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Building K-Theories

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INTRODUCTION

This paper is a much expanded account of a talk we gave at the Conference on Algebraic K-Theory held at the Battelle Institute, Seattle, in the late summer of 1972, and a preliminary version was to have appeared in the Proceedings of that Conference. Unfortunately we discovered an error at the last moment and decided to withdraw the article, now publishing a hopefully correct account together with results obtained subsequently.

For a cotriple E from rings to rings and a functor F from rings to groups, there are at least two ways to set up a theory of derived functors of F with respect to E. The first, and by now rather current method, is to take the standard simplicial cotriple resolution of a ring R, apply the functor F and define the homotopy groups of the resulting augmented simplicial group as the derived functors of F evaluated at R. These homotopy groups can be calculated as the homology groups of the associated Moore complex. For F is the general linear group GL, this is the point of view adopted by Gersten in [4] and [5] and we briefly recall this theory in Section 7.

In this paper we try out a different approach and take the canonical resolution of the cotriple due to Eilenberg and Moore [2], apply F to

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this complex of rings and take homology of the resulting complex of groups. For $E = E^{\mathrm{I}}$, the free cotriple, this amounts to taking a canonical free resolution of R and indeed it is rather easy to show that some of the procedures of abelian homological algebra carry through, provided we impose restrictions on F, satisfied by the general linear group. We consider three different cotriples and for the functor $F = GL$ call the resulting K-theories K^* in order to distinguish them from the ones mentioned above, Sections 1–6.

We do not investigate further the intermediate theory II, but show that in the "polynomial" theory $K^{\mathrm{III}*} = K^{\mathrm{III}}$, Section 8. This fact that Gersten's theory [5] yields the Karoubi–Villamayor theory of [11] was of course known [6], but out proof brings out that this only depends on the cotriple E^{III} being a left exact functor. For the free cotriple E^{I}, which is not left exact, we go on to show in Section 9 that $K_n^{\mathrm{I}*} \neq K_n{}^{\mathrm{I}}$ for $n \geqslant 2$, which can be viewed as "failure of excision."

Having established that they differ in general, we wish to compare the two theories more closely. After some more preliminaries on the Moore complex in Section 10, we introduce in Section 11 a double complex of groups. This gives rise in the next section to an exact sequence which describes the behavior of the K-functors on the rings $\Omega^n R$ used to define K^*-theory. Here we use spectral sequence techniques developed in the preceding paper [19], bearing out our contention that rather more of homological algebra can be carried out for arbitrary groups than is commonly realized. Specifically, the E_2 terms of the spectral sequence measure the deviation of K^*-theory from K-theory.

Section 13 discusses the connection between $K_2{}^{\mathrm{I}}$ and Milnor's $K_2{}^M$, which are now known to be equal. In Section 14 we review the notion of polynomial homotopy due to [11] and abstract this to each of our cotriples. In this rarefied algebraic setting, we mimick procedures from homotopy theory and generalize in Section 15 ideas of [11] and [10], attaching to a functor from rings to groups a covering functor with respect to a given cotriple. In particular, we exhibit $K_2{}^M$ as the fundamental group of GL with respect to the free cotriple E^{I}.

We conclude this introduction by mentioning a number of reasons why the K^{I}-theory of Swan and Gersten should be preferred to our tentative $K^{\mathrm{I}*}$-theory. The latter was, we have been informed, considered by several people previously who did not reach any definite conclusion.

First, $K_n{}^{\mathrm{I}}$ vanishes on free rings, which makes K^{I}-theory a homotopy theory with respect to E^{I}, Section 14. Then free simplicial resolutions are chain homotopic [9], which allows one to use arbitrary free simplicial resolutions in K^{I}-theory. Patently, this is not true of "ordinary" free resolutions in the category of rings, since $K^{\mathrm{I}*}$-theory does not vanish on free rings, Section 9. Further, $K_2{}^{\mathrm{I}}$ equals Milnor's K_2 which is in many ways the "right" K_2, again illustrated by our result $K_2{}^M = \pi_1{}^{\mathrm{I}}GL$ mentioned above. Finally and perhaps most importantly, K^{I}-theory coincides with Quillen's theory of higher K-functors [7], and his is the theory which has yielded by far the most interesting results so far.

1. Free Resolutions for Rings

Let **Rg** be the category of rings (without necessarily an identity element) and $\mathbf{S}_*$ the category of sets with base point. The forgetful functor $V\colon \mathbf{Rg} \to \mathbf{S}_*$ has a left adjoint which can be described as follows. Let $S \in \mathbf{S}_*$, and take for each $s \in S$, $s \neq 0$, a polynomial variable X_s. Then LS is the ring of polynomials in the noncommuting variables X_s with integral coefficients but without constant terms. In other words, LS is the augmentation ideal of the "free ring with unit" on S with respect to $\mathbf{Z}$. Then $E = LS$ is an endofunctor on **Rg** which associates to every ring R the free ring ER on R. Clearly, sending X_r to r for $r \in R$ induces a surjection $\epsilon(R)\colon ER \to R$ which yields a morphism of functors $\epsilon\colon E \to I_{\mathbf{Rg}}$. Define $\Omega = \operatorname{Ker} \epsilon$.

For a given ring R, we now naively build a free resolution in the following way. Map ER onto R by $\epsilon(R)$; the kernel is ΩR onto which we map the free ring $E\Omega R$ by $\epsilon(\Omega R)$. Continuing like this, we obtain the free resolution

$$\begin{tikzcd}[column sep=small]
 & & & \Omega^n R \arrow[dr] & & & \\
\mathscr{E}R\colon & \cdots \arrow[r] & E\Omega^n R \arrow[ur, dash] \arrow[rr] & & E\Omega^{n-1}R \arrow[r] & \cdots &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & \Omega^2 R \arrow[dr] & & \Omega R \arrow[dr] & & & \\
\cdots \arrow[r] & E\Omega^2 R \arrow[ur] \arrow[rr] & & E\Omega R \arrow[ur] \arrow[rr] & & ER \arrow[r, two heads] & R \arrow[r] & 0.
\end{tikzcd}$$

Recall first how functors like GL, the general linear group, and EL, the elementary group, are extended to rings R without identity [15, Section 8]. In particular $EL(R)$ is generated by all conjugates of elements $I + re_{ij}$, $i \neq j$ natural numbers, $r \in R$, under the action of elements $u \in EL(\mathbf{Z})$. This action is given by considering R and $\mathbf{Z}$ as an ideal, respectively, subring of the unital ring R^+ under the embeddings $r \mapsto (0, r)$, respectively, $m \mapsto (m, 0)$. Here e_{ij}, as usual, is the denumerably infinite square matrix in which the only nonnull entry is 1 at the intersection of the ith row and the jth column. For future convenience, note that, by utilizing the well-known relations between elementary matrices, we need only choose the elements r from a set of generators of the ring R. Conjugating with an invertible $\mathbf{Z}$-matrix of determinant -1 does not take us out of the group just described, as is easily verified. It follows that we may equally well conjugate with $GL(\mathbf{Z})$, cf. [17, p. 122].

For unital rings the Whitehead Lemma asserts that $EL(R)$ is precisely the commutator subgroup of $GL(R)$. For rings without identity, $EL(R)$ by the above description turns out to be the normal subgroup of $EL(R^+)$ generated by all elementary matrices with entries in R. In the notation of Bass this is called $EL(R^+, R)$ and the relative Whitehead identity [1, Section V.2] shows it to contain the commutator subgroup of $GL(R^+, R) = GL(R)$. The (abelian) factor group $K_1(R)$ is then Bass' relative group $K_1(R^+, R)$.

This being said, apply the functor GL to the resolution $\mathscr{E}R$. There results a complex of groups

$$GL(\mathscr{E}R): \cdots$$

$$\begin{array}{ccccccccccccc}
\longrightarrow G_{n+1} & \xrightarrow{d_{n+1}} & G_n & \longrightarrow \cdots \longrightarrow & G_3 & \xrightarrow{d_3} & G_2 & \xrightarrow{d_2} & G_1 & \xrightarrow{d_1} & G_0 & \to & 1 \\
\| & & \| & & \| & & \| & & \| & & \| & & \\
\to GL(E\Omega^n R) & \to & GL(E\Omega^{n-1}R) & \to \cdots \to & GL(E\Omega^2 R) & \to & GL(E\Omega R) & \to & GL(ER) & \to & GL(R) & \to & 1
\end{array}$$

in which the differentials d_n are just $GL(\epsilon(\Omega^{n-1}R))$, $n \geqslant 1$. This is an honest "complex" of non-abelian groups in the sense that $\operatorname{Im} d_n$ is a normal subgroup of $\operatorname{Ker} d_{n-1}$; this follows from Gersten's basic observation that GL of a free ring is generated by elementary matrices [3] under conjugation by $GL(\mathbf{Z})$, hence $G_n = EL(E\Omega^{n-1}R)$ so $\operatorname{Im} d_{n-1} = EL(\Omega^{n-1}R) \lhd GL(\Omega^{n-1}R)$; but the latter group equals $\operatorname{Ker} d_{n-1}$ because the general linear group is a left exact functor. We can therefore take homology and define

$$K_n{}^*(R) = H_{n-1}(GL(\mathscr{E}R)) = K_1(\Omega^{n-1}R), \qquad n \geqslant 1.$$

2. The Canonical Resolution of an Endofunctor

Suppose the function E: $\mathbf{Rg} \to \mathbf{Rg}$ comes equipped with a morphism $\epsilon: E \to I_{\mathbf{Rg}}$ which is surjective on all rings. Put $\Omega = \operatorname{Ker} \epsilon$. Then the complex $\mathscr{E}R$ is again exact and we can conceive of it as a resolution of R in terms of rings EA, $A \in \mathbf{Rg}$. The retracts of such EA's will be the "projectives" or the "contractible rings" of our theory. In fact, $\mathscr{E}$ is the canonical resolution of $I_{\mathbf{Rg}}$ with respect to E as introduced by Eilenberg and Moore [2, Section 4]. (The connection with their work was pointed out to one of the authors by M. Tierney).

Let F: $\mathbf{Rg} \to \mathbf{Gr}$ be a left exact functor to groups with the property that for all rings R we have $\operatorname{Im} F(\epsilon(R)) \lhd F(R)$ with abelian factor group. Though, since we are doing algebraic K-theory, our main concern will be with the general linear group GL, we formulate the theory for such suitable functors F. Then $F(\mathscr{E}R)$ is a complex of groups and we can take homology. A ring homomorphism $f: B \to C$ is called an F-fibration when $F(E\Omega^n f)$ is surjective for all $n \geqslant 0$. Thus a fibration yields a morphism $F(\mathscr{E}f): F(\mathscr{E}B) \to F(\mathscr{E}C)$ between group complexes which is surjective except perhaps at the zeroth level. Write $\mathscr{E}_f = \operatorname{Ker} \mathscr{E}f$ and $F(\mathscr{E})_f = \operatorname{Ker} F(\mathscr{E}f)$. We must ascertain that $(F\mathscr{E})_f$ is a complex of groups.

The cycles of $\mathscr{E}R$ at level n are $\Omega^n R$; write S_n for the chains and T_n for the cycles of the kernel complex $\mathscr{E}_f$. Because F is left exact the ladder

$$\begin{array}{ccccccccc}
1 & \to & F(S_{n+1}) & \to & F(E\Omega^n B) & \to & F(E\Omega^n C) & \to & 1 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
1 & \longrightarrow & F(T_n) & \longrightarrow & F(\Omega^n B) & \longrightarrow & F(\Omega^n C) & &
\end{array}$$

is exact. We must verify that the image of the left-hand column is a normal subgroup of $F(T_n)$. By functoriality $F(ET_n)$ maps into $F(E\Omega^n B)$ and its image in $F(E\Omega^n C)$ is 1. Therefore $F(ET_n)$ maps into the kernel $F(S_{n+1})$ and the composition with d is just the homomorphism $F(\epsilon(T_n))$: $F(ET_n) \to F(T_n)$. In the chain of subgroups

$$d(F(ET_n)) < d(F(S_{n+1})) < F(T_n)$$

the smallest, by assumption, contains the commutator subgroup of $F(T_n)$. Thus $d(F(S_{n+1})) \lhd F(T_n)$—even with abelian factor group—and $(F\mathscr{E})_f$ is a complex of groups.

From the short exact sequence of group complexes

$$1 \to (F\mathscr{E})_f \to F(\mathscr{E}B) \to F(\mathscr{E}C) \to 1,$$

we obtain a long exact homology sequence

$$\cdots \to H_n((F\mathscr{E})_f) \to H_n(F(\mathscr{E}B)) \to H_n(F(\mathscr{E}C)) \to H_{n-1}((F\mathscr{E})_f)$$
$$\cdots \to H_1(F(\mathscr{E}C)) \to H_0((F\mathscr{E})_f) \to H_0(F(\mathscr{E}B)) \to H_0(F(\mathscr{E}C)).$$

There does not appear to exist a standard reference in the literature for homology in the category of groups rather than abelian groups. The diagram lemma's are treated abstractly in [20] and [21]. Alternatively, one may convince oneself that the usual elementwise proofs of the abelian case go through provided one takes good care that "complex" always means that $\operatorname{Im} d_n \lhd \operatorname{Ker} d_{n-1}$. Thus we have attached to every F-fibration f a long exact sequence of homology groups connected by the relative terms $H_n((F\mathscr{E})_f)$. Clearly $H_n(F(\mathscr{E}R)) = H_0(F(\mathscr{E}\Omega^{n-1}R)$ and $H_n((F\mathscr{E})_f) = H_0((F\mathscr{E})_{\Omega^{n-1}f})$. By our assumption on F all these groups are abelian and in case f is split surjective the long sequence splits into short ones.

3. Setting up K-Theories

We shall now confine ourselves to the functor $F = GL$, but study concurrently three endofunctors E: $\mathbf{Rg} \to \mathbf{Rg}$.

I. $E = LS$ if the free ring functor of Section 1.

II. Let V' be the forgetful functor from rings to the underlying abelian groups and L' its left adjoint. Thus for an abelian group A, the ring $L'A$ is the polynomial ring over $\mathbf{Z}$ in noncommuting variables x_a, $a \in A$, $a \neq 0$, subject to the relations $x_a + x_b = x_{a+b}$ but without constant terms. Another description is as the positively graded part of $T_{\mathbf{Z}}(A)$, the tensor algebra of the $\mathbf{Z}$-module A. Put $E = L'V'$; for any ring R, this ER is just the free ring on R modulo the ideal generated by all elements $X_a + X_b - X_{a+b}$. Define $\epsilon(R)$: $ER \to R$ by $\epsilon(x_r) = r$.

III. Let $R[t]$ be the polynomial ring over R in one variable t and write ER for the principal ideal $tR[t]$. Define $\epsilon(R)$ by $\epsilon(\Sigma r_i t^i) = \Sigma r_i$.

We can also consider variants of these if we restrict attention to commutative rings. Then E is an endofunctor on commutative rings defined respectively by

I^c. ER is the polynomial ring over $\mathbf{Z}$ without constants in the variables X_r, one for every non zero $r \in R$.

II^c. ER is the above modulo the ideal generated by all polynomials $X_r + X_s - X_{r+s}$ (i.e., the augmentation ideal of the symmetric algebra $S_{\mathbf{Z}}(R)$).

III^c. As III, but we only consider commutative rings R.

4. Identifying the Fibrations

Clearly, this is our first task. We need the following:

Lemma 4.1. *Let E be an endofunctor on* **Rg** *such that $\epsilon: E \to I$ is surjective on all rings. If E preserves surjections, so does every monomial in E and Ω.*

Proof. It manifestly suffices to prove that if $f: B \to C$ is a surjection of rings, so is Ωf. Write $A = \operatorname{Ker} f$ and chase the diagram

$$\begin{tikzcd}
 & \Omega B \arrow[r, "\Omega f"] \arrow[d, tail] & \Omega C \arrow[d, tail] \\
EA \arrow[r] \arrow[d, two heads] & EB \arrow[r, two heads, "Ef"] \arrow[d, two heads] & EC \arrow[d, two heads] \\
A \arrow[r, tail] & B \arrow[r, two heads, "f"] & C
\end{tikzcd}$$

in which the bottom row and both columns are exact.

From this we immediately get that in case I every surjection f is a GL-fibration. Indeed, all $E\Omega^n f: E\Omega^n B \to E\Omega^n C$, $n \geqslant 0$, are surjective and, since the targets are free rings, split. By Gersten's result the general linear group GL on these free rings is just the elementary group EL and so the $GL(E\Omega^n f)$: $EL(E\Omega^n B) \to EL(E\Omega^n C)$ are all surjective.

Wherever necessary we distinguish the theories by attaching superscripts to the functors, thus E^{I} is the free ring functor, etc. In order to apply the theory of Section 2 to GL, we must verify that $\operatorname{Im} GL(\epsilon(R)) \lhd GL(R)$ with abelian factor group for every ring R. For case I we have seen in Section 1 that $\operatorname{Im} GL(\epsilon(R)) = EL(R)$. For every ring R, there are surjections

$$\begin{tikzcd}
E^{\mathrm{I}}R \arrow[r, two heads] \arrow[dr, two heads, "\epsilon^{\mathrm{I}}(R)"'] & E^{\mathrm{II}}R \arrow[r, two heads] \arrow[d, two heads, "\epsilon^{\mathrm{II}}(R)"'] & E^{\mathrm{III}}R \arrow[dl, two heads, "\epsilon^{\mathrm{III}}(R)"] \\
 & R &
\end{tikzcd}$$

given by $X_r \mapsto x_r$ and $x_r \mapsto rt$, respectively, which make the above diagram commute. This implies that (signs $<$, $\lhd$, $\subset$ are not meant to exclude $=$)

$$\operatorname{Im} GL(\epsilon^{\mathrm{I}}(R)) < \operatorname{Im} GL(\epsilon^{\mathrm{II}}(R)) < \operatorname{Im} GL(\epsilon^{\mathrm{III}}(R)).$$

Since the smallest of these groups $EL(R)$ always contains the commutator subgroup of $GL(R)$, all three of the subgroups are normal in $GL(R)$ with abelian factor groups. We can therefore take homology and define $K_n{}^*(R) = H_{n-1}(GL(\mathscr{E}R)) = K_1{}^*(\Omega^{n-1}R)$ for every ring R and $K_{nf}^* = H_{n-1}((GL\mathscr{E})_f) = K_1{}^*(\Omega^{n-1}f)$ for every fibration f, $n \geqslant 1$. In all theories these groups are abelian.

More generally, the maps $E^{\mathrm{I}} \twoheadrightarrow E^{\mathrm{II}} \twoheadrightarrow E^{\mathrm{III}}$ imply maps $\Omega^{\mathrm{I}} \twoheadrightarrow \Omega^{\mathrm{II}} \twoheadrightarrow \Omega^{\mathrm{III}}$ hence induce homomorphisms $K_n^{\mathrm{I}*} \to K_n^{\mathrm{II}*} \to K_n^{\mathrm{III}*}$ which, whenever $f\colon B \twoheadrightarrow C$ is a fibration common to all three theories, yield maps $K^{\mathrm{I}*} \to K^{\mathrm{II}*} \to K^{\mathrm{III}*}$ between connected sequences of functors $\mathbf{Rg} \to \mathbf{Ab}$ which are surjective at the K_1-level.

Similar considerations apply to the commutative theories, and if we restrict our attention to commutative rings, a common fibration evidently gives rise to a commuting diagram

$$\begin{tikzcd} K^{\mathrm{I}*} \arrow[r] \arrow[d] & K^{\mathrm{II}*} \arrow[r] \arrow[d] & K^{\mathrm{III}*} \arrow[d, shift right] \arrow[d, phantom, shift left, "\wr"] \\ K^{\mathrm{I}^c*} \arrow[r] & K^{\mathrm{II}^c*} \arrow[r] & K^{\mathrm{III}^c*}. \end{tikzcd}$$

In case III, it is known that $K_1{}^*(R) = GL(R)/UN(R)$ where $UN(R)$ is the subgroup of $GL(R)$ generated by unipotent matrices $I + N$ with N a nilpotent. This description holds for rings with identity; without, one extends the definition of the functor UN in the usual manner, see [11, Section 3] or [5, Proposition 5.6]. Here we have $\Omega R = (t^2 - t)\, R[t]$ and since $K_n{}^*(R) = K_1{}^*(\Omega^{n-1}R)$, we recover the K^{-n} of Karoubi–Villamayor [11, Section 5]: $K_n^{\mathrm{III}*} = K^{-n}$.

Their definition of fibration was slightly modified by Gersten, but both versions are equivalent. He calls a surjection $f\colon B \to C$ a fibration when $GL(E^n f)\colon GL(E^n B) \to GL(E^n C)$ is surjective for $n \geqslant 1$. It is known that such fibrations are preserved by the functor Ω [11, Proposition 2.10]. Hence all the $GL(E\Omega^n f)$ are surjective for $n \geqslant 0$, hence f is also a fibration in our theory III. The converse is also true. A proof by induction may be given which we omit. The maps $\epsilon(R)$ are always fibrations [5, Theorem 3.5]. Not all surjections are fibrations [11, Theorem 2.6], the trouble often stemming from nilpotents in Ker f [18, Section 4].

Though we have not yet investigated case II closely, we do know that there too, nilpotents in the kernel can prevent a surjection from being a fibration. The commutative theories can be examined in a similar way, but we leave these aside for the moment.

5. Excision

In each theory we have attached to a fibration f a long exact sequence of $K_n{}^*$'s connected by the relative terms $H_n((GL\mathscr{E})_f)$ which we call K^*_{nf}. Excision is said to hold for a K-theory provided $K^*_{nf} = K_n{}^*(A)$ for all n where $A = \ker f$. Since $K_1^{\mathrm{I}*} = K_1{}^B$, the K_1 of Bass, the work of Swan [16] tells us that excision cannot be valid in theory I.

In theory III the functors E and Ω are exact so if we write $\mathscr{E}_f = \operatorname{Ker} \mathscr{E}f$: $\mathscr{E}B \to \mathscr{E}C$, we have $\mathscr{E}_f = \mathscr{E}A$, the canonical resolution of A. Because the functor GL is left exact ,

$$GL(\mathscr{E}A) = GL(\mathscr{E}_f) = \operatorname{Ker} GL(\mathscr{E}f)\colon (GL(\mathscr{E}B) \to GL(\mathscr{E}C)) = (GL\mathscr{E})_f\,,$$

and passing to homology establishes that $K_n{}^*(A) = K^*_{nf}$. This proves excision in the Karoubi–Villamayor theory III, a fact implied in [11] and made explicit in [17, p. 131]. Our proof however brings out that excision solely depends on E^{III} being left exact. In combination with Swan's negative excision result for $K_1{}^B$ [16], our argument for instance shows that there exists no left exact functor E: $\mathbf{Rg} \to \mathbf{Rg}$ with ϵ: $E \to I_{\mathbf{Rg}}$ such that $\operatorname{Im} GL(\epsilon(R)) = EL(R)$ for all rings R and such that all surjections are GL-fibrations.

In case II excision remains undecided.

6. Connecting with K_0

In theory I, since we know that $K_1{}^* = K_1{}^B$, we can for each surjection continue our long exact sequence to the right with

$$K_1(B) \xrightarrow{K_1{}^*f} K_1(C) \xrightarrow{\delta} K_{0f} \longrightarrow K_0(B) \xrightarrow{K_0f} K_0(C)$$

[5, Theorem 5.7]. From its definition, it is not hard to prove that the connecting homomorphism δ factors through $K_1^{\mathrm{III}*}(C)$ [5, Theorem 5.10] and that in fact we have a commuting diagram

$$\begin{tikzcd}
K_1^{\mathrm{I}}(B) \arrow[r] \arrow[d] & K_1^{\mathrm{I}}(C) \arrow[d] \arrow[dr, "\delta"] & & & \\
K_1^{\mathrm{II}*}(B) \arrow[r] \arrow[d] & K_1^{\mathrm{II}*}(C) \arrow[d] & K_{0f} \arrow[r] & K_0(B) \arrow[r] & K_0(C). \\
K_1^{\mathrm{III}*}(C) \arrow[r] & K_1^{\mathrm{III}*}(C) \arrow[ur, dashed] & & &
\end{tikzcd}$$

This induces $K_1^{\mathrm{II}*}(C) \to K_{0f}$ and from the exactness of the I-sequence the exactness of the other two follows (this for III of course being known [11, Théorème 3.8], [5, Theorem 5.10]).

Now take for f the surjection $\epsilon(R)$ and notice that by results of Gersten and Stallings not only $K_1{}^{\mathrm{I}}$ but also K_0 vanishes on the free ring ER [1, Section XII. 11]. Hence $K_1{}^{\mathrm{I}}(R) = K_{0\epsilon(R)}$. The latter group is, in the more classical notation of [1], $K_0((ER)^+, R^+)$. However, for K_0 excision is known to hold, and so $K_0((ER)^+, R^+) = K_0((\Omega R)^+, \mathbf{Z})$ which group is by definition $K_0(\Omega R)$. Hence in theory I we have $K_1(R) = K_0(\Omega R)$ and we may base the theory on K_0 and apply Ω to obtain $K_n{}^*(R) = K_1(\Omega^{n-1}R) = K_0(\Omega^n R)$.

In theory III we do not in general have $K_n{}^*(R) = K_0(\Omega^n R)$ [18]; the intermediate theory II as usual awaits further investigation.

7. The Standard Simplicial Cotriple Resolution

Each of our endofunctors E is a cotriple in the sense of [2], i.e., there exists morphisms $\mu\colon E \to E^2$ such that $E\epsilon \circ \mu = 1_E = \epsilon E \circ \mu$ and $E\mu \circ \mu = \mu E \circ \mu$. These morphisms are determined by $\mu(X_r) = Y_{X_r}$, $\mu(x_r) = y_{x_r}$ and $\mu(t) = st$ in cases, I, II and III, respectively. There is a standard way to associate a simplicial resolution $\underset{\sim}{\mathscr{E}}$ with such a cotriple. The face operators $\epsilon_i\colon E^n \to E^{n-1}$ are defined as $E^i\epsilon E^{n-i-1}$ and the degeneracy operators $\mu_j : E^n \to E^{n+1}$ as $E^j\mu E^{n-j-1}$, $i, j = 0,\ldots, n-1$, $n \geqslant 1$, which satisfy the standard simplicial relations:

$$\begin{aligned}
\epsilon_i \circ \epsilon_j &= \epsilon_{j-1} \circ \epsilon_i && \text{if } i < j \\
\mu_i \circ \mu_j &= \mu_{j+1} \circ \mu_i && \text{if } i \leqslant j \\
\epsilon_i \circ \mu_j &= \mu_{j-1} \circ \epsilon_i && \text{if } i < j \\
\epsilon_j \circ \mu_j &= \epsilon_{j+1} \circ \mu_j = \mathrm{Id}. && \\
\epsilon_i \circ \mu_j &= \mu_j \circ \epsilon_{i-1} && \text{if } i > j + 1.
\end{aligned}$$

Augment with $I \to 0$ and apply GL to obtain an augmented simplicial group functor $GL(\underset{\sim}{\mathscr{E}})$. For every ring R one may then define $K_n(R)$'s as the simplicial homotopy groups of $GL(\underline{\mathscr{E}}R)$. To a "fibration" of rings, i.e., $f\colon B \to C$ such that $GL(E^n f)\colon GL(E^n B) \to GL(E^n C)$ is surjective for $n \geqslant 1$, is associated a long exact sequence of these homotopy groups connected by relative terms which is then interpreted as the long exact sequence of K-theory. We shall call these simplicially defined functors—which also go to abelian groups—K_n, in accordance with the standard terminology, distinguished when needs be by superscripts I, II or III.

This is (after a slight modification) the procedure adopted by Gersten in [4] with regard to cotriple I; the fibrations are all the surjections. These $K_n{}^{\mathrm{I}}$'s coincide with an earlier definition of higher K's by Swan [15]. The proof is furnished in [17]. A third way to obtain that theory is described in [9].

With cotriple III, Gersten has defined higher K's this way [5] and he has shown that one obtains the same theory when one uses the definition $K_n{}^*(R) = K_1(\Omega^{n-1}R)$ as was done by Karoubi and Villamayor [11], in our terminology: $K^{\mathrm{III}*} = K^{\mathrm{III}}$. This will also become apparent in the next section.

The functors K all vanish on the projectives ER of the theory, in theory I and II even on free rings LS, S a set, respectively, $L'A$, A an abelian group. The proof goes by showing that for these rings the complex $\underline{\mathscr{E}}R$ is simplicially contractible, cf. [4, Section 7]. Again, the commutative theories function similarly.

There exists a useful device for computing the homotopy groups of an (augmented) simplicial group $\underline{G}$. To $\underline{G}$ one attaches its normalized subcomplex or Moore complex. This is a chain complex $M(\underline{G})$ of (non-abelian) groups in the sense discussed in Section 2. The point is now that the homotopy of $\underline{G}$ equals the homology of $M(\underline{G})$. For the details, the reader is referred to [12, Section VII. 5]. For the simplicial group of our interest, $GL(\underline{\mathscr{E}}R)$, we write $M_n(R)$ for the nth chain group of the Moore complex $M(GL(\underline{\mathscr{E}}R))$. The construction then shows that $M_n(R) = \bigcap_{i=1}^{n-1} \operatorname{Ker} GL(\epsilon_i(R))\colon (GL(E^n R) \to GL(E^{n-1}R))$ and the differential $d\colon M_n(R) \to M_{n-1}(R)$ is given by $GL(\epsilon_0(R))$, and $K_n(R) = H_{n-1}(M(GL(\underline{\mathscr{E}}R)))$, $n \geqslant 1$.

8. Comparing K with K^*

Write $C_n = \bigcap_{i=1}^{n-1} \operatorname{Ker} \epsilon_i\colon (E^n \to E^{n-1})$, $n \geqslant 2$, $d = \epsilon_0 \mid C_n$. In particular $C_1 R = ER$.

Proposition 8.1. *For every ring R,*

$$\mathscr{C}R\colon \cdots \xrightarrow{d} C_{n+1}R \xrightarrow{d} C_nR \xrightarrow{d} C_{n-1}R \xrightarrow{d} \cdots$$
$$\cdots \xrightarrow{d} C_3R \xrightarrow{d} C_2R \xrightarrow{d} C_1R \xrightarrow{d} R \longrightarrow 0$$

is an acyclic resolution of R. One obtains the Moore complex $M(GL(\underline{\mathscr{E}}R))$ by applying GL to it.

Proof. The last statement is easy: since GL is a left exact functor, it commutes with kernels and intersections. In proving the first, we choose a fixed ring R but omit it from notation. The cycles in the chain ring C_n are $Z_n = \bigcap_{i=0}^{n-1} \operatorname{Ker} \epsilon_i\colon (E^n \to E^{n-1})$; write $\eta_n\colon Z_n \to E^n$ for the embedding. Then

$$EZ_n \xrightarrow{E\eta_n} E^{n+1} \xrightarrow{\epsilon_i} E^n$$

are nullsequences for $i = 1,\dots, n$. Hence $E\eta_n$ factors through C_{n+1}, which is the infimum of the $\epsilon_1 ,\dots, \epsilon_n$. The composite

$$EZ_n \longrightarrow C_{n+1} \xrightarrow{d} C_n$$

has its image in Z_n and in fact it is just $\epsilon(Z_n)$ which for our E's is surjective for all rings. Thus $\operatorname{Im} d = Z_n$ and the complex $\mathscr{C}R$ is acyclic. From the fact that the homology $H_{n-1}(M(GL(\underline{\mathscr{E}}R))) = GL(Z_{n-1}R)/d(M_n(R))$ and $d(M_n(R)) \supset EL(Z_{n-1}R)$, we see that this homology $K_n(R)$ is an abelian group.

THEOREM 8.2. *There is a map between resolutions $\alpha\colon \mathscr{E} \to \mathscr{C}$ which induces maps $K_n^* \to K_n$, $n \geqslant 1$.*

Proof. Since the composites $E\Omega^{n-1} \to EE^{n-1} \xrightarrow{\epsilon_i} E^{n-1}$ are nullmaps for $i = 1,\dots, n-1$, there is a unique map $\alpha_n\colon E\Omega^{n-1} \to C_n$. In both complexes the differential d is essentially given by ϵ_0. From functoriality and the fact that $C_{n-1} \to E^{n-1}$ is a monomorphism it follows that $\alpha_{n-1} \circ d = d \circ \alpha_n$, establishing the map $\alpha\colon \mathscr{E} \to \mathscr{C}$.

Since $GL(\mathscr{C})$ is by Proposition 8.1 the Moore complex $M(GL(\underline{\mathscr{E}}))$, the map $GL\alpha\colon GL(\mathscr{E}) \to GL(\mathscr{C})$ induces maps $K_n^* \to K_n$ on homology.

COROLLARY 8.3. *In case of a fibration common to all theories, there is a commuting diagram of morphism between connected sequences of functors* **Rg** $\to$ **Ab**

$$\begin{array}{ccccc} K^{\mathrm{I}*} & \rightarrow & K^{\mathrm{II}*} & \rightarrow & K^{\mathrm{III}*} \\ \downarrow & & \downarrow & & \downarrow \\ K^{\mathrm{I}} & \longrightarrow & K^{\mathrm{II}} & \longrightarrow & K^{\mathrm{III}}. \end{array}$$

Proof. Observe that the morphisms $E^{\mathrm{I}} \rightarrow E^{\mathrm{II}} \rightarrow E^{\mathrm{II}}$ of Section 4 are actually maps between cotriples, i.e., commute with the μ's. Hence they induce $K^{\mathrm{I}} \rightarrow K^{\mathrm{II}} \rightarrow K^{\mathrm{III}}$; combine with Theorem 8.2 to finish the proof.

Here K^{I} is, we repeat, Gersten's version [4] of Swan's theory, $K^{\mathrm{III}*}$ is the theory of Karoubi and Villamayor [11] of which K^{III} is Gersten's treatment in [5].

Remark. Patently, a split surjection $f: B \rightarrow C$ is a common fibration. It can also be shown that surjections onto regular noetherian rings are fibrations in all theories.

As announced, we prove the known:

THEOREM 8.4. $K^{\mathrm{III}} = K^{\mathrm{III}*}$.

Proof. As with the proof of excision in Section 5 the distinctive feature of cotriple III is that both E and Ω are (left) exact functors. This entails that $Z_2R = E\Omega R \cap \Omega ER = \Omega^2 R$. Proceeding by induction, we see that $C_n R = E\Omega^{n-1}R$. Thus the defining complex $GL\mathscr{E}$ of $K^{\mathrm{III}*}$-theory and the Moore complex $M(GL\underline{\mathscr{E}})$ whose homology describes K^{III}-theory, coincide. This proof is really no different from that of Eilenberg and Moore, who state the result for pre-additive categories, [2, Proposition 4.1].

The commutative theories can of course be treated in the same way.

9. THE MAP FROM $K_2{}^*$ TO K_2

Let E be one of our cotriples. Since the two lowest terms of the complexes $\mathscr{E}$ and $\mathscr{C}$ are identifical, we not only have $K_1{}^*(R) = K_1(R)$ for every ring R, but also $K_{1f}^* = K_{1f}$ for every surjection of rings f.

Given a ring R, consider the mapping $\epsilon(R): ER \rightarrow R$ which is a fibration in all theories. From Corollary 8.3 we have a commuting diagram of abelian groups with exact rows

$$\begin{array}{ccccccc} K_2{}^*(ER) & \longrightarrow & K_2{}^*(R) & \longrightarrow & K_{1\epsilon}^* & \longrightarrow & K_1{}^*(ER) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_2(ER) & \longrightarrow & K_2(R) & \longrightarrow & K_{1\epsilon} & \longrightarrow & K_1(ER). \end{array}$$

Now $K_2(ER) = K_1(ER) = K_1{}^*(ER) = 0$. This yields the following.

PROPOSITION 9.1. *There is an exact sequence*

$$K_2{}^*(ER) \to K_2{}^*(R) \to K_2(R) \to 0.$$

For the remainder of this section we shall discuss the cotriple $E = E^{\mathrm{I}}$ and usually drop the superscripts. Going back to the defining complex, we see that in this case the homomorphism $K_2{}^*(R) \to K_2(R)$ is just $GL(\Omega R)/EL(\Omega R) \to GL(\Omega R)/dM_2(R)$.

Let $f\colon B \to C$ be a surjection of unital rings and write $A = \operatorname{Ker} f$. We have the following proposition.

PROPOSITION 9.2. $K_{1f} = GL(A)/[GL(B), GL(A)] = K_1(B, A)$.

Here the last term is Bass' relative group. The result is proved in [17, Theorem 3.1] and [9, Theorem 14.16]; since we shall use this fact, we briefly review the latter proof. This depends on the relative Whitehead identity

$$[GL(B), GL(A)] = [EL(B), EL(B, A)] = EL(B, A)$$

which holds for any ideal A in a unitary ring B. The last term is by definition the normal subgroup of $EL(B)$ generated by all elementary matrices with off-diagonal entry in A, [1, Section V.2].

Consider the exact ladder of rings

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] \arrow[d, "d"] & EB \arrow[r, "Ef"] \arrow[d, "d"] & EC \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r, "f"'] & C \arrow[r] & 0,
\end{tikzcd}$$

where we have written $D = \operatorname{Ker} Ef = \operatorname{Ker}(Ef)^+\colon (EB)^+ \to (EC)^+$. Apply GL to the split ring homomorphism $(Ef)^+$. There results a split short exact sequence of abelian groups [1, VIII, Lemma 2.6]:

$$1 \to \frac{GL(D)}{[GL(EB)^+, GL(D)]}$$
$$\to \frac{GL(EB)^+}{[GL(EB)^+, GL(EB)^+]} \to \frac{GL(EC)^+}{[GL(EC)^+, GL(EC)^+]} \to 1.$$

For the free EB we have

$$GL(EB)^+/[GL(EB)^+, GL(EB)^+] = GL(EB)^+/EL(EB)^+ = K_1(EB)^+ = K_1(\mathbf{Z});$$

similarly for EC. We find that $GL(D) = EL((EB)^+, D)$ using the Whitehead identity recalled above. The map f being surjective, so is $\Omega f\colon \Omega B \to \Omega C$ by Lemma 4.1. An easy chase around our ladder shows that the differential d maps the ring D onto A. Hence an elementary matrix with off-diagonal entry in A can be lifted to such a one in D, and the differential maps $GL(D)$ onto $EL(B, A)$, from which the proposition follows. In case B and C do not possess unit elements, we adjoin these and find $K_{1f} = K_1(B^+, A)$.

COROLLARY 9.3. $K_2{}^{\mathrm{I}}(R) = GL(\Omega R)/EL((ER)^+, \Omega R)$.

This follows from the identifications $K_2{}^{\mathrm{I}}(R) = K^{\mathrm{I}}_{1\epsilon} = K_1{}^{\mathrm{I}}((ER)^+, \Omega R)$.

THEOREM 9.4. *For every ring R with more than two elements, the surjection $K_2^{\mathrm{I}*}(R) \to K_2{}^{\mathrm{I}}(R)$ is not an isomorphism.*

Proof. It suffices to exhibit an element in $EL((ER)^+, \Omega R)$ which does not live in $EL(\Omega R) = EL((\Omega R)^+, \Omega R)$ or in other words show that the excision surjection $K_1((\Omega R)^+, \Omega R) \to K_1((ER)^+, \Omega R)$ is not injective.

According to the addendum at the end of this section, there exist $\omega \in \Omega R$ and $\rho \in ER$ such that $\omega\rho - \rho\omega \notin (\Omega R)^2$. The product of matrices

$$\begin{pmatrix} 1 & 0 \\ \rho & 1 \end{pmatrix}\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\rho & 1 \end{pmatrix} = \begin{pmatrix} 1-\omega\rho & \omega \\ -\rho\omega\rho & 1+\rho\omega \end{pmatrix} = M$$

is clearly in $EL((ER)^+, \Omega R)$. Now consider the ring $\Omega R/(\Omega R)^2 = C$; this is commutative because all products are null (we are indebted to H. Bass for reminding us of this old trick.) The matrix M is in $GL((\Omega R)^+, \Omega R)$; under the residue class map $(\Omega R)^+ \to C^+$ it goes to $\bar{M} \in GL(C^+, C)$. Following Swan [16, Theorem 1.1] we claim that $\bar{M} \notin EL(C^+, C)$, the point being that in the commutative ring C^+ one can take $\det(\bar{M})$ and observe that this is $1 - \bar{\omega}\bar{\rho} + \bar{\rho}\bar{\omega} \neq 1$. We conclude that $M \notin EL((\Omega R)^+, \Omega R)$.

COROLLARY 9.5. *$K_n^{\mathrm{I}*}(R) \neq 0$ for all rings R and $n \geq 2$, except that $K_2^{\mathrm{I}*}(\mathbf{F}_2) = 0$. In particular, $K_n^{\mathrm{I}*}$ does not vanish on free rings.*

Proof. The formula $K_n{}^*(R) = K_2{}^*(\Omega^{n-2}R)$ reduces it to the above, because the ring $\Omega^{n-2}R$ has more than two elements, save for the exceptional case. But $E^{\mathrm{I}}\mathbf{F}_2 = E^{\mathrm{III}}\mathbf{F}_2$ and $\Omega^{\mathrm{I}}\mathbf{F}_2 = \Omega^{\mathrm{III}}\mathbf{F}_2$ so $K_2^{\mathrm{I}*}(\mathbf{F}_2) = K_1{}^{\mathrm{I}}(\Omega^{\mathrm{III}}\mathbf{F}_2)$ and the latter is known to be 0, e.g., [18, Section 6].

Remark. It would be nice to treat similarly the map $K_2^{\mathrm{I}^c*}(R) \to K_2^{\mathrm{I}^c}(R)$ in the commutative theory, but obstructions to excision for K_1 in commutative rings are much harder to come by. The known examples so far [16, Section 3] depend on the results of Bass–Milnor–Serre for totally imaginary number fields. We have not managed to take advantage of this approach to prove inequality in the commutative theory.

Addendum. We now demonstrate the seemingly innocent fact used in the proof of Theorem 9.4, i.e., for every ring R with more than two elements there are $\omega \in \Omega R$, $\rho \in ER$ such that $\omega\rho - \rho\omega \notin (\Omega R)^2$.

Let R be a ring, M a left R-module, then we give a ring structure to the module $R \oplus M$ by defining $(r, m)(r'm') = (rr', rm')$. This ring will be called $R \mp M$.

Consider the ring epimorphism β: $R \mp M \to R$ given by $\beta(r, m) = r$. Since ER is free with free generators X_r we can define a ring homomorphism θ: $ER \to R \mp M$ by putting $\theta(X_r) = (r, m_r)$ with arbitrary elements $m_r \in M$, for each generator X_r.

This map clearly satisfies $\beta \circ \theta = \epsilon$, hence θ maps ΩR into M and $(\Omega R)^2$ into $M^2 = (0)$. So, to prove the assertion we will build a ring $R \mp M$ such that $\theta(\omega\rho - \rho\omega) \neq 0$ for a suitable choice of θ, ω and ρ.

Take $M = R$ if R has a unit, otherwise take $M = R^+$, and define $\theta(X_r) = (r, 1)$ for every $r \neq 0$. Consider now two distinct nonzero elements a and b, so $a - b \neq 0$ and take $\omega = X_{a-b} + X_b - X_a$, $\rho = X_a$. Thus $\theta(\omega) = (0, 1)$ and $\theta(\omega\rho - \rho\omega) = (0, -a) \neq 0$.

10. More About the Moore Complex

In the case of cotriple I we can determine the chain modules $M_n(R) = GL(C_nR)$ in the Moore complex in terms of elementary matrices, which enables us to give a description of the $K_n{}^{\mathrm{I}}(R)$'s.

THEOREM 10.1. *For any ring R, $M_n{}^{\mathrm{I}}(R) = EL((E^{\mathrm{I}n}R)^+, C_n{}^{\mathrm{I}}R)$, for $n \geqslant 1$.*

Proof. In proving the theorem we drop the superscript I. Define $C_{n,k}R$ as the ideal of E^nR consisting of those elements which go to zero under ϵ_i, $i = k$, $k + 1,\ldots, n - 1$, for $n \geqslant 2$, $C_{1,1}R = ER$. Thus

$C_{n,k}R = \bigcap_{i=k}^{n-1} \operatorname{Ker} \epsilon_i$, $k = 0,..., n-1$. Then $C_{n,0}R = Z_nR$, $C_{n,1}R = C_nR$ in the notation of Section 8, and if we put $C_{n,n}R = E^nR$ then we may conceive of

$$C_{n,0}R \subset C_{n,1}R \subset \cdots \subset C_{n,n-1}R \subset C_{n,n}R$$

as a chain of ideals in the free ring with identity $(ER)^+$.

Now suppose $\alpha \in M_n(R) = GL(C_nR)$. Since GL is left exact, $GL(C_nR) \subset GL(E^nR) = EL((E^nR)^+, C_{n,n}R)$. Assume therefore we have obtained $\alpha \in EL((E^nR)^+, C_{n,k+1}R)$ and wish to prove that

$$\alpha \in EL((E^nR)^+, C_{n,k}R), \qquad 1 \leqslant k \leqslant n-1.$$

Write $x^y = yxy^{-1}$, then $\alpha = \prod_{h=1}^{t} (I + p_h e_{ij}^{(h)})^{u_h}$ where $p_h \in C_{n,k+1}R$ and $u_h \in EL(E^nR)^+$.

By the choice of α we know $\epsilon_k(\alpha) = 1$. Consider $1 = \beta = \mu_{k-1}\epsilon_k(\alpha^{-1}) = \prod_{h=t}^{1} (I - \mu_{k-1}\epsilon_k(p_h)\, e_{ij}^{(h)})^{u_h}$ then

$$\alpha = \alpha\beta = \prod_{h=1}^{t-1} (I + p_h e_{ij}^{(h)})^{u_h} \cdot \sigma \cdot \prod_{h=t-1}^{1} (I - \mu_{k-1}\epsilon_k(p_h)\, e_{ij}^{(h)})^{u_h}$$

where $\sigma = (I + (p_t - \mu_{k-1}\epsilon_k(p_t))\, e_{ij}^{(t)})^{u_t}$. For $s \geqslant k+1$ we have

$$\begin{aligned} \epsilon_s(p_t - \mu_{k-1}\epsilon_k(p_t)) &= \epsilon_s(p_t) - \epsilon_s\mu_{k-1}\epsilon_k(p_t) = -\mu_{k-1}\epsilon_{s-1}\epsilon_k(p_t) \\ &= -\mu_{k-1}\epsilon_k\epsilon_s(p_t) = 0 \end{aligned}$$

because $p_t \in C_{n,k+1}R$. On the other hand,

$$\begin{aligned} \epsilon_k(p_t - \mu_{k-1}\epsilon_k(p_t)) &= \epsilon_k(p_t) - \epsilon_k\mu_{k-1}\epsilon_k(p_t) \\ &= \epsilon_k(p_t) - \epsilon_k(p_t) = 0. \end{aligned}$$

Therefore $(p_t - \mu_{k-1}\epsilon_k(p_t)) \in C_{n,k}R$.
Hence we can write

$$\alpha = \prod_{h=1}^{t-1} (I + p_h e_{ij}^{(h)})^{u_h} \cdot \prod_{h=t-1}^{1} (I - \mu_{k-1}\epsilon_k(p_h)\, e_{ij}^{(h)})^{u_h} \cdot \sigma^v$$

with $v \in EL(E^nR)^+$. Continuing in this way, we can eventually write α as a product of such elements σ^v, consequently $\alpha \in EL((E^nR)^+, C_{n,k}R)$. By induction on k we obtain $\alpha \in EL((ER)^+, C_nR)$, since we can get as far as $C_{n,1}R = C_nR$ because we have μ_0 available.

The following generalizes Corollary 9.3. Compare with [9, Theorem 14.10].

COROLLARY 10.2. *For every ring R we have*

$$K_n^{\mathrm{I}}(R) = GL(Z_{n-1}^{\mathrm{I}}R)/EL((E^{\mathrm{I}n-1}R)^+, Z_{n-1}^{\mathrm{I}}R)$$

for $n \geqslant 2$.

Proof. According to the description in Section 8 it remains to prove that $d(M_n(R)) = EL((E^{n-1}R)^+, Z_{n-1}R)$. But the complex of rings $\mathscr{C}R$ is exact, therefore $d(C_nR) = Z_{n-1}R$ while of course $\epsilon_0 = d\colon E^nR \to E^{n-1}R$ is surjective, hence $M_n(R) = EL((E^nR)^+, C_nR)$ maps onto $EL((E^{n-1}R)^+, Z_{n-1}R)$ under the differential of the Moore complex.

For use in the next section we need the following proposition.

PROPOSITION 10.3. *For any cotriple E and every ring R, the homomorphism $GL(\epsilon_n(R))\colon M_n(ER) \to M_n(R)$ is surjective for $n \geqslant 1$.*

Proof. Keep the notation $C_{n,k}$ introduced above and write $M_{n,k}(R) = GL(C_{n,k}R)$. For simplicity we shall write ϵ_i, μ_j without specifying the ring, even after applying GL.

Consider the statements

$A(p, i)$: The homomorphism $\epsilon_i\colon M_{p,i+1}(R) \to M_{p-1,i}(R)$ is split by μ_{i-1}

and

$B(p, i)$: The homomorphism $\epsilon_p\colon M_{p,i}(ER) \to M_{p,i}(R)$ is surjective.

We shall prove that $A(p, i)$ and $B(p, i)$ hold for all $p \geqslant 2$ and $i = 1,..., p - 1$, respectively, all $p \geqslant 1$ and $i = 1,..., p$. Clearly, $B(n, 1)$ is the statement of the proposition.

The proof proceeds by induction on both statements at once; we first induce on the index p, then on i.

Observe now that the statements $A(p + 1, p)$ and $B(p, p)$ are evident and in fact identical: for $p \geqslant 1$ the map $\epsilon_p\colon E^{p+1}R = M_{p+1,p+1}(R) = M_{p,p}(ER) \to M_{p,p}(R) = E^pR$ is split by the degeneracy μ_{p-1}, hence surjective. Thus $A(2, 1)$ and $B(1, 1)$ are true and we are entitled to assume as an induction hypothesis that $A(p, i)$ holds for $2 \leqslant p \leqslant n$, $i = 1,\ldots, p-1$ and $B(p, i)$ for $1 \leqslant p \leqslant n-1$, $i = 1,\ldots, p$.

We know that $A(n+1, n)$ and $B(n, n)$ are true and assume further that $A(n+1, i)$ has been verified for $i = k+1,\ldots, n$, $2 \leqslant k \leqslant n-1$, and $B(n, i)$ for $i = k+1,\ldots, n$, $2 \leqslant k \leqslant n-1$. Bearing in mind that the functor GL is left exact, consider the ladder

$$\begin{tikzcd}
1 \arrow[r] & M_{n,k}(ER) \arrow[r] \arrow[d, "\epsilon_n"] & M_{n,k+1}(ER) \arrow[r, shift right, "\epsilon_k"'] \arrow[d, "\epsilon_n"] & M_{n-1,k}(ER) \arrow[r] \arrow[d, "\epsilon_{n-1}"] \arrow[l, shift right, "\mu_{k-1}"'] & 1 \\
1 \arrow[r] & M_{n,k}(R) \arrow[r] & M_{n,k+1}(R) \arrow[r, shift left, "\epsilon_k"] & M_{n-1,k}(R) \arrow[r] \arrow[l, shift left, "\mu_{k-1}"] & 1.
\end{tikzcd}$$

Because of $A(n, k)$ the ladder is exact and split by μ_{k-1}. Taking vertical kernels, we establish $A(n+1, k)$. By $B(n, k+1)$ and $B(n-1, k)$ the two right hand columns are surjective. Apply the Snake Lemma to conclude that the left hand column is then also surjective, establishing $B(n, k)$. Continuing by induction, we can get as far as $A(n+1, 1)$ and $B(n, 1)$ because μ_0 is the last splitting at our disposal.

11. The Double Complex and the Spectral Sequence

As we have seen, K- and K^*-theories are different for a general cotriple E, and we want to study this difference by using the spectral sequence developed in [19].

First of all, consider the canonical resolution of the ring R

$$\xrightarrow{\epsilon_0} E\Omega^n R \xrightarrow{\epsilon_0} \cdots \xrightarrow{\epsilon_0} E\Omega R \xrightarrow{\epsilon_0} ER \xrightarrow{\epsilon_0} R \xrightarrow{\epsilon_0} 0$$

and build the Moore complex above each one of these rings. Then we have a double complex of rings $\{A_{i,j}\}$ where $A_{i,j} = C_i(E\Omega^{j-1}R)(i > 0, j \geqslant 1)$, $A_{i,0} = C_i(R)(i > 0)$ and $A_{0,0} = R$, $A_{0,j} = E\Omega^{j-1}R$ $(j \geqslant 1)$, $A_{i,j} = (0)(i < 0, j < 0)$. We obtain, then, the following picture of the complex of rings

$$\begin{tikzcd}
\vdots \arrow[d, "\epsilon_0"] & \vdots \arrow[d, "\epsilon_0"] & \vdots \arrow[d, "\epsilon_0"] & \vdots \arrow[d, "\epsilon_0"] & \\
C_3(R) \arrow[d, "\epsilon_0"] & C_3(ER) \arrow[l, "\epsilon_3"'] \arrow[d, "\epsilon_0"] & C_3(E\Omega R) \arrow[l, "\epsilon_3"'] \arrow[d, "\epsilon_0"] & C_3(E\Omega^2 R) \arrow[l, "\epsilon_3"'] \arrow[d, "\epsilon_0"] & \cdots \arrow[l, "\epsilon_3"'] \\
C_2(R) \arrow[d, "\epsilon_0"] & C_2(ER) \arrow[l, "\epsilon_2"'] \arrow[d, "\epsilon_0"] & C_2(E\Omega R) \arrow[l, "\epsilon_2"'] \arrow[d, "\epsilon_0"] & C_2(E\Omega^2 R) \arrow[l, "\epsilon_2"'] \arrow[d, "\epsilon_0"] & \cdots \arrow[l, "\epsilon_2"'] \\
ER \arrow[d, "\epsilon_0"] & E^2R \arrow[l, "\epsilon_1"'] \arrow[d, "\epsilon_0"] & E^2\Omega R \arrow[l, "\epsilon_1"'] \arrow[d, "\epsilon_0"] & E^2\Omega^2 R \arrow[l, "\epsilon_1"'] \arrow[d, "\epsilon_0"] & \cdots \arrow[l, "\epsilon_1"'] \\
R & ER \arrow[l, "\epsilon_0"'] & E\Omega R \arrow[l, "\epsilon_0"'] & E\Omega^2 R \arrow[l, "\epsilon_0"'] & \cdots . \arrow[l, "\epsilon_0"']
\end{tikzcd}$$

Since $A_{i,j}$ is a subring of $E^{i+j}R$, the maps indicated in the picture are induced by those in $E^{i+j}R$.

Now apply the functor GL and call $G_{i,j} = GL(A_{i,j})$, so we have a positive double complex of groups with maps $d = \epsilon_i \colon G_{i,j} \to G_{i,j-1}$, $\delta = \epsilon_0 \colon G_{i,j} \to G_{i-1,j}$.

Since the rings in the zeroth row are all contractible except R, all but the zeroth column in $\{G_{i,j}\}$ are exact, see Section 7, and we can use the results obtained for this case, namely, that the homology of the double complex is naturally isomorphic to the homology of the zeroth column, i.e., to the K-groups of R [19, Section 2].

Remark. Recall that $M_i(A) = GL(C_iA)$. Since $d\colon M_i(E\Omega^{j-1}R) \to M_i(E\Omega^{j-2}R)$ is ϵ_i and GL is left exact, we have $\operatorname{Ker} d = M_{i+1}(\Omega^{j-1}R)$. A similar consideration is valid for the low indexed columns.

If $\delta\colon M_i(A) \to M_{i-1}(A)$ call $Y_i(A) = GL(Z_iA) = \operatorname{Ker} \delta$.

Define a filtration in the following way: W_p is the subcomplex obtained by replacing $G_{i,j}$ by (1) for $i > p$. We can then build a spectral sequence as in the preceding paper [19], of which we freely use the notation and results.

To compute $E_2^{p,n-p}$ we have to consider homogeneous strings of total degree n contained in W_p, i.e., sequences $(a_0, \ldots, a_p, 1, \ldots, 1)$ with $da_p = 1$, $da_{p-1} = \delta a_p$, $a_i \in G_{i,n-i}$, which form the group $Z_2^{p,n-p}$. The equivalence relation is given by multiplication by the subgroup $Z_1^{p-1,n-p+1} = Z_2^{p,n-p} \cap W_{p-1}$, that is, sequences $(c_0, \ldots, c_{p-1}, 1, \ldots, 1)$ with $dc_{p-1} = 1$ and by β-action from $Z_1^{p+1,n-p}$.

Since every sequence of the form $(a_0, \ldots, a_{p-2}, 1, \ldots, 1)$ is in $Z_1^{p-1,n-p+1}$

and it is then equivalent to 1, we may only consider the pair (a_{p-1}, a_p) for an element of $Z_2^{p,n-p}$ (with the previous conditions), $(c_{p-1}, 1)$ in $Z_1^{p-1,n-p+1}$ and the β-action by (b_{p-1}, b_p, b_{p+1}) (with $db_{p+1} = 1$).

Then two pairs (a_{p-1}, a_p) and (a'_{p-1}, a_p') in $Z_2^{p,n-p}$ will be equivalent if there are $(c_{p-1}, 1)$ $(dc_{p-1} = 1)$ and (b_{p-1}, b_p, b_{p+1}) $(db_{p+1} = 1$, the b_i's of total degree $n + 1)$ such that

$$\begin{aligned}(a'_{p-1}, a_p') &= (db_{p-1}^{-1}c_{p-1}a_{p-1}\delta b_p, \delta b_{p+1}^{-1}a_p db_p) \text{ or} \\ &= (\delta b_p^{-1}c_{p-1}a_{p-1}db_{p-1}, db_p^{-1}a_p\delta b_{p+1})\end{aligned}$$

according to p being odd or even.

Since $(db_{p-1}, 1) \in Z_1^{p-1,n-p+1}$ we can replace (b_{p-1}, b_p, b_{p+1}) by the pair (b_p, b_{p+1}) and forget about b_{p-1}.

In our situation $a_p \in G_{p,n-p} = M_p(E\Omega^{n-p-1}R)$, but $da_p = 1$ implies that $a_p \in M_{p+1}(\Omega^{n-p-1}R)$. Also $a_{p-1} \in M_{p-1}(E\Omega^{n-p}R)$ and $da_{p-1} = \delta a_p$.

For the same reason $c_{p-1} \in M_p(\Omega^{n-p}R)$ and $b_{p+1} \in M_{p+2}(\Omega^{n-p-1}R)$. Since $d: G_{i,1} \to G_{i,0}$ $(i > 0)$ are surjective, see Proposition 10.3, $E_2^{i,0} = \{*\}$ $(i > 0)$ [19, Proposition 16].

Remark. For the cotriple E^{III} it can easily be seen that the nonzero rows are exact, so the homology of the double complex also coincides with the homology of the zeroth row. For the same reason all $E_2^{p,q} = \{*\}$ for $p > 0$. This furnishes yet another, admittedly roundabout proof that $K^{\text{III}} = K^{\text{III}*}$.

12. The Exact Sequence

$$K_p(\Omega^{n-p+1}R) \to K_{p+1}(\Omega^{n-p}R) \to E_2^{p-1,n-p+1} \to K_{p-1}(\Omega^{n-p+1}R) \to K_p(\Omega^{n-p}R).$$

(a) *The map*

$$\psi: K_{p+1}(\Omega^{n-p}R) \to E_2^{p-1,n-p+1} \qquad \text{for} \quad n \geqslant p \geqslant 1.$$

Let $\bar{a} \in K_{p+1}(\Omega^{n-p}R)$, hence we can choose a cycle $a_{p-1} \in Y_p(\Omega^{n-p}R)$, (i.e., a cycle in the Moore complex over $\Omega^{n-p}R$) representing $\bar{a}$.

Hence $a_{p-1} \in M_p(\Omega^{n-p}R) \subset M_{p-1}(E\Omega^{n-p}R) = G_{p-1,n-p+1}$ and $da_{p-1} = \delta a_{p-1} = 1$. So the pair $(1, a_{p-1}) \in Z_2^{p-1,n-p+1}$ and it gives an element $\psi(\bar{a})$ of $E_2^{p-1,n-p+1}$.

LEMMA 12.1. *ψ is well defined.*

Proof. If a'_{p-1} is another representative of $\bar{a}$, then $a'_{p-1} = \delta b_p^{-1} a_{p-1}$ for some $b_p \in M_{p+1}(\Omega^{n-p}R)$, so $db_p = 1$ and $b = (1, 1, b_p) \in Z_1^{p,n-p+1}$ hence, if p is even, $\beta(b) \cdot (1, a_{p-1}) = (1, a'_{p-1})$. For p odd, use the fact that the boundaries, Im δ, form a normal subgroup of the cycles Y_p, hence we may think δb_p multiplies on the right and so on.

Therefore both images are equivalent, giving the same element of $E_2^{p-1,n-p+1}$, so ψ is well defined.

Remark. The image Im ψ will be the set of those equivalence classes of $Z_2^{p-1,n-p+1}$ which contain elements of the form $(1,\ldots, 1, a_{p-1}, 1,\ldots, 1)$ with $da_{p-1} = \delta a_{p-1} = 1$.

LEMMA 12.2. *If* $(c_{p-2}, c_{p-1}) \in Z_2^{p-1,n-p+1}$, *then* $(c_{p-2}, c_{p-1}) \sim (1, a_{p-1})$ *for some* a_{p-1} *if and only if there are*

$$b_{p-1} \in M_{p-1}(E\Omega^{n-p+1}R), f_{p-2} \in M_{p-1}(\Omega^{n-p+1}R)$$

such that $c_{p-2} = f_{p-2}\delta b_{p-1}$ (*respectively* $c_{p-2} = \delta b_{p-1}^{-1} f_{p-2}$) *if* p *is even* (*respectively odd*).

Proof. Suppose $(c_{p-2}, c_{p-1}) \sim (1, a_{p-1})$, then there are

$$(g_{p-2}, 1) \in Z_1^{p-2,n-p+2}$$

(so $g_{p-2} \in M_{p-1}(\Omega^{n-p+1}R)$) and $(b_{p-2}, b_{p-1}, b_p) \in Z_1^{p,n-p+1}$ such that

$$\begin{aligned}(c_{p-2}, c_{p-1}) &= (db_{p-2}^{-1} g_{p-2}\, \delta b_{p-1}, \delta b_p^{-1} a_{p-1}\, db_{p-1}) \text{ or}\\ &= (\delta b_{p-1}^{-1} g_{p-2}\, db_{p-2}, db_{p-1}^{-1} a_{p-1}\, \delta b_p)\end{aligned}$$

according to the parity of p.

Since $db_{p-2} \in M_{p-1}(\Omega^{n-p+1}R)$, call $f_{p-2} = db_{p-2}^{-1} g_{p-2}$ (or $f_{p-2} = g_{p-2} db_{p-2}$) and the conditions of the lemma are satisfied.

Conversely, assume the existence of f_{p-2} and b_{p-1} verifying the conditions stated in the lemma, then $f = (f_{p-2}, 1) \in Z_1^{p-2,n-p+2}$ and $b = (1, b_{p-1}, 1) \in Z_1^{p,n-p+1}$ (call $b = (1, b_{p-1}^{-1}, 1)$ if p is odd). Call $a_{p-1} = db_{p-1}c_{p-1}$ is p is odd, $a_{p-1} = c_{p-1} db_{p-1}^{-1}$ if p is even. Then $(1, a_{p-1}) \sim \beta(b) \cdot (f(1, a_{p-1})) = (f_{p-2}\delta b_{p-1}, a_{p-1} db_{p-1}) = (c_{p-2}, c_{p-1})$ if p is even and similarly if p is odd.

As explained before, if $a_{p-1} \in Y_p(\Omega^{n-p}R)$, then $(1, a_{p-1}) \in Z_2^{p-1,n-p+1}$. Call $\xi\colon Y_p(\Omega^{n-p}R) \to Z_2^{p-1,n-p+1}$ this map, which is obviously a group homomorphism.

LEMMA 12.3. *If* $(c_{p-2}, c_{p-1}) \in Z_2^{p-1,n-p+1}$, *then* $(c_{p-2}, c_{p-1}) \sim (1, a_{p-1})$ *for some* a_{p-1} *if and only if there exists* $f_{p-2} \in M_{p-1}(\Omega^{n-p+1}R)$ *such that* $\delta(c_{p-2}f_{p-2}) = 1$ *provided* $p < n+2$.

Proof. In fact, if $(c_{p-2}, c_{p-1}) \sim (1, a_{p-1})$ there is, according to the previous lemma, an $f_{p-2} \in M_{p-1}(\Omega^{n-p+1}R)$ such that $c_{p-2}f_{p-2} = \delta b_{p-1}$ (change f_{p-2} to f_{p-2}^{-1} since $M_{p-1}(\Omega^{n-p+1}R)$ is a normal subgroup of $M_{p-2}(E\Omega^{n-p+1}R)$. Hence $\delta(c_{p-2}f_{p-2}) = 1$.

Conversely, if $\delta(c_{p-2}f_{p-2}) = 1$, since the column $n - p + 2$ is exact, there is $b_{p-1} \in M_{p-1}(E\Omega^{n-p+1}R)$ such that $c_{p-2}f_{p-2} = \delta b_{p-1}$ and we can apply the previous lemma.

LEMMA 12.4. *The saturation of the image of* Im $(\xi\colon Y_p(\Omega^{n-p}R) \to Z_2^{p-1,n-p+1})$ *under the equivalence relation defining* $E_2^{p-1,n-p+1}$ *is a normal subgroup.*

Proof. The saturation will mean the set of all elements equivalent to elements in Im ξ. Call this set Q.

Suppose (c_{p-2}, c_{p-1}) is in this saturation, (a_{p-2}, a_{p-1}) arbitrary in $Z_2^{p-1,n-p+1}$, hence $a_{p-2} \in M_{p-2}(E\Omega^{n-p+1}R)$. Now consider

$$h = (a_{p-2}c_{p-2}a_{p-2}^{-1}, a_{p-1}c_{p-1}a_{p-1}^{-1}).$$

According to the previous lemma, there is an element

$$f_{p-2} \in M_{p-1}(\Omega^{n-p+1}R)$$

such that $\delta(c_{p-2}f_{p-2}) = 1$. But, since $M_{p-1}(\Omega^{n-p+1}R)$ is a normal subgroup of $M_{p-2}(E\Omega^{n-p+1}R)$, there is a g_{p-2} such that $a_{p-2}^{-1}g_{p-2} = f_{p-2}a_{p-2}^{-1}$, so

$$\begin{aligned}\delta(a_{p-2}c_{p-2}a_{p-2}^{-1}g_{p-2}) &= \delta(a_{p-2}c_{p-2}f_{p-2}a_{p-2}^{-1})\\ &= \delta(a_{p-2})\,\delta(c_{p-2}f_{p-2})\,\delta(a_{p-2}^{-1}) = 1.\end{aligned}$$

Hence h is also in the saturation of the image of ξ.

(b) *The map*

$$\theta\colon E_2^{p-1,n-p+1} \to K_{p-1}(\Omega^{n-p+1}R), \qquad p > 1.$$

Given $(a_{p-2}, a_{p-1}) \in Z_2^{p-1,n-p+1}$ consider $k_{p-3} = \delta a_{p-2}$ with

$$k_{p-3} \in M_{p-3}(E\Omega^{n-p+1}R).$$

Since $dk_{p-3} = d\delta a_{p-2} = \delta d a_{p-2} = \delta\delta a_{p-1} = 1$, we know

$$k_{p-3} \in M_{p-2}(\Omega^{n-p+1}R).$$

From $\delta k_{p-3} = \delta\delta a_{p-2} = 1$ it follows that $k_{p-3} \in Y_{p-2}(\Omega^{n-p+1}R)$ and so it gives an element of $K_{p-1}(\Omega^{n-p+1}R)$. Hence we obtain a map $Z_2^{p-1,n-p+1} \to K_{p-1}(\Omega^{n-p+1}R)$.

To prove this map induces a map $E_2^{p-1,n-p+1} \to K_{p-1}(\Omega^{n-p+1}R)$ we have to show that equivalent elements are mapped into the same element of the K-group. We divide the proof in two parts:

(i) Consider (a_{p-2}, a_{p-1}) and $(a_{p-2}c_{p-2}, a_{p-1})$ for

$$c_{p-2} \in M_{p-1}(\Omega^{n-p+1}R).$$

Call $k'_{p-3} = \delta(a_{p-2}c_{p-2}) = \delta a_{p-2}\delta c_{p-2}$. Since $c_{p-2} \in M_{p-1}(\Omega^{n-p+1}R)$, δc_{p-2} is a boundary in the Moore complex of $\Omega^{n-p+1}R$. We obtain that δa_{p-2} and $\delta a_{p-2}\delta c_{p-2}$ define the same element of $K_{p-1}(\Omega^{n-p+1}R)$.

(ii) Consider now (a_{p-2}, a_{p-1}) and

$$(db_{p-2}^{-1}a_{p-2}\,\delta b_{p-1},\ \delta b_p^{-1}a_{p-1}\,db_{p-1})$$

or $(\delta b_{p-1}^{-1}a_{p-2}db_{p-2},\ db_{p-1}^{-1}a_{p-1}\delta b_p)$ according to the parity of p.

In both cases, the factor db_{p-2} can be eliminated by the proof of part (i), so we have only to consider δa_{p-2} and $\delta(a_{p-2}\delta b_{p-1}) = \delta a_{p-2}\delta\delta b_{p-1} = \delta a_{p-2}$, and similarly for the other case.

Hence the map θ: $E_2^{p-1,n-p+1} \to K_{p-1}(\Omega^{n-p+1}R)$ is well defined.

Remark. As we have seen, the map θ: $E_2^{p-1,n-p+1} \to K_{p-1}(\Omega^{n-p+1}R)$ is induced by a map $\tilde\theta$: $Z_2^{p-1,n-p+1} \to K_{p-1}(\Omega^{n-p+1}R)$, which is a group homomorphism since $\tilde\theta(a_{p-2}, a_{p-1}) = \delta a_{p-2}$. Moreover $\operatorname{Ker}\tilde\theta \supset Q$ which is a normal subgroup of $Z_2^{p-1,n-p+1}$; therefore, $\tilde\theta$ induces a group homomorphism $Z_2^{p-1,n-p+1}/Q \to K_{p-1}(\Omega^{n-p+1}R)$.

(c) *The map*

$$\phi: K_{p-1}(\Omega^{n-p+1}R) \to K_p(\Omega^{n-p}R) \qquad \text{for}\quad n \geqslant p > 2.$$

Consider $\bar a \in K_{p-1}(\Omega^{n-p+1}R)$ and $a_{p-3} \in Y_{p-2}(\Omega^{n-p+1}R)$ representing $\bar a$. Then $a_{p-3} \in M_{p-3}(E\Omega^{n-p+1}R)$ with $da_{p-3} = \delta a_{p-3} = 1$. Since $M_{p-3}(E\Omega^{n-p+1}R)$ is in the column $n + p - 2$ which, by the restriction

$n - p + 2 > 0$, is exact, there is a $b_{p-2} \in M_{p-2}(E\Omega^{n-p+1}R)$ such that $\delta b_{p-2} = a_{p-3}$.

Take $h_{p-2} = db_{p-2} \in M_{p-2}(E\Omega^{n-p}R)$, hence $dh_{p-2} = 1$, so

$$h_{p-2} \in M_{p-1}(\Omega^{n-p}R).$$

Since $\delta h_{p-2} = \delta db_{p-2} = d\delta b_{p-2} = da_{p-3} = 1$ we have $h_{p-2} \in Y_{p-1}(\Omega^{n-p}R)$, hence defining an element of $K_p(\Omega^{n-p}R)$.

If we take another element b'_{p-2} with $\delta b'_{p-2} = a_{p-3}$ then $\delta(b'_{p-2}b^{-1}_{p-2}) = 1$, so $b'_{p-2}b^{-1}_{p-2} = \delta t_{p-1}$ and if $h'_{p-2} = db'_{p-2}$, then $h'_{p-2}h^{-1}_{p-2} = d\delta t_{p-1} = \delta dt_{p-1}$, with $dt_{p-1} \in M_p(\Omega^{n-p}R)$. Thus δdt_{p-1} is a boundary in $Y_{p-1}(\Omega^{n-p}R)$, so both elements give the same image in $K_p(\Omega^{n-p}R)$.

If we choose a different representative, say a'_{p-3} of $\bar{a}$, then $a'_{p-3} = a_{p-3}\delta c'_{p-2}$ for some $c'_{p-2} \in M_{p-1}(\Omega^{n-p+1}R)$, hence $dc'_{p-2} = 1$. By choosing $b'_{p-2} = b_{p-2}c'_{p-2}$ we obtain $h''_{p-2} = db'_{p-2} = db_{p-2}dc'_{p-2} = db_{p-2} = h_{p-2}$.

Thus the map is well defined by taking $\phi(\bar{a}) = \bar{h}$, i.e., the element of $K_p(\Omega^{n-p}R)$ represented by h_{p-2}.

If we consider $\bar{a}$, $\bar{c}$ and $\bar{a}\bar{c}$ in $K_{p-1}(\Omega^{n-p+1}R)$ and if a_{p-3}, c_{p-3} are representatives of $\bar{a}$ and $\bar{c}$ respectively, we can take $a_{p-3}c_{p-3}$ as a representative of $\bar{a}\bar{c}$. If we have chosen b_{p-2} and f_{p-2} such that $\delta b_{p-2} = a_{p-3}$ and $\delta f_{p-2} = c_{p-3}$, then $\delta(b_{p-2}f_{p-2}) = a_{p-3}c_{p-3}$, so $\phi(\bar{a}\bar{c}) = d(b_{p-2}f_{p-2}) = db_{p-2}df_{p-2} = \phi(\bar{a})\phi(\bar{c})$. Hence ϕ is a group homomorphism between abelian groups.

(d) Up to now, we have shown the existence of maps

$$K_p(\Omega^{n-p+1}R) \xrightarrow{\phi} K_{p+1}(\Omega^{n-p}R) \xrightarrow{\psi} E_2^{p-1,n-p+1} \xrightarrow{\theta} K_{p-1}(\Omega^{n-p+1}R)$$
$$\xrightarrow{\phi} K_p(\Omega^{n-p}R)$$

and we want to prove this is an exact sequence of group-dominated sets.

LEMMA 12.5. $\operatorname{Ker}\phi = \operatorname{Im}\theta$.

Proof. (i) $\operatorname{Im}\theta \subset \operatorname{Ker}\phi$.

Consider $(a_{p-2}, a_{p-1}) \in Z_2^{p-1,n-p+1}$, hence $da_{p-1} = 1$ and $da_{p-2} = \delta a_{p-1}$. We have defined $\tilde{\theta}(a_{p-2}, a_{p-1})$ as the image of δa_{p-2} in $K_{p-1}(\Omega^{n-p+1}R)$. Now to compute $\phi\theta$ we have to choose an element b_{p-2} such that $\delta b_{p-2} = \delta a_{p-2}$, so we can take $b_{p-2} = a_{p-2}$, hence $h_{p-2} = da_{p-2} = \delta a_{p-1}$.

But, since $da_{p-1} = 1$, we have $a_{p-1} \in M_p(\Omega^{n-p}R)$, so h_{p-2} is a boundary, giving zero in $K_p(\Omega^{n-p}R)$.

(ii) $\operatorname{Im} \theta \supset \operatorname{Ker} \phi$.

If $\bar{a} \in \operatorname{Ker} \phi$, consider a representative $a_{p-3} \in Y_{p-2}(\Omega^{n-p+1}R)$, so, by choosing b_{p-2} such that $\delta b_{p-2} = a_{p-3}$ and calling $h_{p-2} = db_{p-2}$, there is $c_{p-1} \in M_p(\Omega^{n-p}R)$ such that $\delta c_{p-1} = h_{p-2}$, because $\phi(\bar{a}) = 0$. Then the pair (b_{p-2}, c_{p-1}) satisfies $dc_{p-1} = 1$, $db_{p-2} = \delta c_{p-1}$ so

$$(b_{p-2}, c_{p-1}) \in Z_2^{p-1,n-p+1}$$

and $a_{p-3} = \delta b_{p-2}$ shows that $\bar{a} = \tilde{\theta}(b_{p-2}, c_{p-1})$ hence $\bar{a} \in \operatorname{Im} \tilde{\theta} = \operatorname{Im} \theta$.

LEMMA 12.6. *Suppose* $\bar{a}, \bar{a}' \in K_{p+1}(\Omega^{n-p}R)$. *If there exists* $\bar{c} \in K_p(\Omega^{n-p+1}R)$ *such that* $\bar{a}' = \phi(\bar{c})\bar{a}$, *then* $\psi(\bar{a}') = \psi(\bar{a})$.

Proof. Let $a_{p-1} \in Y_p(\Omega^{n-p}R)$ represent $\bar{a}$, $c_{p-2} \in Y_{p-1}(\Omega^{n-p+1}R)$ represent $\bar{c}$, hence if $\delta b_{p-1} = c_{p-2}$ and $h_{p-1} = db_{p-1}$ then $h_{p-1}a_{p-1}$ represents $\bar{a}'$. So $\psi(\bar{a})$ is the image of $(1, a_{p-1})$ in $E_2^{p-1,n-p+1}$ and $\psi(\bar{a}')$ is the image of $(1, h_{p-1}a_{p-1})$. Since $(1, b_{p-1}^{-1}, 1) \in Z_1^{p,n-p+1}$, then

$$(1, b_{p-1}^{-1}, 1).(1, a_{p-1}) = (\delta b_{p-1}, db_{p-1}a_{p-1}) = (\delta b_{p-1}, h_{p-1}a_{p-1}).$$

On the other hand, since $d\delta b_{p-1} = dc_{p-2} = 1$, then

$$(\delta b_{p-1}, 1) \in Z_1^{p-2,n-p+2},$$

so $(1, a_{p-1}) \sim (1, h_{p-1}a_{p-1})$. This proof has been carried out for the case p is odd. A similar one works for p even. Hence $\psi(\bar{a}) = \psi(\bar{a}')$.

LEMMA 12.7. *Suppose* $\bar{a}, \bar{a}' \in K_{p+1}(\Omega^{n-p}R)$ *and* $\psi(\bar{a}) = \psi(\bar{a}')$. *Then there is a* $\bar{c} \in K_p(\Omega^{n-p+1}R)$ *such that* $\bar{a}' = \phi(\bar{c})\bar{a}$.

Proof. Suppose $a_{p-1}, a'_{p-1} \in Y_p(\Omega^{n-p}R)$ represent $\bar{a}$ and $\bar{a}'$ respectively, then $\psi(\bar{a})$ and $\psi(\bar{a}')$ are represented by $(1, a_{p-1})$ and $(1, a'_{p-1})$.

Suppose $(1, a_{p-1}) \sim (1, a'_{p-1})$ in $Z_2^{p-1,n-p+1}$. This means there exist $(b_{p-1}, b_p) \in Z_1^{p,n-p+1}$, c_{p-2} such that $db_p = dc_{p-2} = 1$, $db_{p-1}^{-1}a_{p-1}\delta b_p = a'_{p-1}$, $\delta b_{p-1}^{-1} = c_{p-2}$.

Since $dc_{p-2} = \delta c_{p-2} = 1$, c_{p-2} represents an element $\bar{c}$ of $K_p(\Omega^{n-p+1}R)$. On the other hand, $b_p \in M_p(\Omega^{n-p}R)$ because $db_p = 1$, and $\phi(\bar{c})$ is the image of db_{p-1}^{-1} in $K_{p+1}(\Omega^{n-p}R)$. Since δb_p is a boundary, it follows from the previous formulae that $\bar{a}' = \phi(\bar{c})\bar{a}$.

LEMMA 12.8. $\operatorname{Im} \psi = \operatorname{Ker} \theta$.

Proof. (i) $\operatorname{Im} \psi \subset \operatorname{Ker} \theta$.

Let $\bar{a} \in K_{p+1}(\Omega^{n-p}R)$, and choose $a_{p-1} \in Y_p(\Omega^{n-p}R)$ representing $\bar{a}$, then $\psi(\bar{a})$ is represented by the pair $(1, a_{p-1}) \in Z_2^{p-1,n-p+1}$ and $\theta\psi(\bar{a})$ is represented by $k_{p-3} = \delta 1 = 1$, hence $\psi(\bar{a})$ is in $\operatorname{Ker} \theta$.

(ii) $\operatorname{Im} \psi \supset \operatorname{Ker} \theta$.

Let $\bar{a} \in E_2^{p-1,n-p+1}$, and assume (a_{p-2}, a_{p-1}) is a representative in $Z_2^{p-1,n-p+1}$. Then $\theta(\bar{a})$ is represented by $k_{p-3} = \delta a_{p-2}$,

$$k_{p-3} \in Y_{p-2}(\Omega^{n-p+1}R).$$

Suppose $\bar{a} \in \operatorname{Ker} \theta$, then there exists $f_{p-2} \in M_{p-1}(\Omega^{n-p+1}R)$ such that $\delta f_{p-2} = k_{p-3}$, hence $\delta(a_{p-2}k_{p-3}) = 1$ and, in virtue of Lemma 12.3, (a_{p-2}, a_{p-1}) is equivalent to an element of the form $(1, x_{p-1})$, i.e., $\bar{a} \in \operatorname{Im} \psi$.

THEOREM 12.9. *The sequence*

$$K_p(\Omega^{n-p+1}R) \xrightarrow{\phi} K_{p+1}(\Omega^{n-p}R) \xrightarrow{\psi} E_2^{p-1,n-p+1} \xrightarrow{\theta} K_{p-1}(\Omega^{n-p+1}R)$$
$$\xrightarrow{\phi} K_p(\Omega^{n-p}R)$$

is exact for $n \geqslant p > 2$.

Remark. $\phi: K_p(\Omega^{n-p+1}R) \to K_{p+1}(\Omega^{n-p}R)$ is a group homomorphism. From Lemma 12.7 it follows that $\operatorname{Ker} \psi = (0)$ implies ψ is injective. Since the sequence is exact, $\operatorname{Ker} \theta$ is the image of Q in $E_2^{p-1,n-p+1}$, hence, if $\operatorname{Ker} \theta = \{*\}$, $E_2^{p-1,n-p+1} \cong Z_2^{p-1,n-p+1}/Q$ and θ becomes a group homomorphism, which is also injective, cf. [19, Section 3].

Remark. For $p = 1$ the maps ϕ and θ are not defined, and ψ is an isomorphism. For $p = 2$ the second ϕ and θ are not defined, but the sequence $K_2(\Omega^{n-1}R) \xrightarrow{\phi} K_3(\Omega^{n-2}R) \xrightarrow{\psi} E_2^{1,n-1}$ is still exact.

13. $K_2{}^{\mathrm{I}} = K_2{}^M$

Of course, the oldest of the K_2's is Milnor's. For a given unital ring R this is defined as the kernel in the extension of groups

$$\text{(A)} \qquad 1 \to K_2{}^M(R) \to ST(R) \to EL(R) \to 1,$$

which is in fact a universal central extension of the group $EL(R)$; furthermore, $K_2{}^M(R)$ is the center of $ST(R)$ so certainly abelian. For a ring with 1, the Steinberg group $ST(R)$ has generators $x_{ij}(r)$, $i \neq j$ natural numbers, $r \in R$, which satisfy the relations

$$\begin{aligned}
&\text{(a)} \quad x_{ij}(r)\, x_{ij}(x) = x_{ij}(r+s)\\
&\text{(b)} \quad [x_{ij}(r), x_{kl}(s)] = 1 \qquad \text{for } i \neq l,\\
&\qquad\qquad\qquad\qquad\quad\; = x_{il}(rs) \quad \text{for } i \neq l,\, j = k.
\end{aligned}$$

These imply $[x_{ij}(r), x_{ki}(s)] = x_{kj}(-sr)$ for $j \neq k$, where $[u, v]$ stands for the commutator $uvu^{-1}v^{-1}$.

The surjection $ST(R) \to EL(R)$ is given by $x_{ij}(r) \mapsto I + re_{ij}$ (this is a homomorphism because the Steinberg relations are just abstracted from those which hold for every ring in the elementary group) and the definitions are extended in the usual manner to rings without identity [15, Lemma 8.4]. Explicitly, Swan has shown that for any ring R the group $ST(R)$ may be described as the group generated by elements $x_{ij}(r)$, $r \in R$, under conjugation by $ST(\mathbf{Z})$ as a group of operators. We tacitly regard $\mathbf{Z}$ and R as a subring, respectively, an ideal, of R^+. Besides the relations (a) and (b) one has, for $m \in \mathbf{Z}$, $r \in R$

$$\begin{aligned}
&\text{(c)} \quad x_{ij}(m)\, x_{ij}(r)\, x_{ij}(-m) = x_{ij}(r),\\
&\text{(d)} \quad x_{ij}(m)\, x_{kl}(r)\, x_{ij}(-m) = x_{kl}(r) \text{ when } i \neq l,\, j \neq k\\
&\qquad\qquad\qquad\qquad\qquad\qquad\; = x_{il}(mr)\, x_{kl}(r) \text{ when } i \neq l,\, j = k.
\end{aligned}$$

These imply $x_{ij}(m)\, x_{ki}(r)\, x_{ij}(-m) = x_{kj}(-mr)\, x_{ki}(r)$ when $j \neq k$. For all rings, (A) is a central extension of groups (the group of operators $ST(\mathbf{Z})$ goes to $EL(\mathbf{Z})$).

Consider for any of our cotriples E the cokernel $\widetilde{GL}(R)$ of the composite map $GL(E\Omega R) \xrightarrow{GL\epsilon\Omega} GL(\Omega R) \to GL(ER)$. Thus $\widetilde{GL}(R)$ is the group $GL(ER)$ modulo its normal subgroup generated by the image of $GL(E\Omega R)$. We now choose $E = E^{\mathrm{I}}$ and for the balance of this section usually drop the superscript. In this case

$$\widetilde{GL}(R) = EL(ER)/EL((ER)^+, \Omega R)$$

and the map given by $X_r \mapsto r$ induces a surjection $\widetilde{GL}(R) \to EL(R)$. Since GL is left exact, its kernel is $K_2{}^{\mathrm{I}}(R)$ in virtue of Corollary 9.3.

THEOREM 13.1. *For every ring R, there exists a unique morphism $\Theta(R)$:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2{}^M(R) & \longrightarrow & ST(R) & \longrightarrow & EL(R) & \longrightarrow & 1 \\ \Theta(R) & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & K_2{}^{\mathrm{I}}(R) & \longrightarrow & \widetilde{GL}(R) & \longrightarrow & EL(R) & \longrightarrow & 1 \end{array}$$

between central extensions of $EL(R)$. This map is surjective and functorial in R.

Proof. We map $ST(R)$ to $\widetilde{GL}(R)$ by sending $x_{ij}(r)$ to $I + X_r e_{ij}$ and the operator group $ST(\mathbf{Z})$ to $EL(\mathbf{Z})$ by sending $x_{ij}(m)$ to $I + me_{ij}$. We must verify that the Steinberg relations (a) and (b) are satisfied and that conjugation in $ST(R)$ by $ST(\mathbf{Z})$ is taken to conjugation in $\widetilde{GL}(R)$ by $EL(\mathbf{Z})$, (c) and (d), cf. [17, Theorem 3.1].

(a)

$$(I + X_r e_{ij})(I + X_s e_{ij})(I + X_{r+s} e_{ij})^{-1} = I + (X_r + X_s - X_{r+s})\, e_{ij} \in EL(\Omega R)$$

since $X_r + X_s - X_{r+s} \in \Omega R$.

(b) The commutator $[I + X_r e_{ij}, I + X_s e_{kl}] = 1$ when $i \neq l$, $j \neq k$, while for $i \neq l$ we find $[I + X_r e_{ij}, I + X_s e_{jl}]. (I + X_{rs} e_{il})^{-1} = I + (X_r X_s - X_{rs})\, e_{il} \in EL(\Omega R)$ since $X_r X_s - X_{rs} \in \Omega R$.

(c) $[I + me_{ij}, I + X_r e_{ij}] = 1$.

(d) $[I + me_{ij}, I + X_r e_{kl}] = 1$ whenever $i \neq l$, $j \neq k$ while for $i \neq l$ we find

$$[I + me_{ij}, I + X_r e_{jl}]\, (I + X_{mr} e_{il})^{-1} = I + (mX_r - X_{mr}) \in EL(\Omega R)$$

since $mX_r - X_{mr} \in \Omega R$.

The map we obtain is clearly surjective since $\widetilde{GL}(R)$ is generated by the images of the $I + X_r e_{ij}$ under conjugation by $EL(\mathbf{Z})$.

It follows that the bottom extension of the elementary group is also central. To prove uniqueness of the map Θ, there is a standard argument. Remark that the existence of a second satisfactory map Θ': $ST(R) \to \widetilde{GL}(R)$ gives rise to a map χ: $ST(R) \to \widetilde{GL}(R)$ defined by sending $u \in ST(R)$ to $\chi(u) = \Theta'(u)\, \Theta^{-1}(u)$. This image is in Ker $(\widetilde{GL}(R) \to EL(R))$

which is the central subgroup $K_2(R)$ of $\widetilde{GL}(R)$. It follows that χ is a homomorphism $ST(R) \to K_2(R)$, because

$$\chi(uv) = \Theta'(u)\,\Theta'(v)\,\Theta^{-1}(v)\,\Theta^{-1}(u)$$

and $\Theta'(v)\,\Theta^{-1}(v) \in K_2(R)$, so $\chi(uv) = \chi(u)\,\chi(v)$. Now every element of $ST(R)$ is a product of commutators, while $K_2(R)$ is abelian. The homomorphism χ is consequently trivial and $\Theta = \Theta'$. Functoriality of Θ is obvious.

Following a lecture by D. Quillen, we were able to prove in our talk at the Seattle conference that Θ is an isomorphism on commutative rings. The next day, R. G. Swan had a proof which works for all rings, and the day after S. M. Gersten claimed an even stronger result which implies this fact as a special case. All three proofs depend heavily on the work of Quillen as well as, in linear progression, certain facts about free rings. Gersten's result will appear in [7, Theorem 3.13], so we refrain from detailing our partial contribution. Suffice it to state the following theorem:

THEOREM 13.2. *The map Θ is an isomorphism of functors; $\widetilde{GL}^{\mathrm{I}} \cong ST$ and $K_2{}^{\mathrm{I}} \cong K_2{}^{M}$. For every unital ring R, the group $\widetilde{GL}^{\mathrm{I}}(R)$ is a universal central extension of the elementary group $EL(R)$, whose center is $K_2{}^{\mathrm{I}}(R)$.*

Since $K_2{}^{\mathrm{I}}$ is Swan's K_2, this affirmatively solves a problem of several years' standing [15, Corollary 8.6], [4, Section 0]. It is a tribute to the power of Quillen's methods that no direct proof has so far been devised.

14. HOMOTOPY WITH RESPECT TO A COTRIPLE

If one wishes to ape the topological theory and define homotopy between ring homomorphisms, one is at a loss to define the product of the unit interval [0, 1] with a ring. However, there exists a plausible candidate for "algebraic" mappings from [0, 1] to a ring R, viz. the polynomial ring $R[t]$. A polynomial q assigns to each $r \in R$ an element $q(r) \in R$ and in particular to the endpoints 0 and 1 the values $q(0)$ and $q(1)$ respectively. In case R has no identity element, $q(1)$ is of course just shorthand for the sum of the coefficients of q. Such a polynomial can be regarded as a path on R. One therefore has projections p_0 and p_1: $R[t] \rightrightarrows R$ which to each path q on R associate its beginning and endpoints $q(0)$ and $q(1)$.

This is exactly dual to the ideas developed in [13] for schemes and it is what led Karoubi and Villamayor to call two ring homomorphisms f, g: $B \rightrightarrows C$ simply homotopic whenever there is a homotopy on C connecting f and g, i.e., there is an s: $B \to C[t]$ with $p_0 s = f$, $p_1 s = g$, thus:

$$\begin{tikzcd} & C[t] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ B \arrow[ur, "s"] \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \end{tikzcd} .$$

This relation is reflexive and symmetric; by making it transitive one defines a homotopy relation $\sim$ between maps which is preserved by composition [11, Definition 2.1] and [6, Lemma 1.1].

A functor F: $\mathbf{Rg} \to \mathbf{Gr}$ is called a homotopy functor if $f \sim g$ implies $Ff = Fg$. Of course, it is enough to demand this when f and g are simply homotopic. It is easy to see that F is a homotopy functor if and only if for every ring R the maps $Fp_i(R)$: $F(R[t]) \rightrightarrows F(R)$ are isomorphisms, $i = 0, 1$.

We argue that most of this theory can be adequately formulated solely in terms of the cotriple E^{III}. Indeed, a ring homomorphism f is simply homotopically trivial if it is simply homotopic to the null map, i.e., if there exists a map s: $B \to EC$ such that

$$\begin{tikzcd} & EC \arrow[d, "\epsilon(C)"] \\ B \arrow[ur, "s"] \arrow[r, "f"'] & C \end{tikzcd}$$

commutes. Here $EC = E^{\mathrm{III}}C = tC[t] = \operatorname{Ker} p_0$ and $\epsilon = p_1 \mid E$. In words there exists a homotopy which connects f with the null map. A ring R is contractible if 1_R is simply homotopically trivial, i.e., is split by $\epsilon(R)$. Paths with basepoint on R are in this language the elements of ER, loops those of $\Omega R = t(1 - t)\, R[t] = \operatorname{Ker} \epsilon(R)$. A homotopy functor clearly vanishes on contractible rings; because of the splitting μ: $E \to E^2$ the path rings ER are all contractible, and vanishing on all ER is sufficient to make a functor F a homotopy functor provided for every ring R

$$F(ER) \to F(R[t]) \xrightarrow{Fp_0(R)} F(R)$$

is exact. This mild condition is certainly satisfied by GL, K_0, $K_1{}^B$ and others [18, Section 1].

We offer this review of the ideas behind the $K^{\mathrm{III}*}$-theory of Karoubi and Villamayor as a justification for proceeding by analogy and agreeing on the following terminology. Whenever ϵ: $E \to I$ is a morphism between endofunctors on **Rg**, we spak of ER as the paths (with base point) on R and we call Ker $\epsilon(R) = \Omega R$ the loops on R. In the balance of the paper we propose to show that this purely formal convention, in the cases of cotriples I and II no less than III, provides a heurstic tool of some penetration.

Let F be a functor from **Rg** to **Gr**. We call FE the paths (starting in the origin) on F, then $\operatorname{Im} F\epsilon: FE \to F$ (set theoretic image) is the connected component F° and the cosets form $\pi_0 F$. This is a homogeneous space, but in the case $F = GL$, $\operatorname{Im} F\epsilon(R)$ is normal in $F(R)$ so then $\pi_0 F$ is a functor to groups. The functor F is connected when $\pi_0 F = 1$. Loops on F is the name for the functor $F\Omega$; in the case of the K^*-theories, we have $K_1{}^* = \pi_0 GL$ while $K_n{}^* = K_1{}^* \Omega^{n-1} = (\pi_0 GL)\, \Omega^{n-1} = \pi_{n-1}^* GL$ by definition, where the higher homotopy groups $\pi_n{}^*$ are introduced just as the Hurewicz groups in topology. They should not be confused with the simplicial homotopy groups π_n of Section 7.

Another remark is that the K-theories are trivially homotopy theories: K_n vanishes on ER hence on contractible (= projective) rings for $n \geqslant 1$, see Section 7. (In Case I we even have $K_0(ER) = 0$, see Section 6). Thus the K-theories are homotopy theories with respect to the defining cotriple, e.g., Swan's theory K^{I} satisfies the homotopy axiom with respect to E^{I} rather than E^{III} as considered *a priori* desirable in his lecture to the Seattle conference.

15. Covering Functors with Respect to a Cotriple

Covering groups have made their appearance in the theory of K_2 starting with C. Moore's work. Karoubi and Villamayor take another point of view in their description of $K^{-2} = K_2^{\mathrm{III}*}$ [11, Appendix 6]. Independently and more fully covering groups in this theory were investigated by M. I. Krusemeyer [10, Section 3] but in part unpublished. We here take up these questions in a more functorial setting and attach coverings with regard to an arbitrary cotriple E, using our new dictionary.

Let E: **Rg** $\to$ **Rg** be a cotriple and F: **Rg** $\to$ **Gr** a functor. We do not have the notion of a continuous map, but we can still imitate covering spaces in topology. Call a functorial morphism $\phi: H \to F$ a covering of F provided (a), (b) and (c) hold:

(a) H is connected, i.e., $H\epsilon$ is always surjective or $\pi_0 H = 1$,

(b) H covers the connected component of F, i.e., $\operatorname{Im}\phi = F^\circ$. The split morphism $E\epsilon: E^2 \to E$ shows that $FE\epsilon: FE^2 \to FE$ is surjective, i.e., FE is a connected functor, hence (b) implies that $\phi E: HE \to FE$ is surjective.

(c) We want the fiber $N = \operatorname{Ker}\phi$ to be discrete, i.e., there are no paths on N, i.e., $NE = 1$. Because of the above this is equivalent to $\phi E: HE \to FE$ being an isomorphism (unique lifting of paths).

Given F, we write $Y_1 = \operatorname{Ker}(F\epsilon: FE \to F)$ and $M_2 = \operatorname{Ker}(FE\epsilon: FE^2 \to FE)$. In the commuting diagram

$$\begin{tikzcd}
1 \arrow[r] & M_2 \arrow[r] \arrow[d, "d"'] & FE^2 \arrow[r, "FE\epsilon"] \arrow[d, "F\epsilon E"'] & FE \arrow[r] \arrow[d, "F\epsilon"] & 1 \\
1 \arrow[r] & Y_1 \arrow[r] & FE \arrow[r, "F\epsilon"'] & F &
\end{tikzcd}$$

the rows are exact; we have written d for the induced map between kernels. We claim that $\operatorname{Im} d$ is normal in FE. Indeed, let R be any ring and take $m \in M_2(R)$ and $y \in F(ER)$. The element $F\mu(y)\, mF\mu(y^{-1})$ lives in $M_2(R)$ and maps to $z = yd(m)\, y^{-1}$ under d; since $z \in \operatorname{Im} d$ we have proved our contention.

Now write $\hat{F}$ for the cokernel of the composite map $M_2 \xrightarrow{d} Y_1 \to FE$. Then $\hat{F}$ is again a functor from **Rg** to **Gr**. The map $F\epsilon: FE \to F$ factors through $\hat{F}$. This induces a morphism $\Lambda: \hat{F} \to F$ which evidently satisfies conditions (a) and (b).

LEMMA 15.1. *Let $\phi: H \to F$ satisfy conditions* (a) *and* (b). *If ϕ satisfies* (c), *then $\hat{\phi}: \hat{H} \to \hat{F}$ is an isomorphism. The converse holds if $\Lambda: \hat{F} \to F$ is a covering of F.*

Proof. If ϕ is a covering, ϕE and ϕE^2 are isomorphisms. From the functoriality of the defining diagram it follows that $\hat{\phi}: \hat{H} \to \hat{F}$ is an isomorphism.

On the other hand, the commuting square of surjections

$$\begin{tikzcd}
\hat{H}E \arrow[r, two heads, "\Lambda(H)E"] \arrow[d, "\hat{\phi}E"'] & HE \arrow[d, "\phi E"] \\
\hat{F}E \arrow[r, two heads, "\Lambda(F)E"'] & FE
\end{tikzcd}$$

becomes a square of isomorphisms once $\Lambda(F)E$ and $\hat{\phi}E$ are isomorphisms, showing that $\phi: H \to F$ satisfies (c).

DEFINITION. Given a cotriple E, a covering $\psi: G \to F$ of F is called universal if for every covering $\phi: H \to F$ there exists a morphism $\alpha: G \to H$ such that $\phi \circ \alpha = \psi$.

Using functoriality, it is easy to see that the morphism α is then a covering of H and is uniquely determined. Two universal coverings of F (if they exist) are isomorphic in the obvious way.

LEMMA 15.2. *If $\Lambda: \hat{F} \to F$ is a covering, it is a universal covering of F.*

Proof. Since $\hat{\phi}: \hat{H} \to \hat{F}$ is an isomorphism by the previous lemma, $\alpha = \Lambda(H) \circ \phi^{-1}: \hat{F} \to H$ fills the requirements.

THEOREM 15.3. *For every functor F: $\mathbf{Rg} \to \mathbf{Gr}$, the morphism $\Lambda: \hat{F} \to F$ is a universal covering with respect to the given cotriple. It is at the same time a universal covering of the connected component F°.*

Proof. It suffices to prove that $\Lambda E: \hat{F}E \to FE$ is injective (on all rings). With the notations introduced above the defining diagram for $\hat{F}E$

$$\begin{tikzcd}
1 \arrow[r] & M_2E \arrow[r] \arrow[d, "dE"'] & FE^3 \arrow[d, "F\epsilon E^2"'] & FE^2 \arrow[l, shift right, "FE\epsilon E"'] \arrow[l, leftarrow, shift left, "FE\mu"] \arrow[r] \arrow[d, "F\epsilon E"] & 1 \\
1 \arrow[r] & Y_1E \arrow[r] & FE^2 & FE \arrow[l, shift right, "F\mu"'] \arrow[l, leftarrow, shift left, "F\epsilon E"] \arrow[r] & 1
\end{tikzcd}$$

commutes and has split exact rows. We have already seen that $\operatorname{Im} dE \lhd Y_1E$; call the cokernel TE. The splitting implies that the sequence of vertical kernels splits, and the Snake Lemma then shows that $TE = 1$. It follows that the induced map $\Lambda E: \hat{F}E \to FE$ is an isomorphism and the composite $\Lambda E \circ F\mu$ is the identity on FE. Furthermore, $\hat{F} \to F^\circ$ is clearly a covering. But then $\hat{\hat{F}} \to \hat{F}^\circ$ is an isomorphism by Lemma 15.1 and by the same token $\hat{\hat{F}} \cong \hat{F}$, which finishes the proof.

In analogy to the universal covering of a topological group, one may define $\pi_1F = \operatorname{Ker} \Lambda$, and it is easily checked that coverings of F correspond to subfunctors of this fundamental group. Moreover, since $\hat{\Lambda}: \hat{\hat{F}} \to \hat{F}$ is an isomorphism by Lemma 15.1, we know that its fiber $\pi_1\hat{F} = 1$, i.e., the universal covering functor $\hat{F}$ is simply connected.

The $\pi_1 F$ just defined coincides with the simplicial fundamental group of F using the standard cotriple resolution. Indeed, by comparing the defining diagram for $\hat{F}$ with the Moore complex $M(F\underline{\mathscr{E}})$ described in Sections 7 and 8, we see that $\pi_1 F = H_1(M(F\underline{\mathscr{E}}))$ and this first homology group is the simplicial fundamental group of F, cf. [4, Section 1], [9, Definition 9.4].

Recall that in Theorem 8.4 we proved $K^{\mathrm{III}} = K^{\mathrm{III}*}$ using besides left exactness of E^{III}: $\mathbf{Rg} \to \mathbf{Rg}$ only that GL is left exact. In case III we therefore have for every left exact F: $\mathbf{Rg} \to \mathbf{Gr}$ that $M(F\underline{\mathscr{E}}) = F\mathscr{E}$, the functor F applied to the canonical resolution of Section 2. As in Section 13, write $\tilde{F}$ for the cokernel of the composite map $FE\Omega \to FE\Omega \to F\Omega$. Since $E\Omega = C_2$, for left exact F we know by the discussion of $\hat{F}$ that $\operatorname{Im} FE\Omega \lhd FE$ and that $\hat{F} = \tilde{F}$. Moreover

$$\pi_1 F = H_1(M(F\underline{\mathscr{E}})) = H_1(F\mathscr{E}) = (\pi_0 F)\Omega = \pi_1{}^* F.$$

The next proposition recaptures a result of Krusemeyer. It was an attempt to interpret certain manipulations with polynomials in unpublished notes of his, which gave us the idea for this section.

Proposition 15.4. *For $E = E^{\mathrm{III}}$ and F: $\mathbf{Rg} \to \mathbf{Gr}$ a left exact functor, the morphism $\tilde{F}^{\mathrm{III}} \to F$ induced by $F\epsilon$ is a universal covering of F. Its fiber is $\pi_1^{\mathrm{III}} F = (\pi_0^{\mathrm{III}} F)\Omega = \pi_1^{\mathrm{III}*} F$.*

Proof. The above and Theorem 15.3. It can be proved that $(\pi_1^{\mathrm{III}*} F)(R)$ is a central subgroup of $\tilde{F}^{\mathrm{III}}(R)$ for all rings R, [10, Proposition 3.6].

Proposition 15.5. *Let $E = E^{\mathrm{I}}$. Then $\widetilde{GL}^{\mathrm{I}} \to GL$ is a universal covering and $K_2{}^{\mathrm{I}} = \pi_1{}^{\mathrm{I}} GL$ is its fiber.*

Proof. From Corollary 9.3 it follows that $\widetilde{GL} = \widehat{GL}$ and then use Theorem 15.3.

In these cases, the analogy with the coverings of a topological group becomes even more striking. The universal covering $\tilde{F}$ is "paths with base point on F modulo homotopically trivial loops" and $\pi_1 F$ is "homotopy classes of loops on F."

Combining this with Theorem 13.2 we prove the following result.

THEOREM 15.6. *The morphism $ST \to GL$ is a universal covering of the general linear group with respect to the free cotriple E^{I}. Its fiber is $\pi_1{}^{\mathrm{I}}GL = K_2{}^M$.*

Sometimes a group is called connected when it equals its commutator subgroup and a universal central extension of such a group is referred to as a universal covering [8]. For rings with identity, $ST(R) \to EL(R)$ is now universal in both senses. It would be interesting to know how general a phenomenon this is in the context of algebraic groups. Universal central extensions of Chevalley groups are discussed in [14].

Note added in proof. (1) The remark on commutative excision at the end of Section 9 is not quite true. There is a counter example, also due to Swan, which does not depend on number theory. It is mentioned in M. R. Stein, Relativizing functors on rings and algebraic K-theory, *J. Algebra* **19** (1971), 140–152. (2) Results concerning the query which ends the paper have been obtained for GL_n by the junior author, to appear.

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