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RENORMALIZATION GROUP SOLUTION OF THE ONE-DIMENSIONAL CLASSICAL HEISENBERG MODEL

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It is shown that the method of Van Leeuwen[1,2]) and Nauenberg and Nienhuis[3]) in the application of the renormalization theory to Ising-like spin systems, can easily be extended to include all one-dimensional classical spin systems with nearest neighbor interactions. The series for the free energy converges very rapidly towards the known exact value (for Heisenberg interaction), provided that the temperature is not too close to the critical temperature $T = 0$.

1. Introduction

Up till now the renormalization theory in statistical mechanics has been used mainly to obtain critical properties, and so far, its applications to calculate the free energy, or other thermodynamic functions, have been restricted to systems in which the dynamical variables are constrained to discrete values[1,3,4,5]). It is to be hoped that eventually also systems with continuous variables can be treated in this way and that it will be possible to study two- and three-dimensional gases and liquids with this method. We consider this paper as a step in that direction, since we shall show that with the renormalization theory we can obtain a rapidly convergent infinite series for the free energy of one-dimensional classical spin systems with nearest neighbor interactions. As a sequel to this work, the authors plan to extend these results to certain one-dimensional gases.

The renormalization transformation that is constructed, is exact and the ensuing results can be checked for a general class of interactions for which the solution

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is known analytically from conventional statistical mechanics. We obtain a rapidly convergent infinite series for the free energy; the way in which the renormalization transformation is set up is an extension of the one used in ref. 6 and by Nauenberg[5]), who calculated the free energy of the one-dimensional Ising model by means of the renormalization theory. In the latter case the interaction does not change "functionally" in the sense that it can always be expressed as a 2×2 matrix, whereas it does change functionally for a chain of classical spins, as we shall show below.

Consider a one-dimensional classical spin system (with $|\boldsymbol{S}_i| = 1$), governed by the hamiltonian

$$-\beta H_N = h(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2) + h(\boldsymbol{S}_2 \cdot \boldsymbol{S}_3) + \cdots h(\boldsymbol{S}_{N-1} \cdot \boldsymbol{S}_N) + h(\boldsymbol{S}_N \cdot \boldsymbol{S}_1), \tag{1}$$

where the last term obviously implies periodic boundary conditions. $h(u)$ is an arbitrary function of the scalar product $u = \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}$. The more general case where $h = h(\boldsymbol{S}_i, \boldsymbol{S}_{i+1})$ (*e.g.*, the anisotropic X–Y model) and where also an external field is taken into account, will shortly be discussed in the third section of this paper. The partition function of $2N$ spins is a functional of the interaction h and is given by

$$Q_{2N}(h) = \int \cdots \int \mathrm{d}\boldsymbol{S}_1 \cdots \mathrm{d}\boldsymbol{S}_{2N} \exp\left[h(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2) + \cdots + h(\boldsymbol{S}_N \cdot \boldsymbol{S}_1)\right], \tag{2}$$

where each $\boldsymbol{S}_i$ is integrated over the unit sphere. From the partition function (2) the free energy per spin $f(h)$ is calculated as

$$\beta f(h) = \hat{f}(h) = -\lim_{N\to\infty} \frac{1}{2N} \ln Q_{2N}[h]. \tag{3}$$

We now define a new interaction, and thereby a renormalization transformation, and an energy shift $g[h]$ which is a functional of the original interaction, through the eqs. (4) and (5):

$$\exp\left[g[h] + h_1(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)\right] = \int \mathrm{d}\boldsymbol{S} \exp\left[h(\boldsymbol{S}_1 \cdot \boldsymbol{S}) + h(\boldsymbol{S} \cdot \boldsymbol{S}_2)\right] \tag{4}$$

with

$$h_1(1) = h(1) \equiv \bar{h}. \tag{5}$$

Taking $\boldsymbol{S}_1 = \boldsymbol{S}_2$ in eq. (4) and using eq. (5) leads to

$$\mathrm{e}^{g[h]} = \mathrm{e}^{-\bar{h}} \int \mathrm{d}\boldsymbol{S} \exp\left[2h(\boldsymbol{S}_1 \cdot \boldsymbol{S})\right] = 2\pi\, \mathrm{e}^{-\bar{h}} \int_{-1}^{1} \mathrm{e}^{2h(u)}\, \mathrm{d}u. \tag{6}$$

From this last equation it follows that $g[h]$ indeed is uniquely determined for any function $h(u)$. Since the right-hand side of eq. (4) is a function of the scalar pro-

duct $u = \boldsymbol{S}_1 \cdot \boldsymbol{S}_2$, the function $h_1(u)$ is also completely determined. Eqs. (4) and (5) determine the renormalization transformation

$$h(u) \to h_1(u). \tag{7a}$$

By repeated application one obtains

$$e^{g_n} = 2\pi\, e^{-\bar{h}} \int_{-1}^{1} e^{2h_n(u)}\, du \tag{7b}$$

and

$$\exp\,[h_{n+1}\,(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)] = e^{-g_n} \int d\boldsymbol{S} \cdot \exp\,[h_n\,(\boldsymbol{S}_1 \cdot \boldsymbol{S}) + h_n\,(\boldsymbol{S} \cdot \boldsymbol{S}_2)], \tag{7c}$$

where $g_n \equiv g(h_n)$ and $h_0(u) \equiv h(u)$. Eqs. (7) determine the renormalization transformations, which can be used for the calculation of the free energy. As is already indicated in eq. (7c) this is done in the usual way by integrating over the even spins in the expression (2) for the partition function. With eq. (4) we can write

$$Q_{2N}[h] = e^{Ng[h]}\, Q_N[h_1] \tag{8}$$

from which follows

$$\hat{f}[h] = -\tfrac{1}{2}g\,[h] + \tfrac{1}{2}\hat{f}[h_1]. \tag{9}$$

Repeated application of the renormalization transformation (7) leads to the following expression for the free energy

$$\hat{f}[h] = -\sum_{n=0}^{M} \frac{g_n}{2^{n+1}} + \frac{\hat{f}[h_{M+1}]}{2^{M+1}}, \tag{10}$$

where M is any integer. The fact that $\hat{f}[h]$ can be represented in the form of a series is due to the inhomogeneous term in eq. (9), which is also present in Nauenberg's[5]) equivalent of eq. (9) [his eq. (14)]. In our case it can be traced back to the condition (5), which at first appeared rather arbitrary, whereas for the Ising model such a condition is missing. Then, however, it is of no importance, since the inhomogeneous term arises from the rather accidental fact that the transformed interaction is of the same form as the original interaction.

It is immediately seen that the fixed point of the transformation (7) is given by

$$h_n(u) \to \bar{h} \qquad \text{and} \qquad g_n \to \bar{h} + \ln 4\pi. \tag{11}$$

Since these one-dimensional classical spin systems, with no long-range forces, cannot show a phase transition, all functions $h(u)$ belong to the domain of attraction and it therefore seems safe to assume that the series $(h_n(u), g_n)$ always approaches the fixed point. This implies that for sufficiently large M the last term

can be neglected and (11) can be written as an infinite series

$$\hat{f}[h] = -\sum_{n=0}^{\infty} \frac{g_n}{2^{n+1}}, \tag{12}$$

If $|g_n - (\bar{h} + \ln 4\pi| \ll 1$ for $n \geqslant M$ this becomes

$$\hat{f}[h] \cong \sum_{n=0}^{M} \frac{g_n}{2^{n+1}} - \frac{1}{2^M}(\bar{h} + \ln 4\pi). \tag{13}$$

In the next section we will show amongst others, that a small value of M is sufficient to obtain highly accurate results for $\hat{f}[h]$.

2. Numerical results

In order to study the convergence properties of the series (12), we have performed some numerical calculations. A convenient way to do this was to represent the function $e^{h_n(u)}$ by its expansion in Legendre polynomials $P_l(u)$

$$e^{h_n(u)} = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l + 1)\, \lambda_l(n)\, P_l(u), \tag{14}$$

where the numbers

$$\lambda_l(n) = 2\pi \int_{-1}^{1} e^{h_n(u)} P_l(u)\, du \tag{15}$$

are the eigenvalues of the integral equation

$$\int d\boldsymbol{S}\, e^{h_n(\boldsymbol{S}_1 \cdot \boldsymbol{S})}\, Y_{l,m}(\boldsymbol{S}) = \lambda_l(n)\, Y_{l,m}(\boldsymbol{S}_1). \tag{16}$$

The largest eigenvalue $\lambda_0(n)$ is positive and non-degenerate since the kernel of (16) is symmetric, positive definite and of Hilbert–Schmidt type. Substitution of the expansion (14) into the expression (2) for the partition function leads to the following[6]) exact result for the free energy

$$\hat{f}[h] = -\ln \lambda_0 \tag{17}$$

which will serve as a comparison for our expansion of eq. (12). By inserting the expansion (14) into the renormalization transformation formulas (7) we obtain

$$e^{g_n} = \frac{1}{4\pi} e^{-\bar{h}} \sum_{l=0}^{\infty} (2l + 1)\, \lambda_l^2(n) \tag{18a}$$

and

$$\lambda_l\,(n+1) = \mathrm{e}^{-g_n}\,\lambda_l^2(n). \tag{18b}$$

Starting from the eigenvalues $\lambda_l(0)$, we can, therefore, by repeated application of eqs. (18), generate as many g_n's as we wish and in this way evaluate the series of eq. (12). Of course, once the eigenvalues $\lambda_l(0)$ are known (and in particular $\lambda_0(0)$, higher order $\lambda_l(0)$'s contain information about correlation functions, to which we will not pay any attention here), there is no need for calculating the free energy in the round about way of eq. (12), since eq. (17) can be applied directly. *We repeat, however, that for the moment we are only interested in the convergence properties of the series (12) obtained by the renormalization transformation, in the hope that this new technique may eventually be extended to the more interesting two- and three-dimensional systems, where simple formulas like eq. (17) do not exist.*

The mechanism underlying the converge of (12) is the following. From eqs. (18) it follows that

$$\lambda_l\,(n+1) = 4\pi\,\mathrm{e}^{\bar{h}}\,\lambda_l^2(n)\Big/\sum_{l=0}^{\infty}(2l+1)\,\lambda_l^2(n) \equiv P_n(l). \tag{19}$$

For each n, the right-hand side of eq. (19) can be considered as an (unnormalized) distribution $P_n(l)$ in l which has its maximum at $l = 0$ since $\lambda_0(n)$ is the largest eigenvalue. Because of the squaring of the eigenvalues in eq. (19) the distribution in $P_n(l)$ will become sharper with increasing n and eventually only the term with $l = 0$ will survive. From that we see that

$$\lim_{n\to\infty}\lambda_0\,(n+1) = 4\pi\,\mathrm{e}^{\bar{h}} \tag{20}$$

which, when inserted into eq. (18a), yields the limit eq. (11) for g_n when n tends to infinity. The rate of convergence towards the limit (20) is determined by the factor $4\pi\,\mathrm{e}^{\bar{h}}$ in eq. (19) and since $\bar{h}$ is inversely proportional to the temperature, we expect this convergence to slow down for decreasing temperatures.

In the case of the classical Heisenberg chain[7]), the nearest neighbor interaction is given by the function

$$h(u) = h_0(u) = Ku \tag{21}$$

and the eigenvalues become

$$\lambda_l(0) = 4\pi\,(\pi/2K)^{\frac{1}{2}}\,I_{l+\frac{1}{2}}(K), \tag{22}$$

TABLE I

$(1/4\pi)\,\lambda_l(n)$ for $K = 0.01$

l \ n	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
0	1.000	1.010	1.010	1.010	1.010	1.010	1.010	1.010	1.010	1.010
1	0.3338E−02	0.1122E−04	0.1247E−09	0.1539E−19	0.2346E−39	0.5450E−79	0.2940E−158	0.0	0.0	0.0
2	0.6667E−05	0.4489E−10	0.1995E−20	0.3941E−41	0.1538E−82	0.2340E−165	0.0	0.0	0.0	0.0
3	0.9524E−08	0.9161E−16	0.8309E−32	0.6836E−64	0.4626E−128	0.2119E−256	0.0	0.0	0.0	0.0
4	0.1058E−10	0.1131E−21	0.1266E−43	0.1588E−87	0.2497E−175	0.0	0.0	0.0	0.0	0.0
5	0.9620E−14	0.9347E−28	0.8650E−56	0.7408E−112	0.5433E−224	0.0	0.0	0.0	0.0	0.0
6	0.7400E−17	0.5531E−34	0.3029E−68	0.9081E−137	0.8165E−274	0.0	0.0	0.0	0.0	0.0
7	0.4933E−20	0.2458E−40	0.5982E−81	0.3543E−162	0.0	0.0	0.0	0.0	0.0	0.0
8	0.2902E−23	0.8506E−47	0.7163E−94	0.5080E−188	0.0	0.0	0.0	0.0	0.0	0.0
9	0.1527E−26	0.2356E−53	0.5496E−107	0.2991E−214	0.0	0.0	0.0	0.0	0.0	0.0
10	0.7273E−30	0.5343E−60	0.2826E−120	0.7908E−241	0.0	0.0	0.0	0.0	0.0	0.0
11	0.3162E−33	0.1010E−66	0.1010E−133	0.1010E−267	0.0	0.0	0.0	0.0	0.0	0.0
12	0.1265E−36	0.1616E−73	0.2585E−147	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.4685E−40	0.2217E−80	0.4865E−161	0.0	0.0	0.0	0.0	0.0	0.0	0.0
14	0.1615E−43	0.2636E−87	0.6878E−175	0.0	0.0	0.0	0.0	0.0	0.0	0.0
15	0.5211E−47	0.2743E−94	0.7448E−189	0.0	0.0	0.0	0.0	0.0	0.0	0.0
16	0.1579E−50	0.2519E−101	0.6280E−203	0.0	0.0	0.0	0.0	0.0	0.0	0.0
17	0.4512E−54	0.2056E−108	0.4185E−217	0.0	0.0	0.0	0.0	0.0	0.0	0.0
18	0.1219E−57	0.1502E−115	0.2233E−231	0.0	0.0	0.0	0.0	0.0	0.0	0.0

TABLE II

$(1/4\pi)\,\lambda_l(n)$ for $K = 1.0$

l \ n	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
0	1.175	2.070	2.642	2.718	2.718	2.718	2.718	2.718	2.718	2.718
1	0.3679	0.2029	0.2537E−01	0.2506E−03	0.2311E−07	0.1965E−15	0.1420E−31	0.7417E−64	0.2024E−128	0.1507E−257
2	0.7156E−01	0.7677E−02	0.3633E−04	0.5138E−09	0.9716E−19	0.3473E−38	0.4437E−77	0.7244E−155	0.0	0.0
3	0.1007E−01	0.1519E−03	0.1422E−07	0.7867E−16	0.2278E−32	0.1909E−65	0.1341E−131	0.6616E−264	0.0	0.0
4	0.1107E−02	0.1838E−05	0.2082E−11	0.1687E−23	0.1048E−47	0.4040E−96	0.6005E−193	0.0	0.0	0.0
5	0.9996E−04	0.1498E−07	0.1383E−15	0.7447E−32	0.2041E−64	0.1533E−129	0.8643E−260	0.0	0.0	0.0
6	0.7650E−05	0.8773E−10	0.4745E−20	0.8764E−41	0.2827E−82	0.2941E−165	0.0	0.0	0.0	0.0
7	0.5080E−06	0.3889E−12	0.9227E−25	0.3315E−50	0.4044E−101	0.6016E−203	0.0	0.0	0.0	0.0
8	0.2979E−07	0.1330E−14	0.1091E−29	0.4636E−60	0.7910E−121	0.2302E−242	0.0	0.0	0.0	0.0
9	0.1564E−08	0.3667E−17	0.8290E−35	0.2676E−70	0.2635E−141	0.2554E−283	0.0	0.0	0.0	0.0
10	0.7433E−10	0.8281E−20	0.4228E−40	0.6958E−81	0.1782E−162	0.0	0.0	0.0	0.0	0.0
11	0.3226E−11	0.1560E−22	0.1500E−45	0.8763E−92	0.2826E−184	0.0	0.0	0.0	0.0	0.0
12	0.1289E−12	0.2489E−25	0.3818E−51	0.5675E−103	0.1186E−206	0.0	0.0	0.0	0.0	0.0
13	0.4766E−14	0.3405E−28	0.7148E−57	0.1989E−114	0.1456E−229	0.0	0.0	0.0	0.0	0.0
14	0.1642E−15	0.4040E−31	0.1006E−62	0.3941E−126	0.5717E−253	0.0	0.0	0.0	0.0	0.0
15	0.5291E−17	0.4196E−34	0.1085E−68	0.4585E−138	0.7737E−277	0.0	0.0	0.0	0.0	0.0
16	0.1602E−18	0.3846E−37	0.9119E−75	0.3237E−150	0.0	0.0	0.0	0.0	0.0	0.0
17	0.4573E−20	0.3135E−40	0.6058E−81	0.1429E−162	0.0	0.0	0.0	0.0	0.0	0.0
18	0.1235E−21	0.2287E−43	0.3223E−87	0.4045E−175	0.0	0.0	0.0	0.0	0.0	0.0

TABLE III

$(1/4\pi)\ \lambda_l(n)$ for $K = 1.0$

l \ n	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
0	1101.0	2203.0	4292.0	8027.0	0.1386E+05	0.1997E+05	0.2195E+05	0.2203E+05	0.2203E+05	0.2203E+05
1	991.2	1784.0	2816.0	3455.0	2569.0	685.6	25.88	0.3062E−01	0.4256E−07	0.8225E−19
2	804.0	1174.0	1219.0	647.4	90.17	0.8445	0.3927E−04	0.7051E−13	0.2257E−30	0.2313E−65
3	589.2	630.5	351.6	53.88	0.6245	0.4051E−04	0.9038E−13	0.3734E−30	0.6332E−65	0.1820E−134
4	391.5	278.4	68.55	2.048	0.9022E−03	0.8456E−10	0.3937E−24	0.7086E−53	0.2280E−110	0.2359E−225
5	236.8	101.9	9.180	0.3672E−01	0.2901E−06	0.8741E−17	0.4207E−38	0.8091E−81	0.2972E−166	0.0
6	131.0	31.16	0.8591	0.3216E−03	0.2226E−10	0.5145E−25	0.1457E−54	0.9711E−114	0.4281E−232	0.0
7	66.54	8.041	0.5721E−01	0.1426E−05	0.4375E−15	0.1988E−34	0.2176E−73	0.2165E−151	0.0	0.0
8	31.18	1.766	0.2758E−02	0.3314E−08	0.2364E−20	0.5803E−45	0.1854E−940	0.1572E−193	0.0	0.0
9	13.54	0.3327	0.9792E−04	0.4178E−11	0.3756E−26	0.1465E−56	0.1182E−117	0.6388E−240	0.0	0.0
10	5.465	0.5423E−01	0.2602E−05	0.2949E−14	0.1871E−32	0.3637E−69	0.7283E−143	0.2425E−290	0.0	0.0
11	2.060	0.7704E−02	0.5251E−07	0.1201E−17	0.3105E−39	0.1001E−82	0.5519E−170	0.0	0.0	0.0
12	0.7273	0.9606E−03	0.8164E−09	0.2904E−21	0.1814E−46	0.3420E−97	0.6439E−199	0.0	0.0	0.0
13	0.2414	0.1058E−03	0.9907E−11	0.4277E−25	0.3935E−54	0.1609E−112	0.1425E−229	0.0	0.0	0.0
14	0.7554E−01	0.1036E−04	0.9498E−13	0.3930E−29	0.3324E−62	0.1148E−128	0.7252E−262	0.0	0.0	0.0
15	0.2235E−01	0.9067E−06	0.7274E−15	0.2305E−33	0.1143E−70	0.1358E−145	0.0	0.0	0.0	0.0
16	0.6265E−02	0.7129E−07	0.4496E−17	0.8808E−38	0.1669E−79	0.2894E−163	0.0	0.0	0.0	0.0
17	0.1669E−02	0.5059E−08	0.2265E−19	0.2235E−42	0.1074E−88	0.1199E−181	0.0	0.0	0.0	0.0
18	0.4234E−03	0.3256E−09	0.9378E−22	0.3832E−47	0.3160E−98	0.1307E−200	0.0	0.0	0.0	0.0

where $I_\nu(z)$ is the modified Bessel function, which for $\nu = \frac{1}{2}$ is

$$I_{\frac{1}{2}}(z) = \left(\frac{2z}{\pi}\right)^{\frac{1}{2}} \frac{\sinh z}{z}. \tag{23}$$

With (17) the exact free energy therefore is

$$\hat{f}(K) = -\ln\left[\frac{4\pi \sinh K}{K}\right]. \tag{24}$$

Starting from $\lambda_l(0)$ as given by eq. (22) we have calculated $\lambda_l(n)$ using eq. (19). For three temperatures the results are given in tables I, II and III.

Notice that for $K = 0.01$ all eigenvalues have practically reached their asymptotic values (*i.e.* zero for $l > 0$) after only one step in n. For $K = 1.0$ this is the case after three steps and for $K = 10.0$ after seven steps. For large K the two

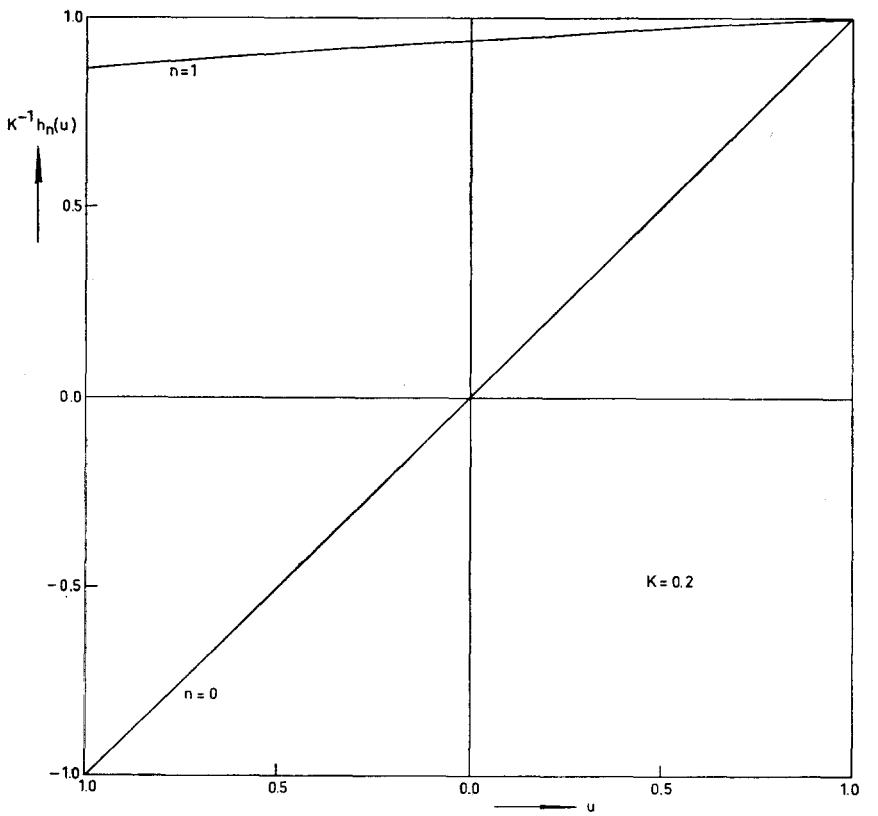

Fig. 1. $K^{-1}h_n(u)$ *versus* u for $K = 0.2$ and $n = 0, 1$.

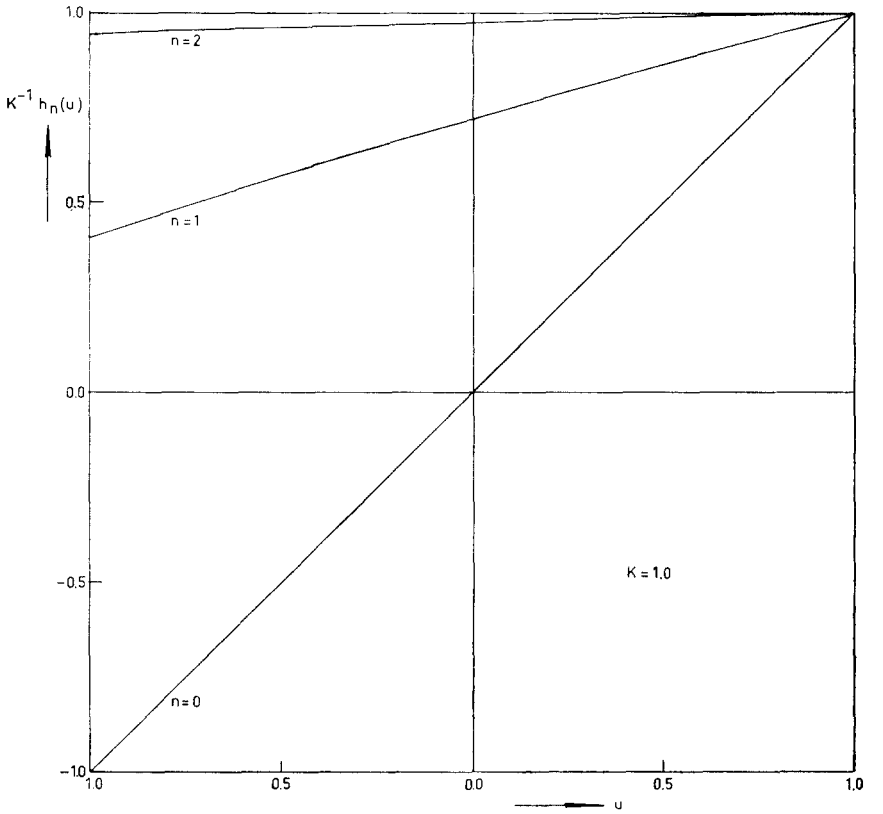

Fig. 2. $K^{-1}h_n(u)$ *versus* u for $K = 1.0$ and $n = 0, 1, 2$.

eigenvalues $\lambda_0(0)$ and $\lambda_1(0)$ are almost degenerate indicating that we are approaching the quasi transition temperature $T = 0$. Since this transition takes place at $T = 0$, where renormalization transformations always have a fixed point, we will not discuss it here in the context of the renormalization technique, the issue being the convergence of the series (12).

Using the eigenvalues $\lambda_l(n)$ and the g_n, obtained *via* eq. (18a) we have calculated the free energy as given by eq. (13) with $M = 15$. Complete agreement with the exact free energy of eq. (24) was obtained for a number of K values ranging between 0.01 and 100. The same agreement is obtained with much smaller values of N ($N = 1$, 3 and 7 for $K = 0.01$, 1.0 and 10.0, respectively) provided the remainder in eq. (13) is taken into account.

In order to see how the interaction functions $h_n(u)$ changed from one to another when the transformations of eq. (7) were applied and how fast they approached their fixed value $\bar{h}$ ($= K$ for the classical Heisenberg chain) a number of them were calculated, using eq. (14). For three values of K the functions $K^{-1}h_n(u)$ have been

plotted in figs. 1, 2 and 3. The curves are shown only for those values of n for which $1 - K^{-1}h_n(-1) > 0.03$. In order to check the accuracy of the calculations we used $K^{-1}h_n(1) = 1$ and

$$K^{-1}h_1(-1) = 1 - K^{-1}\ln\left[\frac{\sinh 2K}{2K}\right] \tag{25}$$

which can easily be derived from eq. (7). Eq. (25) reduces to the limiting cases

$$\begin{aligned} K^{-1}h_1(-1) &= 1 - \tfrac{2}{3}K + \cdots \qquad \text{for} \qquad K \ll 1, \\ K^{-1}h_1(-1) &= -1 + \cdots \qquad \text{for} \qquad K \gg 1. \end{aligned} \tag{26}$$

The fact that $h_n(u)$ approaches a constant $\bar{h}$ means that after a few renormalization transformations (7) the new system behaves as a collection of free spins. This seems to be quite different from Kadanoff's picture[8]), in which the interactions

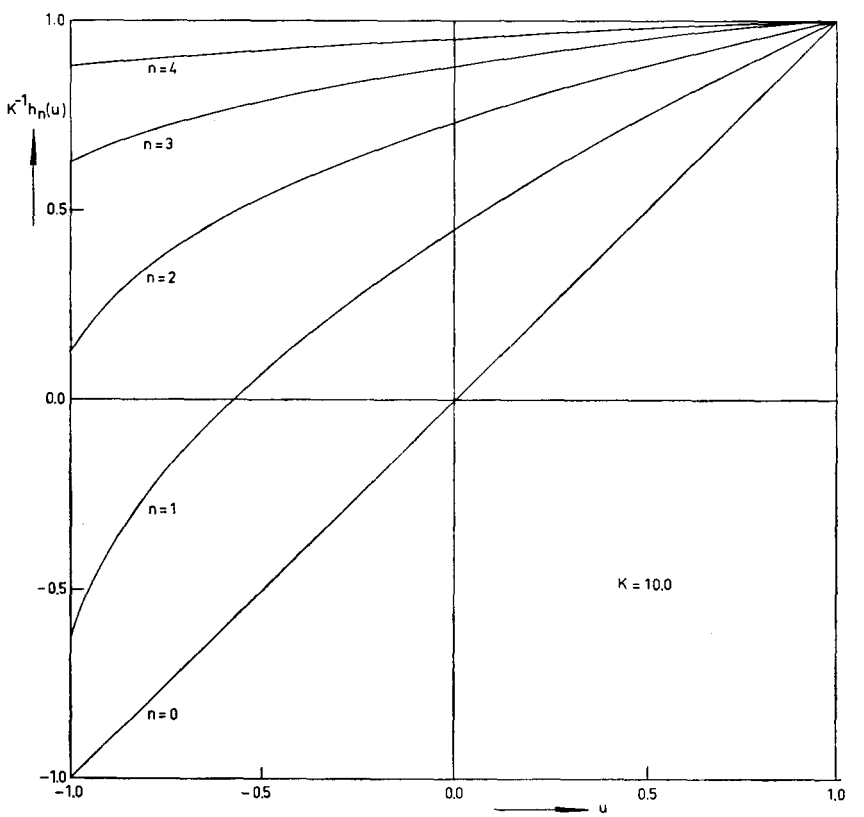

Fig. 3. $K^{-1}h_n(u)$ *versus* u for $K = 10.0$ and $n = 0, 1, 2, 3, 4$.

for a system close to the critical temperature does not change under a renormalization transformation. This difference, however, is only superficially there, since it can be shown that for $K \gg 1$ and for u close to 1, *i.e.* for the ferromagnetic state of two neighboring spins, the first transform of $h(u) = Ku$, is given by $h_1(u) \cong \frac{1}{2}(u + 1)K$. The linearity near the fixed point, therefore, is maintained.

3. External fields

In this section we shall show qualitatively how one should extend the renormalization theory to one-dimensional spin systems in an external field. Let the hamiltonian be given by

$$-\beta H_N = h(\boldsymbol{S}_1, \boldsymbol{S}_2) + \cdots + h(\boldsymbol{S}_N, \boldsymbol{S}_1) + k(\boldsymbol{S}_1) + \cdots + k(\boldsymbol{S}_N), \tag{27}$$

where $h(\boldsymbol{S}_1, \boldsymbol{S}_2)$ is symmetric in the arguments $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$.

The partition function of $2N$ spins now becomes a functional, not only of h, but also of k:

$$Q_{2N}[h, k] = \int \mathrm{d}\boldsymbol{S}_1 \cdots \mathrm{d}\boldsymbol{S}_N \, \mathrm{e}^{-\beta H_{2N}}. \tag{28}$$

We now define, analogous to eqs. (4), (5) and (7), $g[h, k]$, $h_1(\boldsymbol{S}_1, \boldsymbol{S}_2)$ and $k_1(\boldsymbol{S}_1)$ through the following three equations:

$$\exp\{g[h_1, k_1] + h_1(\boldsymbol{S}_1, \boldsymbol{S}_2) + \tfrac{1}{2}k_1(\boldsymbol{S}_1) + \tfrac{1}{2}k_1(\boldsymbol{S}_2)\}$$
$$= \exp\tfrac{1}{2}\{k_1(\boldsymbol{S}_1) + k_1(\boldsymbol{S}_2)\} \int \mathrm{d}\boldsymbol{S} \exp\{h(\boldsymbol{S}_1, \boldsymbol{S}) + h(\boldsymbol{S}_1, \boldsymbol{S}_2) + k(\boldsymbol{S})\} \tag{29a}$$

with

$$h_1(\boldsymbol{S}, \boldsymbol{S}) = h(\boldsymbol{S}, \boldsymbol{S}) \equiv \bar{h}(\boldsymbol{S}) \tag{30}$$

and

$$k_1(\hat{z}) = k(\hat{z}) \equiv \bar{k}, \tag{31}$$

where $\hat{z}$ is a unit vector in an arbitrary direction, which we shall call the z-direction. In order to show that g, h_1 and k_1 are indeed completely specified by these equations, take $\boldsymbol{S}_1 = \boldsymbol{S}_2$ in eq. (29a). This yields

$$\exp\{g[h, k] + \bar{h}(\boldsymbol{S}_1) + k_1(\boldsymbol{S}_1)\}$$
$$= \exp\{k(\boldsymbol{S}_1)\} \cdot \int \mathrm{d}\boldsymbol{S} \exp\{2h(\boldsymbol{S}_1, \boldsymbol{S}) + k(\boldsymbol{S})\}, \tag{29b}$$

where eq. (30) has been used. Now substitute $\boldsymbol{S}_1 = \hat{z}$ into eq. (29b). With eq. (31) this then leads to

$$\exp\{g[h, k] + \bar{h}(\hat{z})\} = \int \mathrm{d}\boldsymbol{S} \exp\{2h(\hat{z}, \boldsymbol{S}) + k(\boldsymbol{S})\}. \tag{29c}$$

The eqs. (29) then indeed determine $g[h, k]$, $k_1(\boldsymbol{S})$ and $h_1(\boldsymbol{S}_1, \boldsymbol{S}_2)$. This process again can be repeated indefinitely (for $N = \infty$!), so that the eqs. (29) define a renormalization semi-group of transformations.

The partition function (28) can now be written as

$$Q_{2N}[h, k] = \mathrm{e}^{Ng[h, k]} Q_N[h_1, k_1] \tag{32}$$

from which we again derive

$$\hat{f}[h, k] = -\tfrac{1}{2} g[h, k] + \tfrac{1}{2} \hat{f}[h_1, k_1]. \tag{33}$$

As in section 1 we can write this as

$$\hat{f}[h, k] \cong -\sum_{n=0}^{M} \frac{g_n}{2^{n+1}} - \frac{\bar{g}}{2^M} \tag{34}$$

with

$$\bar{g} = \lim_{n \to \infty} g_n \tag{35}$$

and with g_n, $k_n(\boldsymbol{S})$ and $h_n(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2)$ defined by the equations

$$\mathrm{e}^{g_n} = \mathrm{e}^{-\bar{h}(\hat{z})} \int \mathrm{d}\boldsymbol{S}\, \mathrm{e}^{2[\bar{h}_n(\hat{z} \cdot \boldsymbol{S}) + \frac{1}{2} k_n(\boldsymbol{S})]}, \tag{36a}$$

$$\mathrm{e}^{g_n + \bar{h}(\boldsymbol{S}_1) + k_{n+1}(\boldsymbol{S}_1)} = \mathrm{e}^{k_n(\boldsymbol{S}_1)} \int \mathrm{d}\boldsymbol{S}\, \mathrm{e}^{2[h_n(\boldsymbol{S}_1, \boldsymbol{S}) + \frac{1}{2} k_n(\boldsymbol{S})]}, \tag{36b}$$

$$\mathrm{e}^{g_n + h_{n+1}(\boldsymbol{S}_1, \boldsymbol{S}_2) + \frac{1}{2} k_{n+1}(\boldsymbol{S}_1) + \frac{1}{2} k_{n+1}(\boldsymbol{S}_2)} = \mathrm{e}^{\frac{1}{2} k_n(\boldsymbol{S}_1) + \frac{1}{2} k_n(\boldsymbol{S}_2)} \int \mathrm{d}\boldsymbol{S}\, \mathrm{e}^{[h_n(\boldsymbol{S}_1, \boldsymbol{S}) + h_n(\boldsymbol{S}, \boldsymbol{S}_2) + k_n(\boldsymbol{S})]}. \tag{36c}$$

The fixed point of this transformation satisfying the conditions (30) and (31), is given by

$$k_n(\boldsymbol{S}) \to k^*(\boldsymbol{S}), \qquad h_n(\boldsymbol{S}_1, \boldsymbol{S}_2) \to \tfrac{1}{2}[\bar{h}(\boldsymbol{S}_1) + \bar{h}(\boldsymbol{S}_2)]$$

and

$$g_n \to \bar{g} = \ln \int \mathrm{d}\boldsymbol{S}\, \mathrm{e}^{\bar{h}(\boldsymbol{S}) + k^*(\boldsymbol{S})} \tag{37}$$

as can easily be verified upon substitution. After a number of transformations the system therefore behaves as a collection of mutually non-interacting spins with an external interaction

$$k_{\text{ext}}(\boldsymbol{S}) = \bar{h}(\boldsymbol{S}) + k^*(\boldsymbol{S}). \tag{38}$$

This is qualitatively the same conclusion as at the end of section 2, but now in the presence of a homogeneous external field, which is not transformed away by this particular renormalization transformation.

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