Milatz, J. M. W.
van Zolingen, J. J.
1953

Physica XIX
181–194

THE BROWNIAN MOTION OF ELECTROMETERS

by J. M. W. MILATZ and J. J. VAN ZOLINGEN *)

Physisch Laboratorium der Universiteit Utrecht, Nederland

Synopsis

In § 2 the equation of motion of an electrometer appears to be of the 3rd order. For sufficient high labilisation, critical damping occurs at two values of the resistance R of the measuring circuit.

The Brownian motion is shown to be due for the greatest part to the irregular molecular collissions of the surrounding air, the thermal voltage fluctuations contributing only very slightly.

From the expression

$$\overline{\Delta Q^2} = 2kT\tau/R + 2kT\,(AC_0 + B^2)\,C_0/B^2$$

it follows that for a Hoffmann electrometer the root of mean square error at charge measurements, caused by the molecular collisions appears to be seventy times larger than the error arising from the thermal voltage fluctuations. Both the fluctuating torque exerted by the molecular collisions and the air damping, are aspects of the interaction of the instrument with the surrounding air. Air damping not being essential for the proper working of the electrometer, the Brownian motion may be reduced substantially by evacuating the instrument and applying an artificial damping which introduces almost no fluctuations.

In a following paper †) the experimental verification of this method, reducing the Brownian energy by a factor of one hundred, will be described.

§ 1. *Introduction.* In statistical mechanics it is shown that all bodies have a mean thermal energy of $\frac{1}{2}kT$ for each degree of freedom, where k is Boltzman's constant and T the absolute temperature. This is the law of equipartition of energy which, as may be expected, is valid for the random deflections of an electrometer as

*) Now at the laboratory of N.V. KEMA, Arnhem.

†) The reduction of the Brownian motion of electrometers, by J. M. W. Milatz, J. J. van Zolingen and B. B. van Iperen. This paper will be referred to as II.

well. Indeed, E g g e r s [1]) who measured the Brownian motion of a Hoffmann electrometer has been able to verify the relation:

$$\overline{\tfrac{1}{2} A\varphi^2} = \tfrac{1}{2}kT, \tag{1.1}$$

φ = deflection angle,
A = directional force of the suspension wire,

equating the mean value of the potential energy of the electrometer-system to the thermal energy $\frac{1}{2}kT$. According to E g g e r s [2]), the Brownian motion has to be ascribed completely to the fortuitous collisions of the air molecules with the system. However, this may be true only, when the influence of the electrical circuit is so small that it can be neglected.

In our theoretical analysis of the problem, we start by drawing up the equation of motion of the instrument. From this equation it will appear that the resistance R of the measuring circuit or subsequently the leakage resistance of the instrument, brings about an electrical damping, by converting into heat the induction currents produced by the motion of the electrometersystem. However, the design of many electrometers and the circuits used is such, that this intrinsic electrical damping is rather small compared with the air damping.

It will be known that energy dissipation (damping) is always accompanied by energy fluctuations: in fact, the damping may be considered to be the systematical part of the transformation of energy into heat and the fluctuations the insystematical part, originating from the discontinuous structure and the thermal motion of matter and electric charge. Thus the Brownian motion of an electrometer cannot be of molecular origine only and the thermal voltage fluctuations in the resistance R must contribute also to the fortuitous deflections.

It may be shown that the intensity of the respective energy fluctuations is proportional to the corresponding damping. In the present paper expressions are derived for the precision to be obtained when the damping air and the resistance are not at the same temperature. These expressions show the large increase in precision to be expected, if it would be possible to replace the air damping, which in fact is not essential for the proper functioning of the instrument, by a damping which introduces no fluctuations [3]).

In § 4 and § 5 formulae are derived and discussed describing the reduction of the Brownian energy and the increase in measuring precision respectively, when this special kind of artificial damping is applied.

Experiments in which the Brownian motion of an electrometer is strongly reduced by artificial damping [2]) will be described in a following paper.

§ 2. *The equation of motion of the electrometer.* The electrometer under consideration cf. fig. 1 consists mainly of a system, which is suspended on a thin fibre, and of the stationary electrodes E_1 and E_2, which are placed opposite to the electrometersystem. To increase the sensitivity, the system is maintained at a fixed auxiliary potential V_H against earth. We will assume that the mechanical and

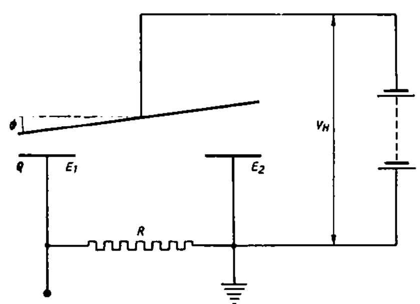

Fig. 1. Schematic representation of the electrometer.

electrical zeros of the instrument coïncide. In the calculations, two cases have to be distinguished:

a) The resistance R between the electrodes is so high ($R > 10^{14}\,\Omega$) or so low ($R < 10^7\,\Omega$), that it has a negligeable influence on the motion of the system. This corresponds to measurements with constant charge or with constant voltage on the measuring electrode respectively.

b) The value of R is such that its influence is no more to be neglected.

a) Let us assume that the capacities between each one of the stationary electrodes and the system may be described to a sufficient degree of approximation as quadratic functions of the deflection angle φ. Because of the symmetry supposed, we have:

$$\begin{aligned} C_1 &= C_0 + a\varphi + \tfrac{1}{2}b\varphi^2 \\ C_2 &= C_0 - a\varphi + \tfrac{1}{2}b\varphi^2, \end{aligned} \tag{2.1}$$

C_1 and C_2 being the capacities between the system and E_1, and between the system and E_2 respectively. The electrostatic torque K acting on the electrometersystem is given by

$$K = \tfrac{1}{2}V_1^2\, \mathrm{d}C_1/\mathrm{d}\varphi + \tfrac{1}{2}V_2^2\, \mathrm{d}C_2/\mathrm{d}\varphi \tag{2.2}$$

where V_1 and V_2 are the potential differences between the system and the electrodes E_1 and E_2 respectively.

Assuming the potential v to be measured and also the corresponding deflection angle φ to be small, elementary calculation yields for the resulting electrostatic torque K:

$$K = aV_H\, v + bV_H^2\, \varphi. \tag{2.3}$$

The equation of motion of the moving system is generally given by

$$I\ddot{\varphi} + \beta\dot{\varphi} + D\varphi = K, \tag{2.4}$$

where I is the moment of inertia of the system around the axis of rotation, β the coefficient of mechanical damping and D the directional torque of the suspension fibre.

Substituting (2.3) and inserting:

$$D - bV_H^2 = A \text{ and } aV_H = \mathrm{B} \tag{2.5}$$

we have

$$I\ddot{\varphi} + \beta\dot{\varphi} + A\varphi = Bv \tag{2.6}$$

This equation describes the motion of the electrometer, when the potential v to be measured is kept constant .The voltage sensitivity proves to be

$$S_v = aV_H/(D - bV_H^2) \equiv B/A \tag{2.7}$$

From (2.7) one observes the well known fact that the sensitivity S_v may be increased infinitely by raising the potential V_H to the value $V_H = (D/b)^{1/2}$.

When the electrometer is used for charge measurements, the charge Q applied to the collecting electrode E_1 remains constant while R is very large. Now, the potential v of E_1 is a function of the angle of deflection φ. An elementary calculation shows, that

$$v = Q/C_0 - (aV_H/C_0)\, \varphi \tag{2.8}$$

Substituting (2.8) in (2.6) one obtains the equation of motion of the electrometer, when used for measuring charges,

$$I\ddot{\varphi} + \beta\dot{\varphi} + (AC_0 + B^2)/C_0\, \varphi = (B/C_0)\, Q \tag{2.9}$$

The charge sensitivity S_Q is

$$S_Q = B/(AC_0 + B^2) \qquad (2.10)$$

Eq. (2.10) and (2.9) show, that an increase in sensitivity is accompanied by an increase of the period of the instrument. The period when measuring charge is given by

$$\tau_Q = 2\pi(S_Q I C_0/B)^{1/2} \qquad (2.11)$$

b) Next the case is investigated where the influence of the measuring circuit, in fig. 1 represented by the resistance R, may no more be neglected.

The current in the circuit may be described by the equation,

$$RC_0 \dot{v} + BR\dot{\varphi} + v = 0, \qquad (2.12)$$

as is easily verified by applying(2.1). Eliminating v between (2.6) and (2.12), one obtains the general equation of motion of the electrometer system

$$RC_0 I \dddot{\varphi} + (RC_0\beta + I)\ddot{\varphi} + (\beta + B^2R + RC_0A)\dot{\varphi} + A\varphi = 0 \qquad (2.13)$$

It is clear that the equation of motion becomes one of the third order now, because the reactance (capacitance) of the circuit has been taken into account. Comparing this to the equation of motion of a galvanometer, one observes readily that only by neglecting the inductance of the circuit, the well known second order differential equation for that instrument could have been derived.

However, in many cases in practice, for the electrometer as well as for the galvanometer the third order equation may be replaced by one of the second order to a sufficient degree of approximation, as can be shown by inserting the actual values of the constants involved.

For instance for the electrometer we used in the experiments to be described later, with the constants shown in table I, (2.6) is a good approximation.

When, however, the instrument is employed for measuring charges ($R > 10^{15}\,\Omega$) then (2.9) may be used.

Now the electrical damping, caused by dissipation of energy in the resistance R, will be investigated [2]). In order to simplify the calculation the mechanical damping β will be assumed to be zero tempo-

TABLE I

Constant	Hoffmann electrometer	Milatz electrometer	units
I	5×10^{-6}	$6{,}2 \times 10^{-2}$	g.cm
A	10^{-7}	$6{,}6 \times 10^{-2}$ *)	dyne cm rad^{-1}
B	$1{,}3 \times 10^{-3}$	$4{,}1 \times 10^{-2}$ *)	
C_u	2,9	3	cm
a	$3{,}3 \times 10^{-2}$	$8{,}25 \times 10^{-1}$	cm rad^{-1}
b	$9{,}5 \times 10^{-3}$	6,55	cm rad^{-2}
β_{min}	$7{,}2 \times 10^{-9}$	$1{,}1 \times 10^{-2}$	dyne cm sec. rad^{-1}
β_{critic}	$4{,}54 \times 10^{-7}$	$1{,}28 \times 10^{-1}$	id.
β_{max}	—	1,9	id.

*) Corresponding to the rather low auxiliary potential $V_H = 15$ V applied in our experiments †).

For this value of V_H, labilisation is still very small. Complete labilisation occurs at $V_H = 33$ V. The values given for the Hoffmann electrometer apply to the optimal sensitivity i.e. for a relatively high value of $V_H . \beta_{max}$ was the maximal damping obtainable with the artificial damping we used.

rally. Moreover we introduce the frequency **) $\nu_0 = (A/I)^{1/2}$ of the electrometer with $R = 0$ and the frequency $\nu_1 = [(A/I) + (B^2/IC_0)]^{1/2}$, when $R = \infty$ then (2.13) becomes

$$\dddot{\varphi} + (1/RC_0)\,\ddot{\varphi} + \nu_1 2\dot{\varphi} + (\nu_0^2/RC_0)\,\varphi = 0 \tag{2.14}$$

From (2.14) the degree of damping may be calculated as a function of the electrical time constant RC_0 of the circuit and the frequencies ν_1 and ν_0. In order to obtain critical damping it is necessary that two of the real roots of the characteristic equation of (2.14) be equal.

Then, as is shown in the general theory of ordinary differential equations of the 3rd order, the following relation must exist between the coefficients

$$\frac{4}{RC_0\nu_0} - \frac{1}{(RC_0\nu_0)^2}\left(\frac{\nu_1}{\nu_0}\right)^4 - \frac{18}{(RC_0\nu_0)^2}\left(\frac{\nu_1}{\nu_0}\right)^2 + 4\left(\frac{\nu_1}{\nu_0}\right)^6 + \frac{27}{(RC_0\nu_0)^2} = 0. \tag{2.15}$$

From fig. 2, one finds the type of motion for different values of the resistance R and the auxiliary tension V_H, the latter determining ν_0 and ν_1/ν_0.

Here, $(\nu_1/\nu_0)^2$ is plotted against the corresponding values of $RC_0\nu_0$ according to (2.15), thus the curves represent the locus of the values of these constants in the case of critical damping. The area between

†) Comp. II 8).

**) Reciprocal value of the period of the moving system.

them corresponds to an overdamped motion, the area outside to periodic motion.

§ 3. *Calculation of the Brownian deflections of electrical and of molecular origine.* Now the mean square of the deflections $\overline{\varphi_M^2}$ and $\overline{\varphi_E^2}$ caused by the molecular collisions and by the electrical voltage fluctuations respectively will be calculated separately. Then the condition of equal temperatures of the air and the resistance R may be avoided. The results thus obtained will be used in later chapters to derive expressions for the ultimate precision when measuring charges and to indicate methods to reduce the Brownian motion generally.

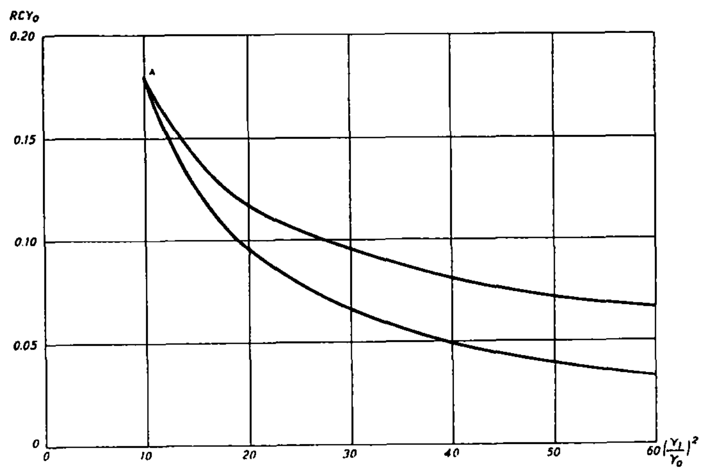

Fig. 2. Diagram representing the influence on the type of motion of $RC\,\nu_0$ and $(\nu_1/\nu_0)^2$.

One may start by introducing the fluctuation into (2.6) and (2.3) Substitution of ther andom mechanical torque, represented by K into (2.6) yields

$$I\ddot{\varphi} + \beta\dot{\varphi} + A\varphi = Bv + K \qquad (3.1)$$

When the thermal voltage fluctuations from the damping resistance R are designed by F, then (2.12) becomes

$$RC_0\dot{v} + RV_H a\dot{\varphi} + v = F \qquad (3.2)$$

By eliminating v from (3.1) and (3.2) one obtains

$$\frac{RC_0 I}{B}\dddot{\varphi} + \frac{(RC_0\beta + I)}{B}\ddot{\varphi} + \frac{(\beta + B^2R + RC_0 A)}{B}\dot{\varphi} + \frac{A}{B}\varphi =$$

$$= F + \frac{RC_0}{B}\dot{K} + \frac{1}{B}K \qquad (3.3)$$

or when replacing for simplification in (3.3) the coefficients of $\dddot{\varphi}$, $\ddot{\varphi}$, $\dot{\varphi}$ and φ by a, b, c and d respectively and those of $\dot{K}$ and K by g and f respectively, one obtains

$$a\dddot{\varphi} + b\ddot{\varphi} + b\dot{\varphi} + d\varphi = F + g\dot{K} + fK \qquad (3.3a)$$

From (3.3) the mean squares of the deflections of electrical and of mechanical origin, indicated by $\overline{\varphi_E^2}$ and by $\overline{\varphi_M^2}$ respectively, may be calculated separately with S c h o t t k y's method [5]), if the spectral distribution of F and K are known.

According to N y q u i s t [4]) the spectral intensity of F at a temperature T_R of the resistance R is given in the $\omega = 2\pi\nu$ scale by

$$I_{F^2}(\omega) = 2RkT_R/\pi \qquad (3.4)$$

The spectral intensity $I_{M^2}(\omega)$ of the fluctuating torque K, caused by the air molecules and corresponding to the macroscopic air damping β_M, at an absolute temperature T_M of the damping air is according to M i l a t z [3]), in the ω scale

$$I_{M^2}(\omega) = 2\beta_M kT_M/\pi \qquad (3.5)$$

The air damping β_M which occurs in this spectrum of mechanical noise has to be distinguished from the total damping β occurring in (3.3) and thus into the frequency characteristics (3.6) and (3.9). This will be discussed below.

In order to calculate $\overline{\varphi_E^2}$, let us assume $K = 0$ and $\dot{K} = 0$. Then from (3.3) or (3.3a) the frequency characteristic (or amplification factor) $g(\omega)$ for deflections of electrical origin, may be calculated readily [2]). One obtains

$$g_F^2(\omega) = 1/\{(d - b\omega^2) + (c - a\omega^3)^2\} \qquad (3.6)$$

The mean square of the corresponding deflections is given by

$$\overline{\varphi_E^2} = \int_0^\infty g_F^2(\omega)\, I_{F^2}(\omega)\, \mathrm{d}\omega \qquad (3.7)$$

Inserting (3.4) and (3.6) and evaluating the integral by Cauchy's method one obtains

$$\overline{\varphi_E^2} = \frac{RkT_R}{A} \cdot \frac{B^2 (RC_0 \beta + I)}{R^2 B^2 C_0 \beta + RB^2 I + I\beta + RC_0 \beta^2 + R^2 C_0^2 A \beta} \quad (3.8)$$

The calculation of $\overline{\varphi_M^2}$, the mean square of the deflections caused by the molecular collisions, may be carried out in exactly the same way as that of $\overline{\varphi_E^2}$. Here in (3.3) one assumes the fluctuating voltage F to be zero. For the corresponding amplification factor $g_M(\omega)$ for molecular collisions one obtains [2]) now from (3.3*a*)

$$g_M^2(\omega) = \frac{f^2 + g^2\omega^2}{(d - b\omega^2) + (c\omega - a\omega^3)^2} \quad (3.9)$$

Then

$$\varphi_M^2 = \int_0^\infty g_M^2(\omega)\, I_M(\omega)\,.\,\mathrm{d}\omega = \frac{2\beta_M kT_M}{\pi} \int_0^\infty \frac{f^2 + g^2\omega^2}{(d - b\omega^2) + (c - a\omega^3)^2}\,.\,\mathrm{d}\omega \quad (3.10)$$

after substituting the constants of (3.3) and (3.3*a*), the integral in the third member of (3.10) yields

$$\frac{\pi}{2} \cdot \frac{I + RC_0\beta + R^2C_0^2A}{A(R^2B^2C_0\beta + RB^2I + I\beta + RC_0\beta^2 + R^2C_0^2A\beta)}$$

which expression, being equal to $\int_0^\infty g_M^2(\omega)\,.\,\mathrm{d}\omega$, demonstrates the influence of the amplification factor $g_M^2(\omega)$ in which figures the total damping β.

Finally

$$\overline{\varphi_M^2} = \frac{\beta_M kT_M}{A} \cdot \frac{I + RC_0\beta + R^2C_0^2A}{R^2B^2C_0\beta + RB^2I + I\beta + RC_0\beta^2 + R^2C_0A\beta} \quad (3.11)$$

The expressions for $\overline{\varphi_E^2}$ and $\overline{\varphi_M^2}$ may readily be checked by assuming equal temperatures for the resistance and the air viz. $T_R = T_M = T$ and thus $\beta_M = \beta$, and adding (3.8) and (3.11). According to expectations one obtains then $\overline{\varphi_{total}^2} = \overline{\varphi_E^2} + \overline{\varphi_M^2} = kT/A$, i.e. the law of equipartition.

§ 4. *The introduction of artificial (cold) damping.* We have purposely introduced in (3.8) and (3.11) two different characters for the damping, viz. the total damping which determines the amplification factors, indicated with the letter β, and the mechanical damping β_M

which determines the mechanical fluctuation and to which $\overline{\varphi_M^2}$ in (3.11) appears to be proportional.

The significance hereof becomes clear, when the mechanical damping β_M is reduced, for instance by evacuating the electrometer, and when at the same time the total damping is kept constant by supplying a suitable amount of a special kind of artificial damping which is able to damp the motion of the system without introducing fluctuations. Such a method of damping, the experimental realisation of which is to be described in a following paper, may be called cold damping.

From (3.11) one observes immediately that $\overline{\varphi_M^2}$ is then proportionally reduced and eventually completely nullified with β_M and thus a considerable reduction of the deflections may be achieved. Thus the same result is obtained as by refrigerating the damping air within the electrometer. Of course these measures will not influence $\overline{\varphi_E^2}$ in any respect.

For large values of the resistance R, (3.11) reduces to

$$\varphi_M^2 = (\beta_M/\beta)\ kT_M/(A+B^2/C_0) \tag{4.1}$$

From (4.1) it follows that a reduction of the air-damping β_M by a factor 10^2, which may be achieved readily, and application of cold damping until β obtains its original value, has the same result as refrigerating the air to 3°K.

§ 5. *The error in charge measurements and methods for increasing the precision.* The statistical error due to the Brownian fluctuations will be calculated for two different methods of observation, viz.

a) When the deflection φ of the system is observed at moments lying τ seconds apart. Here we shall calculate the mean value

$$\overline{(\Delta\varphi)^2} = \overline{[\varphi(t+\tau) - \varphi(t)]^2} \tag{5.1}$$

b) When the mean square $\overline{\varphi_\tau^2}$ is taken over averages of the deflection, over intervals of τ seconds, thus

$$\overline{\varphi_\tau^2} = \overline{[\tau^{-1} \int_0^\tau \varphi(t)\, \mathrm{d}t]^2} \tag{5.2}$$

a) It is obvious that (5.1) is identical with

$$\overline{(\Delta\varphi)^2} = 2\,[\overline{\varphi^2(t)} - \overline{\varphi(t+\tau)\,\varphi(t)}] \tag{5.3}$$

Because of the absence of correlation between the deflections of mechanical origin φ_M and of electrical origin φ_E, we have

$$\overline{(\Delta\varphi)^2} = \overline{(\Delta\varphi_M)^2} + \overline{(\Delta\varphi_E)^2} \tag{5.4}$$

For the first as well as for the second term in the right member of (5.4) an equation similar to (5.3) applies. Thus

$$\overline{(\Delta\varphi)^2} = 2[\overline{\varphi_M^2(t)} - \overline{\varphi_M(t+\tau)\,\varphi_M(t)} + \overline{\varphi_E^2(t)} - \overline{\varphi_E(t+\tau)\,\varphi_E(t)}] \tag{5.5}$$

$\overline{\varphi_M^2}$ and $\overline{\varphi_E^2}$ have been calculated earlier cf (3.8) and (3.11). The product terms in (5.5) may be found by observing that for a critically damped instrument holds

$$\overline{\varphi_M(t+\tau)\,\varphi_M(t)} = \overline{\varphi_M^2}e^{-r_M\tau} \tag{5.6}$$

where r_M corresponds to the double root of the characteristic equation of the equation of motion. Hence for values of τ, large compared with the period of the system $\overline{\varphi_M(t+\tau)\,\varphi_M(t)}$ tends to zero. This condition may be met in practice readily.

For the random deflections of electrical origin we have

$$\overline{\varphi_E(t+\tau)\,\varphi_E(t)} = \overline{\varphi_E^2}\,e^{-r_E\tau} \tag{5.7}$$

but here in practice, τ is never so large that $r_E\tau \gg 1$.

Calculation of (5.7) may be accomplished with the aid of Fourier integrals comp. [2]) viz. e.g. Bernamont [6]). Starting from (3.7) we find $r_E = A/R(AC_0 + B^2)$, where R is the resistance in the measuring circuit.

As $$\overline{(\Delta\varphi_E)^2} = \overline{\varphi_E^2}(1 - e^{-r_E\tau}),$$

we obtain, after neglecting terms of orders higher than the second and conversion into units of charge cf. (2.10) for the error in the charge ΔQ_E caused by electrical fluctuations only

$$\overline{(\Delta Q_E)^2} = (2kT/R)\,\tau \tag{5.8}$$

Taking into account the influence of the mechanical fluctuations by observing that $\overline{(\Delta\,\varphi_M)^2}$ is determined by $\overline{\varphi_M^2}$ only, while the mechanical correlation time is so small that $e^{-r_M\tau}$ and thus (5.6) tends to zero, and taking into account (5.8) for the electrical fluctuations we find for the total error in the measurement of charge within a time of observation τ and for critical damping

$$\overline{(\Delta Q)^2} = 2kT\tau/R + 2kT\,(AC_0 + B^2)C_0/B^2 \tag{5.9}$$

It will be clear that the second term of the right member, which describes the influence of the mechanical fluctuations, has been obtained according to (5.5) and (5.6) from the expression (3.11) for $\overline{\varphi_M^2}$ by substituting $R = \infty$.

It has been remarked before (M i l a t z [3])) that for most electrometers the second term of (5.9) exceeds the first by tens of times. Thus, for a Hoffmann electrometer $[\overline{(\Delta Q_M)^2}]^{1/2}$ is seventy times larger than $[\overline{(\Delta Q_E)^2}]^{1/2}$. The actual quantities being 650 and 8.5 electron charges respectively.

However this applies solely when the instrument is used in the usual way. If it is supplied with an artificial damping which introduces no fluctuations, as indicated in § 4, then one should start from expression (4.1). This yields

$$\overline{(\Delta Q)^2} = 2kT\tau/R + 2(\beta_M/\beta)\, kT\, (AC_0 + B^2)C_0/B^2 \qquad (5.10)$$

where β stands for the total damping i.e. normal damping plus cold damping.

From (5.10) it follows that, if it were possible to provide so much artificial damping and reduce β_M so far that $\beta_{crit} \gg \beta_M$, then it should be possible to remove the mechanical fluctuations completely and achieve the ultimate precision to be obtained with an electrometer, viz. an error of only 8.5 electroncharges for a Hoffmann electrometer.

b) Next the error will be computed when the mean square $\overline{Q_\tau^2}$ is taken over averages of the deflections over intervals of time τ. Assuming the deflections to be expressed in units of charge, we have for the quantity $\overline{Q_\tau^2}$ concerned

$$\overline{Q_\tau^2} = \overline{[\tau^{-1} \textstyle\int_0^\tau Q_E(t)\, \mathrm{d}t]^2} + \overline{[\tau^{-1} \textstyle\int_0^\tau Q_M(t)\, \mathrm{d}t]^2} \qquad (5.11)$$

While Q_E and Q_M are the deflections arising from electrical and mechanical fluctuations respectively, the first and second term of (5.11), which we will denote by $\overline{Q_{\tau E}^2}$ and $\overline{Q_{\tau M}^2}$ respectively describe the corresponding m.s. in the errors involved.

$\overline{\varphi_{\tau E}^2}$ has been calculated earlier by Z e r n i k e [7]) with Ornsteins method. He obtained

$$\overline{Q_{\tau E}^2} = 2kT\tau/R \qquad (5.12)$$

In order to calculate $\overline{Q_{\tau M}^2}$ we observe that the fortuitous collisions of the air molecules which give rise to this error, impart a fluctuating mechanical torque to the system. We denote by K_τ the mean value of the torque $k(t)$ taken over a time τ.

Thus

$$\overline{K_\tau^2} = \overline{[\tau^{-1} \int_0^\tau k(t) \, \mathrm{d}t]^2}. \tag{5.13}$$

While the spectral intensity of the torque is given by $I_M(\nu) = 4\beta_M kT$, where β_M is the mechanical damping, $\overline{K_\tau^2}$ may be calculated by using the general relation,

$$\overline{[\tau^{-1} \int_0^\tau k(t) \, \mathrm{d}t]^2} = I_M(\nu)/2\tau. \tag{5.14}$$

Thus we find

$$\overline{K_\tau^2} = 2\beta_M kT/\tau. \tag{5.15}$$

The m.s. $\overline{Q_{\tau M}^2}$ of the resulting deflections expressed in units of charge is then given by

$$\overline{Q_{\tau M}^2} = (C_0/B)^2 . 2\beta_M kT/\tau \tag{5.16}$$

From (5.11), (5.12) and (5.16) we find for the resulting mean square of the error when averaging over a time τ

$$\overline{Q_\tau^2} = 2kT\tau/R + (C_0/B)^2 \, 2\beta_M kT/\tau \tag{5.17}$$

While here, as in (5.10) the second term of the right member is very much — for a Hoffmann electrometer about seventy times — greater than the first, it is of importance that (5.17) indicates that by reducing the mechanical damping β_M and averaging the deflections one may gain the same considerable factor in precision.

Increasing the auxiliary tension V_H and thus $B = aV_H$ provides but few, if any, real gain while the reduction in error being proportional to B, is accompanied by an increase in the period of the instrument proportional to $(AC_0+B^2)^{1/2}$, which approximates B when the effective directional force A is reduced to zero. This occurs when V_H and thus B are approaching the lability value.

The error occurring in practice will be about twice as large as follows from (5.17) because one has to record not only the deflections of the charged system, but the zero position also.

It may be observed that (5.17) is not only valid when the deflections are averaged during a time τ, but still after the slight change of the factors 2 by π when one takes single deflections of a critically damped electrometer having a period τ.

§ 6. *Conclusion.* We will summarize which methods are indicated by the theory developed in the preceding chapters and leading finally to (5.10) and (5.17), for reducing the Brownian motion or subsequently for increasing the measuring precision.

As has been remarked before, the instability arising from the thermal voltage fluctuations and represented by the first term in the right members of (5.10) and (5.17), contributes only very slightly, in most cases only up to a few percents, to the final error. In principle it may be further reduced by increasing R, but in practice it seems very difficult to surpass the excellent insulators quarts, amber or polystyrene already used generally.

Fortunately it appears that several means may be indicated to reduce the influence of the mechanical fluctuations, represented by the second term of the right members of (5.10) and (5.17), which provide the most important contribution to the resulting instability. These are

1. Reduction of the capacity C_0 of the system. In this respect Millikans "oildrop electrometer" seems to be insurpassable!

2. Substitution of the airdamping β_M by an artificial damping without fluctuation. This will enable in principle to remove the mechanical fluctuations completely and to increase the precision very considerably, comp. (5.10).

When the deflections are averaged during a certain time the precision seems to be raised also by

3. Evacuation of the electrometer i.e. reducing β_M to zero, comp. (5.17), and using the instrument undamped. By averaging over some periods of the system, the intrinsic error of only a few electroncharges seems to be obtainable.

In the following article II [8]) experiments will be described in which a considerable reduction of the Brownian motion has been obtained indeed.

Received 13-12-52.

REFERENCES

1) Eggers, W., Ann. Physik **7** (1930) 833.
2) Van Zolingen, J. J., Thesis, Utrecht 1948.
3) Milatz, J. M. W., Ned. T. Natuurk. **8** (1941) 19.
4) Nyquist, H., Phys. Rev. **32** (1928) 110.
5) Schottky, W., Ann. Physik **57** (1918) 541.
6) Bernamont, J., Ann. Physique **7** (1937) 87.
7) Zernike, F., Z. Phys. **40** (1926) 628.
8. Milatz, J. M. W., Van Zolingen, J. J., and Van Iperen, B. B., Physica **19** (1953) 195.