

Accurate Estimates of Solutions of Second Order Recursions

R. M. M. Mattheij

Mathematisch Instituut

Rijksuniversiteit Utrecht—De Uithof, Netherlands

Recommended by James Wilkinson

ABSTRACT

Two important types of two dimensional matrix-vector and second order scalar recursions are studied. Both types possess two kinds of solutions (to be called forward and backward dominant solutions). For the directions of these solutions sharp estimates are derived, from which the solutions themselves can be estimated.

1. INTRODUCTION

In this paper we investigate the real solutions of the *matrix-vector recursion* (MVR for short):

$$\begin{pmatrix} x_{i+1}^1 \\ x_{i+1}^2 \end{pmatrix} = A_i \begin{pmatrix} x_i^1 \\ x_i^2 \end{pmatrix}, \quad \text{where } A_i = \begin{pmatrix} b_i & c_i \\ d_i & e_i \end{pmatrix}, \quad i \geq 0; \quad (1.1)$$

b_i, c_i, d_i and e_i are real numbers.

We shall pay some special attention to the important special case of the *three term scalar recursion* (SR for short):

$$x_{i+2} = e_i x_{i+1} + d_i x_i, \quad i \geq 0. \quad (1.2)$$

The SR (1.2) can in fact be understood as an MVR:

$$\begin{pmatrix} x_{i+1} \\ x_{i+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ d_i & e_i \end{pmatrix} \begin{pmatrix} x_i \\ x_{i+1} \end{pmatrix}. \quad (1.3)$$

Note that $x_i = x_i^1$ and $x_{i+1} = x_i^2$.

As is well known, the solutions of MVR's can quite often be divided into two classes: the dominant and dominated solutions [5,6]. Our purpose is to give a theory which facilitates the derivation of quite accurate estimates of those solutions. This will be done by estimating the "directions" x_i^2/x_i^1 of

solutions of MVR's to some accuracy, and then it will be easy to estimate the "growth factors" x_{i+1}^1/x_i^1 and x_{i+1}^2/x_i^2 . In order to see how the "directions" may be estimated, we recall that in the case of constant coefficients the iterated vector will approach the dominant eigenvector. In the case of non-constant coefficients we shall then expect the iterated vector to follow the "local" dominant eigenvector rather closely. Special attention will therefore be paid to cases where these coefficients are slowly varying.

In Sec. 2 we give definitions and some properties, which will be used in what follows. In Sec. 3 we indicate the solutions to be estimated; the estimating will be done in Sec. 4 (rough) and Sec. 5 (sharp). Section 6 deals with SR's. A useful means of practical application can be the so-called co-diagram (Sec. 7). Finally, examples can be found in Sec. 8.

There is quite a close connection between the present work and the work reported in [6]. The main differences between the two approaches are that we do not need the $T^{-1}T$ transformations used in [6], and that we look more accurately at the non-linear recursion for the direction. On the other hand, our results may lead to more accurate applications of [3].

2. DEFINITIONS, CONVENTIONS AND AUXILIARY PROPERTIES

2.1. The Growth Factor

We shall call a sequence

$$\left\{ \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix}, \begin{pmatrix} x_1^1 \\ x_1^2 \end{pmatrix}, \dots \right\} \stackrel{D}{=} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

a *solution* of the MVR (1.1) if $\left\{ \begin{pmatrix} x_i^1 \\ x_i^2 \end{pmatrix} \right\}_{i \geq 0}$ obeys the recursion (1.1).

Similarly a sequence $\{x_0, x_1, \dots\} \stackrel{D}{=} x$ is a *solution* of the SR (1.2) if $\{x_i\}_{i \geq 0}$ obeys the relation (1.2).

The *growth factors* (g.f.) of a solution $\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$ of an MVR are defined as

$$\rho_i^j = \frac{x_{i+1}^j}{x_i^j}, \quad i=0, 1, \dots \text{ and } j=1, 2. \quad (2.1)$$

By ρ^j ($j=1, 2$) we shall mean the sequence $\{\rho_0^j, \rho_1^j, \dots\}$ ($j=1, 2$). The sequences ρ^1 and ρ^2 will also be called *growth factors* of $(x^1, x^2)^T$ for short.

Once the g.f.'s have been estimated, we can get estimates for solutions as follows:

$$x_i^j = x_0^j \prod_{k=0}^{i-1} \rho_k^j, \quad j = 1, 2. \quad (2.2)$$

Similarly to (2.1), we define the *growth factors* of a solution x of an SR as

$$\rho_i = \frac{x_{i+1}}{x_i}, \quad i = 0, 1, \dots \quad (2.3)$$

ρ will indicate the sequence $\{\rho_0, \rho_1, \dots\}$.

2.2. The Direction

The *direction* of the vector $(x_i^1, x_i^2)^T$ is defined as

$$\delta_i = \frac{x_i^2}{x_i^1}. \quad (2.4)$$

We shall call $\delta = \{\delta_0, \delta_1, \dots\}$ the direction of the solution $\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$.

REMARK 2.1. For a solution x of an SR the direction will virtually coincide with the g.f. Indeed, if

$$\delta_i = \frac{x_{i+1}}{x_i}$$

(see (1.3) and (2.4)), then $\rho_i = \delta_i$.

REMARK 2.2. Due to division by 0 the quantities δ_i , ρ_i^1 and ρ_i^2 may become undefined. Translation to (1.1) shows, however, that this has no special meaning for our problem. We shall therefore allow these quantities to be $\pm \infty$.

From (1.1) we obtain the following important non-linear recursion:

$$\delta_{i+1} = \frac{d_i + e_i \delta_i}{b_i + c_i \delta_i}. \quad (2.5)$$

Once δ is known, we can obtain ρ^1 and ρ^2 from

$$\rho_i^1 = b_i + c_i \delta_i, \quad (2.6)$$

$$\rho_i^2 = \frac{d_i}{\delta_i} + e_i. \quad (2.7)$$

2.3. Eigenvalues and Characteristic Directions of A_i

We denote the eigenvalues of A_i by λ_i and μ_i , and assume that $|\lambda_i| \geq |\mu_i|$.

From (2.5) we see that the directions of the eigenvectors of A_i (also called the *characteristic directions* of A_i) are the fixed points of the functions Φ_i , defined as follows:

$$\Phi_i(x) = \frac{d_i + e_i x}{b_i + c_i x}, \quad i = 0, 1, \dots \quad (2.8)$$

Hence they are the roots of

$$c_i x^2 + (b_i - e_i)x - d_i = 0. \quad (2.9)$$

Denote a real root of (2.9), for which $|(d/dx)\Phi_i(x)| \leq 1$ [$|(d/dx)\Phi_i(x)| \geq 1$] by α_i [β_i] (N. B. this definition always makes sense, since the graph of Φ_i is a hyperbola with asymptotes parallel to the axes—see Sec. 3—which is to be intersected by the line $y = x$.)

PROPERTY 2.1. α_i and β_i are the directions of the eigenvectors belonging to λ_i and μ_i , respectively, and the following relations hold:

- (a) $\lambda_i = c_i \alpha_i + b_i$,
- (b) $\mu_i = c_i \beta_i + b_i$,
- (c) $\lambda_i = e_i - c_i \beta_i$,
- (d) $\mu_i = e_i - c_i \alpha_i$.

Proof. Note that the directions of the eigenvectors have to be the roots of (2.9). Let y be the direction of the eigenvector belonging to λ_i ; then (see

(1.1)) $c_i y = \lambda_i - b_i$. Since

$$\frac{d}{dx} \Phi_i(x) = \frac{\det A_i}{(b_i + c_i x)^2},$$

we obtain

$$\left| \frac{d}{dx} \Phi_i(x) \right|_{x=y} = \left| \frac{\mu_i}{\lambda_i} \right| \leq 1;$$

hence $y = \alpha_i$. This proves (a). Similarly β_i corresponds to μ_i . The other relations follow from the observation that $\alpha_i + \beta_i = (e_i - b_i)/c_i$. ■

We shall distinguish two types of Φ_i :

DEFINITION 2.1. Φ_i is of type I [II] if $(d/dx)\Phi_i(x) > 0$ [$(d/dx)\Phi_i(x) < 0$] except for $x = -b_i/c_i$.

PROPERTY 2.2. Φ_1 is of type I [II] if and only if the eigenvalues of A_1 have the same [opposite] sign.

This follows from the observation that $(d/dx)\Phi_i(x) = (\det A_i)/(b_i + c_i x)^2$. ■

2.4. The Two Types of MVR's; Basic Assumptions

In the remainder of this paper it will be useful to suppose that $\forall_i \alpha_i$ and β_i are finite, which is equivalent to $\forall_i c_i \neq 0$.

REMARK 2.3. If $\forall_i c_i = 0$, we have in fact a scalar first order (inhomogeneous) recursion, which can be solved explicitly.

REMARK 2.4. If some of the c_i are zero our theory may be applied piecewise, i.e., to sequences of values of i for which $c_i \neq 0$.

REMARK 2.5. A more convenient way to overcome problems if some of the c_i are zero is to find a transformation T such that none of the (1,2) elements of $T A_i T^{-1}$ will be zero for $i < \infty$.

By a simple transformation of the recursion we can see that the requirement $\forall_i c_i \neq 0$ is no more restrictive than $\forall_i c_i = 1$.

For the matrices A_i we shall assume the following:

$$\forall_i c_i = 1, \quad (2.10)$$

$$(b_i - e_i)^2 > -4d_i, \quad (2.11a)$$

$$\text{sign}(\det A_i) \text{ is independent of } i, \quad (2.11b)$$

$$\text{sign}(b_i + e_i) = +1 \quad \text{for all } i. \quad (2.11c)$$

REMARK 2.6. These conditions imply:

- (i) the eigenvalues of A_i are real and distinct;
- (ii) for a given MVR Φ_i is either of type I or of type II for all i (cf. (2.11b));
- (iii) $\forall_i \lambda_i > 0$ and $\text{sign}(\mu_i)$ is independent of i (cf. (2.11b) and (2.11c));
- (iv) $\forall_i \alpha_i > \beta_i$ (cf. (iii) and Property 2.1).

REMARK 2.7. Actually the condition (2.11c) is no more restrictive than the condition $\text{sign}(b_i + e_i) = \text{constant}$, since a simple transformation transforms a recursion with $\text{sign}(b_i + e_i) = -1$ for all i into one with $\text{sign}(b_i + e_i) = 1$ for all i .

DEFINITION 2.2. An MVR is of type I [II] if the assumptions (2.11) hold with $\text{sign}(\det A_i) = +1$ [$\text{sign}(\det A_i) = -1$] for all i .

REMARK 2.8. We shall restrict our investigation to MVR's of type I and type II. In all other cases the reader should consult the paper of van der Sluis [6]. Type I and type II do form an important class of MVR's, and in particular most of the familiar SR's are either of type I or of type II.

2.5. Segments

To facilitate the notation in the next sections we define for a set of numbers p_1, \dots, p_k :

DEFINITION 2.3. $[p_1; \dots; p_k]$ is the smallest segment containing p_1, \dots, p_k .

DEFINITION 2.4. Let S_1 and S_2 be two segments of the real line. Then $S_1 < S_2 \leftrightarrow \forall_{x \in S_1} \forall_{y \in S_2} x < y$.

DEFINITION 2.5. Let S be a segment and c some real number. Then $cS = \{cx | x \in S\}$.

3. FORWARD AND BACKWARD DOMINANT SOLUTIONS

Before indicating the kinds of solutions which will be considered in what follows, we have a closer look at the possible graphs of Φ_i .

We shall always draw the line $y = x$ in connection with those graphs in order to be able to use the graphical method of successive substitution for the successive δ_i . (N.B. The points of intersection of this line with Φ_i correspond to α_i and β_i .) In Figs. 1 and 2 we have drawn type I and type II, respectively, and several choices of pairs ' δ_i, δ_{i+1} '. It can be seen there that α_i "attracts" the direction by recurring forward (i.e., δ_{i+1} is obtained from δ_i), while β_i "attracts" the direction by recurring backward (i.e., δ_i is obtained from δ_{i+1}) in accordance with the power method theory (cf. the Introduction).

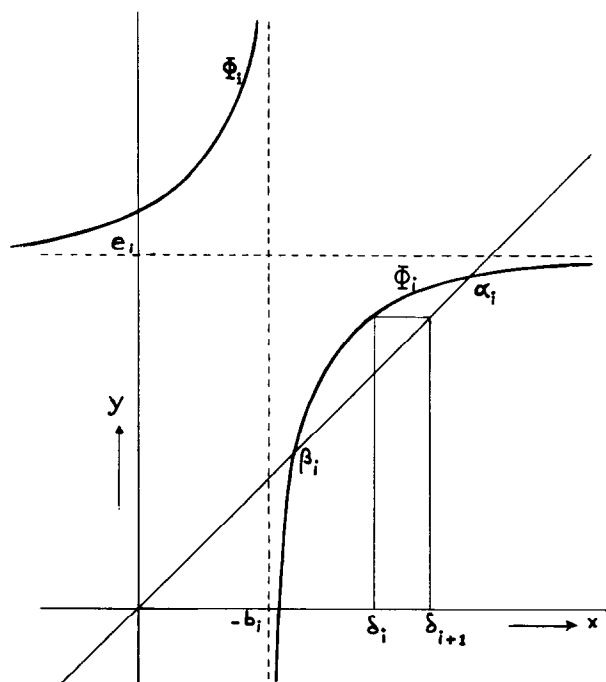


FIG. 1. Type I.

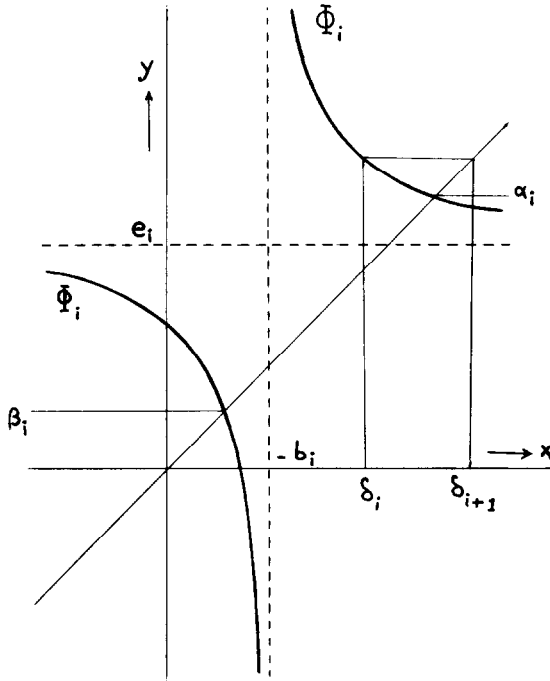


FIG. 2. Type II.

For type I the approach to the characteristic direction is monotonic, while for type II it is cobweblike. This difference in behavior is the reason for our distinction into two types.

REMARK 3.1. It should be noted that by the assumption (2.11c), every graph of a Φ_i belonging to an MVR of type I [II] is a hyperbola of the type shown in Fig. 1 (Fig. 2).

Since we expect that in the case of slowly varying coefficients the direction will follow the successive characteristic directions rather closely, we shall study the solutions indicated below:

DEFINITION 3.1.

Solutions with direction ψ "close" to the successive α_i . They will be called *forward dominant*. A special kind of this type of solutions are solutions with direction ψ^n determined by $\psi_n^n = \alpha_n$.

Solutions with direction ϕ "close" to the successive β_i . They will be called *backward dominant*—in particular, solutions with direction ϕ^n determined by $\phi_{n+1}^n = \beta_n$.

If $\lim_{n \rightarrow \infty} \phi^n$ exists (elementwise), then this limit is again a direction. We shall indicate it by ω and call the corresponding solution the *dominated solution*. Any other solution with direction ψ , $\psi_0 \neq \omega_0$, will then be called *dominant*.

REMARK 3.2. Sufficient conditions for the existence of a dominated solution will be given in Secs. 4.2 and 5.4. (See also [4, 5, 6].)

4. ROUGH ESTIMATES OF THE DIRECTIONS

4.1. An Estimation Theory

In this section we shall derive fairly general, though rough, estimates. In Sec. 5 we shall see, however, that the estimates will become much sharper with some additional conditions.

Let n, N denote integers, $0 \leq n \leq N$.

THEOREM 4.1. *Let the MVR be of type I. If p and q are such that for $i = n, \dots, N$, $[\alpha_n; \dots; \alpha_i; p] \geq \beta_i$ and $\alpha_i \geq [\beta_i; \dots; \beta_N; q]$, then for the directions η defined by $\eta_n = p$ and ζ defined by $\zeta_{N+1} = q$ the following estimates hold:*

$$\eta_i \in [\alpha_n; \dots; \alpha_{i-1}; p] \quad (n+1 \leq i \leq N+1)$$

and

$$\zeta_i \in [\beta_i; \dots; \beta_N; q] \quad (n \leq i \leq N).$$

Proof. (Induction.) If $\beta_i \leq \eta_i \leq \alpha_i$, then $\eta_i \leq \eta_{i+1} \leq \alpha_i$; if $\alpha_i \leq \eta_i$, then $\alpha_i \leq \eta_{i+1} \leq \eta_i$ (see Fig. 1). The reasoning for ζ is similar. ■

COROLLARY 4.1. *If the MVR is of type I and if*

$$F(n, i) = [\alpha_n; \dots; \alpha_i] \geq \beta_i \quad (n \leq i \leq N)$$

and

$$B(i, N) = [\beta_i; \dots; \beta_N] \leq \alpha_i \quad (n \leq i \leq N),$$

then we have

$$\psi_i^n \in F(n, i-1) \quad \text{for } i = n+1, \dots, N+1$$

and

$$\phi_i^n \in B(i, N) \quad \text{for } i = n, \dots, N.$$

For MVR's of type II, rough estimates similar to those of 4.1 cannot be given. Indeed, in contrast with the situation for type I, where for a direction δ we always had either $\delta_i \leq \delta_{i+1} \leq \alpha_i$ or $\alpha_i \leq \delta_{i+1} \leq \delta_i$, we now have either $\delta_i \leq \alpha_i \leq \delta_{i+1}$ or $\delta_{i+1} \leq \alpha_i \leq \delta_i$ in a neighborhood of α_i . Similarly with β_i instead of α_i . This fact prevents us from using the characteristic directions as bounds. Under some conditions, however, one can apply Theorem 4.1 to type II in a slightly transformed version. Since $\det(A_{i+1}A_i) > 0$, we know that at least $\Phi_{i+1}\Phi_i$ is of type I. Now suppose $b_{i+1} + e_i > 0$ for a certain MVR of type II. Let T_i be the matrix

$$T_i = \text{diag}(1, b_{i+1} + e_i), \quad (4.1)$$

and define

$$\hat{A}_i = T_{i+2}A_{i+1}A_iT_i^{-1}. \quad (4.2)$$

One can then easily establish that the MVR's

$$\begin{pmatrix} z_{i+2}^1 \\ z_{i+2}^2 \end{pmatrix} = \hat{A}_i \begin{pmatrix} z_i^1 \\ z_i^2 \end{pmatrix} \quad (4.3)$$

for i even and for i odd, respectively, are of type I. Combination of the estimates for the solutions of (4.3) with even and odd index, respectively, then yields estimates for the corresponding solutions of the original MVR, using

$$\begin{pmatrix} z_i^1 \\ z_i^2 \end{pmatrix} = T_i \begin{pmatrix} x_i^1 \\ x_i^2 \end{pmatrix}, \quad (4.4)$$

and supposing, of course, that the other assumptions of Theorem 4.1 are fulfilled for (4.3).

REMARK 4.1. The condition $b_{i+1} + e_i > 0$ indicates a kind of slow variation (cf. (2.11c) where $e_i + b_i > 0$).

4.2. Existence of Dominated Solutions

We now give some sufficient conditions for which $\omega = \lim_{n \rightarrow \infty} \phi^n$ exists. In this case estimates for ω can be derived from estimates for ϕ^n using a limit argument.

THEOREM 4.2. *Let the MVR be of type I. If $\inf_{i \geq i_0} \beta_i > -\infty$ and $\sup_{i \geq i_0} \beta_i \leq \alpha_i$ for all i then there exists a dominated solution.*

Proof. Application of Corollary 4.2 shows that $\forall_{n \geq 0} |\phi_0^n|$ is bounded; hence $\{\phi_0^n\}_{n \geq 0}$ contains a convergent subsequence $\{\phi_0^{n_k}\}$. A limit argument yields the existence of ω (cf. Sec. 9 of [6]). ■

THEOREM 4.3. *Let the MVR be of type II. If $\forall_{i \geq 0} b_{i+1} + e_i > 0$, $\inf_{i \geq 0} \Phi_i^{-1}(-b_{i+1}) > -\infty$ and $\sup_{i \geq 0} -b_i < \infty$, then there exists a dominated solution.*

Proof. If $\beta_i \leq \delta_{i+1} \leq -b_{i+1}$ (hence $\delta_{i+1} < e_i$), then $\Phi_i^{-1}(-b_{i+1}) \leq \delta_i \leq \beta_i$. If $\Phi_i^{-1}(-b_{i+2}) \leq \delta_{i+1} \leq \beta_i$, then $\beta_i \leq \delta_i \leq -b_i$. Induction and application of this property for ϕ^n shows that $\forall_{n \geq 0} |\phi_0^n|$ is bounded, etc. ■

5. SHARPER ESTIMATES OF THE DIRECTIONS

5.1. Estimates Presupposing Monotonicity of $\{\alpha_i\}$ and/or $\{\beta_i\}$

We summarize our main results in two theorems of a fairly general nature. The proofs will be given afterwards. Theorems 5.3 and 5.4 will demonstrate some applications of them.

J , M and N will always denote integers.

THEOREM 5.1. *Suppose the MVR is of type I.*

(a) *Let $M \geq 2$. If $\{\alpha_i\}$ is monotonically $\begin{smallmatrix} \text{in} \\ \text{de} \end{smallmatrix}$ creasing and*

$$\alpha_i \begin{matrix} \geq \\ \leq \end{matrix} \Phi_i(\alpha_{i-2}) \begin{matrix} \geq \\ \leq \end{matrix} \alpha_{i-1} \quad \text{for } i \geq M \text{ and } \psi_M \in [\alpha_{M-2}; \alpha_{M-1}], \quad (5.1)$$

then for $i \geq M$, $\psi_i \in [\alpha_{i-2}; \alpha_{i-1}]$.

(b) *Let $N \geq 0$ and $J > N$. If $\{\beta_i\}$ is monotonically $\begin{smallmatrix} \text{in} \\ \text{de} \end{smallmatrix}$ creasing and*

$$\Phi_i(\beta_{i+1}) \begin{matrix} \geq \\ \leq \end{matrix} \beta_{i+2} \quad \text{for } J-1 \geq i \geq N \text{ and } \phi_J \in [\beta_J; \beta_{J+1}], \quad (5.2)$$

then for $J \geq i \geq N$, $\phi_i \in [\beta_i; \beta_{i+1}]$.

THEOREM 5.2. Suppose the MVR is of type II.

(a) Let $M \geq 1$. If $\{\alpha_i\}$ is monotonically $\begin{smallmatrix} \text{in} \\ \text{de} \end{smallmatrix}$ creasing and

$$\alpha_i \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} \Phi_i(\alpha_{i-1}) \begin{smallmatrix} \leq \\ \geq \end{smallmatrix} \alpha_{i+1} \quad \text{for } i \geq M \text{ and } \psi_M \in [\alpha_{M-1}; \alpha_M], \quad (5.3)$$

then for $i \geq M$, $\psi_i \in [\alpha_{i-1}; \alpha_i]$.

(b) Let $N \geq 2$, $J > N$. If $\{\beta_i\}$ is monotonically $\begin{smallmatrix} \text{in} \\ \text{de} \end{smallmatrix}$ creasing and

$$\Phi_i(\beta_{i-1}) \begin{smallmatrix} \geq \\ \leq \end{smallmatrix} \beta_{i+1} \quad \text{for } J-1 \geq i \geq N \text{ and } \phi_J \in [\beta_{J-1}; \beta_J], \quad (5.4)$$

then for $J \geq i \geq N$, $\phi_i \in [\beta_{i-1}; \beta_i]$.

We now come to the proofs of these theorems; they will be indicated only briefly.

Proof of Theorem 5.1. For (a), Suppose, e.g., $\{\alpha_i\}$ increasing. Let $\alpha_{i-2} \leq \psi_i \leq \alpha_{i-1}$. From Fig. 3 we see that, because of (5.1), $\alpha_{i-1} \leq \psi_{i+1} \leq \alpha_i$. So it holds for all $i \geq M$.

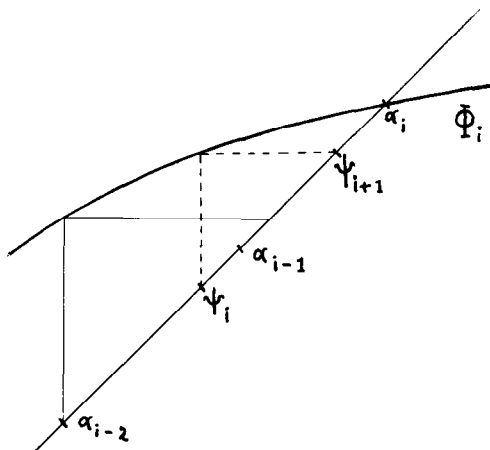


FIG. 3.

Suppose $\{\beta_i\}$ increasing. Let $i < J$; if $\beta_{i+1} \leq \phi_{i+1} \leq \beta_{i+2}$, we see with aid of Fig. 4 and (5.2) that $\beta_i \leq \phi_i \leq \beta_{i+1}$.

Suppose $\{\alpha_i\}$ increasing. We then have a situation like in Fig. 5, from which the induction step is easily seen.

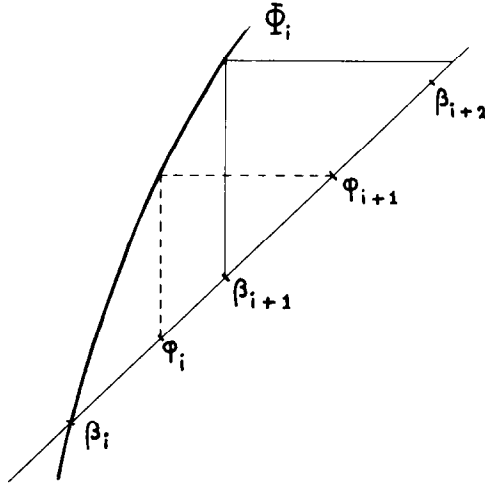


FIG. 4.

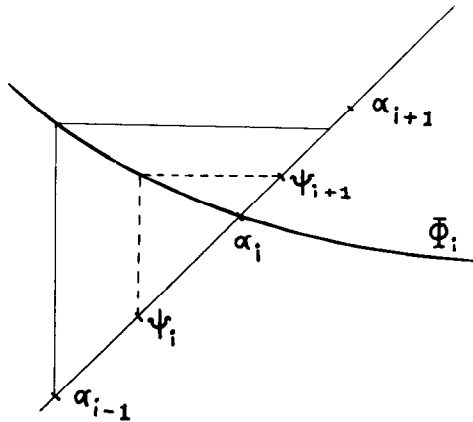


FIG. 5.

The proof of part (b) is quite similar to the proof of part (a), except for the index, which is now decreasing in the induction step. ■

REMARK 5.1. If $\lim \phi^n = \omega$ exists (see Sec. 5.4), then estimates for ω can be derived from the preceding theorems, by replacing ϕ^n with ω in the corresponding places. The proof follows immediately from a limit argument.

REMARK 5.2. If the monotonicity conditions are satisfied, then the theorems apply to ψ^{M-1} and ϕ^N (M and N obeying the conditions of the theorems), since $\psi_{M-1}^{M-1} = \psi_M^{M-1} = \alpha_{M-1}$ and $\phi_{N+1}^N = \phi_N^N = \beta_N$.

The usefulness of Theorems 5.1 and 5.2 will now be demonstrated in the following theorems, showing under which (weak) conditions requirements like (5.1) are fulfilled.

THEOREM 5.3. Suppose the MVR is of type I.

(a) Let $M \geq 2$, and let there exist a positive sequence $\{\sigma_i\}$ such that for $i \geq M$

$$0 \leq \frac{\mu_i}{\lambda_i + \alpha_{i-2} - \alpha_i} \leq \frac{\sigma_i}{1 + \sigma_i} \quad (5.5)$$

and

$$|\alpha_i - \alpha_{i-1}| \geq \sigma_i |\alpha_{i-1} - \alpha_{i-2}|. \quad (5.6)$$

Then $\alpha_i \geq \Phi_i(\alpha_{i-2}) \geq \alpha_{i-1}$ for $i \geq M$ if $\{\alpha_i\}$ is monotonically $\frac{\text{in}}{\text{de}}$ creasing.

(b) Let $N \geq 0$, and let there exist a positive sequence $\{\tau_i\}$ such that for $i \geq N$

$$\frac{\lambda_i}{\mu_i + \beta_{i+1} - \beta_i} \geq \frac{1 + \tau_i}{\tau_i} \quad (5.7)$$

and

$$|\beta_{i+1} - \beta_i| \geq \tau_i |\beta_{i+2} - \beta_{i+1}|. \quad (5.8)$$

Then $\Phi_i(\beta_{i+1}) \geq \beta_{i+2}$ for $i \geq N$, if $\{\beta_i\}$ is monotonically $\frac{\text{in}}{\text{de}}$ creasing.

THEOREM 5.4. Suppose the MVR is of type II.

(a) Let $M \geq 1$, and let there exist a positive sequence $\{\sigma_i\}$ such that for $i \geq M$

$$0 \geq \frac{\mu_i}{\lambda_i + \alpha_{i-1} - \alpha_i} \geq -\sigma_i \quad (5.9)$$

and

$$|\alpha_{i+1} - \alpha_i| \geq \sigma_i |\alpha_i - \alpha_{i-1}|. \quad (5.10)$$

Then $\alpha_i \leq \Phi_i(\alpha_{i-1}) \leq \alpha_{i+1}$ for $i \geq M$, if $\{\alpha_i\}$ is monotonically $\frac{\text{in}}{\text{de}}$ creasing.

(b) Let $N \geq 2$, and let there exist a positive sequence $\{\tau_i\}$ such that for $i \geq N$

$$\frac{-\lambda_i}{\mu_i + \beta_{i-1} - \beta_i} \geq \tau_i \quad (5.11)$$

and

$$|\beta_i - \beta_{i-1}| \geq \frac{1}{\tau_i} |\beta_{i+1} - \beta_i|. \quad (5.12)$$

Then $\Phi_i(\beta_{i-1}) \geq \beta_{i+1}$ for $i \geq N$, if $\{\beta_i\}$ is monotonically $\frac{\text{in}}{\text{de}}$ creasing.

Proof of Theorem 5.3(a). Using Property 2.1 we can write

$$\Phi_i(x) = \alpha_i + \frac{\mu_i(x - \alpha_i)}{\lambda_i + x - \alpha_i}. \quad (5.13)$$

Suppose $\{\alpha_i\}$ is increasing. From (5.10) we obtain $(\alpha_i - \alpha_{i-1}) \geq \sigma_i(\alpha_{i-1} - \alpha_{i-2})$. Hence

$$(\alpha_{i-2} - \alpha_i) \geq \frac{1 + \sigma_i}{\sigma_i} (\alpha_{i-1} - \alpha_i).$$

Substitution of this inequality and of (5.5) in (5.13) for $x = \alpha_i$ yields $\Phi_i(\alpha_{i-2}) \geq \alpha_i + \alpha_{i-1} - \alpha_i = \alpha_{i-1}$. Since $\mu_i(\alpha_{i-2} - \alpha_i)/(\lambda_i + \alpha_{i-2} - \alpha_i) > 0$, we obtain $\Phi_i(\alpha_{i-2}) \leq \alpha_i$. ■

Proof of Theorem 5.4(a). Suppose $\{\alpha_i\}$ is increasing. From (5.10) we obtain $(\alpha_{i-1} - \alpha_i) \geq (1/\sigma_i)(\alpha_i - \alpha_{i+1})$. Substitution again in (5.13) (now for $x = \alpha_{i-1}$) yields $\Phi_i(\alpha_{i-1}) \leq \alpha_i - \alpha_i + \alpha_{i+1} = \alpha_{i+1}$. Since $\mu_i(\alpha_{i-1} - \alpha_i)/(\lambda_i + \alpha_{i-1} - \alpha_i) \geq 0$, we have $\Phi_i(\alpha_{i-1}) \geq \alpha_i$. ■

Proof of Theorem 5.3(b). Using Property 2.1 we can also write

$$\Phi_i(x) = \beta_i + \frac{\lambda_i(x - \beta_i)}{\mu_i + x - \beta_i}. \quad (5.14)$$

With (5.14) we can prove parts (b) of Theorems 5.3 and 5.4 in a similar way. ■

REMARK 5.3. From Theorems 5.3 and 5.4 one can easily see that if $|\mu_i/\lambda_i|$ is fairly small and/or $\{|\alpha_i - \alpha_{i+1}|\} [\{|\beta_i - \beta_{i+1}|\}]$ is strongly increasing [decreasing], Theorems 5.1 and 5.2 are always applicable.

REMARK 5.4. In a sense the conditions for type II, as formulated in Theorem 5.4, are less rigid than for type I in Theorem 5.3. Supposing, e.g., $\sigma_i \equiv \tau_i \equiv 1$ for some slowly varying MVR, one has to require $|\mu/\lambda| \lesssim \frac{1}{2}$ for type I and only $|\mu/\lambda| \lesssim 1$ for type II, the latter condition being almost trivial.

REMARK 5.5. For MVR's of type I, for which $|(d/dx)\Phi_i(x)| \approx 1$ for $x = \alpha_i$ and/or $x = \beta_i$, Theorem 5.1 may not be applicable. In these cases one can indicate conditions (analogues of Theorem 5.3) for which one can prove $\psi_i \in [\alpha_{i-2-p}; \alpha_{i-1-p}]$ and $\phi_i \in [\beta_{i-p}; \beta_{i+1-p}]$ with p fixed for a certain MVR; this is intuitively clear from a diagram. We shall not enter, however, into further details.

5.2 Refinements of the Estimates of Sec. 5.1

The estimates of Sec. 5.1 can even be sharpened if one has information about the "bulge" of the graph of Φ_i —More concretely, if $|(d/dx)\Phi_i(x)|$ is known to be fairly small [large] in the neighborhood of α_i [β_i] (see below).

THEOREM 5.5. Suppose the MVR is of type I and satisfies the requirements of Theorem 5.1.

- (a) If $\forall_{x \in [\alpha_{i-2}; \alpha_i]} (d/dx)\Phi_i(x) \leq p$, then $|\psi_{i+1} - \alpha_i| < p|\alpha_{i-2} - \alpha_i|$.
- (b) If $\forall_{x \in [\beta_i, \beta_{i+2}]} (d/dx)\Phi_i(x) \geq q$, then $|\phi_i - \beta_i| < (1/q)|\beta_{i+2} - \beta_i|$.

THEOREM 5.6. *Suppose the MVR is of type II and satisfies the requirements of Theorem 5.2.*

- (a) *If $\forall_{x \in [\alpha_{i-1}, \alpha_i]} |(d/dx)\Phi_i(x)| \leq p$, then $|\psi_{i+1} - \alpha_i| < p|\alpha_{i-1} - \alpha_i|$.*
 (b) *If $\forall_{x \in [\beta_{i+1}, \beta_i]} |(d/dx)\Phi_i(x)| \geq q$, then $|\phi_i - \beta_i| < (1/q)|\beta_{i+1} - \beta_i|$.*

Proof of Theorem 5.5(a). $|\psi_{i+1} - \alpha_i| = |\Phi_i(\psi_i) - \Phi_i(\alpha_i)| \leq p|\psi_i - \alpha_i| < p|\alpha_{i-2} - \alpha_i|$ (cf. Fig. 3). ■

REMARK 5.6. If p is rather small and q rather large, we expect that Theorem 5.5 will give an overestimation by about a factor $|\alpha_{i-2} - \alpha_i|/|\alpha_{i-1} - \alpha_i|$ for (a) and $|\beta_{i+2} - \beta_i|/|\beta_{i+1} - \beta_i|$ for (b), since $\psi_i \approx \alpha_{i-1}$ and $\phi_i \approx \beta_i$, respectively—i.e., by about a factor 2 if α_i and β_i move slowly to $\lim \alpha_i$ and $\lim \beta_i$, respectively.

5.3. More General Estimates

The theorems of this subsection can be regarded as a kind of generalization of Theorems 5.1 and 5.2. Their most important feature is that they no longer require monotonicity.

Define:

$$k_i = |\alpha_i - \alpha_{i-1}|, \quad K_i = [\alpha_{i-1} - k_i, \alpha_{i-1} + k_i], \quad (5.15)$$

$$l_i = |\beta_i - \beta_{i-1}|, \quad L_i = [\beta_i - l_i, \beta_i + l_i]. \quad (5.16)$$

THEOREM 5.7. *Let $M \geq 1$. If there exists a sequence of positive numbers $\{\sigma_i\}$ such that for $i \geq M$,*

$$\sigma_i k_i < k_{i+1}, \quad \forall_{x \in K_i} \left| \frac{d}{dx} \Phi_i(x) \right| < \frac{\sigma_i}{2} \quad \text{and} \quad \psi_M \in K_M, \quad (5.17)$$

then $\psi_i \in K_i$ for $i \geq M$.

THEOREM 5.8. *Let $N \geq 2$, $J > N$. If there exists a sequence of positive numbers $\{\tau_i\}$ such that for $J-1 \geq i \geq N$,*

$$l_i > \tau_i l_{i+1}, \quad \forall_{x \in L_i} \left| \frac{d}{dx} \Phi_i(x) \right| > \frac{2}{\tau_i} \quad \text{and} \quad \phi_J \in L_J, \quad (5.18)$$

then $\phi_i \in L_i$ for $J \geq i \geq N$.

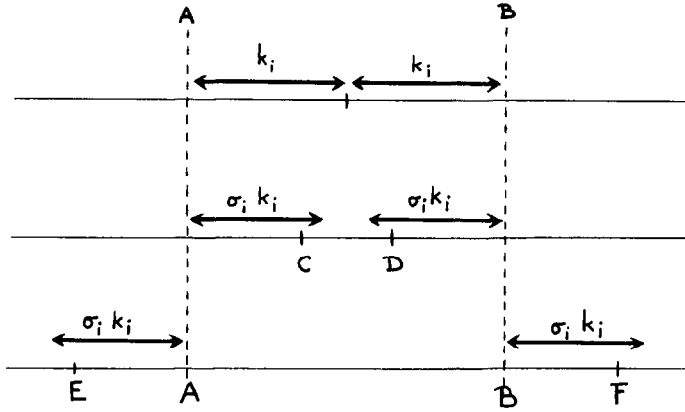


FIG. 6.

Proof of Theorem 5.7. In Fig. 6, A and B are the possible positions of α_i with respect to α_{i-1} . Suppose the MVR is of type I and $\alpha_i = A$, and $\Phi_i(K_i) = [A, C]$. Since $|C - A| < \sigma_i k_i < k_{i+1}$, we obtain $[A, C] \subset K_{i+1}$. If $\alpha_i = B$, we obtain $\Phi_i(K_i) = [D, B] \subset K_{i+1}$. For type II we have, similarly, $\Phi_i(K_i) = [E, A] \subset K_{i+1}$ and $\Phi_i(K_i) = [B, F] \subset K_{i+1}$, respectively. ■

The proof of Theorem 5.8 is essentially the same.

REMARK 5.7. Since the theorems of Secs. 5.1, 5.2 and 5.3 are much simpler in the case of an SR (see Sec. 6), we shall only give examples of SR's to demonstrate them (see Sec. 8).

5.4. Existence of Dominated Solutions

As in Section 4.1, we can derive sufficient conditions for the MVR to possess a dominated solution with direction ω . They are that all requirements either of Theorem 5.1(b) or of Theorem 5.2(b) or of Theorem 5.8 have to be fulfilled for arbitrary J and fixed N . We are able then to prove that $|\phi_N^n|_{n>N}$ is bounded, which is sufficient for the existence of a dominated solution (see Sec. 4.2).

6. THE SCALAR RECURSION (SR)

The theory developed in the previous sections simplifies somewhat when applied to SR's. In this case $\forall_{i>0} b_i = 0$, i.e. $\forall_{i>0} A_i$ is a companion matrix.

We obtain (see Sec. 2.2 and (1.3))

$$x_i^2 = x_{i+1}^1 = x_{i+1}, \quad (6.1)$$

$$\rho_{i+1} = \rho_{i+1}^1 = \delta_{i+1} = \frac{d_i}{\rho_i^1} + e_i = \frac{d_i}{\rho_i} + e_i. \quad (6.2)$$

The estimates for δ therefore apply to ρ .

Another simplifying fact is that $\lambda_i = \alpha_i$ and $\mu_i = \beta_i$ for an SR. Since $d_i = -\alpha_i \beta_i$ and $e_i = \alpha_i + \beta_i$ (see (2.9)), we can easily verify the following properties. (N. B. The assumptions of (2.11) still hold.)

PROPERTY 6.1. *An SR is of type I if and only if $\forall_{i \geq 0} e_i > 0$ and $\forall_{i \geq 0} d_i > 0$.*

PROPERTY 6.2. *An SR is of type II if and only if $\forall_{i \geq 0} e_i > 0$ and $\forall_{i \geq 0} d_i > 0$.*

REMARK 6.1. The condition $b_{i+1} + e_i > 0$ for type II in Sec. 4 is always satisfied for SR's of type II, since $b_{i+1} = 0$ and $e_i > 0$.

The conditions of Theorems 5.3 and 5.4 can be rewritten in simpler forms:

PROPERTY 6.3. *In case of an SR the conditions of Theorems 5.3 and 5.4 coalesce as follows:*

$$(5.5) \wedge (5.6) \rightarrow \frac{\beta_i}{\alpha_{i-2} - \beta_i} \leq \frac{|\alpha_i - \alpha_{i-1}|}{|\alpha_{i-1} - \alpha_{i-2}|},$$

$$(5.7) \wedge (5.8) \rightarrow \frac{\beta_{i+1}}{\alpha_i - \beta_{i+1}} \leq \frac{|\beta_{i+1} - \beta_i|}{|\beta_{i+2} - \beta_{i+1}|},$$

$$(5.9) \wedge (5.10) \rightarrow \frac{-\beta_i}{\alpha_{i-1}} \leq \frac{|\alpha_{i+1} - \alpha_i|}{|\alpha_i - \alpha_{i-1}|},$$

$$(5.11) \wedge (5.12) \rightarrow \frac{-\beta_{i-1}}{\alpha_i} \leq \frac{|\beta_i - \beta_{i-1}|}{|\beta_{i+1} - \beta_i|}.$$

We note that in the case of slowly varying $\{\alpha_i\}$ and $\{\beta_i\}$ the first two requirements are only slightly stronger than $\beta_i < \frac{1}{2}\alpha_i$, whereas the last two requirements are only slightly stronger than $|\beta_i| < \alpha_i$. Hence these requirements will often be satisfied.

7. THE CO-DIAGRAM

In the preceding sections we sometimes assumed *a priori* knowledge about the monotonicity of $\{\alpha_i\}$ and $\{\beta_i\}$. This is not always easy to establish. The diagram described below could be a helpful means to investigate this and also to obtain a rough idea of the magnitudes of the successive α_i and β_i .

Define the abscissa as p -axis and the ordinate as q -axis. Consider the parabola

$$P = \{p, q \mid q^2 + 4p = 0\}. \quad (7.1)$$

From every $[p^*, q^*] = [d_i, e_i - b_i]$, which is outside the parabola, one can draw two tangents to P (cf. Fig. 7).

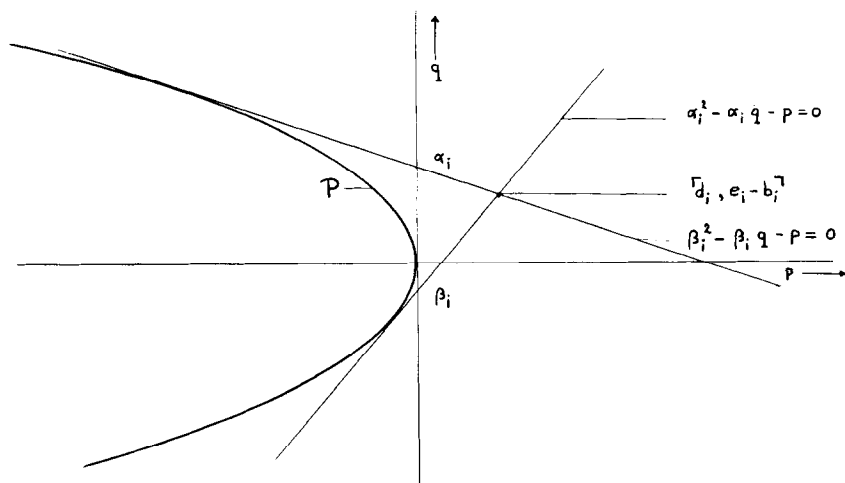


FIG. 7.

PROPERTY 7.1. These tangents intersect the q -axis at the roots α_i and β_i of the equation $x^2 - (e_i - b_i)x - d_i = 0$.

It may now be easy to establish the monotonicity of $\{\alpha_i\}$ and $\{\beta_i\}$ on sight.

REMARK 7.1. Note that if Φ_i is of type I, $[d_i, e_i - b_i]$ will be in the left half plane, and if Φ_i is of type II, $[d_i, e_i - b_i]$ will be in the right half plane.

The diagram described in this section will be referred to as a co-diagram.

8. NUMERICAL EXAMPLES

We shall restrict ourselves to two examples of SR's, derived from [2, pp. 46 and 72 respectively]. The first one is the well known SR of the Bessel functions of integer order. The second one has the property that the characteristic directions α_i and β_i are asymptotically equal, a case in which the theorems of Perron et al. [2, p. 34] cannot be applied, but where nevertheless a dominant and dominated solution can be indicated.

EXAMPLE 8.1. The Bessel functions $Y_i(x)$ and $J_i(x)$ obey the SR

$$u_{i+2}(x) = \frac{2}{x}(i+1)u_{i+1}(x) - u_i(x), \quad u = Y, J. \quad (8.1)$$

Suppose $x \in (0, 1]$; then α_i, β_i exist for all $i \geq 0$. We find

$$\alpha_i(x) = \frac{2(i+1)}{x} - \frac{1}{2} \frac{x}{i+1} + O\left(\frac{x^3}{(i+1)^3}\right) \quad (8.2)$$

$$\beta_i(x) = \frac{1}{2} \frac{x}{i+1} + O\left(\frac{x^3}{(i+1)^3}\right)$$

From the co-diagram (see Fig. 8), we see that (8.1) is of type I and that for fixed x , $\alpha_i(x)$ is increasing, while $\beta_i(x)$ decreases ($i \rightarrow \infty$). We put

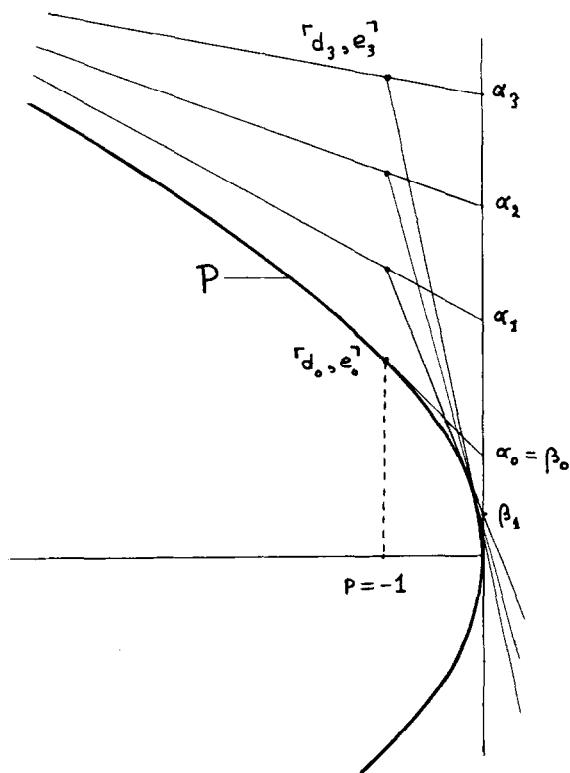
$$\frac{Y_{i+1}(x)}{Y_i(x)} = \psi_i(x) \quad \text{and} \quad \frac{J_{i+1}(x)}{J_i(x)} = \omega_i(x). \quad (8.3)$$

(a) $x=0.4$. A short examination (e.g., with the aid of a diagram) shows that both $\psi_0(0.4)$ and $\omega_1(0.4)$ are smaller than $\alpha_0(0.4)$, while $\alpha_0(0.4) \leq \psi_2(0.4) \leq \alpha_1(0.4)$. It can easily be verified that Theorem 5.3 can be applied with $M=2$, $N=0$, $\sigma_i \equiv \tau_i \equiv 1$; hence we obtain from Theorem 5.3

$$\alpha_{i-2}(0.4) \leq \psi_i(0.4) \leq \alpha_{i-1}(0.4), \quad i \geq 2$$

and

$$\beta_{i+1}(0.4) \leq \omega_i(0.4) \leq \beta_i(0.4), \quad i \geq 0.$$

FIG. 8. $x = 1$.

The result is shown in Table 1 ($\psi(0.4)$ and $\omega(0.4)$ are compiled from [1]).

TABLE 1

i	$\psi(0.4)$	$\alpha(0.4)$	$\omega(0.4)$	$\beta(0.4)$
0	2.939	4.791	0.20411	0.20871
1	4.660	9.899	0.10067	0.10102
2	9.785	14.934	0.06689	0.06696
3	14.897	19.950	0.05010	0.05013
4	19.934	24.960	0.04005	0.04006

(b) $x=1$. We now have $\psi_0(1) < 0 < \alpha_0(1)$. In a diagram like Fig. 1 we find $\alpha_0(1) < 2 < \psi_1(1) < \alpha_1(1)$; hence $\alpha_0(1) \leq \psi_2(1) \leq \alpha_1(1)$. We obtain

$$\alpha_{i-2}(1) \leq \psi_i(1) \leq \alpha_{i-1}(1), \quad i \geq 2.$$

(See Table 2.)

TABLE 2

i	$\psi(1)$	$\alpha(1)$
0	-8.852	1.000
1	+2.113	3.732
2	3.527	5.828
3	5.716	7.873
4	7.825	9.899

(c) Using Theorem 5.5 we can give even sharper estimates. We obtain

$$\forall_{t \in [\alpha_{i-2}, \alpha_i]} \quad \frac{d}{dt} \Phi_i(t) \lesssim \frac{x^2}{4(i^2 - 1)},$$

while

$$|\alpha_{i-2} - \alpha_i| \approx \frac{4}{x}.$$

Hence

$$0 \leq \alpha_{i-1}(x) - \psi_i(x) \leq \frac{x}{i^2} + O\left(\frac{1}{i^3}\right).$$

Similarly,

$$\forall_{x \in [\beta_i, \beta_{i+1}]} \quad \frac{d}{dx} \Phi_i(x) \gtrsim \frac{4}{x^2} (i+1)^2,$$

while

$$|\beta_{i+2} - \beta_i| \approx \frac{x}{(i+3)(i+1)},$$

giving

$$0 \leq \beta_i(x) - \omega_i(x) \leq \frac{x^3}{4} \frac{1}{(i+1)^3(i+3)} + O\left(\frac{1}{i^5}\right).$$

These results (which are very sharp indeed) agree nicely with Tables 1 and 2. (d) for n large we find

$$\bar{Y}_n(x) = \prod_{i=1}^{n-2} \alpha_i(x) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ex}{2n}\right)^{-n} \quad \text{and}$$

$$\bar{J}_n(x) = \prod_{i=1}^{n-1} \beta_i(x) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ex}{2n}\right)^n.$$

$\bar{Y}_n(x)$ only differs by a factor 2 from $Y_n(x)$ asymptotically, while the expression for $\bar{J}_n(x)$ is asymptotically equal to $J_n(x)$ (cf. [2, p. 46]).

EXAMPLE 8.2. Repeated integrals of the error function:

$$I_i(x) = \int_x^\infty I_{i-1}(t) dt, \quad I_0 = \operatorname{erf}(x).$$

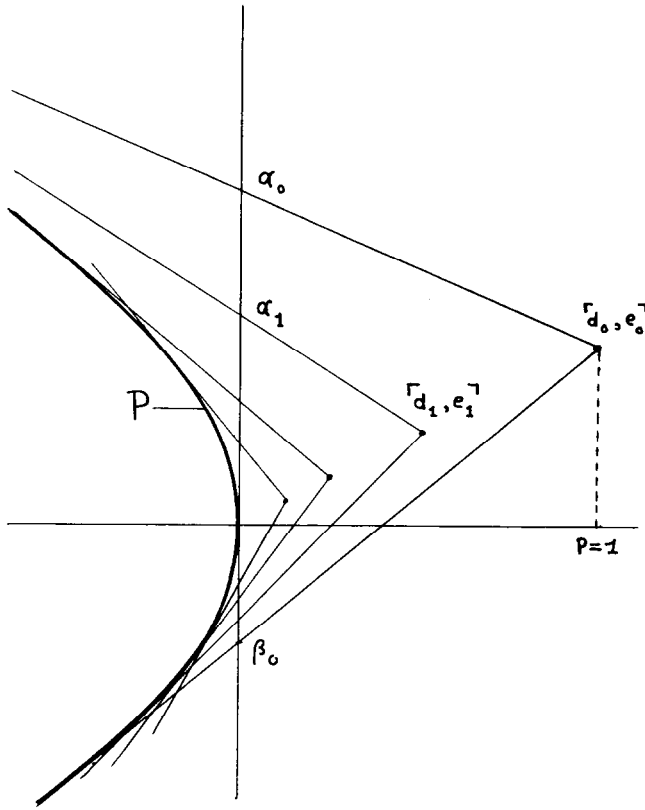
$$I_{i+2}(x) = -\frac{x}{i+2} I_{i+1}(x) + \frac{1}{2(i+1)} I_i(x). \quad (8.4)$$

Suppose $0 < x < 2$. Substitute $u_i(x) = (-1)^i I_i(x)$. We then obtain

$$u_{i+2}(x) = \frac{x}{i+2} u_{i+1}(x) + \frac{1}{2(i+2)} u_i(x). \quad (8.5)$$

This SR is of type II. The co-diagram (see Fig. 9 for $x=1$) shows that both $\alpha_i(x)$ and $\beta_i(x) \rightarrow 0$ ($i \rightarrow \infty$), while

$$\lim_{i \rightarrow \infty} \frac{\alpha_i(x)}{\beta_i(x)} = 1.$$


 FIG. 9. $x=1$.

For $\beta_i(x)$ we obtain

$$\beta_i(x) = \frac{-\sqrt{2}}{2\sqrt{i+2}} \left\{ 1 - \frac{x}{\sqrt{2i+4}} + \frac{x^2}{4i+8} + O\left(\frac{x^4}{(i+2)^2}\right) \right\}. \quad (8.6)$$

Define

$$-\omega_i(x) = \frac{I_{i+1}(x)}{I_i(x)}. \quad (8.7)$$

(a) $x=1$. Investigation reveals that Theorem 5.4 can be applied with $N=2$, $\tau_i \equiv 1$; hence we obtain from Theorem 5.2

$$\beta_{i-1}(1) \leq \omega_i(1) \leq \beta_i(1) \quad (i \geq 1).$$

(See Table 3; ω_i is compiled from [1].)

TABLE 3

i	$-\omega(1)$	$\beta(1)$
0	-0.319	-0.309
1	-0.283	-0.274
2	-0.257	-0.250
3	-0.237	-0.232
4	-0.222	-0.217

(b) For n large we find the asymptotic result

$$\underline{I}_n(x) = \prod_{i=1}^{n-2} -\beta_i(x) \sim \sqrt[4]{\frac{n}{2}\pi} \frac{1}{2^n \Gamma(n/2 + 1)} \exp(-\sqrt{2n} x)$$

$\underline{I}_n(x)$ asymptotically differs from $I_n(x)$ by a factor $\sqrt[4]{(n/2)\pi} e^{-\frac{1}{2}x^2}$ (cf. [2, p. 73]). This result is less sharp than in Example 8.1, due to the fact that the $\beta_i(x)$ there approximated the growth factors far better.

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