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ANGULAR DISTRIBUTION AND POLARIZATION OF GAMMA RADIATION EMITTED BY ORIENTED NUCLEI

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Synopsis

The angular distribution and polarization of γ-radiation emitted by oriented nuclei is calculated assuming pure multipole transitions. For the description of the orientation of the nuclei $2j$ independent parameters f_k are used; these are essentially the statistical tensors introduced by Fano (j is the spin quantum number of the nuclei). The state of polarization of the γ-radiation is characterized by the degree of polarization P and a real three-dimensional polarizationvector ξ_0. If these quantities f_k, P and ξ_0 are used, the formulae take a relatively simple form. For the cases of dipole and quadrupole radiation expressions are given which are suitable for numerical calculation by easy substitutions.

§ 1. *Introduction.* If we have an ensemble of oriented nuclei, the angular distribution of γ-radiation emitted by these nuclei no longer has spherical symmetry. Spiers [1]) discussed this effect and Steenberg [2]) extended his considerations, both giving formulae which are practical only in the case of small nuclear orientation. In the following sections we shall derive explicit formulae for the angular distribution of γ-radiation from oriented nuclei, valid for any degree of orientation of the nuclei. Furthermore we shall calculate the polarization of the emitted γ-radiation.

The theory of γ-radiation from *oriented nuclei* is closely related to the theory of angular correlation of pairs of successive γ-quanta from nuclei *oriented at random*. In the latter case the angular correlation function can be considered to be the angular distribution function for the second radiation emitted by nuclei which have been

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oriented by the emission of the first radiation in a fixed direction. A short review of the relation between these problems will be given in this section. The theory of angular correlation of successive γ-quanta emitted by nuclei oriented at random was given by Hamilton [3]). Recent contributions have been made by Falkoff, Ling, Uhlenbeck [4]) [5]), Racah [6]), Lloyd [7]) and Alder [8]).

We shall begin with some general formulae concerning the angular distribution of radiation from oriented nuclei. We assume the nuclei to be oriented in such a way that an axis $\boldsymbol{\eta}$ of rotational symmetry exists. The nuclear angular momentum quantum number and magnetic quantum number are j and m respectively (m determines the component of the nuclear angular momentum in the direction $\boldsymbol{\eta}$, which we call the axis of quantization). The orientation of the nuclei is then characterized by the numbers a_m, these being the probabilities of the states specified by j, m ($\Sigma a_m = 1$).

$I_{j_i}^{m_i}(\vartheta)$ is the angular distribution of the radiation from a nucleus in the state j_i, m_i.

$W(\vartheta)$ is the angular distribution of the observed radiation.

ϑ is the angle between the direction of emission of the observed radiation and $\boldsymbol{\eta}$.

Now $W(\vartheta)$ is given by

$$W(\vartheta) = \Sigma_{m_i} a_{m_i} I_{j_i}^{m_i}(\vartheta). \tag{1}$$

If the angular distribution of radiation with angular momentum quantum number L and magnetic quantum number M is denoted by $F_L^M(\vartheta)$, we can express $I_{j_i}^{m_i}(\vartheta)$ in terms of $F_L^M(\vartheta)$ by

$$I_{j_i}^{m_i}(\vartheta) = \Sigma_M G_{m_i, m_i - M}^{j_i L j_f} F_L^M(\vartheta). \tag{2}$$

The $G_{m_i, m_i - M}^{j_i L j_f}$ are the squared transformation coefficients for the addition of angular momenta (see formula 60).

In general there will be a β or γ-transition $(j_0, m_0) \to (j_i, m_i)$, which precedes the radiation under consideration. In this case the a_{m_0} give the initial orientation of the nuclei and we have to calculate the relative populations a_{m_i} of the levels m_i in order to compute $W(\vartheta)$ for the radiation emitted by the nuclei with spin j_i. If $P(m_0, m_i)$ gives the relative transition probability for the transition $(j_0, m_0) \to (j_i, m_i)$, the expression for a_{m_i} becomes

$$a_{m_i} = \Sigma_{m_0} a_{m_0} P(m_0, m_i), \tag{3}$$

$$\Sigma_{m_i} P(m_0, m_i) = 1. \tag{4}$$

We add some formulae on the angular correlation of successive radiations emitted by nuclei which are oriented at random. $P(m_1, m_2, \vartheta)$ is the angular distribution of the radiation emitted in the transition $(j_1, m_1) \rightarrow (j_2, m_2)$. ϑ is the angle between the direction of the radiation and the axis of quantization. This quantity $P(m_1, m_2, \vartheta)$ is related to $P(m_1, m_2)$, $I_{j_1}^{m_1}(\vartheta)$ and $F_L^M(\vartheta)$ by

$$P(m_1, m_2) = \int P(m_1, m_2, \vartheta)\, \mathrm{d}\Omega, \tag{5}$$

$$I_{j_1}^{m_1}(\vartheta) = \Sigma_{m_2} P(m_1, m_2, \vartheta), \tag{6}$$

$$P(m_1, m_2, \vartheta) = G_{m_1, m_2}^{j_1\ Lj_2} F_L^M(\vartheta), \ (m_1 = m_2 + M). \tag{7}$$

The angular correlation function $W(\vartheta)$, giving the relative probability for the angle ϑ between the successive radiations, is given by (cf. [5]))

$$W(\vartheta) = C\, \Sigma_{m_0, m_i, m_f} P(m_0, m_i, \vartheta{=}0)\,.\,P(m_i, m_f, \vartheta). \tag{8}$$

Here the successive states of the nucleus are (j_0, m_0), (j_i, m_i) and (j_f, m_f). With (6) we can rewrite (8) as

$$W(\vartheta) = \Sigma_{m_i} a_{m_i} I_{j_i}^{m_i}(\vartheta), \tag{9}$$

$$a_{m_i} = C\, \Sigma_{m_0} P(m_0, m_i, \vartheta = 0). \tag{10}$$

These formulae allow the following interpretation. The direction $\vartheta = 0$ of the first radiation is an axis $\boldsymbol{\eta}$ of rotational symmetry in the problem of the angular distribution of the second radiation. The first radiation causes an orientation of the nuclei (j_i, m_i) given by (10). From this point of view the angular correlation is a special case of the angular distribution of radiation from oriented nuclei. For a derivation of these formulae we refer to [3]) and [5]). It is essential in our interpretation of the formulae that (8) allows a "splitting of the process into two parts". Cf. [9]) for a discussion of this point.

In § 2 is discussed how the orientation of atomic nuclei can be characterized. In § 3 the use of different parameters to characterize the polarization of γ-radiation is treated. With §§ 1, 2 and 3 as a basis the calculation of the angular distribution and polarization of γ-radiation of oriented nuclei is rather straightforward. This is discussed in § 4. With the aid of more advanced mathematics, namely, a method developed by R a c a h [10]), we can derive formulae for arbitrary multipole order in § 5. Though the results of § 5 include the results of § 4, we have thought it worth while to discuss this first method, as it gives a clearer understanding of some features and relations in this field. Explicit formulae for the angular distribution are given in § 6, for the polarization in § 7 (in these expressions the general expressions have been evaluated as far as possible). A short note containing some of the results has appeared earlier [11]). Later we shall discuss the ways in which these formulae can be used for the treatment of experiments with oriented nuclei [12]).

§ 2. *The description of the orientation of atomic nuclei by the parameters* f_k. Generally the state of orientation of an ensemble of nuclei, all with angular momentum j (or of one nucleus of which our knowledge is incomplete), cannot be described by means of a single wave function. We therefore use a density matrix (or statistical operator) ϱ with $(2j+1)^2$ matrix elements (compare e.g. [13]), part II and IV). ϱ is hermitian and normalized to

$$\Sigma_m \varrho_{mm} = 1. \tag{11}$$

With each state $$\psi = \Sigma_m c_m \psi_m, \tag{12}$$

we associate a matrix $$(\varrho_\psi)_{mm'} = c_m c^*_{m'}. \tag{13}$$

The probability of finding a system described by the density matrix ϱ in the state ψ is then given by

$$W = \mathrm{Tr}\,(\varrho\varrho_\psi).\ \ (\mathrm{Tr: Trace}) \tag{14}$$

If an axis $\boldsymbol{\eta}$ of rotational symmetry exists, and $\boldsymbol{\eta}$ has been chosen as axis of quantization, then ϱ is a diagonal matrix (compare [9])). Now the probability of finding a state ψ_m is given by

$$a_m = \mathrm{Tr}(\varrho\varrho_{\psi_m}) = \varrho_{mm}. \tag{15}$$

In this case ϱ is determined by $2j+1$ numbers a_m. As $\Sigma_m a_m = 1$ on account of (11), there are only $2j$ independent numbers a_m which characterize the orientation. As in the results of the calculations in the following sections a_m will appear always in combinations of the form $\Sigma_m m^k a_m$ (often called moments), we therefore define $2j$ independent combinations of this form, which are equivalent to the set of $2j$ numbers a_m.

$$f_k = \Sigma_{\nu=0}^{k} \alpha_{k,\nu}\, \Sigma_m m^\nu a_m, \tag{16a}$$

$$f_k = 0 \text{ if } a_m = \Sigma_{p=0}^{k-1} A_p m^p, \tag{16b}$$

$$\alpha_{k,k} = j^{-k}. \tag{16c}$$

By taking $a_m = m^p$ ($p = 0 \dots . k - 1$) in (16*b*) we get k linearly independent relations for the $k + 1$ coefficients $a_{k,0} \dots . a_{k,k}$. It is easily seen that these coefficients become entirely determined by the additional condition (16*c*). This condition has been chosen in such a way that the f_k for totally oriented nuclei remain finite if we let $j \to \infty$ (cf. (19*a*).... (19*d*). Hence the f_k are uniquely defined by (16). If an arbitrary orientation is given by a_m, the dependence of a_m on m can always be expressed by a polynomial of degree $2j$:

$$a_m = \Sigma_{p=0}^{2j} A_p \, m^p. \tag{17}$$

Hence we obtain the following interesting property of the f_k directly from the definition (16): *all* f_k *with* $k \geqslant 2j + 1$ *vanish identically.*

Since $f_0 = 1$ cannot vary there are $2j$ parameters f_k left which are independent and suffice for a complete description of the orientation. These f_k will be used as they give simple forms to our formulae. Explicit expressions for f_1, f_2, f_3 and f_4 are:

$$f_1 = j^{-1} \Sigma_m \, m a_m, \tag{18a}$$

$$f_2 = j^{-2} [\Sigma_m \, m^2 a_m - \tfrac{1}{3} j(j + 1)], \tag{18b}$$

$$f_3 = j^{-3} [\Sigma_m \, m^3 a_m - \tfrac{1}{5} (3j^2 + 3j - 1) \, \Sigma_m \, m a_m], \tag{18c}$$

$$f_4 = j^{-4} [\Sigma_m m^4 a_m - \tfrac{1}{7} (6j^2 + 6j - 5) \, \Sigma_m \, m^2 a_m + \tfrac{3}{35} j(j-1)(j+1)(j+2)]. \tag{18d}$$

If the nuclei are totally oriented ($a_m = \delta_{mj}$), we calculate from (18)

$$f_1 = 1 \tag{19a}$$

$$f_2 = (2j - 1)/3j \tag{19b}$$

$$f_3 = (j - 1)(2j - 1)/5j^2 \tag{19c}$$

$$f_4 = 2(j - 1)(2j - 1)(2j - 3)/35j^3 \tag{19d}$$

For numerical calculations it can be useful to approximate a distribution by a polynomial in order to calculate the f_k. If we have for example a Boltzmann distribution

$$a_m = C' \exp(\beta m) \quad (\text{with } \beta = \mu B/kTj), \tag{20a}$$

we can write approximately

$$a_m = C \, (1 + \beta m + \tfrac{1}{2}\beta^2 m^2 + \tfrac{1}{6} \beta^3 m^3 + \tfrac{1}{24} \beta^4 m^4) \quad \text{if} \quad \beta j \ll 1. \tag{20b}$$

Now if we have in general a distribution

$$a_m = C \, (1 + A_1 m + A_2 m^2 + A_3 m^3 + A_4 m^4), \tag{21}$$

it is readily shown that

$$f_1 = [\tfrac{1}{3}(j + 1) \, A_1 + \tfrac{1}{15} (j + 1)(3j^2 + 3j - 1) \, A_3] \, n, \tag{22a}$$

$$f_2 = [\tfrac{1}{45j}(2j-1)(j+1)(2j+3)A_2 + \tfrac{1}{315j}(2j-1)(j+1)(2j+3)(6j^2+6j-5)A_4]n, \qquad (22b)$$

$$f_3 = (\tfrac{1}{175}j^{-2})(j-1)(2j-1)(j+1)(2j+3)(j+2)A_3\,n, \qquad (22c)$$

$$f_4 = \tfrac{4}{(105)^2}j^{-3}(2j-3)(j-1)(2j-1)(j+1)(2j+3)\,.\,(j+2)(2j+5)A_4\,n. \qquad (22d)$$

$$C = 1/n = 1 + \tfrac{1}{3}j(j+1)A_2 + \tfrac{1}{15}j(j+1)(3j^2+3j-1)A_4 \qquad (22e)$$

We can derive an explicit expression for the f_k by comparing them with the statistical tensors introduced by Fano [14]) according to the definition

$$\langle|(jj)kq\rangle = \Sigma_{mm'}\langle m|\varrho|m'\rangle(-1)^{j-m}\langle jm', j-m|(jj)kq\rangle, \qquad (23)$$

where $\langle m|\varrho|m'\rangle$ are the matrix elements of the density matrix ϱ, and $\langle jm', j-m|(jj)kq\rangle$ are the transformation coefficients for the addition of angular momenta. In the case in which the ensemble of nuclei has an axis of rotational symmetry $\boldsymbol{\eta}$ we write

$$\bar{f}_k = \langle|(jj)\,ko\rangle = \Sigma_m\langle m|\varrho|m\rangle(-1)^{j-m}\langle jm, j-m|(jj)ko\rangle. \qquad (24)$$

The density matrix ϱ is then entirely determined by the probabilities $a_m = \langle m|\varrho|m\rangle$ and we can write

$$\bar{f}_k = \Sigma_m(-1)^{j-m}\langle jm, j-m|(jj)\,ko\rangle\,a_m \qquad (25)$$

We now show that the $\bar{f}_k$ introduced in this way differ only by a constant factor from the f_k defined by (16), i.e.,

$$f_k = w_k(j)\bar{f}_k. \qquad (26)$$

To prove (26), we observe the $\bar{f}_k$ to be of the form

$$f_k = \Sigma_m R_k(m)\,a_m, \qquad (27a)$$

$$R_k(m) = (-1)^{j-m}\langle jm, j-m|(jj)\,ko\rangle. \qquad (27b)$$

$R_k(m)$ is a polynomial in m of degree k, as may be seen from the explicit formula for $\langle jmj-m|(jj)\,ko\rangle$ (Cf. [10]) formula (16)),

$$\langle jmj-m|(jj)|\,ko\rangle = \left[\frac{(2k+1)\,(2j-k)\,!}{(2j+k+1)\,!}\right]^{\frac{1}{2}}(k!)^2(j+m)!\,(j-m)!\,\times$$

$$\times\,\Sigma_z(-1)^z[z!\,(2j-k-z)!\,\{(j-m-z)!\}^2\{(k-j+m+z)!\}^2]^{-1}, \qquad (27c)$$

by rewriting with this expression formula (27b) as

$$R_k(m) = \left[\frac{(2k+1)\,(2j-k)\,!}{(2j+k+1)\,!}\right]^{\frac{1}{2}}\binom{2k}{k}[m^k + u_1 m^{k-1} + \ldots\, u_0]. \qquad (28)$$

From the orthogonality relations for $\langle jmj - m|(jj)\ ko\rangle$ it follows that

$$\Sigma_m R_k(m)\, R_p(m) = 0, \qquad p \leqslant k-1. \tag{29}$$

From (28) and (29) it is easily deduced that

$$\Sigma_m R_k(m) m^p = 0, \qquad p \leqslant k-1. \tag{30}$$

If a_m is given by $a_m = \Sigma_{p=0}^{k-1} A_p m^p$ it follows from (30) that

$$\bar{f}_k = \Sigma_m R_k(m)\, a_m = 0. \tag{31}$$

So the $\bar{f}_k$ are of the form (16*a*) and satisfy the condition (16*b*). Hence the $\bar{f}_k$ and f_k can only differ by a constant factor, which may be obtained from (28) and (16*c*)

$$f_k/\bar{f}_k = w_k(j) = \binom{2k}{k}^{-1} j^{-k} \left[\frac{(2j+k+1)!}{(2k+1)\,(2j-k)!}\right]^{\frac{1}{2}}. \tag{32}$$

From (26), (27*a*), (27*b*) and (27*c*) an explicit expression for f_k follows

$$f_k = \binom{2k}{k}^{-1} j^{-k} \Sigma_m \Sigma_{\nu=0}^{k} (-1)^\nu \frac{(j-m)!\,(j+m)!}{(j-m-\nu)!\,(j+m-k+\nu)!} \binom{k}{\nu}^2 a_m. \tag{33}$$

For totally oriented nuclei ($a_m = \delta_{mj}$) f_k has the value (a generalization of (19*a*) .... (19*d*))

$$f_k = \binom{2k}{k}^{-1} j^{-k} \frac{(2j)!}{(2j-k)!} \tag{34}$$

§ 3. *Characterization of the polarization of electro-magnetic radiation.* For a plane electro-magnetic wave which is propagated in the direction $\mathbf{k}(|\mathbf{k}| = 2\pi/\lambda)$ we can write for the complex vector potential $\mathbf{A}$ and electric field strength $\mathbf{E}$

$$\mathbf{A} = (c_1\mathbf{e}_1 + c_2\mathbf{e}_2)\, A, \quad A = \hat{A} \text{ exp. } i\,(\mathbf{k}.\mathbf{r} - \omega t), \tag{35}$$

$$\mathbf{E} = (c_1\mathbf{e}_1 + c_2\mathbf{e}_2)\, E, \quad E = -(\partial A/\partial t)/c = i|\mathbf{k}|A. \tag{36}$$

Here $\mathbf{k}/|\mathbf{k}|$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are mutually perpendicular unit vectors. We assume c_1 and c_2 to be normalized to

$$|c_1|^2 + |c_2|^2 = 1. \tag{37}$$

The state of polarization is characterized by the complex vector

$$\mathbf{c} = (c_1, c_2). \tag{38}$$

Essentially two (real) parameters (e.g. the ratio $|c_1|/|c_2|$ and the phase difference of c_1 and c_2) are needed for the description of a totally polarized wave (or photon). The state of polarization being described by $\mathbf{c}$, we define a density matrix $\varrho(\mathbf{c})$ ([13]) II formula 4):

$$\varrho(\mathbf{c}) = \begin{Vmatrix} |c_1|^2 & c_1c_2^* \\ c_1^*c_2 & |c_2|^2 \end{Vmatrix}. \qquad (39)$$

This special form of the density matrix occurs only in the case of total polarization. The states of polarization of a partially polarized beam of photons is described by a hermitian density matrix ϱ:

$$\varrho = \begin{Vmatrix} \varrho_{11} & \varrho_{12} \\ \varrho_{21} & \varrho_{22} \end{Vmatrix}, \qquad (40a)$$

$$\varrho_{11} + \varrho_{22} = 1. \qquad (40b)$$

In this case essentially three parameters are needed. Now the probability W of finding a photon in the state of polarization described by $\mathbf{c}$ is given by

$$W = \mathbf{c}^* \varrho\, \mathbf{c} = \mathrm{Tr}\,[\varrho\varrho(\mathbf{c})]. \qquad (41)$$

If $\varrho = \varrho(\mathbf{c}')$ (totally polarized beam), (41) becomes

$$W = \mathrm{Tr}\,[\varrho(\mathbf{c}')\,\varrho(\mathbf{c})] = |c_1'c_1^* + c_2'c_2^*|^2. \qquad (42)$$

Another description of the state of total polarization, discussed by Fano [15]), makes use of parameters α and β. These are related to $\mathbf{c}$ by

$$c_2/c_1 = (\sin\alpha\cos\beta + i\cos\alpha\sin\beta)/(\cos\alpha\cos\beta - i\sin\alpha\sin\beta) =$$
$$= (\sin 2\alpha\cos 2\beta + i\sin 2\beta)/(1 + \cos 2\alpha\cos 2\beta). \qquad (43)$$

α and β are the angles which determine the setting of an ideal analyzer (consisting of a $\lambda/4$ plate and a Nicol prism) with respect to $\mathbf{e}_1$ for the case of maximum transmission. A real three dimensional vector $\boldsymbol{\xi}$, the polarization vector, can be defined as follows with the aid of α and β:

$$\begin{aligned} \xi_1 &= \cos 2\beta\cos 2\alpha, \\ \xi_2 &= \cos 2\beta\sin 2\alpha, \\ \xi_3 &= \sin 2\beta. \end{aligned} \qquad (44)$$

We can describe a state of total polarization by this vector $\boldsymbol{\xi}$. The relation between $\boldsymbol{\xi}$ and $\mathbf{c}$, following from (43) and (44), is

$$\begin{aligned} \xi_1 &= |c_1|^2 - |c_2|^2, \\ \xi_2 &= c_1c_2^* + c_1^*c_2, \\ \xi_3 &= i\,(c_1c_2^* - c_1^*c_2). \end{aligned} \qquad (45)$$

In the case of a partially polarized beam, which is described by the density matrix ϱ (40), we can always write ϱ in the form

$$\varrho = \tfrac{1}{2}(1 - P)\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} + P\begin{Vmatrix} |c_1|^2 & c_1 c_2^* \\ c_1^* c_2 & |c_2|^2 \end{Vmatrix}, \tag{46a}$$

$$0 \leqslant P \leqslant 1. \tag{46b}$$

This means a decomposition of ϱ into two parts, corresponding to a totally polarized and an unpolarized part. P is called the degree of polarization of the beam. $\boldsymbol{\xi}_0$ is the polarization vector of the beam and is determined by $\mathbf{c}$ of formula (46*a*). To describe the polarization of an arbitrary beam we shall use P and $\boldsymbol{\xi}_0$, containing again three real parameters. With (41) and (46) it follows that the probability W of finding a photon with polarization vector $\boldsymbol{\xi}$ in a beam described by P and $\boldsymbol{\xi}_0$ is given by

$$W = \tfrac{1}{2}(1 + P\,\boldsymbol{\xi} \cdot \boldsymbol{\xi}_0). \tag{47}$$

For the special case of a totally polarized beam, with $P = 1$, we find a formula corresponding to (42):

$$W = \tfrac{1}{2}(1 + \boldsymbol{\xi} \cdot \boldsymbol{\xi}_0). \tag{48}$$

The formalism which has been described, for the polarization of electromagnetic radiation, is to a high degree analogous to the formalism for the polarization of electrons, developed in [13]) part II and IV. Compare e.g., formula (47) with formula (70) of [13]) part IV.

§ 4. *A calculation method of the angular distribution and polarization of 2^L-pole γ-radiation (especially for $L = 1$ and $L = 2$).* The angular distribution of γ-radiation of a certain (L, M) pole character results from the spherical eigenwave solution of Maxwell's equation [16]) [17]) [4]). We have used the solutions for the vector potential (in the gauge of zero scalar potential) of the electric and magnetic (L, M) pole radiation in the form in which they are listed in [4]), table I (For the physical quantity $\mathbf{A}$ the real part of the complex quantity must be taken). The electric and magnetic field strengths are obtained from the vector potential $\mathbf{A}$ according to

$$\mathbf{E} = -(\partial \mathbf{A}/\partial t)/c, \tag{49}$$

$$\mathbf{B} = \text{rot}\,\mathbf{A}. \tag{50}$$

The angular distribution of the radiation of a given (L, M) pole character is given by the magnitude of the Poynting vector $\mathbf{S}_L^M$.

If $\mathbf{n}$ is a unit vector in the direction of $\mathbf{S}_L^M$ then the magnitude is given by

$$\mathbf{n} \cdot \mathbf{S}_L^M = (ck^2/8\pi)\, \mathbf{A} \cdot \mathbf{A}^*. \tag{51}$$

We define

$$F_L^M(\vartheta) = 32\pi^3 r^2 c^{-1}\, \mathbf{n} \cdot \mathbf{S}_L^M. \tag{52}$$

Then $F_L^M(\vartheta)$ is normalized to

$$\int F_L^M(\vartheta)\, \mathrm{d}\Omega = 8\pi. \tag{53}$$

From (49), (51) and (52) it follows that

$$F_L^M(\vartheta) = 4\pi^2 r^2 k^2\, \mathbf{A} \cdot \mathbf{A}^*, \tag{54}$$

$$F_L^M(\vartheta) = 4\pi^2 r^2\, \mathbf{E} \cdot \mathbf{E}^*. \tag{55}$$

From the general expression for $\mathbf{A}$ one obtains an expression for $F_L^M(\vartheta)$ according to (54) (Y_L^M are the spherical harmonics):

$$F_L^M(\vartheta) = (4\pi/L(L+1))\,[2M^2\,|Y_L^M|^2 + (L-M)\,(L+M+1)\,|Y_L^{M+1}|^2 + \\ + (L+M)\,(L-M+1)\,|Y_L^{M-1}|^2]. \tag{56}$$

This expression becomes, for $L = 1$ and $L = 2$,

$$F_1^0(\vartheta) = 3(1 - \cos^2\vartheta), \tag{57a}$$

$$F_1^{\pm 1}(\vartheta) = \tfrac{3}{2}\,(1 + \cos^2\vartheta), \tag{57b}$$

$$F_2^0(\vartheta) = \tfrac{5}{2}\,(6\cos^2\vartheta - 6\cos^4\vartheta), \tag{58a}$$

$$F_2^{\pm 1}(\vartheta) = \tfrac{5}{2}\,(1 - 3\cos^2\vartheta + 4\cos^4\vartheta), \tag{58b}$$

$$F_2^{\pm 2}(\vartheta) = \tfrac{5}{2}\,(1 - \cos^4\vartheta). \tag{58c}$$

These formulae apply for electric as well as magnetic multipole radiation. With these results we have calculated the angular distribution of 2^L-pole γ-radiation emitted by oriented nuclei according to the formula (compare (1) and (2)).

$$W(\vartheta) = \Sigma_{m_i, M}\, a_{m_i}\, G^{j_i L j_f}_{m_i\, m_i - M} \cdot F_L^M(\vartheta), \tag{59}$$

$$G^{j_i L j_f}_{m_i\, m_i - M} = |\,\langle j_f m_f\, LM |\, j_f L j_i m_i \rangle|^2. \tag{60}$$

In [18]), pages 76 and 77, one finds tables of transformation coefficients $\langle j_1 m_1 j_2 m_2 | j_1 j_2 j m \rangle$. Results obtained from (59), (60), and (57), and (58) for dipole and quadrupole radiation respectively, are given in § 6. Since

$$\Sigma_{m_2}\, |\,\langle j_1, m_1, j_2\, m_2\, |\, j_1\, j_2\, j m \rangle|^2 = 1, \tag{61}$$

it follows from (53) and $\Sigma_m a_m = 1$, that $W(\vartheta)$ is normalized to

$$\int W(\vartheta)\, \mathrm{d}\Omega = 8\pi. \tag{62}$$

TABLE I

electric		magnetic		2^L-pole radiation; $kr \gg 1$. $C = \frac{1}{r}\left(\frac{2}{\pi}\right)^{\frac{1}{2}} (-i)^{L+1}\, e^{ik(r-ct)}$.
E	B_r	Er	B	0
$-E_\varphi$	B_ϑ	E_ϑ	B_φ	$\left\{\frac{1}{2}\sqrt{\frac{(L-M)(L+M+1)}{L(L+1)}}\, Y_L^{M+1} \cos\vartheta\, e^{i\varphi} + \frac{M}{\sqrt{L(L+1)}}\, Y_L^M \sin\vartheta + \frac{1}{2}\sqrt{\frac{(L+M)(L-M+1)}{L(L+1)}}\, Y_L^{M-1} \cos\vartheta\, e^{i\varphi}\right\}.C$
E_ϑ	B_φ	E_φ	$-B_\vartheta$	$\left\{-\frac{i}{2}\sqrt{\frac{(L-M)(L+M+1)}{L(L+1)}}\, Y_L^{M+1} e^{-i\varphi} + \frac{i}{2}\sqrt{\frac{(L+M)(L-M+1)}{L(L+1)}}\, Y_L^{M-1} e^{i\varphi}\right\}.C$

In order to calculate the polarization of the emitted radiation we determine first the polarization vector ξ_{0L}^M of the radiation in direction (ϑ) in the case of (L, M)-pole radiation. Now we need the field strengths **E** and **B**, which follow with (49) and (50) from the expressions for **A** given in [4]) table I, for both electric and magnetic (L, M)-pole radiation. We give the components E_r, E_ϑ, E_φ and B_r, B_ϑ, B_φ in our table I for large distance from the origin ($kr \gg 1$) (Fig. 1

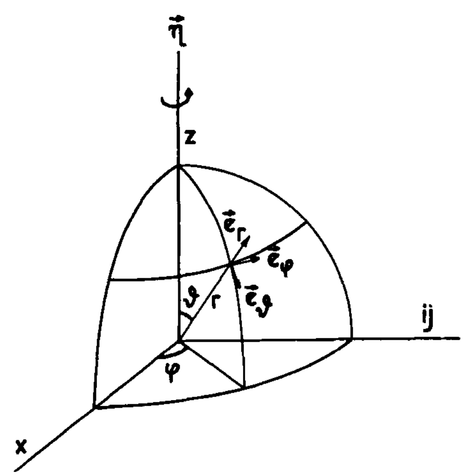

Fig. 1. The coordinates r, ϑ, φ and the axis of quantization $\boldsymbol{\eta}$. $\mathbf{e}_r$ $\mathbf{e}_\vartheta$, $\mathbf{e}_\varphi$, are unit vectors.

TABLE II

Dipole radiation					
electric		magnetic		$L = 1$	
				$M = 0$	$M = \pm 1$
$-E_\varphi$	B_ϑ	E_ϑ	B_φ	0	$\frac{1}{\sqrt{4\pi}} \frac{\sqrt{3}}{2} e^{\pm i\varphi} . C$
E_ϑ	B_φ	E_φ	$-B_\vartheta$	$-\frac{i}{\sqrt{4\pi}} \sqrt{\tfrac{3}{2}} \sin\vartheta . C$	$\pm \frac{i}{\sqrt{4\pi}} \frac{\sqrt{3}}{2} \cos\vartheta \, e^{\pm i\varphi} . C$

illustrates the coordinate system chosen). The tables II and III give these results in a more explicit form for the dipole and quadrupole case ($L = 1$, $L = 2$).

For the calculation of $\boldsymbol{\xi}_{0L}^M$ (for radiation in a direction given by (ϑ, φ)) we take the two unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ defined by

$$\mathbf{e}_1 = \mathbf{e}_\vartheta, \ \mathbf{e}_2 = \mathbf{e}_\varphi. \tag{63}$$

Now $\mathbf{E}$ can be written as

$$\mathbf{E} = E_\vartheta \mathbf{e}_1 + E_\varphi \mathbf{e}_2, \tag{64}$$

and (compare (36))

$$E_\varphi / E_\vartheta = c_2 / c_1 \tag{65}$$

c_2/c_1 can be calculated with the aid of Table II and III (for $L = 1$ and $L = 2$). According to (43) and (44) we then determine α, tg β and $\boldsymbol{\xi}_{0L}^M$. The results are listed in the tables IV and V.

We write $\boldsymbol{\xi}_{0L}^M$ as

$$\boldsymbol{\xi}_{0L}^M = \xi_1 \boldsymbol{\chi}_{\parallel} + \xi_2 \boldsymbol{\chi}_{\perp} + \xi_3 \boldsymbol{\chi}_c. \tag{66}$$

The unit vectors $\boldsymbol{\chi}_{\parallel}$, $\boldsymbol{\chi}_{\perp}$ and $\boldsymbol{\chi}_c$ correspond to the following values of α and β

$$\begin{aligned} \boldsymbol{\chi}_{\parallel} &\to \alpha = 0, \ \beta = 0 \\ \boldsymbol{\chi}_{\perp} &\to \alpha = \pi/4, \ \beta = 0 \\ \boldsymbol{\chi}_c &\to \alpha \text{ arbitrary}, \ \beta = \pi/4. \end{aligned} \tag{67}$$

In connection with the choice of the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ this allows the following interpretation:

$\boldsymbol{\chi}_{\parallel}$ determines the state of linear polarization when the electric vector lies in the plane of $\mathbf{k}$ (direction of propagation) and $\boldsymbol{\eta}$ (axis of rotational symmetry).

TABLE III

Quadrupole radiation						
electric		magnetic		$L = 2$		
				$M = 0$	$M = \pm 1$	$M = \pm 2$
$-E_\varphi$	B_ϑ	E_ϑ	B_φ	0	$\frac{1}{\sqrt{4\pi}} \frac{\sqrt{5}}{2} \cos\vartheta \, e^{\pm i\varphi} . C$	$\pm \frac{1}{\sqrt{4\pi}} \frac{\sqrt{5}}{2} \sin\vartheta \, e^{\pm 2i\varphi} . C$
E_ϑ	B_φ	E_φ	$-B_\vartheta$	$-\frac{i}{\sqrt{4\pi}} \sqrt{\frac{15}{2}} \cos\vartheta \sin\vartheta . C$	$\mp \frac{i}{\sqrt{4\pi}} \frac{\sqrt{5}}{2} (1 - 2\cos^2\vartheta) \, e^{\pm i\varphi} . C$	$\frac{i}{\sqrt{4\pi}} \frac{\sqrt{5}}{2} \sin\vartheta \cos\vartheta \, e^{\pm 2i\varphi} . C$

TABLE V

Sate of polarization of quadrupole radiation									
$L = 2$	M = 0			$M = \pm 1$			$M = \pm 2$		
	α	tg β	ξ_{0L}^M	α	tg β	ξ_{0L}^M	α	tg β	ξ_{0L}^M
electric	0	0	$\chi_{\parallel}$	0	$\mp \frac{\cos\vartheta}{1 - 2\cos^2\vartheta}$	$\frac{1 - 5\cos^2\vartheta + 4\cos^4\vartheta}{1 - 3\cos^2\vartheta + 4\cos^4\vartheta} \chi_{\parallel} \pm \frac{-2\cos\vartheta + 4\cos^3\vartheta}{1 - 3\cos^2\vartheta + 4\cos^4\vartheta} \chi_c$	0	$\pm \frac{1}{\cos\vartheta}$	$-\frac{1 - \cos^2\vartheta}{1 + \cos^2\vartheta} \chi_{\parallel} \pm \frac{2\cos\vartheta}{1 + \cos^2\vartheta} \chi_c$
magnetic	0	∞	$-\chi_{\parallel}$	0	$\mp \frac{1 - 2\cos^2\vartheta}{\cos\vartheta}$	$-\frac{1 - 5\cos^2\vartheta + 4\cos^4\vartheta}{1 - 3\cos^2\vartheta + 4\cos^4\vartheta} \chi_{\parallel} \pm \frac{-2\cos\vartheta + 4\cos^3\vartheta}{1 - 3\cos^2\vartheta + 4\cos^4\vartheta} \chi_c$	0	$\pm \cos\vartheta$	$\frac{1 - \cos^2\vartheta}{1 + \cos^2\vartheta} \chi_{\parallel} \pm \frac{2\cos\vartheta}{1 + \cos^2\vartheta} \chi_c$

TABLE IV

State of polarization of dipole radiation						
$L = 1$	$M = 0$			$M = \pm 1$		
	α	$\mathrm{tg}\,\beta$	$\boldsymbol{\xi}_{0L}^{M}$	α	$\mathrm{tg}\,\beta$	$\boldsymbol{\xi}_{0L}^{M}$
electric	0	0	$\boldsymbol{\chi}_{\parallel}$	0	$\pm \dfrac{1}{\cos\vartheta}$	$-\dfrac{1-\cos^2\vartheta}{1+\cos^2\vartheta}\boldsymbol{\chi}_{\parallel} \pm \dfrac{2\cos\vartheta}{1+\cos^2\vartheta}\boldsymbol{\chi}_c$
magnetic	0	∞	$-\boldsymbol{\chi}_{\parallel}$	0	$\pm\cos\vartheta$	$\dfrac{1-\cos^2\vartheta}{1+\cos^2\vartheta}\boldsymbol{\chi}_{\parallel} \pm \dfrac{2\cos\vartheta}{1+\cos^2\vartheta}\boldsymbol{\chi}_c$

$-\boldsymbol{\chi}_{\parallel}$ gives the state of linear polarization rotated through $\pi/2$ compared with the former.

$\pm\boldsymbol{\chi}_{\perp}$ give the states of linear polarization rotated through $\pi/4$ compared with $\boldsymbol{\chi}_{\parallel}$.

$+\boldsymbol{\chi}_c$, $-\boldsymbol{\chi}_c$ give left and right circular polarized radiation respectively.

The probability $F_L^M(\vartheta, \boldsymbol{\xi})$ of finding a photon with polarization vector $\boldsymbol{\xi}$ in a direction v follows from (55) and (48)

$$F_L^M(\vartheta, \boldsymbol{\xi}) = F_L^M(\vartheta)\cdot\tfrac{1}{2}(1+\boldsymbol{\xi}\cdot\boldsymbol{\xi}_{0L}^M) = 4\pi^2 r^2 \mathbf{E}\cdot\mathbf{E}^*\,\tfrac{1}{2}(1+\boldsymbol{\xi}\cdot\boldsymbol{\xi}_{0L}^M). \tag{68}$$

If we now consider the ensemble of oriented radioactive nuclei, and an axis $\boldsymbol{\eta}$ of rotational symmetry exists, we can calculate in a way similar to (59) the probability $W(\vartheta, \boldsymbol{\xi})$ of finding a γ-quantum in the direction ϑ with polarization vector $\boldsymbol{\xi}$

$$W(\vartheta, \boldsymbol{\xi}) = \Sigma_{m_i M}\, a_{m_i}\, G^{j_i L j_f}_{m_i, m_i - M}\, F_L^M(\vartheta, \boldsymbol{\xi}). \tag{69}$$

We can compute $W(\vartheta, \boldsymbol{\xi})$ as $F_L^M(\vartheta)$ for ($L = 1$ and $L = 2$) is given by (57) and (58), and $\boldsymbol{\xi}_{0L}^M$ is listed in the tables IV and V.

As is clear from the formula (68) for $F_L^M(\vartheta, \boldsymbol{\xi})$ the radiation of a pure (L, M)-pole is totally polarized. The radiation from an ensemble of oriented nuclei, however, is in general partially polarized. $W(\vartheta, \boldsymbol{\xi})$ can also be written as (47)

$$W(\vartheta, \boldsymbol{\xi}) = W(\vartheta)\cdot\tfrac{1}{2}(1 + P\boldsymbol{\xi}\cdot\boldsymbol{\xi}_0). \tag{70}$$

As $W(\vartheta)$ is known from (59), and $W(\vartheta, \boldsymbol{\xi})$ from (69), we can obtain P and $\boldsymbol{\xi}_0$ from (70). For the calculation of the degree of polarization P and the polarization vector $\boldsymbol{\xi}_0$ it is sufficient to know $W(\vartheta)\,P\boldsymbol{\xi}_0$. Therefore we give the results for $W(\vartheta)\,P\boldsymbol{\xi}_0$, instead of for $W(\vartheta, \boldsymbol{\xi})$ in § 7.

From (59), (68), (69) and (70) it follows that

$$W(\vartheta)\,P\boldsymbol{\xi}_0 = \Sigma_{m_i M}\, a_{m_i}\, G^{j_i L j_f}_{m_i m_i - M}\, F_L^M(\vartheta)\,\boldsymbol{\xi}_{0L}^M. \tag{71}$$

§ 5. *Another method of calculation of the angular distribution and polarization of 2^L-pole γ-radiation.* In this section we shall derive formulae for $W(\vartheta)$ and $W(\vartheta)\, P\xi_0$ for arbitrary 2^L-pole γ-radiation, making use of the algebra of tensor operators developed by Racah [10]) [6]). During the derivation we shall drop constant factors without further comment since these factors only affect the normalization which is not essential in the problem. If $\boldsymbol{\eta}$ is the axis of rotational symmetry and of quantization, then the probability of emission of a γ-quantum in the direction $\mathbf{k}$ with polarization described by $\mathbf{c}$ (§ 3) is given by

$$W(\mathbf{k}, \mathbf{c}, \boldsymbol{\eta}) = \Sigma_{m_i m_f} a_{m_i} \,|\langle j_i\, m_i|\, H | j_f\, m_f\rangle|^2. \tag{72}$$

In the case of a pure 2^L-pole radiation we can write for the interaction Hamiltonian $\bar{H}$, in a coordinate system with $\mathbf{k}$ as quantization axis,

$$\bar{H} = \Sigma_M a_{LM}(\mathbf{c})\; T^L_M. \tag{73}$$

T^L_M are the components of an irreducible tensor operator of degree L, which operate on the nucleus. $a_{LM}(\mathbf{c})$ are functions of $\mathbf{c}$, and are thus connected with the polarization of the emitted γ-quantum. Only a_{L1}, and a_{L-1} are different from zero and are expressed by $\mathbf{c}$ as follows [8]), [6]).

$a_{L1} = -(c_1 + ic_2)/\sqrt{2}$, $a_{L-1} = (c_1 - ic_2)/\sqrt{2}$ (electric 2^L-pole radiation) (74*a*)

$a_{L1} = -(c_2 - ic_1)/\sqrt{2}$, $a_{L-1} = (c_2 + ic_1)/\sqrt{2}$(magnetic 2^L-pole radiation (74*b*)

With $\boldsymbol{\eta}$ as quantization axis the Hamiltonian (73) takes the form (with $(\mathbf{k}\cdot\boldsymbol{\eta}) = \cos\vartheta$; cf. [6]) formula (2))

$$H = \Sigma_{M\mu} a_{LM}(\mathbf{c}) T^L_\mu D^L_{\mu M}\;(0, \vartheta, 0). \tag{75}$$

$(0, \vartheta, 0)$ are the three Euler angles associated with the rotation which transforms the coordinate system with $\mathbf{k}$ as z-axis into that whose z-axis is $\boldsymbol{\eta}$. Making use of (72) and (75) we find for $W(\mathbf{k}, \mathbf{c}, \boldsymbol{\eta})$

$$W(\vartheta, \mathbf{c}) = W(\mathbf{k}, \mathbf{c}, \boldsymbol{\eta}) =$$
$$= \Sigma_{m_i m_f M M' \mu\mu'} a_{m_i} a^*_{LM}\, a_{LM'} \langle j_i m_i | T^L_\mu | j_f m_f\rangle^* \langle j_i m_i | T^L_{\mu'} | j_f m_f\rangle D^{L*}_{\mu M} D^L_{\mu' M'}. \tag{76}$$

With the relation [19]) page 203, (16a))

$$D^{L*}_{\mu M} D^L_{\mu' M'} = \Sigma_{\varrho\sigma k} (-1)^{M-\mu} \langle L - \mu L \mu' | LL\, k\varrho\rangle D^k_{\varrho\sigma} \langle LL\, k\sigma | L - M L M'\rangle \tag{77}$$

and the formulae, (16′) and (29) resp., from [10])

$$\langle j_i m_i \,|T^L_\mu|\, j_f m_f \rangle = (-1)^{j_i - m_i} \langle j_i \,||\, T^L \,||\, j_f \rangle \cdot V(j_i j_f L; -m_i m_f \mu), \quad (78)$$

$$\langle L-\mu L\mu' \,|\, LL\,k\varrho \rangle = (-1)^{k+\varrho} (2k+1)^{\frac{1}{2}} V(LLk; -\mu\mu' -\varrho), \quad (79)$$

formula (76) becomes (with summation over m_i, m_f, M, M', μ, μ', k, ϱ and σ)

$$W(\vartheta, \mathbf{c}) = \Sigma\, a_{m_i}\, a^*_{LM}\, a_{LM'}\, (-1)^{k+\varrho+M-\mu} (2k+1)^{\frac{1}{2}} D^k_{\varrho\sigma} \times$$
$$\times \langle LLk\sigma \,|\, L-MLM' \rangle\, V(j_i j_f L; -m_i m_f \mu)\; V(j_i j_f L; -m_i m_f \mu') \times$$
$$\times\; V(LLk;\; -\mu\mu'\varrho). \quad (80)$$

With formula (41) from [10]) the summation over μ, μ' and m_f can be carried out and $W(\vartheta, \mathbf{c})$ becomes

$$W(\vartheta, \mathbf{c}) = \Sigma\, a_{m_i}\, a^*_{LM}\, a_{LM'}\, (-1)^{M-L} D^k_{\varrho\sigma} \langle LLk\sigma|\, L-MLM' \rangle \times$$
$$\times\; W(j_f j_i Lk; Lj_i)\, (-1)^{j_i - m_i} (2k+1)^{\frac{1}{2}} (-1)^{k-\varrho}\, V(j_i j_i\, k; -m_i m_i \varrho). \quad (81)$$

Now we observe that (cf. formula (17) from [10])) that $V(j_i j_i k; -m_i m_i \varrho) = V(j_i j_i k; -m_i m_i o)\, \delta_{0\varrho}$, which makes possible summation over ϱ. From this we obtain with the aid of (79), (25), (26) and (32)

$$W(\vartheta, \mathbf{c}) = \Sigma_{k\sigma}\, C_{k\sigma}(LL)\, W(j_f j_i Lk;\; Lj_i)\, f_k w_k^{-1} D^k_{0\sigma}\, (0, \vartheta, 0), \quad (82)$$

where $C_{k\sigma}(LL)$ is the abbreviation

$$C_{k\sigma}(LL) = \Sigma_{MM'}\, (-1)^{L-M}\, a^*_{LM}\, a_{LM'} \langle LL\, k\sigma \,|\, L-MLM' \rangle. \quad (83)$$

We shall now consider electric 2^L-pole radiation by substituting (74*a*) in (83). Then only $C_{k0}(LL)$ and $C_{k\pm 2}(LL)$ turn out to be different from zero. Making use of (45) we find

$$C_{k0}(LL) = (-1)^{L-1} \langle LLko \,|\, L1L-1 \rangle \text{ if } k \text{ is even}, \quad (84a)$$

$$C_{k0}(LL) = (-1)^{L-1} \langle LLko \,|\, L1L-1 \rangle\, \xi_3, \text{ if } k \text{ is odd}, \quad (84b)$$

$$C_{k2}(LL) + C_{k-2}(LL) = (-1)^{L-1} \langle LLko|L1L1 \rangle. - \xi_1 \text{ if } k \text{ is even}, \quad (84c)$$

$$C_{k2}(LL) + C_{k-2}(LL) = 0 \text{ if } k \text{ is odd}. \quad (84d)$$

With (84) and $D^k_{0\sigma} = Y^\sigma_k(\vartheta, 0)\, (2k+1)^{-\frac{1}{2}}$, (82) becomes

$$W(\vartheta, \boldsymbol{\xi}) = W(\vartheta, \mathbf{c}) =$$
$$= \Sigma_{k\, even}\, W(j_f j_i\, Lk;\; Lj_i)\, f_k\, w_k^{-1} (2k+1)^{-\frac{1}{2}}\, \{ Y^0_k \langle LLk0 \,|\, L1L-1 \rangle -$$
$$- Y^2_k \langle LLk2|L1L1 \rangle \cdot \xi_1 \} +$$
$$+ \Sigma_{k\, odd}\, W(j_f j_i Lk;\; Lj_i)\, f_k w_k^{-1} (2k+1)^{-\frac{1}{2}}\, Y^0_k \langle LLko|L1L-1) \cdot \xi_3. \quad (85)$$

If the polarization is not observed, we can sum over the polarization directions and obtain $W(\vartheta) = W(\vartheta, \boldsymbol{\xi}) + W(\vartheta, -\boldsymbol{\xi})$. From (85)

$$W(\vartheta) = \tag{86}$$
$$= \Sigma_{k\ even}\, 2W(j_f j_i Lk; Lj_i)\, f_k w_k^{-1}(2k+1)^{-\frac{1}{2}} Y_k^0(\vartheta, 0)\, \langle LLko|L1L-1\rangle.$$

Making use of (86) and of the representation

$$\boldsymbol{\xi} = \xi_1 \boldsymbol{\chi}_\parallel + \xi_2 \boldsymbol{\chi}_\perp + \xi_3 \boldsymbol{\chi}_c, \tag{87}$$

it follows that (85) can be written

$$W(\vartheta, \boldsymbol{\xi}) = \tfrac{1}{2} W(\vartheta)\, (1 + P\boldsymbol{\xi}_0 \cdot \boldsymbol{\xi}), \tag{88}$$

with

$$W(\vartheta)P\boldsymbol{\xi}_0 =$$
$$= \Sigma_{k\ odd}\, 2W(j_f j_i Lk; Lj_i)\, f_k w_k^{-1}\, (2k+1)^{-\frac{1}{2}}\, Y_k^0(\vartheta, 0) \langle LLko/L1L-1\rangle\, \boldsymbol{\chi}_c -$$
$$- \Sigma_{k\ even}\, 2W(j_f j_i Lk; Lj_i)\, f_k w_k^{-1}\, (2k+1)^{-\frac{1}{2}}\, Y_k^2(\vartheta, 0) \langle LLk2\,|\,L1L1\rangle\, \boldsymbol{\chi}_\parallel. \tag{89}$$

For magnetic 2^L-pole radiation we must replace $\boldsymbol{\chi}_\parallel$ by $-\boldsymbol{\chi}_\parallel$ in (89), as is easily derived from (74*b*) in the same way as in the electric case. Formula (86) remains unchanged.

$W(\vartheta)$ in formula (86) is not normalized according to (62), but this normalization can be obtained by calculating $\int W(\vartheta)\, d\Omega$.

§ 6. *Explicit formulae for the angular distribution of γ-radiation emitted by oriented nuclei.* We have calculated explicit formulae for the angular distribution function $W(\vartheta)$ (59) and the polarization $W(\vartheta)\, P\boldsymbol{\xi}_0$ (71) for the following cases:

dipole radiation $j_f = j_i \pm 1$, $j_f = j_i$,
quadrupole radiation $j_f = j_i \pm 2$.

j_i and j_f are the angular momentum quantum numbers of the initial and final nuclei. We have assumed that there are no interference effects among radiations of different multipole character. The formulae are then valid for pure dipole or quadrupole radiation. We have made the calculations by methods both of § 4 and of § 5. The method of § 4 does not give rise to long calculations in the dipole case, but for the next order, quadrupole radiation, the amount of labour required already begins to mount. For higher multipole orders the method of § 5 is certainly to be preferred. The computing work to get explicit formulae with the aid of the latter method is now shifted to the evaluation of the general expressions in terms of simple products, etc. Especially, the work to compute the

$W(j_f j_i Lk, Lj_i)$ is considerable. However, once they have been calculated for a number of cases they can be tabulated and can be used for many purposes (tables have been given by K. Alder [8])).

The results for $W(\vartheta)$ are the same for electric and magnetic 2^L-pole radiation. The polarization, however, is different for the electric and magnetic radiations (§ 7). We give here explicit formulae for $W(\vartheta)$ (normalized according to (62)).

Dipole radiation ($L = 1$).

$$j_f = j_i - 1,\quad W(\vartheta) = 2\,(1 + \tfrac{3}{2} N_2 f_2 P_2\ (\cos\vartheta)) \tag{90}$$

$$j_f = j_i\quad ,\quad W(\vartheta) = 2\,(1 - \tfrac{3}{2} K_2 f_2 P_2\ (\cos\vartheta)) \tag{91}$$

$$j_f = j_i + 1,\quad W(\vartheta) = 2\,(1 + \tfrac{3}{2} M_2 f_2 P_2\ (\cos\vartheta)) \tag{92}$$

Quadrupole radiation ($L = 2$)

$$j_f = j_i - 2,\quad W(\vartheta) = 2\,(1 - \tfrac{15}{7} N_2 f_2 P_2\ (\cos\vartheta) - 5 N_4 f_4 P_4(\cos\vartheta)) \tag{93}$$

$$j_f = j_i + 2,\quad W(\vartheta) = 2(1 - \tfrac{15}{7} M_2 f_2 P_2\ (\cos\vartheta) - 5 M_4 f_4 P_4(\cos\vartheta)) \tag{94}$$

with

f_k the degree of orientation of order k ((16), (18)),

$$P_2\ (\cos\vartheta) = \tfrac{3}{2}\,(\cos^2\vartheta - \tfrac{1}{3}) \tag{95a}$$

$$P_4\ (\cos\vartheta) = \tfrac{35}{8}\,(\cos^4\vartheta - \tfrac{6}{7}\cos^2\vartheta + \tfrac{3}{35}) \tag{95b}$$

$$N_k = b_k\, j^k (2j - k)!/(2j)!,\quad M_k = b_k\, j^k\ (2j + 1)!/(2j + k + 1)! \tag{96a}$$

$$b_k = 2^{\frac{1}{2}k} \text{ if } k \text{ is even},\ b_k = 2^{\frac{1}{2}(k+1)} \text{ if } k \text{ is odd}, \tag{96b}$$

$$K_1 = 1/(j_i + 1),\ K_2 = j_i/(j_i + 1). \tag{97}$$

§ 7. *Explicit formulae for the polarization of γ-radiation emitted by oriented nuclei.* We now give explicit formulae for $W(\vartheta)\, P\xi_0$. The formulae given below are valid for *electric* dipole and quadrupole radiation. For the magnetic radiation, the sign of $\chi_{\parallel}$ is changed in the corresponding formulae for the electric case, while the sign of χ_c remains unaltered.

Electric dipole radiation ($L = 1$)

$$j_f = j_i - 1,\quad W(\vartheta)\, P\xi_0 = 3 N_1 f_1 \cos\vartheta\,\chi_c - \tfrac{9}{2} N_2 f_2\ (1 - \cos^2\vartheta)\ \chi_{\parallel} \tag{98}$$

$$j_f = j_i,\quad W(\vartheta)\, P\xi_0 = 3 K_1 f_1 \cos\vartheta\,\chi_c + \tfrac{9}{2} K_2 f_2\ (1 - \cos^2\vartheta)\ \chi_{\parallel} \tag{99}$$

$$j_f = j_i + 1,\quad W(\vartheta)\, P\xi_0 = -3 M_1 f_1 \cos\vartheta\,\chi_c - \tfrac{9}{2} M_2 f_2\,(1 - \cos^2\vartheta)\,\chi_{\parallel} \tag{100}$$

Electric quadrupole radiation ($L = 2$)

$j_f = j_i - 2$

$$W(\vartheta)\, P\xi_0 = \{\tfrac{45}{7} N_2 f_2 (\cos^2\vartheta - 1) + \tfrac{25}{4} N_4 f_4 (-7\cos^4\vartheta + 8\cos^2\vartheta - 1)\}\chi_{\parallel} + + \{2N_1 f_1 \cos\vartheta + 5N_3 f_3 (-5\cos^3\vartheta + 3\cos\vartheta)\}\chi_c. \qquad (101)$$

$j_f = j_i + 2$

$$W(\vartheta)\, P\xi_0 = \{\tfrac{45}{7} M_2 f_2 (\cos^2\vartheta - 1) + \tfrac{25}{4} M_4 f_4 (-7\cos^4\vartheta + 8\cos^2\vartheta - 1)\}\chi_{\parallel} - - \{2M_1 f_1 \cos\vartheta + 5 M_3 f_3 (-5\cos^3\vartheta + 3\cos\vartheta)\}\chi_c. \qquad (102)$$

The angular dependent functions which occur in (98) .... (102) are proportional to $Y_k^{\sigma}(\vartheta, o)$ as also follows from (89). Here f_k, N_k, M_k, K_1 and K_2 have the same meaning as in § 6.

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