter 2

The Bernoulli sieve

2.1 Definition and interpretation

Let $\{\eta_k : k \in \mathbb{N}\}$ be independent copies of an rv η taking values in $(0,1)$. We start with a 'game' interpretation followed by a rigorous description of the model to be discussed. Assume that n persons play the following game. In the first round of the game each of the players tosses once a coin with probability η_1 for tails, then those who threw heads (call them *losers*) are eliminated, while those who threw tails proceed in the second round in which a coin is tossed with probability η_2 for tails, and so on (with probability η_i for tails in the ith round) until there are no players left. If in some round, say jth, all players threw tails, then all of them proceed in the $(j+1)$st round. It is assumed that given η_k the results of players in the kth round are conditionally independent.

In this Section we investigate the asymptotic behavior of the following characteristics of the game:

- U_n– duration of the game;
- $K_{n,0}$– the number of rounds with no losers;
- $K_n = U_n - K_{n,0}$–the number of rounds with some losers;

- Z_n– the number of players in the last round.

In what follows we always assume that

$$\text{the law of} \quad |\log \eta| \quad \text{is non-arithmetic.}$$

Denote by $\{J_k : k \in \mathbb{N}_0\}$ a zero-delayed random walk with a step distributed like $|\log \eta|$. Let $E_1, \ldots, E_n$ be i.i.d. sample from the standard exponential law which is independent of the random walk. Denote by $E_{1,n} \leq E_{2,n} \leq \ldots \leq E_{n,n}$ the corresponding order statistics.

The random walk together with the sample from the standard exponential distgribution define a random occupancy scheme, called *the Bernoulli sieve*, in which n balls $1, 2, \ldots, n$ are thrown into an infinite array of boxes indexed by the integers $1, 2, \ldots$, according to the rule: ball i falls in box k iff the exponential point E_i falls into the interval (J_{k-1}, J_k).

We say that the index of interval (J_{i-1}, J_i) is i and call this interval *occupied*, if it contains at least one exponential points, and *empty*, otherwise. Now the characteristics of the game introduced above can be interpreted as follows:

- $U_n = \inf\{k \in \mathbb{N} : J_k > E_{n,n}\}$, i.e., U_n is the index of the right-most occupied interval;

- $K_{n,0} = \#\{1 \leq k \leq U_n - 1 : (J_{k-1}, J_k) \text{ is empty }\}$, i.e., $K_{n,0}$ is the number of empty intervals with indices not exceeding $U_n - 1$;

- $K_n = \#\{k \in \mathbb{N} : (J_{k-1}, J_k) \text{ is occupied}\}$, i.e., K_n is the number of occupied intervals;

- $Z_n := \#\{1 \leq k \leq n : E_{k,n} \in (J_{U_n-1}, J_{U_n})\}$, i.e., Z_n is the number of points in the right-most occupied interval.

The model just described is additive. Sometimes it is more convenient to work with its multiplicative counterpart obtained by an exponential transformation. More precisely, the exponential sample $E_1, \ldots, E_n$ is replaced by

$e^{-E_1}, \ldots, e^{-E_n}$ which is an i.i.d. sample from the uniform $[0,1]$ law, and the (additive) random walk $\{J_k : k \in \mathbb{N}_0\}$ is replaced by a multiplicative one $\{e^{-J_k} : k \in \mathbb{N}_0\}$.

One of the most prominent applications of the Bernoulli sieve is that it provides a model of random compositions. Recall that a random combinatorial structure which captures the occupancy of boxes is the *weak composition* C_n^* comprised of nonnegative integer *parts* summing up to n. One speaks of *weak* composition meaning that zero parts are allowed, for instance the sequence $(2,3,0,1,0,0,1,0,0,0,\ldots)$ (padded by infinitely many 0's) is a possible value of C_7^*. There is a number of ways to produce a composition as any allocation of n balls into infinitely many boxes leads to a composition of n. However, the set of all models of random compositions contains more or less adequate representatives. The weak compositions C_n^* arising in the Bernoulli sieve verify two distinguished consistency properties which secures their attractiveness and adequateness.

(SC) Sampling consistency: if one of n points is chosen uniformly at random and removed from the interval it occupies, the resulting weak composition has the same probability law as C_{n-1}^*.

(DP) Deletion property: if the first interval is inspected and it turns out that it contains k points, then deleting the first interval yields a weak composition with the same probability law as C_{n-k}^*.

The main results of this Section are concerned with the weak convergence of the functionals acting on the Bernoulli sieve. In particular, Theorem 19 establishes an ultimate criterion for the existence of limiting law for, properly normalized and centered, U_n. Under a side condition, Theorem 21 proves an analogous result for K_n. Among other things, this condition ensures a more delicate result given in Theorem 20: $K_{n,0}$ weakly converges without normalization. Finally, under natural assumptions on the law of η, Theorem 22 investigates the weak convergence of Z_n.

Theorem 19. *The following assertions are equivalent.*

(i) There exist sequences of numbers $\{a_n, b_n : n \in \mathbb{N}\}$ with $a_n > 0$ and $b_n \in \mathbb{R}$ such that, as $n \to \infty$, $(U_n - b_n)/a_n$ converges weakly to a nondegenerate and proper probability law.

(ii) The law of $|\log \eta|$ either belongs to the domain of attraction of a stable law, or $\mathbb{P}\{|\log \eta| > x\}$ slowly varies at ∞.

Set $\mu := \mathbb{E}|\log \eta|$ and $\sigma^2 := \mathrm{Var}\,(\log \eta)$.
(1) If $\sigma^2 < \infty$ then, with $b_n := \mu^{-1} \log n$ and $a_n := (\mu^{-3}\sigma^2 \log n)^{1/2}$, the limiting law is standard normal.
(2) If $\sigma^2 = \infty$ and

$$\int_x^1 \log^2 y \mathbb{P}\{\eta \in \mathrm{d}y\} \sim L(|\log x|), \quad x \to 0,$$

for some L slowly varying at ∞ then, with $b_n := \mu^{-1} \log n$ and $a_n := \mu^{-3/2} c_{[\log n]}$, where $\{c_n : n \in \mathbb{N}\}$ is any non-decreasing sequence of positive numbers satisfying $\lim_{n\to\infty} nL(c_n)/c_n^2 = 1$, the limiting law is standard normal.
(3) Assume that the relation

$$\mathbb{P}\{\eta \le x\} \sim |\log x|^{-\alpha} L(|\log x|), \quad x \to 0 \tag{2.1}$$

holds with $\alpha \in (1, 2)$ and some L slowly varying at ∞. Then, with $b_n := \mu^{-1} \log n$ and $a_n := \mu^{-(\alpha+1)/\alpha} c_{[\log n]}$, where $\{c_n : n \in \mathbb{N}\}$ is any non-decreasing sequence of positive numbers satisfying $\lim_{n\to\infty} nL(c_n)/c_n^\alpha = 1$, the limiting law is α-stable with characteristic function

$$t \mapsto \exp\{-|t|^\alpha \Gamma(1-\alpha)(\cos(\pi\alpha/2) + i \sin(\pi\alpha/2)\,\mathrm{sgn}(t))\},\ t \in \mathbb{R}. \tag{2.2}$$

(4) Assume that the relation (2.1) holds for $\alpha = 1$. Let $c : \mathbb{R}^+ \to \mathbb{R}^+$ be any function satisfying $\lim_{x\to\infty} xL(c(x))/c(x) = 1$, and set

$$\psi(x) := x \int_0^{c(x)} \mathbb{P}\{|\log \eta| > y\} \mathrm{d}y.$$

Let $b : \mathbb{R}^+ \to \mathbb{R}^+$ be any function satisfying $b(\psi(x)) \sim \psi(b(x)) \sim x$, as $x \to \infty$. Then, with $b_n := b(\log n)$ and $a_n := b(\log n)c(b(\log n))/\log n$, the *limiting law is* 1*-stable with characteristic function*

$$t \mapsto \exp\{-|t|(\pi/2 - i \log|t| \operatorname{sgn}(t))\}, \; t \in \mathbb{R}. \tag{2.3}$$

(5) If (2.1) *holds for* $\alpha \in [0,1)$ *then, with* $b_n = 0$ *and* $a_n := \log^\alpha n / L(\log n)$, *the limiting law is the Mittag-Leffler law* θ_α *(exponential, if* $\alpha = 0$*) with moments*

$$\int_0^\infty x^n \theta_\alpha(\mathrm{d}x) = \frac{n!}{\Gamma^n(1-\alpha)\Gamma(1+n\alpha)}, \; n \in \mathbb{N}.$$

An outline of the proof of Theorem 19 is as follows. As $U_n = N_{E_{n,n}}$, where

$$N_t := \inf\{k \in \mathbb{N} : J_k > t\}, \quad t \geq 0, \tag{2.4}$$

and

$$E_{n,n} - \log n \overset{d}{\to} \varrho, \quad n \to \infty, \tag{2.5}$$

where ϱ is an rv with distribution function $F(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, one can anticipate that $\frac{U_n - b_n}{a_n}$ converges weakly to a proper and nondegenerate probability law if and only if the same is true for $\frac{N_{\log n} - b_n}{a_n}$. On the other hand, this equivalence is not automatic and it does require a proof.

Theorem 20. *(a) If* $\nu := \mathbb{E}|\log(1-\eta)| < \infty$ *then, as* $n \to \infty$, $K_{n,0}$ *converges in distribution to a random variable* K_0.
If $\mu = \mathbb{E}|\ln \eta| < \infty$ *then*

$$\mathbb{P}\{K_0 = 0\} = 1 - \frac{1}{\mu} \sum_{j=1}^\infty \frac{\mathbb{E}\eta^j}{j} \mathbb{P}\{K_{j,0} = 0\},$$

$$\mathbb{P}\{K_0 = i\} = \frac{1}{\mu} \sum_{j=1}^\infty \frac{\mathbb{E}\eta^j}{j} \Big(\mathbb{P}\{K_{j,0} = i-1\} - \mathbb{P}\{K_{j,0} = i\} \Big), \quad i \in \mathbb{N},$$

and $\mathbb{E}K_0 = \nu/\mu$.
If $\mu = \infty$ *then* $\mathbb{P}\{K_0 = 0\} = 1$.
(b) If $\mu\nu < \infty$ *then* $K_{n,0}$ *converges to* K_0 *in* L_p *for every* $p \geq 1$. *If* $\nu = \infty$ *and* $\mu < \infty$ *then* $\lim_{n\to\infty} \mathbb{E}K_{n,0} = \infty$.

Taking into account equality $K_n = U_n - K_{n,0}$ along with the result of Theorem 19, we conclude that the boundedness of $K_{n,0}$ in probability would suffice for the implication: if $(U_n - b_n)/a_n$ converges weakly to some probability law then $(K_n - b_n)/a_n$ converges weakly to the same law. According to Theorem 20, condition $\nu < \infty$ ensures that $K_{n,0}$ converges in distribution which is even more than we need.

Theorem 21. *If $\nu < \infty$ then all the assertions of Theorem 19 remain valid on replacing U_n by K_n.*

Theorem 22. *Assume that $\mu = \mathbb{E}|\log \eta| < \infty$. Then, as $n \to \infty$, Z_n converges in distribution to a random variable Z with distribution*

$$\mathbb{P}\{Z = k\} = \frac{\mathbb{E}(1-\eta)^k}{\mu k}, \quad k \in \mathbb{N}.$$

If (2.1) *holds for some $\alpha \in [0,1)$ then*

$$\frac{\log Z_n}{\log n} \overset{d}{\to} \widehat{Z}_\alpha.$$

If (2.1) *holds for $\alpha = 1$ and if $\mu = \infty$ then*

$$\frac{m(\log Z_n)}{m(\log n)} \overset{d}{\to} \widehat{Z}_1,$$

where $m(x) := \int_0^x \mathbb{P}\{|\log \eta| > y\} \mathrm{d}y$, and the law of a random variable $\widehat{Z}_0$ is degenerate at 1*, for $\alpha \in (0,1)$, $\widehat{Z}_\alpha$ has the beta law with parameters $(1-\alpha, \alpha)$, i.e.,*

$$\mathbb{P}\{\widehat{Z}_\alpha \in \mathrm{d}x\} = \frac{\sin \pi\alpha}{\pi} x^{-\alpha}(1-x)^{\alpha-1} 1_{(0,1)}(x)\mathrm{d}x,$$

and $\widehat{Z}_1$ has the uniform law on $[0,1]$.

Examples 23 and 24 illustrate our main results. Example 25 gives an explicit form of the law of η for which Theorem 21 does not apply. Let X_n stand for any of the variables K_n or U_n.

Example 23. Assume that η has the beta law with parameters (a,b), $a, b > 0$, i.e.,

$$\mathbb{P}\{\eta \in \mathrm{d}x\} = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} 1_{(0,1)}(x)\mathrm{d}x,$$

where $B(\cdot,\cdot)$ is the beta function. In this case,

$$\mu = \mathbb{E}|\log\eta| = \Psi(a+b) - \Psi(a) < \infty,$$

$$\nu = \mathbb{E}|\log(1-\eta)| = \Psi(a+b) - \Psi(b) < \infty,$$

$$\sigma^2 = \mathrm{Var}\,(\log\eta) = \Psi'(a) - \Psi'(a+b) < \infty,$$

where Ψ denotes the logarithmic derivative of the gamma function, i.e., $\Psi(x) = \Gamma'(x)/\Gamma(x)$, $x > 0$. Therefore, as $n \to \infty$,

$$\frac{X_n - \mu^{-1}\log n}{(\mu^{-3}\sigma^2\log n)^{1/2}} \overset{d}{\to} \text{normal}\,(0,1).$$

Further $Z_n \overset{d}{\to} Z$ with Z having distribution

$$\mathbb{P}\{Z = k\} = \frac{\Gamma(a+b)}{\mu\Gamma(b)}\frac{\Gamma(k+b)}{k\Gamma(k+b+a)},\ \ k \in \mathbb{N}.$$

The number of empty boxes $K_{n,0}$ converges in distribution and in the mean to a random variable K_0 with some non-degenerate distribution. For $b \neq 1$ an explicit form of the limiting distribution is still a challenge. For $b = 1$ Proposition 26 gives the generating function

$$\mathbb{E}s^{K_0} = \frac{\Gamma(1+a)\Gamma(1+a-as)}{\Gamma(1+2a-as)},\quad s \in [0,1].$$

In particular, for integer a the distribution of K_0 is the convolution of a geometric distributions with parameters $k^{-1}(k+a)$, $k = 1, 2, \ldots, a$.

Example 24. Suppose

$$\mathbb{P}\{\eta \le x\} = \frac{1}{1-\log x},\ \ x \in (0,1),\ \ = 0,\ \ x \le 0,\ \ = 1,\ \ x \ge 1.$$

Then condition (2.1) holds for $\alpha = 1$, and $\mu = \infty$. Since

$$\mathbb{P}\{|\log(1-\eta)| > x\} = \frac{-\log(1-e^{-x})}{1-\log(1-e^{-x})},\ \ x > 0,\ \ = 1,\ \ x \le 0,$$

then, as $x \to \infty$, $\mathbb{P}\{|\log(1-\eta)| > x\} \sim e^{-x}$ which implies that $\nu < \infty$. Therefore, as $n \to \infty$,

$$\frac{(\log\log n)^2}{\log n}X_n - \log\log n - \log\log\log n,\ \ i = 1, 2,$$

weakly converges to the spectrally negative 1-stable law with characteristic function (2.3). Since $\mathbb{P}\{|\log\eta| > x\} = (x+1)^{-1}$ holds for $x > 0$, the normalizing constants in the previous display can be calculated in the same way as in [83, Proposition 2]. Finally, $\frac{\log Z_n}{\log n}$ weakly converges to the uniform $[0,1]$ law, and $K_{n,0}$ converges to zero in probability.

Example 25. Suppose

$$\mathbb{P}\{\eta \le x\} = \frac{|\log(1-x)|}{1+|\log(1-x)|}, \ x \in (0,1), \ = 0, \ x \le 0, \ = 1, \ x \ge 1.$$

Then $\sigma^2 < \infty$, yet $\nu = \infty$, and Theorem 21 is not applicable.

2.2 Relation to linear random recurrences and Markov chains

In this Section we investigate a connection between the functionals acting on the Bernoulli sieve and certain nonincreasing Markov chains. Also we prove that the sequences $\{U_n : n \in \mathbb{N}_0\}$, $\{K_n : n \in \mathbb{N}_0\}$ and $\{K_{n,0} : n \in \mathbb{N}_0\}$ are linear random recurrences.

Denote by $P_k^{(n)}$ and $Q_k^{(n)}$ the number of exponential points that fall outside the first k and respectively the first k occupied intervals. Then $\{P_k^{(n)} : k \in \mathbb{N}_0\}$ and $\{Q_k^{(n)} : k \in \mathbb{N}_0\}$ are Markov chains, the former being nonincreasing and the latter decreasing, that start at point n and have the transition probabilities

$$p_{ij} := \binom{i}{j} \int_0^1 x^j (1-x)^{i-j} \mathbb{P}\{\eta \in \mathrm{d}x\}, \quad j \le i, \ i \in \mathbb{N},$$

and

$$\pi_{ij} := \frac{\binom{i}{j} \int_0^1 x^j (1-x)^{i-j} \mathbb{P}\{\eta \in \mathrm{d}x\}}{1 - \mathbb{E}\eta^i}, \quad j < i, \ i \in \mathbb{N}, \tag{2.6}$$

respectively.

In particular,

$$\mathbb{P}\{P_1^{(n)} = k\} = \binom{n}{k} \int_0^1 x^k (1-x)^{n-k} \mathbb{P}\{\eta \in \mathrm{d}x\}, \quad k = 0, 1, \ldots, n. \tag{2.7}$$

Denote by $\mathrm{Bin}\,(m,x)$ an rv with the binomial distribution

$$\mathbb{P}\{\mathrm{Bin}\,(m,x)=k\}=C_m^k x^k(1-x)^{m-k},\quad k=0,1,\ldots,m.$$

Fix $p_1,p_2\in(0,1)$. If an rv $\mathrm{Bin}\,(n,p_1)$ is independent of an rv $\mathrm{Bin}\,(n,p_2)$ then

$$\mathrm{Bin}\,(\mathrm{Bin}\,(n,p_1),p_2)\overset{d}{=}\mathrm{Bin}\,(n,p_1p_2).$$

Consequently, for $k=0,1,\ldots,n$ and $j\in\mathbb{N}$

$$\mathbb{P}\{P_j^{(n)}=k\}=\binom{n}{k}\int_0^1 x^k(1-x)^{n-k}\mathbb{P}\{\eta_1\eta_2\cdot\ldots\cdot\eta_j\in \mathrm{d}x\}.$$

Since, for $n\in\mathbb{N}$, the absorption time $\inf\{k\in\mathbb{N}:P_k^{(n)}=0\}$ has the same law as U_n then, first, for $k\in\mathbb{N}$

$$\mathbb{P}\{U_n>k\}=\mathbb{P}\{P_k^{(n)}>0\}=\mathbb{E}(1-(1-Q_k)^n),\tag{2.8}$$

where $Q_k:=\eta_1\eta_2\cdot\ldots\cdot\eta_k$; secondly, the marginal distributions of the sequence $\{U_n:n\in\mathbb{N}_0\}$ satisfy the distributional equality

$$U_0:=0,\quad U_n\overset{d}{=}U_{P_1^{(n)}}+1,\quad n\in\mathbb{N},\tag{2.9}$$

where $P_1^{(n)}$ is independent of $\{U_n:n\in\mathbb{N}\}$.

In a similar vein one can check that the marginal distributions of the sequences $\{K_n:n\in\mathbb{N}_0\}$ and $\{K_{n,0}:n\in\mathbb{N}_0\}$ satisfy the following distributional equalities

$$K_0=0,\quad K_n\overset{d}{=}K_{Q_1^{(n)}}+1,\quad n\in\mathbb{N},\tag{2.10}$$

where $Q_1^{(n)}$ is independent of $\{K_i:i\in\mathbb{N}\}$ and has the law

$$\mathbb{P}\{Q_1^{(n)}=k\}=\mathbb{P}\{P_1^{(n)}=k|\ P_1^{(n)}\leq n-1\},k=0,1,\ldots,n-1;$$

and

$$K_{0,0}:=0,\quad K_{n,0}\overset{d}{=}K_{P_1^{(n)},0}+1_{\{P_1^{(n)}=n\}},\quad n\in\mathbb{N},\tag{2.11}$$

where $P_1^{(n)}$ is independent of $\{K_{n,0}:n\in\mathbb{N}_0\}$.

For $m \in \mathbb{N}_0$, set

$$g(n,m) := \mathbb{P}\{\{Q_k^{(n)} : k \in \mathbb{N}_0\} \text{ hits the state } m\}.$$

Under the assumption $\mu < \infty$, in [55] it was shown that

$$\lim_{n\to\infty} g(n,m) = \frac{1-\mathbb{E}\eta^m}{\mu m}, \ m \in \mathbb{N}. \tag{2.12}$$

Notice that when $\mu = \infty$ a similar argument yields $\lim\limits_{n\to\infty} g(n,m) = 0$.

2.3 Index of the last occupied interval. Proof of Theorem 19

In view of Proposition 27 to prove Theorem 19 it suffices to show that $\frac{U_n - b_n}{a_n} \Rightarrow \mu$ iff $\frac{N_{\log n} - b_n}{a_n} \Rightarrow \mu$, where μ is some proper and nondegenerate probability law, and N_t was defined in (2.4).

Assume that $\frac{U_n - b_n}{a_n}$ weakly converges to μ, and $a_n \to \infty$. Then, for $y > 0$,

$$\begin{aligned}
\mathbb{P}\left\{\frac{U_n - b_n}{a_n} > x\right\} &= \mathbb{P}\left\{\frac{N_{E_{n,n}} - b_n}{a_n} > x\right\} \\
&\le \mathbb{P}\left\{\frac{N_{\log n + y} - b_n}{a_n} > x\right\}\mathbb{P}\{E_{n,n} - \log n \le y\} \\
&+ \mathbb{P}\{E_{n,n} - \log n > y\} \le
\end{aligned}$$

Since with probability one $N_{\log n + y} \le N_{\log n} + N'_y$, where N'_y has the same law as N_y, then the last inequality can be continued as follows:

$$\le \mathbb{P}\left\{\frac{N_{\log n} - b_n}{a_n} + \frac{N'_y}{a_n} > x\right\}\mathbb{P}\{E_{n,n} - \log n \le y\} + \mathbb{P}\{E_{n,n} - \log n > y\}.$$

By the selection principle, there exists a sequence $\{n_k : k \in \mathbb{N}\}$ such that $\lim\limits_{k\to\infty} n_k = \infty$, and, as $k \to \infty$, $\frac{N_{\log n_k} - b_{n_k}}{a_{n_k}}$ weakly converges to some μ^*. Letting in the last inequality n go to ∞ along the sequence $\{n_k : k \in \mathbb{N}\}$ and using (2.5), and then sending y to ∞, we obtain $\mu(x,\infty) \le \mu^*(x,\infty)$ at all joint continuity points of μ and μ^*.

Analogously, for $y<0$,

$$\begin{aligned}\mathbb{P}\Big\{\frac{N_{E_{n,n}}-b_n}{a_n}>x\Big\} &\geq \mathbb{P}\Big\{\frac{N_{\log n+y}-b_n}{a_n}>x\Big\}\mathbb{P}\{E_{n,n}-\log n>y\}\\ &\geq \mathbb{P}\Big\{\frac{N_{\log n}-b_n}{a_n}-\frac{N'_{-y}}{a_n}>x\Big\}\mathbb{P}\{E_{n,n}-\log n>y\}.\end{aligned}$$

Once again let first n tend ∞ along $\{n_k : k\in\mathbb{N}\}$, and then send y to $-\infty$ to conclude that $\mu(x,\infty)\geq\mu^*(x,\infty)$ at all joint continuity points of μ and μ^*. Hence, $\mu=\mu^*$. The same argument works for any sequence like $\{n_k : k\in\mathbb{N}\}$ which proves that

$$\frac{N_{\log n}-b_n}{a_n} \;\Rightarrow\; \mu.$$

Recall that $P_1^{(n)}$ is an rv with distribution defined in (2.7). Since $P_1^{(n)}\overset{P}{\to}\infty$ then keeping in mind (2.9) it is plain that U_n cannot converge in distribution. Nor can U_n-b_n, for any unbounded sequence $\{b_n : n\in\mathbb{N}\}$ of real numbers. Indeed, if the convergence were the case, from the limit relation (2.5) and a.s. monotonicity of N_t would follow that $N_{\log n}-b_n$ were bounded in probability which is well-known to be false. Following the same line of reasoning one can prove that $\frac{U_n-b_n}{a_n}$ also cannot converge in distribution if a_n is either bounded or unbounded but does not go to ∞.

To establish the result in the reverse direction, one can argue in a similar manner. We, however, prefer to exploit the multiplicative form of renewal process, widely known as 'stick-breaking'.

For every $\epsilon>0$ define

$$M_n^{(\epsilon)} := \inf\{k\geq 1 : n\eta_1\eta_2\cdot\ldots\cdot\eta_k\leq\epsilon\},\quad n\in\mathbb{N},$$

and note that $M_n^{(1)}=N_{\log n}$. Assume that $\frac{M_n^{(1)}-b_n}{a_n}\;\Rightarrow\;\nu$, where ν is some proper and nondegenerate probability law. According to Proposition 27, ν is continuous distribution, a_n is slowly varying and either $b_n\equiv 0$, or b_n is slowly varying, too. From this it follows that, for every $\epsilon>0$, $\frac{M_n^{(\epsilon)}-b_n}{a_n}\;\Rightarrow\;\nu$, and ν does not depend on ϵ.

For any fixed $x\in\mathbb{R}$ and large enough $n\in\mathbb{N}$ set $k_n:=[a_nx+b_n]$. Recall that

$$Q_k := \eta_1\eta_2\cdot\ldots\cdot\eta_k.$$

Since, for sufficiently large n,

$$\begin{aligned}\mathbb{E}(1-(1-Q_{k_n})^n) &\geq \mathbb{E}(1-(1-Q_{k_n})^n)1_{\{Q_{k_n}>\epsilon/n\}} \\ &\geq (1-(1-\epsilon/n)^n)\mathbb{P}\{M_n^{(\epsilon)} > k_n\},\end{aligned}$$

then letting in (2.8) first n to ∞ and then ϵ to 0 gives

$$\liminf_{n\to\infty} \mathbb{P}\{U_n > k_n\} \geq \nu(x,\infty).$$

On the other hand, for large n,

$$\begin{aligned}\mathbb{E}(1-(1-Q_{k_n})^n) &\leq (1-(1-\epsilon/n)^n)\mathbb{P}\{M_n^{(\epsilon)} \leq k_n\} \\ &+ \mathbb{P}\{M_n^{(\epsilon)} > k_n\}.\end{aligned}$$

Letting in (2.8) first n to ∞ and then ϵ to $+\infty$ leads to

$$\limsup_{n\to\infty} \mathbb{P}\{U_n > k_n\} \leq \nu(x,\infty).$$

Combining the inequalities for the lower and upper limits we obtain $\frac{U_n-b_n}{a_n} \Rightarrow \nu$. The proof is complete.

2.4 Empty intervals and Theorem 20

First we consider an important particular case in which some intrinsic independence will allow us to find an explicit form of the limiting law.

Proposition 26. *Assume that η has the beta law with parameters $a > 0$ and 1. Then, as $n \to \infty$, $K_{n,0}$ converges in distribution to a random variable K_0 with generating function*

$$f(s) := \mathbb{E}s^{K_0} = \frac{\Gamma(1+a)\Gamma(1+a-as)}{\Gamma(1+2a-as)}, \quad s \in [0,1].$$

Thus $K_0 \stackrel{d}{=} \Pi(|\log(1-\eta)|)$, where $\{\Pi(t) : t \geq 0\}$ is a Poisson process with intensity a which is independent of η.

Proof. The process $N^*(t) := \#\{k \geq 1 : J_k \leq t\}$ is a Poisson process with intensity a. Define $M_n^* := \#\{k \geq 1 : J_k \leq E_{1,n}\}$ and

$M_i^* := \#\{k \geq 1 : J_k \in (E_{n-i,n}, E_{n-i+1,n})\}$, $M_i := (M_i^* - 1)^+$, $i = 1, \ldots, n-1$.

We exploit the following equality which holds a.s.:

$$K_{n,0} = M_1 + \ldots + M_{n-1} + M_n^*, \tag{2.13}$$

where all the rvs on the right-hand side are independent,

$$\mathbb{P}\{M_n^* = k\} = \frac{n}{n+a}\Big(\frac{a}{n+a}\Big)^k, \quad k \in \mathbb{N}_0,$$

and, for $i = 1, 2, \ldots, n-1$,

$$\mathbb{P}\{M_i = 0\} = \frac{i}{i+a}\frac{i+2a}{i+a}, \quad \mathbb{P}\{M_i = k\} = \frac{i}{i+a}\Big(\frac{a}{i+a}\Big)^{k+1}, \quad k \in \mathbb{N}.$$

All these facts follow from the following two observations: (a) $\{N^*(t) : t \geq 0\}$ is the Lévy process, (b) $E_{1,n}$, $E_{2,n} - E_{1,n}, \ldots$, $E_{n,n} - E_{n-1,n}$ are independent rvs with exponential laws (with different parameters). It remains to write down an equivalent form of (2.13) in terms of generating functions and let n go to ∞:

$$\mathbb{E}s^{K_{n,0}} = \frac{n}{n+a-as}\prod_{i=1}^{n-1}\frac{i}{i+a}\,\frac{i+2a-as}{i+a-as} \to f(s).$$

Since

$$\mathbb{E}s^{\Pi(t)} = \exp(-at(1-s)) \ \text{ and } \ \mathbb{E}(1-\eta)^u = \frac{\Gamma(1+a)\Gamma(1+u)}{\Gamma(1+a+u)}, \ u > -1,$$

then

$$\begin{aligned}
\mathbb{E}s^{\Pi(|\log(1-\eta)|)} &= \int_0^\infty \mathbb{E}s^{\Pi(t)}\mathbb{P}\{|\log(1-\eta)| \in \mathrm{d}t\} \\
&= \int_0^\infty \exp(-at(1-s))\mathbb{P}\{|\log(1-\eta)| \in \mathrm{d}t\} \\
&= \mathbb{E}(1-\eta)^{a(1-s)} = \frac{\Gamma(1+a)\Gamma(1+a-as)}{\Gamma(1+2a-as)} \\
&= \mathbb{E}s^{K_0}.
\end{aligned}$$

□

To establish the convergence of $K_{n,0}$ in the general case we will first consider a sampling scheme in which exponential points $E_1, E_2, \ldots$ are thrown at the epochs of an independent Poisson process $\{\Pi(t) : t \geq 0\}$ with intensity one and prove the convergence in distribution of $K_{\Pi(t),0}$, as $t \to \infty$. The next step of the proof is *depoissonization.*

Proof of Theorem 20.

(a) *Convergence in the Poisson model.* For $n, i \in \mathbb{N}_0$ and $t \geq 0$ set $a_n^{(i)} := \mathbb{P}\{K_{n,0} = i\}$,

$$f^{(i)}(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} a_k^{(i)} \quad \text{and} \quad g^{(i)}(t) := e^{-t} f^{(i)}(t).$$

Notice that

$$g^{(0)}(t) + e^{-t} = \mathbb{P}\{K_{\Pi(t),0} = 0\}, \quad g^{(i)}(t) = \mathbb{P}\{K_{\Pi(t),0} = i\}.$$

Equality of distributions (2.11) is equivalent to the following equalities

$$a_0^{(0)} = 1, \quad a_n^{(0)} = \sum_{k=0}^{n-1} a_k^{(0)} \mathbb{P}\{P_1^{(n)} = k\}, \ n \in \mathbb{N};$$

$$a_0^{(i)} = 0, \quad a_n^{(i)} = a_n^{(i-1)} \mathbb{E}\eta^n + \sum_{k=0}^{n-1} a_k^{(i)} \mathbb{P}\{P_1^{(n)} = k\}, \ i, n \in \mathbb{N},$$

from which we deduce after some calculations

$$g^{(0)}(t) = \mathbb{E}g^{(0)}(t\eta) + \mathbb{E}e^{-t\eta} - e^{-t} - e^{-t}\mathbb{E}f^{(0)}(t\eta) =: \mathbb{E}g^{(0)}(t\eta) + q(t);$$

$$g^{(i)}(t) = \mathbb{E}g^{(i)}(t\eta) + e^{-t}(\mathbb{E}f^{(i-1)}(t\eta) - \mathbb{E}f^{(i)}(t\eta)), \ i \in \mathbb{N}.$$

Fix any $t_0 \in \mathbb{R}$ and define

$$q_1(t) := 1_{\{t > t_0\}} \Big(\mathbb{E}\exp(-e^t \eta) - \exp(-e^t) \Big),$$

$$q_2(t) := 1_{\{t \leq t_0\}} \Big(\mathbb{E}\exp(-e^t \eta) - \exp(-e^t) \Big),$$

$$q_3(t) := 1_{\{t > t_0\}} \exp(-e^t) \mathbb{E}f^{(0)}(e^t \eta), \ q_4(t) := 1_{\{t \leq t_0\}} \exp(-e^t) \mathbb{E}f^{(0)}(e^t \eta).$$

Since $g^{(0)}$ is bounded and $g^{(0)}(0)=0$,

$$g^{(0)}(e^t)=\int_{\mathbb{R}} q(e^{t-u})\mathrm{d}\Big(\sum_{n=0}^{\infty}\mathbb{P}\{J_n\le u\}\Big).$$

If it were shown that q_j, $j=1,2,3,4$, was directly Riemann integrable (dRi) on $\mathbb{R}$ then since $q(e^t)=q_1(t)+q_2(t)-q_3(t)-q_4(t)$ we could apply the key renewal theorem to conclude that

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}\{K_{\Pi(t),0}=0\} &= \lim_{t\to\infty} g^{(0)}(e^t)=\frac{1}{\mu}\int_0^\infty \frac{q(t)}{t}\mathrm{d}t\\ &= 1-\frac{1}{\mu}\sum_{j=1}^{\infty}\frac{\mathbb{E}\eta^j}{j}\mathbb{P}\{K_{j,0}=0\}. \qquad (2.14)\end{aligned}$$

We will only prove that q_3 and q_4 are dRi, the analysis of q_1 and q_2 being similar. Since q_3 and q_4 are continuous and positive on the sets $\{t\le t_0\}$ and $\{t>t_0\}$ respectively, it suffices to find dRi majorants. We have

$$\begin{aligned}q_3(t) &\le 1_{\{t>t_0\}}\Big(\mathbb{E}\exp(-e^t(1-\eta))-\exp(-e^t)\Big)\\ &\le 1_{\{t>t_0\}}\mathbb{E}\exp(-e^t(1-\eta)) =: q_5(t),\end{aligned}$$

$$\begin{aligned}q_4(t) &\le 1_{\{t\le t_0\}}\Big(\mathbb{E}\exp(-e^t(1-\eta))-\exp(-e^t)\Big)\\ &\le 1_{\{t\le t_0\}}(1-\exp(-e^t)) =: q_6(t).\end{aligned}$$

The functions q_5 and q_6 are dRi, since they are bounded, monotone on the sets $\{t\le t_0\}$ and $\{t>t_0\}$, respectively, and integrable. Integrability of f_5 follows from the condition $\nu<\infty$. This completes the proof of (2.14).

Arguing in the same manner as for the case $i=0$ we conclude that for $i\in\mathbb{N}$

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}\{K_{\Pi(t),0}=i\} &= \lim_{t\to\infty} g^{(i)}(e^t)=\frac{1}{\mu}\int_0^\infty \frac{e^{-t}(\mathbb{E}f^{(i-1)}(t\eta)-\mathbb{E}f^{(i)}(t\eta))}{t}\mathrm{d}t\\ &= \frac{1}{\mu}\sum_{j=1}^{\infty}\frac{\mathbb{E}\eta^j}{j}\Big(\mathbb{P}\{K_{j,0}=i-1\}-\mathbb{P}\{K_{j,0}=i\}\Big).\end{aligned}$$

Assume now that $\mu=\infty$ and $\nu<\infty$. It suffices to prove that, as $t\to\infty$,

$$h(t):=e^{-t}\sum_{k=0}^{\infty}\frac{t^k}{k!}a_k^{(0)}\to 1.$$

Since $h(0)=1$, h is bounded and satisfies

$$h(t)=\mathbb{E}h(t\eta)-e^{-t}\mathbb{E}f^{(0)}(t\eta),$$

we conclude that

$$h(e^t)=1-\int_{\mathbb{R}}\exp(e^{t-u})\mathbb{E}f^{(0)}(e^{t-u}\eta)\mathrm{d}\Big(\sum_{n=0}^{\infty}\mathbb{P}\{J_n\le u\}\Big).$$

In the same way as in the first part of the proof we check that the key renewal theorem applies to yield

$$\lim_{t\to\infty}h(e^t)=1-\frac{1}{\mu}\int_0^{\infty}\frac{e^{-u}\mathbb{E}f^{(0)}(u\eta)}{u}\mathrm{d}u=0$$

the last integral converges in view of the condition $\nu<\infty$). Thus, we have already proved that, as $t\to\infty$, $K_{\Pi(t),0}\overset{d}{\to}K_0$).

If $\mu<\infty$ then

$$\mathbb{E}K_0=\sum_{i=1}^{\infty}\mathbb{P}\{K_0\ge i\}=\frac{1}{\mu}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\mathbb{E}\eta^j}{j}\mathbb{P}\{K_{j,0}=i-1\}=\frac{1}{\mu}\sum_{j=1}^{\infty}\frac{\mathbb{E}\eta^j}{j}=\frac{\nu}{\mu}.$$

Depoissonization. For any fixed $\epsilon\in(0,1)$ and $x>0$ we have

$$\begin{aligned}
&\mathbb{P}\{K_{\Pi(t),0}>x\}\\
\le\ &\mathbb{P}\{K_{\Pi(t),0}>x,[(1-\epsilon)t]\le\Pi(t)\le[(1+\epsilon)t]\}+\mathbb{P}\{|\Pi(t)-t|>\epsilon t\}\le\\
\le\ &\mathbb{P}\{\max_{[(1-\epsilon)t]\le i\le[(1+\epsilon)t]}K_{i,0}>x\}+\mathbb{P}\{|\Pi(t)-t|>\epsilon t\}=\\
=\ &\mathbb{P}\{K_{[(1-\epsilon)t],0}>x\}+\mathbb{P}\{K_{[(1-\epsilon)t],0}\le x,\max_{[(1-\epsilon)t]+1\le i\le[(1+\epsilon)t]}K_{i,0}>x\}+\\
+\ &\mathbb{P}\{|\Pi(t)-t|>\epsilon t\}:=I_1(t)+I_2(t)+I_3(t).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{P}\{K_{\Pi(t),0}\le x\}\le\\
\le\ &\mathbb{P}\{K_{[(1-\epsilon)t],0}\le x\}+\mathbb{P}\{K_{[(1-\epsilon)t],0}>x,\min_{[(1-\epsilon)t]+1\le i\le[(1+\epsilon)t]}K_{i,0}\le x\}+\\
+\ &\mathbb{P}\{|\Pi(t)-t|>\epsilon t\}:=J_1(t)+J_2(t)+I_3(t). \qquad (2.15)
\end{aligned}$$

If exponential points $E_{[(1-\epsilon)t]+1}, \ldots, E_{[(1+\epsilon)t]}$ fall to the left from the point $E_{[(1-\epsilon)t],[(1-\epsilon)t]}$ then

$$\max_{[(1-\epsilon)t]+1\le i\le[(1+\epsilon)t]} K_{i,0} \le K_{[(1-\epsilon)t],\,0}$$

and

$$K_{[(1+\epsilon)t],\,0} \le \min_{[(1-\epsilon)t]\le i\le[(1+\epsilon)t]-1} K_{i,0},$$

which means that neither the event defining $I_2(t)$, nor $J_2(t)$ can hold. Therefore,

$$\begin{aligned} \max(I_2(t), J_2(t)) &\le \mathbb{P}\{\max_{[(1-\epsilon)t]+1\le i\le[(1+\epsilon)t]} E_i > E_{[(1-\epsilon)t],[(1-\epsilon)t]}\} = \\ = \ \mathbb{E}\Big(1-\Big(1-e^{-E_{[(1-\epsilon)t],[(1-\epsilon)t]}}\Big)^{[(1+\epsilon)t]-[(1-\epsilon)t]}\Big) &= 1-\frac{[(1-\epsilon)t]}{[(1+\epsilon)t]}. \end{aligned}$$

By a large deviation result (see, for example, [12]), there exist positive constants δ_1 and δ_2 such that for all $t>0$

$$I_3(t) \le \delta_1 e^{-\delta_2 t}.$$

Select now t such that $(1-\epsilon)t = n \in \mathbb{N}$. Then from the calculations above we obtain

$$\begin{aligned} \mathbb{P}\{K_{\Pi(n/(1-\epsilon)),\,0} > x\} &\le \mathbb{P}\{K_{n,\,0} > x\} \\ &+ 1 - n/[(1+\epsilon)n/(1-\epsilon)] + \delta_1 \exp^{-\delta_2 n/(1-\epsilon)}. \end{aligned}$$

Sending first n to ∞ and then ϵ to 0 we obtain

$$\liminf_{n\to\infty} \mathbb{P}\{K_{n,\,0} > x\} \ge \mathbb{P}\{K_{\infty,0} > x\}$$

at all continuity points x of the right-hand side. The same argument applied to (2.15) establishes the converse inequality for the upper limit.

(b) Assume we can prove that

$$\sup_{n\ge1} \mathbb{E}K_{n,0}^m < \infty \quad \text{for each } m \in \mathbb{N}. \tag{2.16}$$

Then, for each $a > 0$, the sequence $\{K^a_{n,0} : n \in \mathbb{N}\}$ is uniformly integrable. Since, as $n \to \infty$, $K^a_{n,0} \overset{d}{\to} K^a_0$ then, by [29, Theorem 5.4], $\lim\limits_{n\to\infty} \mathbb{E}K^a_{n,0} = \mathbb{E}K^a_0$ and $\mathbb{E}K^a_0 < \infty$.

For $n \in \mathbb{N}_0$ set $b_n := \mathbb{E}K_{n,\,0}$, $r_n := \frac{\mathbb{E}\eta^n}{1-\mathbb{E}\eta^n}$. We have according to (2.11)

$$b_0 = 0,\ b_n = \sum_{m=1}^{n} \pi_{n,n-m} b_{n-m} + r_n,\ n \in \mathbb{N},$$

where π_{ij} were defined in (2.6). Now b_n can be represented as

$$b_n = \sum_{m=1}^{n} g(n,m) r_m, \quad n \in \mathbb{N},$$

where $g(n,m)$ was defined in (2.12) (for convenience we set $g(n,n)$ to be one). Similarly, for $j \in \mathbb{N}$,

$$\mathbb{E}K^j_{n,0} = \sum_{m=1}^{n} g(n,m) r_m \left(\sum_{i=0}^{j-1} \binom{j}{i} \mathbb{E}K^i_{m,0} \right). \tag{2.17}$$

Although there is an ‘explicit’ formula

$$\begin{aligned}
\mathbb{E}K_{n,\,0} &= \mathbb{E}\sum_{k=0}^{\infty} \left(\left(1 - e^{-J_k} + e^{-J_{k+1}}\right)^n - \left(1 - e^{-J_k}\right)^n \right) \\
&= \int_0^\infty \Big(\mathbb{E}(1 - e^{-x}(1-\eta))^n - (1-e^{-x})^n \Big) \mathrm{d}H(x) \\
&= \int_0^\infty \Big(\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} e^{-kx} (1 - \mathbb{E}(1-\eta)^k) \Big) \mathrm{d}H(x) \\
&= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{1 - \mathbb{E}(1-\eta)^k}{1 - \mathbb{E}\eta^k},
\end{aligned}$$

where $H(x) := \sum_{k=0}^{\infty} \mathbb{P}\{J_k \le x\}$ is the renewal function, it is of little help to derive the convergence

$$\lim_{n\to\infty} \mathbb{E}K_{n,0} = \mathbb{E}K_0 \tag{2.18}$$

in case $\mu\nu < \infty$. The only existing proof [59] of (2.18) we are aware of proceeds by exploiting a limiting scheme in which ‘boxes’ are identified with gaps between consecutive points in a stationary renewal process on $\mathbb{R}$ and 'balls' are identified with points of an independent Poisson process with the intensity measure $e^{-x}\mathrm{d}x$, $x \in \mathbb{R}$.

In view of (2.18),

$$\sup_{n\geq 1}\left(\sum_{m=1}^{n} g(n,m)r_m\right) < \infty.$$

Furthermore,

$$\sup_{n\geq 1}\left(\sum_{m=1}^{n} g(n,m)t_m r_m\right) < \infty, \tag{2.19}$$

for any bounded sequence $\{t_n : n \in \mathbb{N}\}$.

The proof of (2.16) runs by induction on m. When $m = 1$, (2.16) follows from (2.18). Assume that (2.16) holds for all positive integers $m \leq m_0 - 1 \in \mathbb{N}$. Setting

$$t_n := \sum_{i=0}^{m_0-1} \binom{m_0}{i} \mathbb{E}K_{n,0}^{i},$$

we conclude that the sequence $\{t_n : n \in \mathbb{N}\}$ is bounded. Therefore, in view of (2.17) and (2.19), $\sup\limits_{n\geq 1} \mathbb{E}K_{n,0}^{m_0} < \infty$ which proves (2.16).

Assuming finally that $\nu = \infty$ and $\mu < \infty$ and using (2.12) along with Fatou's lemma gives

$$\liminf_{n\to\infty} b_n \geq \sum_{m=1}^{\infty} \frac{1-\mathbb{E}\eta^m}{\mu m} \frac{\mathbb{E}\eta^m}{1-\mathbb{E}\eta^m} = \frac{1}{\mu}\sum_{m=1}^{\infty} \frac{\mathbb{E}\eta^m}{m} = \infty,$$

where the last series diverges in view of the condition $\nu = \infty$. The proof is complete. □

2.5 Proof of Theorem 21

Assume that $\nu < \infty$ and that $\frac{U_n - b_n}{a_n}$ weakly converges to some proper and nondegenerate law γ. According to Theorem 19, the latter can occur if one of five conditions (1)-(5) holds and then $a_n \to \infty$. Notice that for each of these conditions there exist distributions that satisfy it together with the condition

$\nu < \infty$ (as e.g. in Examples 23 and (24)). By Theorem 20(a) and Markov inequality

$$\frac{K_{n,0}}{a_n} = \frac{U_n - K_n}{a_n} \xrightarrow{P} 0, \quad n \to \infty.$$

Therefore, $\frac{K_n - b_n}{a_n}$ weakly converges to γ.

Assume now that $\frac{K_n - b_n}{a_n}$ weakly converges to a proper and nondegenerate probability law γ. Essentially in the same way as for U_n (but now using the result of Theorem 20) we can prove that $a_n \to \infty$, and the same argument as above proves that $\frac{U_n - b_n}{a_n}$ weakly converges to γ. The proof is complete.

2.6 Proof of Theorem 22

Case $\mu < \infty$. With $g(n, m)$ defined on p. 63, we have

$$\mathbb{P}\{Z_n = m\} = g(n, m)\mathbb{P}\{Q_1^{(m)} = 0\}.$$

Since, according to (2.6),

$$\mathbb{P}\{Q_1^{(m)} = 0\} = \frac{\mathbb{E}(1-\eta)^m}{1 - \mathbb{E}\eta^m},$$

an appeal to (2.12) completes the proof of this case.

Case $\mu = \infty$. Denote by $\text{Over}\,(z) := \inf\{z - S_n : \ S_n < z, \ n \in \mathbb{N}_0\}$ the undershoot at $z > 0$. For $k \in \{1, 2, \ldots, n\}$ we have

$$\mathbb{P}\{Z_n > k\} = \mathbb{P}\{\text{Over}\,(E_{n,n}) > E_{n,n} - E_{n-k,n}\}. \tag{2.20}$$

Assume first that $\alpha \in [0, 1)$ and for fixed $\epsilon \in (0, 1)$ set $k_n := [n^\epsilon]$. Since $E_{n,n}$ is independent of the undershoot and tends to $+\infty$ in probability, an appeal to [32, Theorem 8.6.3] allows us to conclude that

$$\frac{\text{Over}\,(E_{n,n})}{E_{n,n}} \xrightarrow{d} \widehat{Z}_\alpha.$$

Using (2.5) we obtain that $E_{n,n}/\log n \xrightarrow{P} 1$. Since, for $x > 0$,

$$\mathbb{P}\{E_{n,n} - E_{n-k_n,n} \le x\} = (1 - e^{-x})^{k_n},$$

we can check that

$$(E_{n,n} - E_{n-k_n,n})/\log n \overset{P}{\to} \epsilon.$$

Therefore,

$$\frac{E_{n,n} - E_{n-k_n,n}}{E_{n,n}} \overset{P}{\to} \epsilon.$$

Now the result follows from the relation

$$\begin{aligned}
\mathbb{P}\Big\{\frac{\log Z_n}{\log n} > \epsilon\Big\} &= \mathbb{P}\{Z_n > k_n\} \\
&\overset{(2.20)}{=} \mathbb{P}\Big\{\frac{\mathrm{Over}\,(E_{n,n})}{E_{n,n}} > \frac{E_{n,n} - E_{n-k_n,n}}{E_{n,n}}\Big\} \\
&\to \mathbb{P}\{\widehat{Z}_\alpha > \epsilon\}.
\end{aligned}$$

Indeed, while in the case $\alpha \in (0,1)$, each $\epsilon \in (0,1)$ is a continuity point of the distribution of $\widehat{Z}_\alpha$, in the case $\alpha = 0$ the relation establishes the convergence in probability $\log Z_n/\log n \to 1$ (notice that $\log Z_n/\log n \leq 1$ a.s.).

Consider now the remaining case $\alpha = 1$. For fixed $\epsilon \in (0,1)$ set $k_n := [\exp(m^{-1}(\epsilon m(\log n)))]$, where $m^{-1}(\cdot)$ is the increasing and continuous inverse of $m(x) = \int_0^x \mathbb{P}\{-\log \eta > y\}\mathrm{d}y$, $x > 0$. Using again the independence of $E_{n,n}$ and the undershoot and exploiting [46, Theorem 6] leads to the conclusion

$$\frac{m(\mathrm{Over}\,(E_{n,n}))}{m(E_{n,n})} \overset{d}{\to} \widehat{Z}_1.$$

Fix any $k \in \mathbb{N}$. It is well-known that $m(x)$ is slowly varying at ∞. Therefore, $m^k(\log x)$ is also slowly varying at ∞, and, as $s \downarrow 0$, $m^k(-\log(1-e^{-s})) \sim m^k(-\log s)$. Applying Proposition 1.5.8 and Theorem 1.7.1$'$ from [32] to the equality

$$\mathbb{E}[m^k(E_{n,n})] = n\int_0^\infty m^k(-\log(1-e^{-s}))e^{-ns}\mathrm{d}s$$

we get $\mathbb{E}[m^k(E_{n,n})] \sim m^k(\log n)$.

Similarly,

$$\begin{aligned}
\mathbb{E}m^k(E_{n,n} - E_{n-k_n,n}) &\sim m^i(\log k_n) \\
&\sim m^k(\log\exp(m^{-1}(\epsilon m(\log n)))) = \epsilon^k m^k(\log n).
\end{aligned}$$

The last two relations (with $k = 1$ and $k = 2$) together with Chebyshev's inequality imply that

$$m(E_{n,n})/m(\log n) \xrightarrow{d} 1 \quad \text{and} \quad m(E_{n,n} - E_{n-k_n,n})/m(\log n) \xrightarrow{d} \epsilon.$$

Consequently, $m(E_{n,n} - E_{n-k_n,n})/m(E_{n,n}) \xrightarrow{d} \epsilon$. To finish the proof it remains to note that

$$\begin{aligned}
\mathbb{P}\left\{\frac{m(\log Z_n)}{m(\log n)} > \epsilon\right\} &= \mathbb{P}\left\{Z_n > k_n\right\} \\
&\overset{(2.20)}{=} \mathbb{P}\left\{\frac{m(\mathrm{Over}\,(E_{n,n}))}{m(E_{n,n})} > \frac{m(E_{n,n} - E_{n-k_n,n})}{m(E_{n,n})}\right\} \\
&\to \mathbb{P}\{\widetilde{Z} > \epsilon\}.
\end{aligned}$$

The proof is complete.

2.7 Time of the first exceedance of the threshold

Below is given a criterion of the weak convergence for

$$C_t := \inf\{k \in \mathbb{N} : T_k > t\}, \quad t \geq 0$$

the time of the first exceedance of the threshold t by a zero-delayed random walk with nonnegative steps. Theorem 19 relies heavily on this result.

Proposition 27. *Assume that $T_1 > 0$ almost surely, and the law of T_1 is non-arithmetic.*

The following assertions are equivalent.

(i) There exist functions $\{a(t), b(t) : t \geq 0\}$, $a(t) > 0$ and $b(t) \in \mathbb{R}$ such that, as $t \to \infty$, $(C_t - b(t))/a(t)$ weakly converges to a proper and nondegenerate probability law.

(ii) Either the law of T_1 belongs to the domain of attraction of a stable law, or $\mathbb{P}\{T_1 > x\}$ slowly varies at ∞.

Set $m := \mathbb{E}T_1$ *and* $\sigma^2 := \mathbb{D}T_1$.

(1) If $\sigma^2 < \infty$ *then, with* $b(t) := m^{-1}t$ *and* $a(t) := (m^{-3}\sigma^2 t)^{1/2}$*, the limiting law is standard normal.*

(2) If $\sigma^2 = \infty$ *and*

$$\int_0^x y^2 \mathbb{P}\{T_1 \in \mathrm{d}y\} \sim L(x) \quad x \to \infty,$$

for some L *slowly varying at* ∞*, then, with* $b(t) := m^{-1}t$ *and* $a(t) := m^{-3/2}c(t)$*, where* $c : \mathbb{R}^+ \to \mathbb{R}^+$ *is any function satisfying* $\lim_{t\to\infty} tL(c(t))/c^2(t) = 1$*, the limiting law is standard normal.*

(3) Assume that

$$\mathbb{P}\{T_1 > x\} \sim x^{-\alpha}L(x) \quad x \to \infty, \tag{2.21}$$

for some L *slowly varying at* ∞ *and some* $\alpha \in (1,2)$*. Then, with* $b(t) := m^{-1}t$ *and* $a(t) := m^{-(\alpha+1)/\alpha}c(t)$*, where* $c : \mathbb{R}^+ \to \mathbb{R}^+$ *is any function satisfying* $\lim_{t\to\infty} \dfrac{tL(c(t))}{c^\alpha(t)} = 1$*, the limiting law is* α*-stable with characteristic function* (2.2)*.*

(4) Assume that (2.21) *holds for* $\alpha = 1$*. Let* $c : \mathbb{R}^+ \to \mathbb{R}^+$ *be any function satisfying* $\lim_{x\to\infty} xL(c(x))/c(x) = 1$*, and set*

$$\psi(x) := x \int_0^{c(x)} \mathbb{P}\{T_1 > y\}\mathrm{d}y.$$

Let $b : \mathbb{R}^+ \to \mathbb{R}^+$ *be any function satisfying* $b(\psi(x)) \sim \psi(b(x)) \sim x$*. Then, with the so defined* $b(t)$ *and* $a(t) := b(t)c(b(t))/t$*, the limiting law is* 1*-stable with characteristic function* (2.3)*.*

(5) If (2.21) *holds for some* $\alpha \in [0,1)$ *then, with* $b(t) \equiv 0$ *and* $a(t) := t^\alpha/L(t)$*, the limiting law is the Mittag-Leffler distribution.*

2.8 Bibliographic comments

The *Bernoulli sieve* was introduced by Alexander Gnedin in [55]. This model generalizes at least three classical schemes of Applied Probability.

(a) If the law of η is degenerate at some point from the interval $(0,1)$, the Bernoulli sieve naturally arises in the leader election problem and its variants or equivalently in the problems around the maximum of geometric samples. This setting has received an enormous amount of attention especially in the AofA community ([35, 49, 65, 89, 94] and many others).

(b) Assuming that η has the beta law with parameters $\theta > 0$ and 1 denote by $K_{j,n}$ the number of intervals in the corresponding Bernoulli sieve which contain j points. Consider a random vector $C^{(n)} := (C_{n,1}, \dots, C_{n,n})$, where $C_{n,j}$ is the number of cycles of length j in the θ-biased random permutation of n, and denote by $L_n = \sum_{j=1}^n C_{n,j}$ the total number of cycles (exhaustive information about random permutations and related objects is available in a nice, reader-friendly monograph [2]). Then

$$C^{(n)} \overset{d}{=} (K_{1,n}, \dots, K_{n,n}) \text{ and } L_n \overset{d}{=} K_n = \sum_{j=1}^n K_{j,n}.$$

It is known that the law of $C^{(n)}$ is given by the Ewens sampling formula:

$$\mathbb{P}\{C^{(n)} = (c_1, \dots, c_n)\} = 1_{\{\sum_{j=1}^n jc_j = n\}} \frac{n!}{\theta(n)} \prod_{j=1}^n \left(\frac{\theta}{j}\right)^{c_j} \frac{1}{c_j!}, \quad c_j = 0, 1, \dots, n.$$

Also it is a classics of the field that the numbers of cycles of distinct lengths are asymptotically independent and weakly converge to the Poisson laws, and the total number of cycles, properly normalized and centered, weakly converges to the standard normal law. Thus our Theorem 21 can be seen as a generalization of the latter statement.

As far as we know there are only few papers [55, 58, 59, 60, 61] which do not assume that the law of η is known.

(c) A scheme of allocation of n balls over an infinite array of boxes indexed by positive integers with deterministic probability Q_k of hitting the kth box is called the *infinite occupancy scheme*. Although such a model was introduced and partially studied in [11, 39], the first systematic treatment is due to Samuel Karlin [91]. Among other things the cited paper has proved a CLT for the number of occupied boxes (an ultimate version was obtained in [45]). Modern exposition of the area can be found in a recent survey [56].

The Bernoulli sieve is the Karlin's occupancy scheme in a random environment. The only difference from the original model is that the probability of hitting the kth box (interval) by a ball (exponential point) is random

$$Q_j = \eta_1 \cdots \eta_{j-1}(1-\eta_j), \quad j \in \mathbb{N}.$$

Chapter 2 is based on [58]. Theorem 21 generalizes one Gnedin's result given in [55, Proposition 10] under the assumption $\mathrm{Var}\,(\log \eta) < \infty$. Note that our technique is simpler and more probabilistic than purely analytic approach of [55]. The proof of Theorem 20 exploits a "poissonization-depoissonization" device. This approach is well-known and dates back at least to Marc Kac [90] (see also [91] for an application to the Karlin's occupancy scheme). Section 2.7 is devoted to a criterion of the weak convergence of

$$C_t := \inf\{k \in \mathbb{N} : T_k > t\}$$

the time of the first exceedance of the threshold $t \geq 0$ by a zero-delayed random walk $\{T_n : n \in \mathbb{N}_0\}$ with nonnegative steps. The weak convergence of C_t was investigated by many authors (see, for instance, [7, 30, 47, 69, 73]). However, to our knowledge, separate particular results are scattered over the literature and there is no paper that would contain a complete information concerning the weak asymptotics of C_t. In case when the step of $\{T_n : n \in \mathbb{N}_0\}$ only takes positive integer values, a criterion of the weak convergence of C_t can be derived from Theorem 1.2, Theorem 1.5 and Proposition 3.1 in [84]. Note that the first two cited statements were formulated for rvs other than C_t, but with the same weak asymptotic behavior. Following exactly the same route as in [84], one can check that random walks with steps having nonarithmetic laws enjoy the same weak asymptotics.

Chapter 3

Asymptotics of intrinsic martingales in branching random walks

3.1 Definition of BRW and intrinsic martingales in BRW

Consider a population starting from one progenitor and evolving like a generalized Galton-Watson process, in which individuals may have infinitely many children. We assume that the structure of the branching process is enriched by the locations of individuals on the real line, so that the progentitor is located at the origin, and the displacements of children relative to their mother are described by a point process $\mathcal{Z} = \sum_{i=1}^{N} \delta_{X_i}$ on $\mathbb{R}$.

Thus $N := \mathcal{Z}(\mathbb{R})$ is the total size of offspring of a particular member of the population, and X_i is the displacement of the i-th child. The displacement processes of all population members are supposed to be independent copies of $\mathcal{Z}$. We further assume $\mathcal{Z}(\{-\infty\}) = 0$ and $\mathbb{E}N > 1$ (supercriticality) including the possibility $\mathbb{P}\{N = \infty\} > 0$ as already stated above. If $\mathbb{P}\{N < \infty\} = 1$, then the population size process forms an ordinary Galton-Watson process. Supercriticality ensures survival of the population with positive probability.

For $n = 0, 1, ...$ let $\mathcal{Z}_n$ be the point process that defines the positions on $\mathbb{R}$ of the individuals of the n-th generation, with their total number being given by $\mathcal{Z}_n(\mathbb{R})$. The sequence $\{\mathcal{Z}_n : n = 0, 1, ...\}$ is called *the branching random walk* (BRW).

Let $\mathbf{V} := \bigcup_{n=0}^{\infty} \mathbb{N}^n$ be the infinite Ulam-Harris tree of all finite words $v = v_1...v_n$ (shorthand for sequence $(v_1, ..., v_n)$), with root $\varnothing$, zero generation $\mathbb{N}^0 := \{\varnothing\}$, and edges connecting each $v \in \mathbf{V}$ with its successors vi, $i = 1, 2, ...$ The length of v is denoted as $|v|$. Call v an individual and $|v|$ its generation number. A BRW $\{\mathcal{Z}_n : n = 0, 1, ...\}$ may now be represented as a random labeled subtree of $\mathbf{V}$ with the same root. This subtree $\mathbf{T}$ is obtained recursively as follows: For any $v \in \mathbf{T}$, let $N(v)$ be the number of its successors (children) and $\mathcal{Z}(v) := \sum_{i=1}^{N(v)} \delta_{X_i(v)}$ denote the point process describing the displacements of the children vi of v relative to their mother. By assumption, the $\mathcal{Z}(v)$ are independent copies of $\mathcal{Z}$. The Galton-Watson tree associated with this model is now given by

$$\mathbf{T} := \{\varnothing\} \cup \{v \in \mathbf{V} \backslash \{\varnothing\} : v_i \leq N(v_1...v_{i-1}) \text{ for } i = 1, ..., |v|\},$$

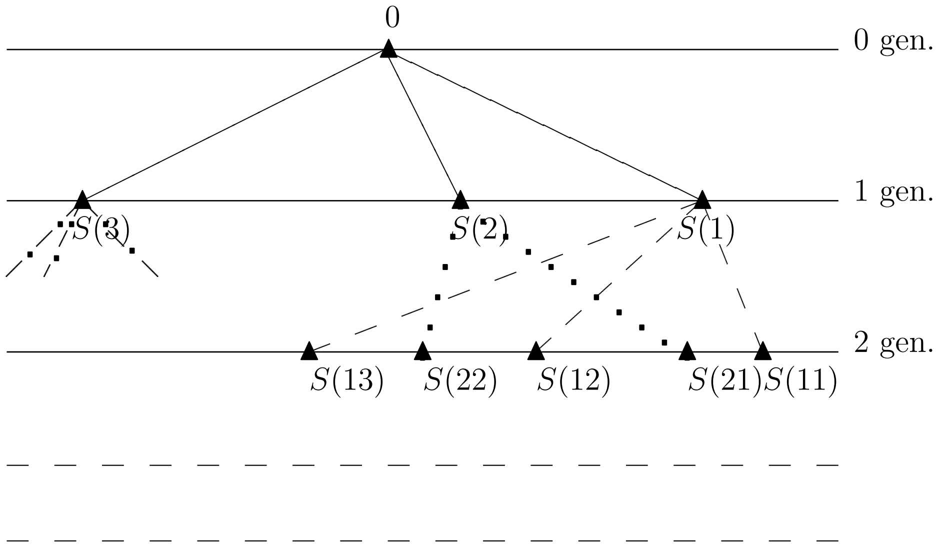
and $X_i(v)$ denotes the label attached to the edge $(v, vi) \in \mathbf{T} \times \mathbf{T}$ and describes the displacement of vi relative to v. Let us stipulate hereafter that $\sum_{|v|=n}$ means summation over all vertices of $\mathbf{T}$ (not $\mathbf{V}$) of length n. For $v = v_1...v_n \in \mathbf{T}$, put $S(v) := \sum_{i=1}^{n} X_{v_i}(v_1...v_{i-1})$. Then $S(v)$ gives the position of v on the real line (of course, $S(\varnothing) = 0$), and $\mathcal{Z}_n = \sum_{|v|=n} \delta_{S(v)}$ for all $n = 0, 1, ...$

On the figure 2.1.1 (a random variable M only takes values 2 and 3) differences $S(11) - S(1)$, $S(12) - S(1)$, $S(13) - S(1)$ and $S(21) - S(2)$, $S(22) - S(2)$ are realizations of two independent copies of $\mathcal{Z}$. Subsequent evolution of the population is similar.

Fig. 4.1.1: A realization of the BRW

Suppose there exists $\gamma > 0$ such that

$$m(\gamma) \;:=\; \mathbb{E} \int_{\mathbb{R}} e^{\gamma x} \, \mathcal{Z}(dx) \in (0, \infty). \tag{3.1}$$



For $n = 1, 2, ...,$ define $\mathcal{F}_n := \sigma(\mathcal{Z}(v) : |v| \le n-1)$, and let $\mathcal{F}_0$ be the trivial σ-field. Put

$$W_n \;:=\; m(\gamma)^{-n} \int_{\mathbb{R}} e^{\gamma x}\, \mathcal{Z}_n(dx) \;=\; m(\gamma)^{-n} \sum_{|v|=n} e^{\gamma S(v)} \;=\; \sum_{|v|=n} L(v), \tag{3.2}$$

where $L(v) := e^{\gamma S(v)}/m(\gamma)^{|v|}$. Notice that the dependence of W_n on γ has been suppressed. The sequence $\{(W_n, \mathcal{F}_n) : \; n \in \mathbb{N}_0\}$ forms a non-negative martingale with mean one, thus converging a.s. to some limiting variable W, satisfying $\mathbb{E}W \le 1$. This martingale is called *intrinsic martingale* of the branching random walk.

It is known (see, for instance, [3]) that $\mathbb{P}\{W > 0\} > 0$ if and only if $\{W_n : n \in \mathbb{N}\}$ is uniformly integrable. While uniform integrability is clearly sufficient, the necessity hinges on the well known fact that W satisfies the stochastic fixed point equation

$$W \;=\; \sum_{|v|=n} L(v) W(v) \quad \text{a.s.} \tag{3.3}$$

for $n \in \mathbb{N}$, where the $W(v)$, $|v| = n$, are i.i.d. copies of W that are also independent of $\{L(v) : |v| = n\}$, see e.g. [26]. In fact $W(v)$ is nothing but the a.s. limit of the martingale $\{\sum_{|w|=m} \frac{L(vw)}{L(v)} : m \in \mathbb{N}_0\}$ which forms the counterpart of $\{W_n : n \in \mathbb{N}_0\}$, but for the subtree of $\mathbf{T}$ rooted at v.

Let (M, Q) be a random variable with distribution defined by

$$\mathbb{P}\{(M,Q) \in A\} = \mathbb{E}\left(\sum_{|v|=1} L(v) 1_A\left(L(v), \sum_{|u|=1} L(u)\right)\right) \tag{3.4}$$

for any Borel set A in $\mathbb{R}^+ \times \mathbb{R}^+$. Notice that the right-hand side of (3.4) does indeed define a probability distribution because $\mathbb{E}\sum_{|v|=1} L(v) = \mathbb{E}W_1 = 1$. Plainly, the distribution of M is given by

$$\mathbb{P}\{M \in B\} \ := \ \mathbb{E}\left[\sum_{|v|=1} L(v)\delta_{L(v)}(B)\right], \tag{3.5}$$

for any Borel subset B of $\mathbb{R}^+$. More generally, we have (see e.g. [26, Lemma 4.1])

$$\mathbb{P}\{\Pi_n \in B\} \ = \ \mathbb{E}\left[\sum_{|v|=n} L(v)\delta_{L(v)}(B)\right], \tag{3.6}$$

for each $n \in \mathbb{N}$, whenever $\{M_k : k \in \mathbb{N}\}$ is a family of independent copies of M, and

$$\Pi_0 := 1, \quad \Pi_n := \prod_{k=1}^{n} M_k, \quad n \in \mathbb{N}.$$

It is important to note that

$$\mathbb{P}\{M = 0\} = 0 \quad \text{and} \quad \mathbb{P}\{M = 1\} < 1. \tag{3.7}$$

The first assertion follows since, by (3.5), $\mathbb{P}\{M > 0\} = \mathbb{E}W_1 = 1$. As for the second, observe that $\mathbb{P}\{M = 1\} = 1$ implies $\mathbb{E}\sum_{|v|=1} L(v)1_{\{L(v)\neq 1\}} = 0$ which in combination with $\mathbb{E}W_1 = 1$ entails that the point process $\mathcal{Z}$ consists of only one point u with $L(u) = 1$. This contradicts the assumed supercriticality of the BRW.

For $x > 0$, define

$$A(x) \ := \ \int_0^x \mathbb{P}\{-\log|M| > y\}\, dy \ = \ \mathbb{E}\min\left(\log^-|M|, x\right) \tag{3.8}$$

and then $J(x) := x/A(x)$. Set $J(0) := \lim_{x\downarrow 0} J(x) = 1/\mathbb{P}\{|M| < 1\}$.

The following criterion of uniform integrability was given in [3].

Theorem 28. *The martingale $\{W_n : n \in \mathbb{N}_0\}$ is uniformly integrable (regular) if and only if the following two conditions hold true:*

$$\lim_{n\to\infty} \Pi_n = 0 \quad a.s. \tag{3.9}$$

and

$$\mathbb{E}W_1 J(\log^+ W_1) \; = \; \int_{(1,\infty)} xJ(\log x)\; \mathbb{P}\{W_1 \in dx\} \; < \; \infty. \tag{3.10}$$

There are three distinct cases in which conditions (3.9) and (3.10) hold simultaneously:
(A1) $\mathbb{E}\log M \in (-\infty, 0)$ and $\mathbb{E}W_1 \log^+ W_1 < \infty$;
(A2) $\mathbb{E}\log M = -\infty$ and $\mathbb{E}W_1 J(\log^+ W_1) < \infty$;
(A3) $\mathbb{E}\log^+ M = \mathbb{E}\log^- M = +\infty$, $\mathbb{E}W_1 J(\log^+ W_1) < \infty$, and

$$\mathbb{E}J\big(\log^+ M\big) \; = \; \int_{(1,\infty)} \frac{\log x}{\int_0^{\log x} \widehat{\mathbb{P}}\{-\log M > y\}\, dy} \; \widehat{\mathbb{P}}\{M \in dx\} \; < \; \infty.$$

The main result of this section proves the power-like tail behavior of the law of $\sup\limits_{n\geq 0} W_n$.

Theorem 29. *Suppose there exists $b > 1$ such that*

$$\mathbb{E}\sum_{|v|=1} L^b(v) = 1, \quad \mathbb{E}\sum_{|v|=1} L^b(v)\log^+ L(v) < \infty \quad \textit{and} \quad \mathbb{E}W_1^b < \infty.$$

(a) If $\log M$ has non-lattice law then there exists a constant $C \in (0,\infty)$ such that

$$\lim_{x\to\infty} x^b \mathbb{P}\{\sup_{n\geq 0} W_n > x\} = C.$$

(b) If $\log M$ has lattice law with the span δ then there exists a positive δ-periodic function $C(x)$ such that, for every $x \in \mathbb{R}$,

$$\lim_{k\to\infty} e^{(\gamma k + x)b}\mathbb{P}\{\sup_{n\geq 0} W_n > e^{\gamma k + x}\} = C(x).$$

Remark 30. (a) The constant C and the function $C(x)$ are known in an explicit, but complicated form (see the proof of Theorem 29).
(b) Under the assumptions of Theorem 29, the martingale W_n converges a.s. and in mean to a random variable W (see the proof of Theorem 29). Arguing in a similar, but simpler way, one can check that the statement of Theorem 29 holds for the law of W with different C and $C(x)$. The first version of this result was given in [77, Proposition 7] and then corrected in [79]. Earlier Liu in [98, Theorem 2.2] obtained a weaker version of this last result by an application of a famous Kesten-Grincevicius' theorem (see [66, Theorem 2.3], [67, Theorem 2], [92, Theorem 5]) to the perpetuity (3.14).

3.2 Properties of the law of W and its connection to perpetuities

First, we point out an inequality that relates the laws of W and $\sup_{n\geq 0} W_n$.

Lemma 31. *Suppose that W_n converges in mean to a random variable W. Then there exists $0 < r < 1$ and $B = B(r) > 1$ such that*

$$\mathbb{P}\{W > t\} \leq \mathbb{P}\{\sup_{n\geq 0} W_n > t\} \leq B\,\mathbb{P}\{W > rt\},$$

whenever $t > 1$.

Proof. In case $N < \infty$ a.s., this result is given in [21, Lemma 2]. Our proof is simpler and does not require that assumption on N.

We only have to check the right-hand side of the inequality. For $t > 1$ and $n \in \mathbb{N}$ define the events

$$E_n := \{\max_{0\leq k\leq n-1} W_k \leq t, W_n > t\}.$$

Then $\mathbb{P}\{\sup_{n\geq 0} W_n > t\} = \sum_{n=1}^{\infty} \mathbb{P}\{E_n\}$. Therefore, for any $r > 0$

$$\mathbb{P}\{W > rt\} \geq \mathbb{P}\{W > rt,\ \sup_{n\geq 0} W_n > t\} = \sum_{n=1}^{\infty} \mathbb{P}\{W > rt|E_n\}\mathbb{P}\{E_n\}. \quad (3.11)$$

For $n \in \mathbb{N}$ and any fixed $d > 0$ set

$$Q_n := \Big\{ \sum_{|v|=n} \frac{L(v)(W(v) \wedge d)}{W_n} > r \Big\}.$$

Using equality (3.3) we obtain

$$\mathbb{P}\{W > rt | E_n\} \geq \mathbb{P}\Big\{ \sum_{|v|=n} \frac{L(v)(W(v) \wedge d)}{W_n} > \frac{rt}{W_n} \Big| E_n \Big\} \geq \tag{3.12}$$

$$\geq \mathbb{P}\{Q_n | E_n\} = (\mathbb{P}\{E_n\})^{-1} \mathbb{E}\mathbb{P}\{Q_n | \mathcal{F}_n\} 1_{E_n} =$$

$$= (\mathbb{P}\{E_n\})^{-1} \mathbb{E}\mathbb{P}\Big\{ \sum_{|v|=n} \frac{L(v)}{W_n}(W(v) \wedge d - r) > 0 \Big| \mathcal{F}_n \Big\} 1_{E_n}.$$

Recall that the $W(v)$, $|v| = n$, are i.i.d. copies of W that are also independent of $\{L(v) : |v| = n\}$.

Let η be a rv with $\mathbb{E}\eta > 0$. By Cauchy-Schwarz inequality

$$\mathbb{P}\{\eta > 0\} \geq \frac{(\mathbb{E}\eta)^2}{\mathbb{E}\eta^2}. \tag{3.13}$$

As $\mathbb{E}W = 1$, we can fix $r \in (0, 1)$ and pick $d = d(r)$ such that $\mathbb{E}(W \wedge d - r) > 0$. Then, according to (3.13),

$$\mathbb{P}\Big\{ \sum_{|v|=n} \frac{L(v)}{W_n}(W(v) \wedge d - r) > 0 \Big| \mathcal{F}_n \Big\} \geq \frac{(\mathbb{E}(W \wedge d - r))^2}{\mathbb{E}(W \wedge d - r)^2} =: \frac{1}{B} \quad \text{a.s.}$$

Applying this inequality to (3.12) gives

$$\mathbb{P}\{W > rt | E_n\} \geq 1/B.$$

Appealing to (3.11) completes the proof. □

The following is an immediate consequence of the preceding result.

Corollary 32. For $\alpha > 0$, $\mathbb{E}W^\alpha < \infty$ if and only if $\mathbb{E}(\sup_{n \geq 0} W_n)^\alpha < \infty$.

In case $\mathbb{P}\{W > 0\} > 0$, our next result shows that the law of W is the law of certain perpetuity.

Lemma 33. *Suppose that the martingale W_n converges in mean to a random variable W. Denote by $\overline{W}$ a random variable with distribution $\mathbb{P}\{\overline{W} \in dx\} = x\mathbb{P}\{W \in dx\}$. Then*

$$\overline{W} \stackrel{d}{=} T\overline{W} + R, \tag{3.14}$$

where (T, R) is a random vector which is independent of $\overline{W}$ with distribution defined by

$$\mathbb{P}\Big\{(T,R) \in A\Big\} = \mathbb{E}\Bigg(\sum_{|v|=1} L(v)1_A\Bigg(L(v), \sum_{\substack{|u|=1\\ u\neq v}} L(u)W(u)\Bigg)\Bigg), \tag{3.15}$$

for any Borel set A in $\mathbb{R}^2$. In particular, the distribution of T coincides with the distribution of M defined in (3.5)*. The random variables $\{W(v) : |v| = 1\}$ in* (3.15) *are i.i.d. copies of W.*

Proof. In the case $N < \infty$ a.s., this result is given in [98, Lemma 4.1]. Our new proof is simpler.

Throughout the proof the record $u \neq v$ only concerns the individuals of *the first generation*, other than v. Let $\varphi(s)$ be the Laplace-Stieltjes transform of W. From equality (3.3) with $n = 1$ it follows that

$$\varphi(s) = \mathbb{E}\prod_{|v|=1} \varphi(sL(v)).$$

For any $h > 0$

$$\begin{aligned} \frac{\varphi(sL(v)) - \varphi((s+h)L(v))}{h} &\leq \frac{\sum_{|v|=1}(\varphi(sL(v)) - \varphi((s+h)L(v)))}{h} \\ &\leq \frac{\sum_{|v|=1}(-\varphi'(sL(v))hL(v))}{h} \leq \sum_{|v|=1} L(v). \end{aligned}$$

Now, using the fact, that $\mathbb{E}\sum_{|v|=1} L(v) = 1$ and the Lebesgue's dominated convergence theorem we can formally differentiate the equality for $\varphi(s)$

$$\varphi'(s) = \mathbb{E}\sum_{|v|=1} \varphi'(sL(v))L(v)\prod_{u\neq v} \varphi(sL(u)). \tag{3.16}$$

To prove this formal differentiating it's enough to show (by Lebegue theorem), that

Denote by $\psi(s)$ the LST of $\overline{W}$. Then

$$\psi(s) = -\varphi'(s).$$

Our aim is to find the LST of the right-hand side of (3.14). Using equality (3.15) with $k(a,b) = e^{-sxa-sb}$, where s and x are fixed, we obtain

$$\begin{aligned} \mathbb{E}e^{-sxT-sR} &= \mathbb{E}\sum_{|v|=1} L(v)e^{-sxL(v)-s\sum_{u\neq v}L(u)W(u)} = \mathbb{E}(\mathbb{E}(\cdot|\mathcal{F}_1)) \\ &= \mathbb{E}\sum_{u} Y_u e^{-sxY_u}\prod_{v\neq u}\varphi(sY_v). \end{aligned}$$

Further

$$\begin{aligned} \mathbb{E}e^{-s(T\overline{W}+R)} &= \int_0^\infty \mathbb{E}e^{-sxT-sR}\mathrm{d}\,\mathbb{P}\{\overline{W}\le x\} \\ &= \mathbb{E}\sum_{|v|=1} L(v)\int_0^\infty e^{-sxL(v)}\mathrm{d}\,\mathbb{P}\{\overline{W}\le x\}\prod_{u\neq v}\varphi(sL(u)) \\ &= \mathbb{E}\sum_{|v|=1}(-\varphi'(sL(v))L(v))\prod_{u\neq v}\varphi(sL(u)). \end{aligned}$$

In view of (3.16) we conclude that

$$-\varphi'(s) = \mathbb{E}e^{-s\overline{W}} = \mathbb{E}e^{-s(T\overline{W}+R)},$$

which proves (3.14).

Finally, (3.15) with $A = B\times\mathbb{R}$ implies that

$$\mathbb{P}\{T\in B\} := \mathbb{E}\left(\sum_{|v|=1} L(v)\delta_{L(v)}(B)\right).$$

Hence, $T \overset{d}{=} M$. $\square$

3.3 Proof of Theorem 29

Set

$$N(x) := \int_0^\infty \frac{1}{z}\mathbb{P}\{\sup_{n\ge 0} W_n > \frac{x}{z}\}\mathrm{d}\,\mathbb{P}\{M\le z\}.$$

The measure U defined by the equality

$$U(\mathrm{d}z) := z^{b-1}\mathbb{P}\{M \in \mathrm{d}z\},$$

is a probability measure, as

$$\int_0^\infty U(\mathrm{d}z) = \mathbb{E}M^{b-1} = \mathbb{E}\sum_{|v|=1} L^b(v) = 1.$$

Our point of departure is the following representation

$$\begin{aligned} x^b\mathbb{P}\{\sup_{n\geq 0} W_n > x\} &= \int_0^\infty \frac{x^b}{z^b}\mathbb{P}\Big\{\sup_{n\geq 0} W_n > \frac{x}{z}\Big\}\mathrm{d}\,U(z) \\ &+ x^b\Big(\mathbb{P}\{\sup_{n\geq 0} W_n > x\} - N(x)\Big). \end{aligned}$$

Integrate this equality over $[0, e^y]$ and then multiply by e^{-y} to obtain

$$\begin{aligned} & e^{-y}\int_0^{e^y} x^b\mathbb{P}\{\sup_{n\geq 0} W_n > x\}\mathrm{d}x \qquad (3.17) \\ = \; & e^{-y}\int_0^{e^y}\int_0^\infty \frac{x^b}{z^b}\mathbb{P}\Big\{\sup_{n\geq 0} W_n > \frac{x}{z}\Big\}\mathrm{d}\,U(z)\mathrm{d}x \\ + \; & e^{-y}\int_0^{e^y} x^b\Big(\mathbb{P}\{\sup_{n\geq 0} W_n > x\} - N(x)\Big)\mathrm{d}\,x. \end{aligned}$$

Set

$$P(y) := e^{-y}\int_0^{e^y} x^b\mathbb{P}\{\sup_{n\geq 0} W_n > x\}\mathrm{d}x,$$

$$Q(y) = e^{-y}\int_0^{e^y} x^b\Big(\mathbb{P}\{\sup_{n\geq 0} W_n > x\} - N(x)\Big)\mathrm{d}x.$$

Since

$$\begin{aligned} & e^{-y}\int_0^{e^y}\int_0^\infty \frac{x^b}{z^b}\mathbb{P}\Big\{\sup_{n\geq 0} W_n > \frac{x}{z}\Big\}\mathrm{d}\,U(z)\mathrm{d}x \\ = \; & e^{-y}\int_0^\infty \mathrm{d}\,U(z) z\int_0^{\frac{e^y}{z}} v^b\mathbb{P}\{\sup_{n\geq 0} W_n > v\}\mathrm{d}v \\ = \; & \int_{-\infty}^\infty e^{-(y-t)}\mathrm{d}\,U(e^t)\int_0^{e^{y-t}} v^b\mathbb{P}\{\sup_{n\geq 0} W_n > v\}\mathrm{d}v \\ = \; & \int_{-\infty}^\infty P(y-t)\mathrm{d}\,U(e^t), \end{aligned}$$

(3.17) is a renewal equation

$$P(y) = \int_{-\infty}^{\infty} P(y-t)\mathrm{d}\,U(e^t) + Q(y).$$

The function $q(x) := \mathbb{E}\sum_{|v|=1} L^x(v)$ is log-convex on the set of finiteness. Thus condition $q(1) = q(b) = 1$ implies that, for $x \in (1, b)$,

$$\mathbb{E}\sum_{|v|=1} L^x(v) < 1. \tag{3.18}$$

By Jensen's inequality,

$$\mathbb{E}\log M = \mathbb{E}\sum_{|v|=1} L(v)\log L(v) < 0.$$

Condition $\mathbb{E}W_1^b < \infty$ implies that $\mathbb{E}W_1 \log^+ W_1 < \infty$. Therefore, by Theorem 28, the martingale $\{W_n : n \in \mathbb{N}_0\}$ converges a.s. and in mean to a rv W. Furthermore, by [77, Proposition 4],

$$\mathbb{E}W^a < \infty \quad \text{for all} \quad a \in (1, b). \tag{3.19}$$

Set

$$g(x) := x^b\Big(\mathbb{P}\{\sup_{n\ge 0} W_n > x\} - N(x)\Big).$$

Then

$$Q(y) = e^{-y}\int_0^{e^y} g(x)\mathrm{d}x = e^{-y}\int_{-\infty}^{y} e^t g(e^t)\mathrm{d}t.$$

Now we intend to find the asymptotics of $P(x)$, as $x \to \infty$, by using the key renewal theorem on the whole line. To this end, we first prove that $Q(y)$ is directly Riemann integrable. For this to be true, according to [66, Lemma 9.2], it suffices to check that $0 < \int_{-\infty}^{\infty} g(e^t)\mathrm{d}t < \infty$ or, equivalently,

$$0 < I(b) := \int_0^{\infty} y^{b-1}\Big(\mathbb{P}\{\sup_{n\ge 0} W_n > y\} - N(y)\Big)\mathrm{d}y < \infty. \tag{3.20}$$

Pick any $a \in (1, b)$. From (3.18) it follows that

$$\mathbb{E}M^{a-1} = \mathbb{E}\sum_{|v|=1} L^a(v) < 1.$$

Furthermore, (3.19) and Corollary 32 together imply that $\mathbb{E}(\sup_{n\geq 0} W_n)^a < \infty$. Therefore,

$$\begin{aligned}\int_0^\infty y^{a-1}\mathbb{P}\{\sup_{n\geq 0} W_n > y\}\mathrm{d}y &= \frac{1}{a}\int_0^\infty y^a \mathrm{d}\,\mathbb{P}\{\sup_{n\geq 0} W_n \leq y\} \\ &= \frac{1}{a}\mathbb{E}(\sup_{n\geq 0} W_n)^a < \infty;\end{aligned}$$

$$\int_0^\infty y^{a-1}N(y)\mathrm{d}y = \frac{1}{a}\mathbb{E}(\sup_{n\geq 0} W_n)^a \mathbb{E}M^{a-1} < \infty;$$

$$0 < I(a) = \frac{1}{a}\mathbb{E}(\sup_{n\geq 0} W_n)^a(1-\mathbb{E}M^{a-1}) < \infty.$$

As $W \leq \sup_{n\geq 0} W_n$ a.s., we have $\mathbb{E}W^a \leq \mathbb{E}(\sup_{n\geq 0} W_n)^a$. By Lemma 31, there exist constants $r \in (0,1)$ and $B > 1$ such that

$$\mathbb{E}(\sup_{n\geq 0} W_n)^a \leq \frac{B}{r^a}\mathbb{E}(W \vee r)^a.$$

Therefore, setting

$$K_a := \mathbb{E}W^a(1-\mathbb{E}M^{a-1}),$$

we conclude that

$$K_a \leq aI(a) \leq \frac{B}{r^a}K_a. \tag{3.21}$$

Recall a known inequality

$$(x+y)^s \leq (2^{s-1}\vee 1)(x^s+y^s), x, y \geq 0, s > 0, \tag{3.22}$$

which follows from the concavity of $x \to x^s$ for $s \in (0,1]$, and the convexity– for $s > 1$. To estimate K_a from above we use equality of distributions (3.14). If $a-1 \in (0,1]$ then

$$\mathbb{E}W^a = \mathbb{E}\overline{W}^{a-1} = \mathbb{E}(T\overline{W}+R)^{a-1} \overset{(3.22)}{\leq} \mathbb{E}T^{a-1}\mathbb{E}W^a + \mathbb{E}R^{a-1};$$
$$K_a \leq \mathbb{E}R^{a-1}.$$

For fixed $c \in (1,b)$ pick $\epsilon > 0$ such that $\mathbb{E}T^{c-1}(1+\epsilon^{-1})^{c-1} < 1$. Suppose that $d := \mathbb{P}\{\epsilon R \geq T\overline{W}\} = 0$. Then $\overline{W} \leq (1+\epsilon^{-1})T\overline{W}$ stochastically, which contradicts the choice of ϵ. Hence, $d > 0$. By Rolle's theorem, for $\epsilon y \geq x > 0$ and $h \in (0,1)$

$$(x+y)^h - x^h \geq h(x+y)^{h-1}y \geq h(1+\epsilon)^{h-1}y^h.$$

Thus

$$\begin{aligned} K_a &\geq \mathbb{E}((T\overline{W}+R)^{a-1} - T^{a-1}\overline{W}^{a-1})1_{\{\epsilon R \geq T\overline{W}\}} \\ &\geq (a-1)(1+\epsilon)^{a-2}\mathbb{E}R^{a-1}1_{\{\epsilon R \geq T\overline{W}\}} > 0. \end{aligned}$$

In case $a-1 > 1$, we use the inequality

$$\begin{aligned} (u+v)^{a-1} &= u^{a-1} + (a-1)\int_0^v (u+t)^{a-2}dt \\ &\leq u^{a-1} + (a-1)(u+v)^{a-2}v \\ &\overset{(3.22)}{\leq} u^{a-1} + (a-1)(2^{a-3} \vee 1)(u^{a-2}v + v^{a-1}), \end{aligned}$$

which holds for $u, v \geq 0$, and Holder's inequality with $p := \frac{a-1}{a-2}$, $q := a-1$, to obtain

$$\begin{aligned} K_a &\leq \text{const}\,(\mathbb{E}W^{a-1}\mathbb{E}T^{a-2}R + \mathbb{E}R^{a-1}) \\ &\leq \text{const}\,(\mathbb{E}W^{a-1}(\mathbb{E}T^{a-1})^{1/p}(\mathbb{E}R^{a-1})^{1/q} + \mathbb{E}R^{a-1}) \\ &= \text{const}\,(\mathbb{E}W^{a-1}(\mathbb{E}R^{a-1})^{1/q} + \mathbb{E}R^{a-1}). \end{aligned}$$

Finally,

$$\mathbb{E}W^a = \mathbb{E}\overline{W}^{a-1} = \mathbb{E}(T\overline{W}+R)^{a-1} \overset{(3.22)}{\geq} \mathbb{E}T^{a-1}\mathbb{E}W^a + \mathbb{E}R^{a-1};$$

$$K_a \geq \mathbb{E}R^{a-1}.$$

By [98, Lemma 4.2],

$$\mathbb{E}R^x \leq \begin{cases} \mathbb{E}W_1^{x+1}, & x \in (0,1], \\ \mathbb{E}W^x\mathbb{E}W_1^{x+1}, & x > 1. \end{cases}$$

whenever $\mathbb{E}R^x < \infty$. Under the assumptions of the theorem, both $\mathbb{E}W^{b-1}$ and $\mathbb{E}W_1^b$ are finite. This implies that $\mathbb{E}R^x \in (0,\infty)$ for all $x \in (0, b-1]$.

Therefore, letting in (3.21) a go to b along some subsequence yields (3.20) and, hence, the direct Riemann integrability of Q.

Since

$$\begin{aligned}\int_{-\infty}^{\infty} t\mathrm{d}\,U(e^t) &= \int_0^{\infty} t^{b-1}\log t\mathbb{P}\{M\in \mathrm{d}t\}\\ &= \mathbb{E}M^{b-1}\log M = \mathbb{E}\sum_{|v|=1} L^b(v)\log L(v) < \infty,\end{aligned}$$

then the key renewal theorem on the whole line ensures that

$$\lim_{x\to\infty} P(x) = (\mathbb{E}\sum_{|v|=1} L^b(v)\log L(v))^{-1}\int_{-\infty}^{\infty} Q(y)\mathrm{d}y =: C = \text{const},$$

provided the law of $\log M$ is non-lattice. The latter implies

$$\lim_{t\to\infty} t^b\,\mathbb{P}\{\sup_{n\geq 0} W_n > t\} = C.$$

If the law of $\log M$ is lattice with the span $\delta > 0$, the renewal theorem only asserts that

$$\lim_{n\to\infty} P(x+\delta n) = (\mathbb{E}\sum_{|v|=1} L^b(v)\log L(v))^{-1}\sum_{k=-\infty}^{\infty} Q(x+\delta k) =: C(x).$$

Thus, for $x_1 \geq x_2$,

$$\begin{aligned}&\lim_{k\to\infty}(e^{x_1}P(x_1+\delta k) - e^{x_2}P(x_2+\delta k))\\ =\;& \lim_{k\to\infty}\int_{e^{x_2}}^{e^{x_1}} e^{\delta bk}s^b\mathbb{P}\{\sup_{n\geq 0} W_n > e^{\delta k}s\}\mathrm{d}s\\ =\;& e^{x_1}C(x_1) - e^{x_2}C(x_2).\end{aligned}$$

Since $\mathbb{P}\{\sup_{n\geq 0} W_n > x\}$ is monotone (in x) then, for fixed s, the last integrand remains bounded. Therefore, there exist constants b_1 and b_2 such that

$$0 < b_1 \leq G_b(x) \leq b_2 < \infty,$$

where $G_b(x) := x^b\mathbb{P}\{\sup_{n\geq 0} W_n > x\}$. Letting k go to ∞ through some sequence $\{k_j : j\in\mathbb{N}\}$ (its existence is granted by the selection principle) we conclude that $e^{\delta bk}s^{\beta}\mathbb{P}\{\sup_{n\geq 0} W_n > e^{\delta k}s\}$ converges to some function $C^*(s)$. The function $x \mapsto x^{-b}C^*(x)$ is non-increasing as the limit of non-increasing functions. Therefore, the number of its discontinuity points is at most denumerable. The equality

$$\int_{e^{x_2}}^{e^{x_1}} C^*(s)ds = e^{x_1}C(x_1) - e^{x_2}C(x_2)$$

uniquely determines $C^*(x)$ at all continuity points which implies that it does not depend on the chosen sequence (k_j). Consequently, as $k \to \infty$,

$$G_b(e^{x+\delta k}) = e^{(x+\delta k)b}\mathbb{P}\{\sup_{n\geq 0} W_n > e^{x+\delta k}\} \to C^*(e^x) = C(x),$$

for all but at most denumerably many x. Applying the diagonalization procedure allows us to conclude that the last equality holds for all $x > 0$. The proof is complete.

3.4 Asymptotics of non-regular intrinsic martingales in the BRW

When the intrinsic martingale $\{W_n : n \in \mathbb{N}_0\}$ is regular, it converges a.s. to a rv W which is positive with positive probability. In this case, seeking for conditions which ensure finiteness of $\mathbb{E}Wf(W)$, for appropriate functions f, forms a classical subject of the branching processes theory. In the opposite situation, when the intrinsic martingale is non-regular, a natural counterpart of the problem above is investigating the asymptotics of $\mathbb{E}W_nf(W_n)$, as $n \to \infty$.

In this Section, we address such a problem for $f(x) = \log^+ x$. An additional motivation for considering these functions stems from the following discussion. As a slight generalization of a classical Doob's inequality [110, p. 71], papers [112] and [80] showed that if $\{Z_n : n \in \mathbb{N}_0\}$ is a non-negative non-regular martingale with $Z_0 = a > 0$ and $\mathbb{E}Z_n \log^+ Z_n < \infty$, $n \in \mathbb{N}$, then

$$\limsup_{n\to\infty} \frac{\mathbb{E}\max_{0\leq k\leq n} Z_k}{\mathbb{E}Z_n \log^+ Z_n} \leq a. \tag{3.23}$$

Although in [80] it was proved that there exist martingales $\{Z_n : n \in \mathbb{N}_0\}$ such that

$$\lim_{n\to\infty} \frac{\mathbb{E} \max_{0\le k\le n} Z_k}{\mathbb{E} Z_n \log^+ Z_n} = c \in [0, a],$$

we believe that, like the critical Galton-Watson processes with finite second moments [9], the non-regular intrinsic martingales satisfy

$$\lim_{n\to\infty} \frac{\mathbb{E} \max_{0\le k\le n} W_k}{\mathbb{E} W_n \log^+ W_n} = 1.$$

In conclusion, we expect that the results derived in this Section may shed some light on the asymptotics of $\mathbb{E} \max_{0\le k\le n} W_k$, as $n \to \infty$.

If the martingale $\{W_n : n \in \mathbb{N}_0\}$ is non-regular then by Vallée-Poussin theorem [104, Theorem T22]

$$\sup_{n\ge 0} \mathbb{E} W_n \log^+ W_n = \infty.$$

Under the assumption $\mathbb{E} W_1 \log^+ W_1 < \infty$, the sequence $\{W_n \log^+ W_n : n \in \mathbb{N}_0\}$ forms a non-negative integrable submartingale. Hence,

$$\lim_{n\to\infty} \mathbb{E} W_n \log^+ W_n = \infty.$$

Theorem 34 investigates the asymptotics of $\mathbb{E} W_n \log^+ W_n$, as $n \to \infty$, under additional assumptions.

Theorem 34. *Assume that*
(a) there exists a sequence of non-negative constants $\{b_n : n \in \mathbb{N}\}$ such that

$$\frac{\log \Pi_n}{b_n} \Rightarrow \rho_\alpha, \quad n \to \infty,$$

where ρ_α is an α- stable law, $\alpha \in (1, 2]$, and Π_n defined on p. 82;
(b) for some $\varepsilon > 0$ $\mathbb{E} W_1 (\log^+ W_1)^{\alpha+\varepsilon} < \infty$.
Then

$$\lim_{n\to\infty} \frac{\mathbb{E} W_n \log^+ W_n}{b_n} = \mathbb{E} \sup_{0\le t\le 1} Z(t) \in (0, \infty), \tag{3.24}$$

where $\{Z(t) : t \ge 0\}$ is an α-stable Lévy process such that the law of $Z(1)$ is ρ_α.

Remark 35. Plainly, condition (a) means that the law of $\log M$ belongs to the domain of attraction of an α-stable law, $\alpha \in (1,2]$ which in particular implies that $\mathbb{E}\log M = 0$. It is well-known (see, for instance, [76]) that $b_n = n^{1/\alpha}L(n)$ for some L slowly varying at ∞ and that

$$\int_{\mathbb{R}} e^{itx}\rho_\alpha(dx) = \exp(-\lambda|t|^\alpha(1+i\beta\,\mathrm{sgn}\,t\tan(\pi\alpha/2))), \quad t \in \mathbb{R},$$

where $\lambda > 0$ and $|\beta| \le 1$.

Remark 36. If $s^2 := \mathbb{E}(\log M)^2 < \infty$, then (3.24) takes the form

$$\lim_{n\to\infty} \frac{\mathbb{E}W_n \log^+ W_n}{\sqrt{n}} = \sqrt{\frac{2}{\pi}}s.$$

Indeed, in this case $b_n = s\sqrt{n}$, and the corresponding Lévy process $\{Z(t) : t \ge 0\}$ is Brownian motion. According to André's reflection principle,

$$\sup_{0\le t\le 1} Z(t) \stackrel{d}{=} |Z(1)|.$$

Hence $\mathbb{E}\sup\limits_{0\le t\le 1} Z(t) = \sqrt{\frac{2}{\pi}}$.

Proof. Let $\{(M_k, Q_k) : k \in \mathbb{N}\}$ be independent copies of a random vector (M, Q) with distribution defined in (3.4). In particular,

$$\mathbb{P}\{Q \in A\} = x\mathbb{P}\{W_1 \in A\}, \tag{3.25}$$

for any Borel set A on $\mathbb{R}^+$.

From the results of [3] it follows that if $g(\cdot)$ is a non-decreasing and concave function, and $h(\cdot)$ is a non-decreasing function then

$$\mathbb{E}W_n g(W_n) \le \mathbb{E}g\left(\sum_{k=1}^n \Pi_{k-1}Q_k\right), \quad n \in \mathbb{N}, \tag{3.26}$$

and

$$\mathbb{E}W_n h(\max_{0\le k\le n} W_k) \ge \mathbb{E}h(\max_{1\le k\le n} \Pi_{k-1}Q_k), \quad n \in \mathbb{N}. \tag{3.27}$$

By using inequalities

$$\log^+ x \le \log(x+1) \le \log^+ x + \log 2,$$

(3.26) with $g(x) = \log(x+1)$, and

$$\sum_{k=1}^{n} \Pi_{k-1} Q_k \le n \max_{1\le k\le n} \Pi_{k-1} Q_k \le n \max_{1\le k\le n} \Pi_{k-1} \max_{1\le k\le n} Q_k,$$

we obtain

$$\begin{aligned} \mathbb{E} W_n \log^+ W_n &\le \mathbb{E} W_n \log(W_n+1) \\ &\le \mathbb{E} \log\left(\sum_{k=1}^{n} \Pi_{k-1} Q_k + 1\right) \\ &\le \log(2n) + \log^+(\max_{1\le k\le n} \Pi_{k-1}) + \log^+(\max_{1\le k\le n} Q_k) \\ &= \log(2n) + \max_{0\le k\le n-1} T_k + \log^+(\max_{1\le k\le n} Q_k), \end{aligned} \tag{3.28}$$

where $T_k := \log \Pi_k$, $k \in \mathbb{N}_0$. As $\{T_n : n \in \mathbb{N}_0\}$ is a martingale then, for $\beta \in (1, \alpha)$, the Doob's $L_\beta-$ inequality holds

$$\mathbb{E}\left(\frac{\max\limits_{0\le m\le n} |T_m|}{b_n}\right)^{\beta} \le \left(\frac{\beta}{\beta-1}\right)^{\beta} \mathbb{E}\left(\frac{|T_n|}{b_n}\right)^{\beta}.$$

Besides that, by [76, Lemma 5.2.2] the moments $\mathbb{E}\left(\frac{|T_n|}{b_n}\right)^{\beta}$ are uniformly bounded (in n).

In view of

$$\mathbb{E}\left(\frac{\max\limits_{0\le m\le n} T_m}{b_n}\right)^{\beta} \le \mathbb{E}\left(\frac{\max\limits_{0\le m\le n} |T_m|}{b_n}\right)^{\beta},$$

the sequence $\left\{\frac{\max\limits_{0\le m\le n} T_m}{b_n} : n \in \mathbb{N}\right\}$ is uniformly integrable.

By [74, Theorem 4],

$$\frac{\max\limits_{0\le m\le n} T_m}{b_n} \xrightarrow{d} \sup_{0\le t\le 1} Z(t), \quad n \to \infty,$$

and now the uniform integrability ensures that

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{\max\limits_{0\le m\le n} T_m}{b_n}\right) = \mathbb{E} \sup_{0\le t\le 1} Z(t) \in (0, \infty). \tag{3.29}$$

Since

$$\mathbb{E}\Big(\frac{\log^+ \max\limits_{1\le k\le n} Q_k}{b_n}\Big)^{\alpha+\varepsilon} = \mathbb{E}\Big(\frac{\max\limits_{1\le k\le n} \log^+ Q_k}{b_n}\Big)^{\alpha+\varepsilon} \le \frac{n}{b_n^{\alpha+\varepsilon}}\mathbb{E}(\log^+ Q_1)^{\alpha+\varepsilon},$$

and recalling that $b_n = n^{1/\alpha}L(n)$, and

$$\mathbb{E}(\log^+ Q_1)^{\alpha+\varepsilon} = \mathbb{E}W_1(\log^+ W_1)^{\alpha+\varepsilon} < \infty$$

by the assumption, we conclude that the right-hand side converges to zero, as $n \to \infty$. This establishes the limit relation

$$\lim_{n\to\infty} \mathbb{E}\Big(\frac{\log^+ \max\limits_{1\le k\le n} Q_k}{b_n}\Big) = 0.$$

Using (3.28) yields

$$\limsup_{n\to\infty} \frac{\mathbb{E}W_n \log^+ W_n}{b_n} \le \mathbb{E} \sup_{0\le t\le 1} Z(t). \tag{3.30}$$

Inequality (3.27) with $h(x) = \log^+ x$ reads

$$\mathbb{E}W_n \log^+ \max_{0\le k\le n} W_k \ge \mathbb{E} \log^+ \max_{1\le k\le n} \Pi_{k-1}Q_k, \quad n \in \mathbb{N}. \tag{3.31}$$

Arguing in the same way as in the proof of a formula in [66, p. 159], we have checked that

$$\mathbb{P}\{\max_{1\le k\le n} (T_{k-1}+\log Q_k) > x\} \ge \mathbb{P}\{\log Q > y\}\mathbb{P}\{\max_{0\le k\le n-1} T_k > x-y\},\ x, y \in \mathbb{R}.$$

Pick $y < 0$ such that $\mathbb{P}\{\log Q > y\} > 0$. Then

$$\begin{aligned}
\mathbb{E}\Big(\frac{\log^+ \max\limits_{1\le k\le n} \Pi_{k-1}Q_k}{b_n}\Big) &= \mathbb{E}\Big(\frac{\max\limits_{1\le k\le n}^+ (T_{k-1} + \log Q_k)}{b_n}\Big) \\
&= \int_0^\infty \mathbb{P}\{\max_{1\le k\le n} (T_{k-1} + \log Q_k) > xb_n\}\mathrm{d}x \\
&\ge \mathbb{P}\{\log Q > y\} \int_0^\infty \mathbb{P}\{\max_{0\le k\le n-1} T_k > xb_n - y\}\mathrm{d}x \\
&= \mathbb{P}\{\log Q > y\} \frac{y + \mathbb{E} \max\limits_{0\le k\le n-1} T_k}{b_n}.
\end{aligned}$$

Recalling (3.29), and sending first $n \to \infty$ and then $y \to -\infty$ gives

$$\liminf_{n\to\infty} \frac{\mathbb{E}\log^+ \max_{1\le k\le n} \Pi_{k-1} Q_k}{b_n} \ge \mathbb{E} \sup_{0\le t\le 1} Z(t).$$

By virtue of (3.31),

$$\liminf_{n\to\infty} \frac{\mathbb{E}W_n \log^+ \max_{0\le k\le n} W_k}{b_n} \ge \mathbb{E} \sup_{0\le t\le 1} Z(t). \tag{3.32}$$

Using working in [80, p. 3014-3015], one can verify that, for any $x_0 > e$,

$$\frac{x_0 - 1}{x_0} \mathbb{E}W_n \log^+ \max_{1\le k\le n} W_k \le \mathbb{E}W_n \log^+ W_n + \frac{\log(x_0 - 1) + 1}{x_0}.$$

Taking into account (3.32), and letting in the last inequality first $n \to \infty$ and then $x_0 \to \infty$, we conclude that

$$\liminf_{n\to\infty} \frac{\mathbb{E}W_n \log^+ W_n}{b_n} \ge \mathbb{E} \sup_{0\le t\le 1} Z(t),$$

which together with (3.30) proves the theorem. $\square$

3.5 Bibliographic comments

The notion of branching random walk, as used in the present work, was introduced by John Biggins in [19]. Before the appearance of article [19] the definition of branching random walk was more restrictive and assumed that the underlying point process $\mathcal{Z}$ has i.i.d. points. The latter processes are sometimes called *homogeneous branching random walks*. The first investigations of the intrinsic martingale defined in (3.35) were undertaken by J.F.C. Kingman [93] and J. Biggins [19].

Until recently most papers concerned with studying the branching random walks have assumed that $N < \infty$ a.s. An incomplete list of papers which make allowance of $\mathbb{P}\{N = \infty\} > 0$ includes [3, 4, 5, 6, 77, 78, 82, 85, 87, 100, 101, 120, 121]. The last two papers investigate the *weighted branching processes* which were introduced in [117]. In case of non-negative weights, the weighted branching process only differs from the BRW in that the point process $\mathcal{Z}_n$ has points $\{L(v) : |v| = n\}$ rather than $\{S(v) : |v| = n\}$, i.e., an additive formulation is replaced by multiplicative one.

The following topics related to the BRW have been receiving a lot of attention over the years:

- a criterion of uniform integrability of the intrinsic martingale (under extra moment conditions– [19, 27, 97, 100]; an ultimate version given in the text as Theorem 28 – in [3] (see also [77]));

- finiteness of f-moments and tail behavior of the law of W, the limiting random variable for the uniformly integrable intrinsic martingales W_n [3, 5, 21, 31, 71, 77, 78, 87, 96, 98, 120];

- convergence of the series $\sum_{n\geq 0} a(n)(W - W_n)$ for exponential or regularly varying sequence $\{a_n : n \in \mathbb{N}_0\}$ [4, 78, 82];

- under the assumption $\mathbb{E}N < \infty$ it was shown in [25] that the law of W is a mixture of an atom at zero and absolutely continuous law; subsequent generalizations of this result to the case $\mathbb{E}N = \infty$, $N < \infty$ a.s. and $N = \infty$ a.s. were derived in [99] and [116], respectively;

- local uniform convergence of $W_n(\gamma)$ (w.r.t. γ) and differentiability of limiting function $W(\gamma)$ [15, 22]; in [24] the results of the latter paper were generalized to the multidimensional BRWs and complex γ;

- in some cases when the intrinsic martingale is non-regular, it was proved in [26] that there exists a sequence $\{c_n : n \in \mathbb{N}\}$ such that W_n/c_n converges in probability to a rv with non-degenerate at zero law (Seneta-Heyde norming); [37, Theorem 14] gives conditions which ensure that the convergence holds a.s.

- a path-wise renewal theory for the BRW [53, 102, 109];

- an a.s. central limit theorem for the BRW [23, 95];

- the asymptotics of the right-most individual in the nth generation of the BRW was studied in [1, 10, 18, 20, 75, 33, 34]; the second article contains many further references to earlier work;
- the BRW with barrier [27, 28].

The results of Chapter 3 (excluding the last section) are based on [85]. The results in Section 3.5 have not yet been published.

BIBLIOGRAPHY

[1] Addario-Berry, L. and Reed, B. (2009). Minima in branching random walks. *Ann. Prob.* **37**, 1044–1079.

[2] Arratia, R., Barbour, A. D. and Tavaré, S. (2003). *Logarithmic combinatorial structures: A probabilistic approach*, Zurich: European Mathematical Society, 363 p.

[3] Alsmeyer, G. and Iksanov, A. (2009). A log-type moment result for perpetuities and its application to martingales in supercritical branching random walks. *Electr. J. Probab.* **14**, 289–313.

[4] Alsmeyer, G., Iksanov, A., Polotsky, S. and Rösler, U. (2009). Exponential rate of L_p-convergence of intrinsic martingales in supercritical branching random walks *Theory Stoch. Proc.* **15(31)**, 1–18.

[5] Alsmeyer, G. and Kuhlbusch, D. (2010+). Double martingale structure and existence of *phi*-moments for weighted branching processes. *Münster J. Math.*, to appear.

[6] Alsmeyer, G. and Meiners, M. (2009). A min-type stochastic fixed-point equation related to the smoothing transformation. *Theory Stoch. Proc.* **15(31)**, 19–41.

[7] Anderson, K. K. and Athreya, K. B. (1988). A note on conjugate Π-variation and a weak limit theorem for the number of renewals. *Stat. Prob. Letters.* **6**, 151–154.

[8] Athreya, K. B. and Ney, P. E. (1972). *Branching processes*, Berlin-Heidelberg-New York: Springer-Verlag, 287 p.

[9] Athreya, K. B. (1988). On the maximum sequence in a critical branching process. *Ann. Prob.* **16**, 502–507.

[10] Bachmann, M. (2000). Limit theorem for the minimal position in a branching random walk with independent logconcave displacements. *Adv.Appl. Prob.* **32**, 159–176.

[11] Bahadur, R. R. (1960). On the number of distinct values in a large sample from an infinite discrete distribution. *Proc. Nat. Inst. Sci. India.* **26A**, 66–75.

[12] Bahadur, R. R. (1971). Some limit theorems in statistics. CBMS Regional conference series in applied mathematics. **4**. Philadelphia: SIAM.

[13] Bai, Z. D., Hwang, H. K. and Liang, W. Q. (1998). Normal approximations of the number of records in geometrically distributed random variables. *Random Struct. Algorithms.* **13**, 319–334.

[14] Baltrunas, A. and Omey, E. (2002). Second order renewal theorem in the finite-means case. *Theory Probab. Appl.* **47**, 127–132.

[15] Barral, J. (2000). Differentiability of multiplicative processes related to branching random walks. *Ann. Inst. H. Poincaré Probab. Statist.* **36**, 407–417.

[16] Barbour, A. D. and Gnedin, A. V. (2006). Regenerative compositions in the case of slow variation. *Stoch. Proc. Appl.* **116**, 1012–1047.

[17] Bertoin, J. and Yor, M. (2001). On subordinators, self-similar Markov processes and some factorization of the exponential variable. *Eletron. Comm. Probab.* **6**, 95–106.

[18] Biggins, J. D. (1976). The first- and last-birth problems for a multitype age-dependent branching processes. *Adv. Appl. Prob.* **8**, 446–459.

[19] Biggins, J. D. (1977). Martingale convergence in the branching random walk. *J. Appl. Prob.* **14**, 25–37.

[20] Biggins, J. D. (1977). Chernoff's theorem in the branching random walk. *J. Appl. Prob.* **14**, 630–636.

[21] Biggins, J. D. (1979). Growth rates in the branching random walk. *Z. Wahrsch. verw Gebiete.* **48**, 17–34.

[22] Biggins, J. D. (1989). Uniform convergence of martingales in the one-dimensional branching random walk. Selected proceedings of the Symposium on applied probability. Sheffield (Great Britain), 159–173.

[23] Biggins, J. (1990). The central limit theorem for the supercritical branching random walk, and related results. *Stoch. Proc. Appl.* **34**, 255–274.

[24] Biggins, J. D. (1992). Uniform convergence of martingales in the branching random walk. *Ann. Prob.* **20**, 137–151.

[25] Biggins, J. D. and Grey, D. R. (1979). Continuity of limit random variables in the branching random walk. *J. Appl. Prob.* **16**, 740–749.

[26] Biggins, J. D. and Kyprianou, A. E. (1997). Seneta-Heyde norming in the branching random walk. *Ann. Prob.* **25**, 337–360.

[27] Biggins, J. D. and Kyprianou, A. E. (2004). Measure change in multitype branching. *Adv. Appl. Prob.* **36**, 544–581.

[28] Biggins, J. D., Lubachevsky, B. D., Shwartz, A. and Weiss, A. (1991). A branching random walk with a barrier. *Ann. Appl. Prob.* **1**, 573–581.

[29] Billingsley, P. (1999). *Convergence of probability measures*, Chichester: Wiley, 277 p.

[30] Bingham, N. H. (1972). Limit theorems for regenerative phenomena, recurrent events and renewal theory. *Z. Wahrsch. verw Gebiete.* **21**, 20–44.

[31] Bingham, N. H. and Doney, R.A. (1975). Asymptotic properties of supercritical branching processes II: Crump-Mode and Jirina processes. *Adv. Appl. Prob.* **7**, 66–82.

[32] Bingham, N. H., Goldie, C. M. and Teugels, J.L. (1989). *Regular variation*, Cambridge: Cambridge University Press, 512 p.

[33] Bramson, M. and Zeitouni, O. (2007). Tightness for the minimal displacement of branching random walk. *J. Stat. Mech. Theory Exp.* **7**, P07010.

[34] Bramson, M. and Zeitouni, O. (2009). Tightness for a family of recursion equations. *Ann. Prob.* **37**, 615–653.

[35] Bruss, F. T. and O'Cinneide, C. A. (1990). On the maximum and its uniqueness for geometric random samples. *J. Appl. Prob.* **27**, 598–610.

[36] Carmona, P., Petit, F. and Yor, M. (1997). On the distribution an asymptotic results for exponential functionals of Lévy processes. In *Exponential functionals and principal values related to Brownian motion*, Madrid, 73–121.

[37] Cohn, H. (2001). Convergence in probability and almost sure with applications. *Stoch. Proc. Appl.* **94**, 135–154.

[38] van Cutsem, B. and Ycart, B. (1994). Renewal-type behaviour of absorption times in Markov chains. *Adv. Appl. Prob.* **26**, 988–1005.

[39] Darling, D. A. (1967). Some limit theorems assiciated with multinomial trials. Proc. Fifth Berkeley Symp. on Math. Statist. and Prob. V. **2**, 345–350.

[40] Delmas, J. F., Dhersin, J. S. and Siri-Jegousse, A. (2008). Asymptotic results on the length of coalescent trees. *Ann. Appl. Prob.* **18**, 997–1025.

[41] Doney, R. A. (1997). One-sided local large deviation and renewal theorems in the case of infinite mean. *Probab. Theory Related Fields* **107**, 451–465.

[42] Drmota, M. (2009). Random trees: An interplay between combinatorics and probability, Vienna: Springer, 458 p.

[43] Drmota, M., Iksanov, A., Möhle, M. and Rösler, U. (2007). Asymptotic results about the total branch length of the Bolthausen-Sznitman coalescent. *Stoch. Proc. Appl.* **117**, 1404–1421.

[44] Drmota, M., Iksanov, A., Möhle, M. and Rösler, U. (2009). A limiting distribution for the number of cuts needed to isolate the root of a random recursive tree. *Random Struct. Algorithms.* **34**, 319–336.

[45] Dutko, M. (1989). Central limit theorems for infinite urn models. *Ann. Prob.* **17**, 1255–1263.

[46] Erickson, K. B. (1970). Strong renewal theorems with infinite mean. *Trans. Amer. Math. Soc.* **151**, 263–291.

[47] Feller, W. (1949). Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* **67**, 98–119.

[48] Feller, W. (1971). *An introduction to probability theory and its applications*, Vol. 2, New-York etc.: John Wiley and Sons, 669 p.

[49] Fill, J. A., Mahmoud, H. M. and Szpankowski, W. (1996). On the distribution for the duration of a randomized leader election algorithm. *Ann. Appl. Probab.* **6**, 1260–1283.

[50] Flajolet, Ph. and Sedgewick, R. (2009). *Analytic combinatorics*, Cambridge: Cambridge University Press, 810 p.

[51] Freund, F. and Möhle, M. (2009). On the time back to the most recent common ancestor and the external branch length of the Bolthausen-Sznitman coalescent. *Markov Process. Related Fields.* **15**, 387–416.

[52] Garsia, A and Lamperti, J. (1963). A discrete renewal theorem with infinite mean. *C.M.H.* **37**, 221–234.

[53] Gatzouras, D. (2000). On the lattice case of an almost-sure renewal theorem for branching random walk. *Adv. Appl. Prob.* **32**, 720–737.

[54] Geluk, J. (2000). Second-order regular variation and the domain of attraction of stable distributions. *Analysis.* **20**, 359–371.

[55] Gnedin, A. V. (2004). The Bernoulli sieve. *Bernoulli.* **10**, 79–96.

[56] Gnedin A., Hansen, A. and Pitman, J. (2007). Notes on the occupancy problem with infinitely many boxes: general asymptotics and power laws. *Probability Surveys.* **4**, 146–171.

[57] Gnedin, A., Iksanov, A. and Möhle, M. (2008). On asymptotics of exchangeable coalescents with multiple collisions. *J. Appl. Prob.* **45**, 1186–1195.

[58] Gnedin, A., Iksanov, A., Negadailov, P. and Rösler, U. (2009). The Bernoulli sieve revisited. *Ann. Appl. Prob.* **19**, 1634–1655.

[59] Gnedin, A., Iksanov, A. and Roesler, U. (2008). Small parts in the Bernoulli sieve. *Discrete Mathematics and Theoretical Computer Science*, Proceedings Series, **AI**, 235–242.

[60] Gnedin, A. and Pitman, J. (2005). Regenerative composition structures. *Ann. Prob.* **33**, 445–479.

[61] Gnedin, A. and Pitman, J. (2005). Self-similar and Markov composition structures. *Zapiski nauchnyh seminarov POMI.* **326**, 59–84.

[62] Gnedin, A., Pitman, J. and Yor, M. (2006). Asymptotic laws for regenerative compositions: gamma subordinators and the like. *Probab. Theory Relat. Fields.* **135**, 576–602.

[63] Gnedin, A., Pitman, J. and Yor, M. (2006). Asymptotic laws for compositions derived from transformed subordinators. *Ann. Prob.* **34**, 468–492.

[64] Gnedin, A. and Yakubovich, Yu. (2007). On the number of collisions in Λ - coalescents. *Electron. J. Probab.* **12**, 1547–1567.

[65] Hitczenko, P. and Goh, W. M. Y. (2007). Gaps in samples of geometric random variables. *Discrete Mathematics.* **307**, 2871–2890.

[66] Goldie, C. M. (1991). Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Prob.* **1**, 126–166.

[67] Grincevicius, A. K. (1975). One limit distribution for a random walk on the line. *Lith. Math. J.* **15**, 580–589.

[68] Gut, A. (1988). *Stopped random walks: limit theorems and applications*, Berlin-Heidelberg-New York: Springer-Verlag, 199 p.

[69] de Haan, L. and Resnick, S. I. (1979). Conjugate Π-variation and process inversion. *Ann. Prob.* **7**, 1028–1035.

[70] Haas, B. and Miermont, G. (2009). Self-similar scaling limits of non-increasing Markov chains, preprint, available at www.arXiv.org.

[71] Hambly, B. M. and Jones, O. D. (2003). Thick and thin points for random recursive fractals. *Adv. Appl. Prob.* **35**, 251–277.

[72] Hinderer, K. and Walk, H. (1972). Anwendung von Erneuerungstheoremen und Taubers itzen fur eine Verallgemeinerung der Erneuerungsprozesse. *Math. Z.* **126**, 95–115.

[73] Heyde, C. C. (1967). A limit theorem for random walks with drift. *J. Appl. Prob.* **1**, 144–150.

[74] Heyde, C. C. (1969). On the maximum of sums of random variables and the supremum functional for stable processes. *J. Appl. Prob.* **6**, 419–429.

[75] Hu, Y. and Shi, Z. (2009). Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees. *Ann. Prob.* **37**, 742–789.

[76] Ibragimov, I. A. and Linnik, Yu. V. (1971). *Independent and stationary sequences of random variables*, Groningen: Wolters-Noordhoff Publishing Company, 443 p.

[77] Iksanov, A. M. (2004). Elementary fixed points of the BRW smoothing transforms with infinite number of summands. *Stoch. Proc. Appl.* **114**, 27–50.

[78] Iksanov, O. M. (2006). On the rate of convergence of a regular martingale related to a branching random walk. *Ukr. Math. J.* **58**, 368–387.

[79] Iksanov, O. M. (2007). *Perpetuities, branching random walk and self-decomposability*, Kiev: Zirka, 192 p. (in Ukrainian).

[80] Iksanov, A. and Marynych, A. (2008). A note on non-regular martingales. *Stat.Prob.Letters.* **78**, 3014–3017.

[81] Iksanov, A., Marynych, A. and Möhle, M. (2009). On the number of collisions in beta(2,b)-coalescents. *Bernoulli.* **15**, 829–845.

[82] Iksanov, A. and Meiners, M. (2010). Exponential rate of almost sure convergence of intrinsic martingales in supercritical branching random walks. *J. Appl. Prob.* **47**, 513–525.

[83] Iksanov, A. and Möhle, M. (2007). A probabilistic proof of a weak limit law for the number of cuts needed to isolate the root of a random recursive tree. *Electron. Comm. Probab.* **12**, 28–35.

[84] Iksanov, A. and Möhle, M. (2008). On the number of jumps of random walks with a barrier. *Adv. Appl. Prob.* **40**, 206–228.

[85] Iksanov, O. and Negadailov, P. (2007). On the supremum of martingale connected with branching random walk. *Theory Probab. Math. Stat.* **74**, 49–57.

[86] Iksanov, A. M. and Negadailov, P. A. (2008). On the number of zero increments of random walks with a barrier. *Discrete Mathematics and Theoretical Computer Science*, Proceedings Series. **AG**, 247–254.

[87] Iksanov, A. M. and Rösler, U. (2006). Some moment results about the limit of a martingale related to the supercritical branching random walk and perpetuities. *Ukr. Math. J.* **58**, 505–528.

[88] Janson, S., Lavault, C. and Louchard, G. (2008). Convergence of some leader election algorithms. *Discrete Mathematics and Theoretical Computer Science.* **10**, 171–196.

[89] Janson, S. and Szpankowski, W. (1997). Analysis of an asymmetric leader election algorithm. *Electronic J. Combin.* **4**, R17.

[90] Kac, M. (1949). On the deviations between theoretical and empirical distributions. *Proc. Nat. Acad. Sci. USA*. **35**, 252–257.

[91] Karlin, S. (1967). Central limit theorems for certain infinite urn schemes. *J. Math. Mech.* **17**, 373–401.

[92] Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta. Math.* **131**, 207–248.

[93] Kingman, J. F. C. (1975). The first birth problem for an age-dependent branching process. *Ann. Prob.* **3**, 790–801.

[94] Kirschenhofer P. and Prodinger, H. (1996). The number of winners in a discrete geometrically distributed sample. *Ann. Appl. Probab.* **6**, 687-694.

[95] Klebaner, C. F. (1982). Branching random walk in varying environments. *Adv. Appl. Probab.* **14**, 359–367.

[96] Liu, Q. (1996). The growth of an entire characteristic function and the tail probabilities of the limit of a tree martingale. In *Progress in Probability*. Basel: Birkhäuser. **40**, 51–80.

[97] Liu, Q. (1997). Sur une équation fonctionnelle et ses applications: une extension du théorème de Kesten-Stigum concernant des processus de branchement. *Adv. Appl.Prob.* **29**, 353–373.

[98] Liu, Q. (2000). On generalized multiplicative cascades. *Stoch. Proc. Appl.* **86**, 263–286.

[99] Liu, Q. (2001). Asymptotic properties and absolute continuity of laws stable by random weighted mean. *Stoch. Proc. Appl.* **95**, 83–107.

[100] Lyons R. (1997). A simple path to Biggins's martingale convergence for branching random walk. In *Classical and modern branching processes*, IMA Volumes in Mathematics and its Applications. Berlin: Springer, **84**, 217–221.

[101] Mauldin, R. D. and Williams, S. C. (1986). Random recursive constructions: asymptotic geometric and topological properties. *Trans. Amer. Math. Soc.* **295**, 325–346.

[102] Meiners, M. (2009). Weighted branching and a pathwise renewal equation. *Stoch. Proc. Appl.* **119**, 2579-2597.

[103] Meir, A. and Moon, J. (1974). Cutting down recursive trees. *Math. Biosci.* **21**, 173-181.

[104] Meyer, P. A. (1966). *Probability and potentials*, Waltham, Mass.-Toronto-London: Blaisdell Publishing Company, 266 p.

[105] Mohan, N. R. (1976). Teugels' renewal theorem and stable laws. *Ann. Prob.* **4**, 863–868.

[106] Möhle, M. (2006). On the number of segregating sites for populations with large family sizes. *Adv. Appl. Prob.* **38**, 750–767.

[107] Negadailov, P. A. (2008). Asymptotic results for the absorption times of random walks with a barrier. *Theor. Probab. Math. Statist.* **79**, 127–138.

[108] Negadailov, P. A. (2008). On the absorption times in random walk with barrier, *Bulletin of Kiev University.* **4**, 149–152 (in Ukrainian).

[109] Nerman, O. (1981). On the convergence of supercritical general (C-M-J) branching processes. *Z. Wahrsch. Gebiete.* **57**, 365–395.

[110] Neveu, J. (1975). *Discrete-parameter martingales*, Amsterdam: North-Holland, 236 p.

[111] Neininger, R. and Rüschendorf, L. (2004). On the contraction method with degenerate limit equation. *Ann. Prob.* **32**, 2838–2856.

[112] Pakes, A. G. (1987). Remarks on the maxima of a martingale sequence with application to the simple critical branching process. *J.Appl.Prob.* **24**, 768–772.

[113] Panholzer, A. (2003). Non-crossing trees revisited: cutting down and spanning subtrees. *Discrete Mathematics and Theoretical Computer Science.* **AC**, 265–276.

[114] Panholzer, A. (2006). Cutting down very simple trees. *Quaest. Math.* **29**, 211–227.

[115] Pitman, J. (1999). Coalescents with multiple collisons. *Ann. Prob.* **27**, 1870–1902.

[116] Polotskiy, S. (2009). On the absolute continuity of fixed points of smoothing transforms. *Theor. Probab. Math. Statist.* **79**, 139–142.

[117] Rösler, U. (1991). A limit theorem for "Quicksort". *RAIRO, Inform. Theor. Appl.* **25**, 85–100.

[118] Rösler, U. (2001). On the analysis of stochastic divide and conquer algorithms. *Algorithmica.* **29**, 238–261.

[119] Rösler, U. and Rüschendorf, L. (2001). The contraction method for recursive algorithms. *Algorithmica.* **29**, 3–33.

[120] Rösler, U., Topchii, V. A. and Vatutin, V. A. (2000). Convergence conditions for the weighted branching process. *Discrete Mathematics and Applications* **10**, 5–21.

[121] Rösler, U., Topchii, V. A. and Vatutin, V. A. (2002). Convergence rate for stable Weighted Branching Processes. In *Mathematics and Computer Science II*, Basel: Birkhauser, 441–453.

[122] Sagitov, S. (1999). The general coalescent with asynchronous mergers of ancestral lines. *J. Appl. Prob.* **36**, 1116–1125.

[123] Sgibnev, M. S. (1981). Renewal theorem in the case of an infinite variance. *Siberian Math. J.* **22**, 787–796.

[124] Sgibnev, M. S. (2009). On a renewal function when the second moment is infinite. *Stat. Prob. Letters.* **79**, 1242–1245.

[125] Vatutin, V. A. and Topchij, V. A. (1997). Maximum of the critical Galton-Watson processes and left-continuous random walks. *Theory Probab. Appl.* **42**, 17–27.

[126] Vervaat, W. (1979). On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. *Adv. Appl. Prob.* **11**, 750–783.

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Samenvatting

In dit proefschrift onderzoeken we het asymptotieke gedrag van de oplossingen van random rekursieve vergelijkingen van de form

$$X_n \stackrel{d}{=} X'_{I_n} + Y_n, \quad X_0 := a \geq 0, \ n \in \mathbb{N}, \tag{3.33}$$

waarbij een random grootheid (index) I_n waarden uit verzameling $\{0, 1, \ldots, n\}$ aanneemt, de vector (I_n, Y_n) niet van de reeks $\{X'_n : n \in \mathbb{N}_0\}$ afhangt, en $\{X'_n : n \in \mathbb{N}_0\}$ een stochastische kopie van $\{X_n : n \in \mathbb{N}_0\}$ is. We beschouwen drie verschillende modellen en functionalen die op een natuurlijke manier bij deze modellen ontstaan.

Het eerste deel is gewijd aan het bestuderen van het asymptotische gedrag van de functionalen, die op de random wandelingen met een barriére zijn gedefinieerd. *Een random wandeling met een barriére* $n \in \mathbb{N}$ is een random reeks van stochasten $\{R_k^{(n)} : k \in \mathbb{N}_0\}$, die aan de volgende rekurrente vergelijking voldoet:

$$R_0^{(n)} := 0, \quad R_k^{(n)} := R_{k-1}^{(n)} + \xi_k 1_{\{R_{k-1}^{(n)} + \xi_k < n\}}, \quad k \in \mathbb{N},$$

waarbij $\{\xi_k : k \in \mathbb{N}\}$ onafhankelijke copiën zijn van de stochastische grootheid ξ met de kansverdeling $p_k := \mathbb{P}\{\xi = k\}, \ k \in \mathbb{N}$. In het vervolg veronderstellen we dat $p_1 > 0$. We bekijken de drie basiskarakteristieken van dit model: M_n – aantal sprongen; V_n – aantal nul-aangroeingen tot het moment van $n - 1$, en het moment van absorptie T_n self. In het bijzonder, we hebben aangetoond dat de laatste twee reeksen voldoen aan de recurrente absorptievergelijkingen van der form (3.33).

Stellingen 4, 5 en 6 geven de volgende voorwaarden (in Stelling 4 zijn deze voorwaarden tevens noodzakelijk), die garanderen dat de reeks T_n, op

een geschikte manier gecentreerd en genormaliseerd, zwak convergeert. Daarbij werden de expliciete uitdrukkingen van de normerende constanten en de mogelijke limietverdelingen gegeven. In het bijzonder zegt Stelling 6 dat, onder vorwaarde van een oneindige gemiddelde stap van de random wandeling, de reeks T_n (op een bepaalde manier genormaliseerd) zwak convergeert naar de exponentiële integraal van een subordinator.

Onder aanname dat ξ tot de attractie-gebied behoort van een stabiele verdeling met parameter $\alpha \in (1, 2)$, worden de eertse twee termen in de asymptotische ontwikkeling voor de momenten van de eerste en tweede order van het absorptie-tijdstip T_n gegeven in Stelling 11. Als gevold daarvan krijgt men ook de asymptotiek van $\operatorname{Var} T_n$. In geval van een eindig verwachting van ξ, gaat Stelling 12 over de zwake convergentie (zonder normalisatie) van de reeks V_n (aantal nul-aangroeiingen tot het absorptie-moment). In geval $\mathbb{E}\xi = \infty$, beschrijft Stelling 13 de zwakke wet van de grote aantallen voor V_n en de asymptotiek van $\mathbb{E}V_n$.

Zij $\{J_n : n \in \mathbb{N}_0\}$ en random wandeling met een stap, die de kansverdeling van $|\log \eta|$ heeft, waarbij de random variabele η nu uit de interval $(0, 1)$ komt. Zij verder $E_1, \dots, E_n$, onafhangelijk van $\{J_n\}$, een steekproef uit de standaard exponentiële verdeling. Noteer $E_{1,n} \leq E_{2,n} \leq \dots \leq E_{n,n}$ de order statistiken gebaseerd op deze steekproef. Een random wandeling, samen met de exponentiële steekproef, bepaalt een platsingsschema dat Bernoulli-zeef wordt genoemd, waarbij n 'ballen' $1, 2, \dots, n$ worden geplaatst in een oneindige reeks genummerde dosen volgens de volgende regel: ball i komt in doos k als exponentieel verdeeld punt E_i in het interval (J_{k-1}, J_k) terecht komt. We noemen een interval *bezet* als het minstens een punt vam $E_1, \dots, E_n$ bevat, en *leeg* andersom.

Het tweede deel van het proefschrift gaat over het onderzoek naar de volgende karakteristieken van de Bernoulli-zeef: U_n - index van het meest rechtse bezette intervaal, $K_{n,0}$ - aantal lege (onder eerste $U_n - 1$) intervallen, K_n - aantal bezette intervallen en Z_n - aantal punten in het laatste (meest rechtse) bezette intervaal. We tonen an dat de reeksen U_n, $K_{n,0}$ K_n de oplossingen van recurrenten vergelijkingen (3.33) zijn.

Hoofdresultaten uit dit deel zijn over het bestuderen van de zwakke convergentie van de bovengenoemde functionalen onder de additionele voorwaarden dat $|\log \eta|$ geen rooster-verdeling geeft.

In het bijzonder, in Stelling 19 wordt het algemene criterium gegeven van het bestaan van een limiet-verdeling voor de op een geschickte manier gecentreerde en genormaliseerde reeks U_n. Een soortgelijk resultaat is gegeven in Stelling 21 voor K_n onder nog meer voorwaarden. Onder anderen, deze voorwaarden impliciren een nog fijnere resultaat, dat in stelling 20 wordt gegeven: $K_{n,0}$ convergeert zwak zonder normalisatie. Tenslotte, beschrijft Stelling 22 de zwakke convergentie van Z_n onder zekere natuurlijke voorwaarden voor de verdeling van η.

We bekijken enkele populatie, die van een ouder begint en evolutioneert volgens het gegeneraliseerde Galton-Watson proces, waarbij individuen onendig veel nakomelingen kunnen hebben. We veronderstellen dat alle individuen worden weergegeven als punten op de reële lijn: het alle eerste individu zit in nul, de aangroeiing van de nakomelingen relatief op hun vader worden beschreven door het puntproces $\mathcal{Z} = \sum_{i=1}^{N} \delta_{X_i}$ op $\mathbb{R}$. Hierbij $N := \mathcal{Z}(\mathbb{R})$ is het totale aantal nakomelingen van de vader, X_i is de positie van de i-de nakomeling. Het opschuiven van de positie van de individuen ten opzichte van hun vader wordt beschreven door een exacte copie van het puntproces $\mathcal{Z}$. In het vervolg veronderstellen we dat $\mathcal{Z}(\{-\infty\}) = 0$ en dat de populatie met een positieve kans overleeft, dat wil zeggen $\mathbb{E}N > 1$ (superkritieke proces).

Voor een $n \in \mathbb{N}_0$ noteer door $\mathcal{Z}_n$ een puntproces dat de posities van de individuen van de n-de generatie beschrijft, waarbij het totale aantal $\mathcal{M}_n(\mathbb{R})$ is. Het stochastische proces $\{\mathcal{Z}_n : n \in \mathbb{N}_0\}$ is een vertakkingsproces. Veronderstel dat er een $\gamma > 0$ bestaat zodanig dat

$$m(\gamma) := \mathbb{E} \int_{\mathbb{R}} e^{\gamma x} \, \mathcal{Z}(\mathrm{d}x) \in (0, \infty). \tag{3.34}$$

Voor $n \in \mathbb{N}$ noteer door $\mathcal{F}_n$ de σ-algebra, die de alle informatie over de eerste n generaties inhoudt en zij $\mathcal{F}_0$ de triviale σ-algebra. Zij

$$W_n := m(\gamma)^{-n} \int_{\mathbb{R}} e^{\gamma x} \, \mathcal{Z}_n(dx). \tag{3.35}$$

De reeks $\{(W_n, \mathcal{F}_n) : \ n \in \mathbb{N}_0\}$ vormt een niet-negatieve martingaal met verwachting 1, die het cetrale object is in het derde deel van het proefschrift.

Stelling 29 beschrijft het gedrag van de staart van de verdeling van $\sup_{n\geq 0} W_n$ voor de uniform integreerbare (reguliere) martingaal $\{W_n : n \in \mathbb{N}_0\}$, onder bepaalde voorwaarden op de momenten. Het laatste resultaat van het proefschrift, Stelling 34, gaat over de asymptotiek van $\mathbb{E}[W_n \log^+ W_n]$, als $n \to \infty$, voor de niet-reguliere martingaal $\{W_n : n \in \mathbb{N}_0\}$.

Curriculum vitae

Pavlo Negadailov was born on June the 4$^{\text{th}}$, 1984 in Chernivtsy, Ukraine. He attended a secondary school in Chernivtsy in 1990-2000. In the years from 2000 to 2006 he studied applied mathematics at the National Taras Shevchenko University of Kiev. He graduated from the University with master's diploma cum laude.

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In the Fall 2010 he will take a tenure-track lectureship in the Faculty of Cybernetics in the University of Kiev.