

## ON THE RESONANT EXCITATION OF PLASMA OSCILLATIONS WITH LASER BEAMS

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### Synopsis

A recent proposal for the resonant excitation of longitudinal plasma oscillations by transverse electromagnetic waves provides an application of non-secular perturbation techniques on the cold plasma equations. The amplitudes and frequencies of the oscillations so excited are calculable in terms of the amplitudes and frequencies of the transverse radiation and the plasma parameters. Some numerical examples are given.

1. *Introduction.* In a recent letter, Kroll, Ron and Rostoker<sup>1)</sup> have made the interesting suggestion that nonlinear resonance of two transverse electromagnetic waves whose frequencies differ approximately by the electron plasma frequency can be used to excite longitudinal electron plasma oscillations. This method appears to possess advantages over previous techniques for exciting electrostatic waves, which have often been both indirect and uncontrolled.

Using a straightforward perturbation procedure on the nonlinear Vlasov equation, these authors calculate the second order charge density associated with two first order transverse electromagnetic waves of frequency difference  $\Delta\omega \approx \omega_p$ . This charge density is expressed as a Laplace transform. When the transform is inverted, the frequencies in the longitudinal wave are conveniently classified according to the locations of the corresponding poles in the complex Laplace transform plane. These poles may be divided into two groups: (i) those from the zeros of the Landau denominator, or plasma dielectric function; and (ii) those from the sum and difference frequencies of the two first order transverse waves. Each type of pole produces terms which vary as  $\exp(i\omega t)$  in the charge density, where  $i\omega$  is the location of the corresponding pole in the complex Laplace transform plane.

However, it is clear that, as the dissipation of the plasma (collisional plus Landau damping) becomes small, poles of the type (i) move arbitrarily close to some of those of type (ii). The large residues which result produce arbitrarily large terms in the second order charge density (and a "secular" term, in the limit that they exactly coincide), making invalid the ordering originally assumed for the perturbation series. In addition, terms which

have been neglected in computing the transverse fields become no longer negligible. A necessary condition for the accuracy of eq. (6) of ref. 1 is, therefore, that the quantity  $(\mathbf{E}_1 \cdot \mathbf{E}_2 / \Gamma \omega_p)^2$ , (where  $\mathbf{E}_1$  and  $\mathbf{E}_2$  are the transverse fields and  $\Gamma \omega_p$  is the damping decrement, collisional or Landau, whichever is larger) be not so large as to make  $4\pi en^{(1)} / \Delta k$  of the same order as  $E_1$  or  $E_2$ . Since the damping decrement  $\Gamma \omega_p$  (which will always be collisional, since this is far larger than the Landau damping for all the cases in which we shall be interested) goes to zero as the temperature increases at a given density, it is clear that the calculation should be supplemented for this regime by one in which the longitudinal field is treated as of the *same* order as the transverse fields. The data taken by Kroll, Ron and Rostoker are such as to be safely inside the region of validity for the assumed expansion, but this statement is not true if one considers kilovolt plasmas instead of the relatively low temperature of 10 eV which they take.

We take as a starting point the equations of the cold electron plasma with uniform immobile positive background; this is a good approximation, since for all wave numbers  $k$  and all frequencies  $\omega$  for the three waves involved,  $(kv_{th}/\omega)^2$  is  $\ll 1$ , where  $v_{th}$  is an electron thermal velocity. Doing this makes this situation fit into a recent<sup>2)</sup> treatment of the cold plasma equations which uses a modified Krylov-Bogoliubov-Mitropolskii<sup>3)</sup> secularity-free perturbation theory. The problem is mathematically related to that of three nonlinearly coupled harmonic oscillators, two of whose frequencies add to give approximately that of the third<sup>4)</sup>. The three oscillators are here the two transverse modes with frequencies  $\omega_2, \omega_3$ , ( $\omega_2, \omega_3 \gg \omega_p$ , the electron plasma frequency) and the longitudinal mode of frequency  $\omega_p$ . We assume  $\omega_3 - \omega_2 = \omega_p + \varepsilon \Delta$ ;  $\varepsilon$  is a formal expansion parameter, which will eventually be set equal to one, and  $\Delta$  is a measure of the mismatch between  $\omega_3 - \omega_2$  and  $\omega_p$ . In practice, experimental limitations will always prohibit setting  $\Delta$  exactly equal to zero. Two of these are nonuniformity in the density, and hence the plasma frequency, and the dissipative properties of the plasma, which are ignored in using the cold plasma equations. The authors of ref. 1 assume that the dissipation is more important; but the net effect of the lack of perfect resonance is the same, and is always important. In the calculation we shall carry  $\Delta$  along as a parameter; later, in the numerical estimates, we shall assign  $\Delta$  the same value as the quantity  $\Gamma_c \omega_p$  of ref. 1, in order to facilitate comparison. (Whether or not  $\Gamma_c \omega_p$  will remain the dominant part of the detuning, at the higher temperatures we shall consider, will depend on the accuracy assumed for the density).

The complicated nature of the partial differential equations has prohibited a formulation of the problem as a rigorously-posed boundary or initial value problem (the same remark applies to ref. 1). Therefore, we state as clearly as possible the physical assumptions which have been used to determine a solution.

(a) The lowest order transverse electromagnetic field consists of two right-travelling waves throughout the plasma, with electric field amplitudes  $2a_2$ ,  $2a_3$ , which are given at a particular plane which is normal to the  $x$  axis. The waves are driven so as to oscillate sinusoidally at frequencies  $\omega_2$ ,  $\omega_3$ . The *spatial* dependence of the waves must follow from the equations of motion.

b) The lowest order plasma oscillation shall be right-travelling and periodic, at *approximately* the plasma frequency, but with a detailed frequency dependence and a spatial dependence which follow from the equations of motion.

It turns out that these two requirements are enough to calculate both the amplitude and frequency of the plasma oscillation, and the spatial dependence of all three waves. In the language of ref. 2, the spatial dependences are interpretable as "wave number shifts", and the plasma oscillation is "frequency shifted" from its lowest order value  $\omega_p$ .

In the next section, we find the solution in second order. Stopping at second order avoids the necessity for a relativistic treatment. Unlike the situation in ref. 2, the interesting effects here occur in the first, rather than the second, correction to the linear theory. In section 3, some numerical estimates are given for sample situations.

2. *Solution to second order.* The dynamical equations for an electron plasma are well known, and we write then in a way which explicitly displays the nonlinear terms:

$$\frac{\partial n}{\partial t} + n_0 \frac{\partial}{\partial x} \cdot \mathbf{v} = -\varepsilon \frac{\partial}{\partial x} \cdot (n\mathbf{v}), \quad (1a)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{e}{m} \mathbf{E} = -\varepsilon \left( \mathbf{v} \cdot \frac{\partial}{\partial x} \mathbf{v} + \frac{e}{mc} \mathbf{v} \times \mathbf{B} \right), \quad (1b)$$

$$\frac{\partial \mathbf{B}}{\partial t} + c \frac{\partial}{\partial x} \times \mathbf{E} = 0, \quad (1c)$$

$$\frac{\partial \mathbf{E}}{\partial t} - c \frac{\partial}{\partial x} \times \mathbf{B} - 4\pi en_0 \mathbf{v} = \varepsilon 4\pi n \mathbf{v}. \quad (1d)$$

The symbols have the following meanings:

- $-e(n_0 + n)$  = electron charge density,
- $+en_0$  = ion charge density (a constant),
- $\mathbf{v}$  = electron fluid velocity,
- $\mathbf{E}, \mathbf{B}$  = electric and magnetic fields.

The two scalar Maxwell equations, if satisfied at any instant, are preserved

for all time by eqs. (1), and so may be simply regarded as constraints upon the initial conditions. The formal expansion parameter  $\varepsilon$  is a bookkeeping device to remind us that the terms on the right hand sides of eqs. (1) are of second order in powers of the departure from the uniform, field-free equilibrium. Eventually, we shall set  $\varepsilon$  equal to one. Note that we have therefore assumed that the detuning from perfect resonance is to be treated as an  $O(\varepsilon)$  quantity.

For situations where there is variation with only one spatial coordinate ( $x$ , say) these equations are readily combined<sup>2)</sup> to give:

$$\left[ \frac{\partial^2}{\partial t^2} - c^2(1 - \delta_{1,i}) \frac{\partial^2}{\partial x^2} + \omega_p^2 \right] E_i = \varepsilon F_i, \quad (2)$$

where  $i = 1, 2$  or  $3$  refers to the  $x, y$  or  $z$  component in a Cartesian coordinate system ( $\hat{e}_x, \hat{e}_y, \hat{e}_z$ ), and  $F_i$  is the  $i^{\text{th}}$  component of the vector

$$\mathbf{F} = -4\pi e \left[ n_0 \left( v_x \frac{\partial}{\partial x} \mathbf{v} + \frac{e}{mc} \mathbf{v} \times \mathbf{B} \right) - \frac{\partial}{\partial t} (n\mathbf{v}) \right]. \quad (3)$$

Following ref. 2, we seek a solution to the system (1) of the form

$$\begin{pmatrix} n \\ \mathbf{v} \\ \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} n^{(0)} \\ \mathbf{v}^{(0)} \\ \mathbf{E}^{(0)} \\ \mathbf{B}^{(0)} \end{pmatrix} + \varepsilon \begin{pmatrix} n^{(1)} \\ \mathbf{v}^{(1)} \\ \mathbf{E}^{(1)} \\ \mathbf{B}^{(1)} \end{pmatrix} + \varepsilon^2 \begin{pmatrix} n^{(2)} \\ \mathbf{v}^{(2)} \\ \mathbf{E}^{(2)} \\ \mathbf{B}^{(2)} \end{pmatrix} + \dots, \quad (4)$$

where

$$\mathbf{E}^{(0)} = (a_1 e^{i\psi_1} + a_1^* e^{-i\psi_1}) \hat{e}_x + (a_2 e^{i\psi_2} + a_2^* e^{-i\psi_2} + a_3 e^{i\psi_3} + a_3^* e^{-i\psi_3}) \hat{e}_y,$$

$$\mathbf{B}^{(0)} = \left[ \frac{ck_2}{\omega_2} (a_2 e^{i\psi_2} + a_2^* e^{-i\psi_2}) + \frac{ck_3}{\omega_3} (a_3 e^{i\psi_3} + a_3^* e^{-i\psi_3}) \right] \hat{e}_z,$$

$$\begin{aligned} \mathbf{v}^{(0)} = & \frac{e}{mi\omega_1} (a_1 e^{i\psi_1} - a_1^* e^{-i\psi_1}) \hat{e}_x + \\ & + \left[ \frac{e}{mi\omega_2} (a_2 e^{i\psi_2} - a_2^* e^{-i\psi_2}) + \frac{e}{mi\omega_3} (a_3 e^{i\psi_3} - a_3^* e^{-i\psi_3}) \right] \hat{e}_y, \end{aligned}$$

$$n^{(0)} = \frac{k_1 \varepsilon n_0}{mi\omega_1^2} (a_1 e^{i\psi_1} - a_1^* e^{-i\psi_1}). \quad (5)$$

Here,

$$\psi_1 = k_1 x - \omega_1 t + \varphi_1,$$

$$\psi_2 = k_2 x - \omega_2 t + \varphi_2,$$

$$\psi_3 = k_3 x - \omega_3 t + \varphi_3,$$

with phase constants  $\varphi_1, \varphi_2, \varphi_3$ , and

$$\omega_1 \equiv \omega_3 - \omega_2,$$

$$k_1 \equiv k_3 - k_2,$$

so that

$$\psi_3 - \psi_2 - \psi_1 = \varphi_3 - \varphi_2 - \varphi_1 = \vartheta = \text{const.},$$

with  $\omega_3, \omega_2$  given, and  $k_3, k_2$  being determined by the dispersion relations

$$\omega_3^2 = c^2 k_3^2 + \omega_p^2 \approx c^2 k_3^2,$$

$$\omega_2^2 = c^2 k_2^2 + \omega_p^2 \approx c^2 k_2^2,$$

$$\omega_p^2 = \frac{4\pi e^2 n_0}{m}.$$

It will turn out that the phase angles  $\varphi_1, \varphi_2, \varphi_3$  are unimportant, and only the difference  $\vartheta$  will enter into the equations of motion.

For  $\varepsilon = 0$ , eqs. (5) are a solution of eqs. (1) with the  $a$ 's all constant. We shall seek solutions for  $\mathbf{E}^{(1)}, \mathbf{B}^{(1)}, n^{(1)}$ , and  $\mathbf{v}^{(1)}$  which are functions of the amplitudes  $a_1, a_2, a_3$  and phases  $\psi_1, \psi_2, \psi_3$ . It will turn out that we can find well-behaved solutions by letting the  $a$ 's vary as functions of  $\varepsilon x, \varepsilon t, \dots$ . This generalized reformulation of the Krylov-Bogoliubov-Mitropolskii theory was first given by Frieman<sup>5</sup>) in connection with the derivation of kinetic equations. The method of ref. 2, where coefficients in the series expansions of  $\partial a/\partial x$  and  $\partial a/\partial t$  for each  $a$  are calculated, could equally well be used, but would lose in compactness what it would gain in being more explicit. The dependence of the  $a$ 's on  $\varepsilon x, \varepsilon t, \dots$ , will be determined by the requirement that the corrections  $n^{(1)}, \mathbf{v}^{(1)}, \mathbf{E}^{(1)}, \mathbf{B}^{(1)}$ , etc., shall remain small, relative to the zeroth order values given in eqs. (5), under the assumptions (a) and (b) made in the introduction. In particular, the assumption (a) implies that  $a_2, a_3$  will have no  $\varepsilon t$  variation.

Under these assumptions, let us compute the left hand side of eq. (2) through terms of  $O(\varepsilon)$ , in terms of the  $a$ 's and  $\psi$ 's:

$$\begin{aligned} & \left( \frac{\partial^2}{\partial t^2} - c^2(1 - \delta_{1,i}) \frac{\partial^2}{\partial x^2} + \omega_p^2 \right) \mathbf{E} = \\ & \varepsilon \left\{ \left[ -2A\omega_p (a_1 e^{i\psi_1} + a_1^* e^{-i\psi_1}) \hat{e}_x - \right. \right. \\ & -2i\omega_p \left( \frac{\partial a_1}{\partial(\varepsilon t)} e^{i\psi_1} - \frac{\partial a_1^*}{\partial(\varepsilon t)} e^{-i\psi_1} \right) \hat{e}_x - \\ & \left. \left. -2ik_2 c^2 \left( \frac{\partial a_2}{\partial(\varepsilon x)} e^{i\psi_2} - \frac{\partial a_2^*}{\partial(\varepsilon x)} e^{-i\psi_2} \right) \hat{e}_y - \right. \right. \end{aligned}$$

$$\begin{aligned}
& -2ik_3c^2 \left( \frac{\partial a_3}{\partial(\varepsilon x)} e^{i\psi_s} - \frac{\partial a_3^*}{\partial(\varepsilon x)} e^{-i\psi_s} \right) \hat{e}_y \Big] + \\
& + \left( \omega_p^2 + \sum_{j,k=1}^3 \omega_j \omega_k \frac{\partial^2}{\partial \psi_j \partial \psi_k} \right) E_x^{(1)} \hat{e}_x + \\
& + \left( \omega_p^2 + \sum_{j,k=1}^3 (\omega_j \omega_k - c^2 k_j k_k) \frac{\partial^2}{\partial \psi_j \partial \psi_k} \right) E_y^{(1)} \hat{e}_y \Big\} + \\
& + O(\varepsilon^2). \tag{6}
\end{aligned}$$

On the left hand side of eq. (6), the notation  $\delta_{1,i}$  means that  $\delta_{1,i} = 1$  when the operator is applied to the  $x$  component of  $\mathbf{E}$ , and  $\delta_{1,i} = 0$  when the operator is applied to the  $y$  or  $z$  component of  $\mathbf{E}$ .

We also need (through terms of  $O(\varepsilon)$ ) the expression on the *right* hand side of eq. (2):

$$\begin{aligned}
& -4\pi e \varepsilon \left[ n_0(v_x^{(0)} \frac{\partial}{\partial x} \mathbf{v}^{(0)} + \frac{e}{mc} \mathbf{v}^{(0)} \times \mathbf{B}^{(0)}) - \frac{\partial}{\partial t} (n^{(0)} \mathbf{v}^{(0)}) \right] = \\
& = -4\pi e \varepsilon (N + \mathbf{R}) + O(\varepsilon^2), \tag{7}
\end{aligned}$$

where

$$\begin{aligned}
N \equiv \hat{e}_x \Big\{ & \frac{3n_0 e^2 k_1}{im^2 \omega_1^2} (a_1^2 e^{2i\psi_1} - a_1^{*2} e^{-2i\psi_1}) + \\
& + \frac{n_0 e^2 k_2}{im^2 \omega_2^2} (a_2^2 e^{2i\psi_2} - a_2^{*2} e^{-2i\psi_2}) + \\
& + \frac{n_0 e^2 k_3}{im^2 \omega_3^2} (a_3^2 e^{2i\psi_3} - a_3^{*2} e^{-2i\psi_3}) + \\
& + \frac{n_0 e^2 (k_2 + k_3)}{im^2 \omega_2 \omega_3} (a_2 a_3 e^{i(\psi_2 + \psi_3)} - a_2^* a_3^* e^{-i(\psi_2 + \psi_3)}) \Big\} + \\
& + \hat{e}_y \Big\{ \frac{k_1 e^2 n_0}{im^2 \omega_1} \left( \frac{1}{\omega_2} - \frac{1}{\omega_1} \right) (a_1^* a_2 e^{i(\psi_2 - \psi_1)} - a_2^* a_1 e^{-i(\psi_2 - \psi_1)}) + \\
& + \frac{k_1 e^2 n_0}{im^2 \omega_1} \left( \frac{1}{\omega_3} + \frac{1}{\omega_1} \right) (a_1 a_3 e^{i(\psi_1 + \psi_3)} - a_1^* a_3^* e^{-i(\psi_1 + \psi_3)}) \Big\}, \tag{8}
\end{aligned}$$

and

$$\begin{aligned}
R \equiv \hat{e}_x \Big\{ & \frac{n_0 e^2 (k_3 - k_2)}{im^2 \omega_2 \omega_3} (a_2 a_3^* e^{-i(\psi_3 - \psi_2)} - a_2^* a_3 e^{i(\psi_3 - \psi_2)}) \Big\} + \\
& + \hat{e}_y \Big\{ \frac{n_0 e^2 k_1}{im^2 \omega_1} \left( \frac{1}{\omega_2} + \frac{1}{\omega_1} \right) (a_1 a_2 e^{i(\psi_1 + \psi_2)} - a_1^* a_2^* e^{-i(\psi_1 + \psi_2)}) + \\
& + \frac{n_0 e^2 k_1}{im^2 \omega_1} \left( \frac{1}{\omega_3} - \frac{1}{\omega_1} \right) (a_1^* a_3 e^{i(\psi_3 - \psi_1)} - a_3^* a_1 e^{-i(\psi_3 - \psi_1)}) \Big\}. \tag{9}
\end{aligned}$$

The grouping is into "resonant" terms  $\mathbf{R}$  and the "nonresonant" terms  $\mathbf{N}$ . The terms in  $\mathbf{R}$  are what, in a straightforward perturbation procedure, would lead to a breakdown in the expansion. The reason is that they contain frequencies and wave numbers which approximately satisfy the zeroth order dispersion relation; this would lead to an arbitrarily small divisor in the corresponding term in  $\mathbf{E}^{(1)}$ .

The first step in the  $O(\epsilon)$  problem in perturbation theory, therefore, reduces to solving the two inhomogeneous linear partial differential equations for  $E_x^{(1)}$  and  $E_y^{(1)}$  which result from

$$\begin{aligned} & \left( \omega_p^2 + \sum_{j,k=1}^3 \omega_j \omega_k \frac{\partial^2}{\partial \psi_j \partial \psi_k} \right) E_x^{(1)} \hat{e}_x + \\ & + \left( \omega_p^2 + \sum_{j,k=1}^3 (\omega_j \omega_k - c^2 k_j k_k) \frac{\partial^2}{\partial \psi_j \partial \psi_k} \right) E_y^{(1)} \hat{e}_y = \\ & = -4\pi e(N + \mathbf{R}) + \\ & + \left[ 2\Delta\omega_p (a_1 e^{i\psi_1} + a_1^* e^{-i\psi_1}) \hat{e}_x + \right. \\ & + 2i\omega_p \left( \frac{\partial a_1}{\partial(\epsilon t)} e^{i\psi_1} - \frac{\partial a_1^*}{\partial(\epsilon t)} e^{-i\psi_1} \right) \hat{e}_x + \\ & + 2ik_2 c^2 \left( \frac{\partial a_2}{\partial(\epsilon x)} e^{i\psi_2} - \frac{\partial a_2^*}{\partial(\epsilon x)} e^{-i\psi_2} \right) \hat{e}_y + \\ & \left. + 2ik_3 c^2 \left( \frac{\partial a_3}{\partial(\epsilon x)} e^{i\psi_3} - \frac{\partial a_3^*}{\partial(\epsilon x)} e^{-i\psi_3} \right) \hat{e}_y \right]. \quad (10) \end{aligned}$$

However, we note that not nearly all the solutions to eq. (10) are well-behaved. The standard recipe would be to seek a solution for the coefficients in

$$\begin{aligned} E_x^{(1)} &= \sum_{j,k=-3}^3 \mathcal{E}_{jk} e^{i(\psi_j + \psi_k)} + \left( \begin{array}{c} \text{complex} \\ \text{conjugate} \end{array} \right), \\ E_y^{(1)} &= \sum_{j,k=-3}^3 \mathcal{J}_{jk} e^{i(\psi_j + \psi_k)} + \left( \begin{array}{c} \text{complex} \\ \text{conjugate} \end{array} \right), \quad (11) \end{aligned}$$

with the coefficients  $\mathcal{E}_{jk}$  and  $\mathcal{J}_{jk}$  being functions of the amplitudes, by substitution into (10), and add to it a solution to the associated homogeneous equation, which would be determined by whatever boundary condition on  $\mathbf{E}^{(1)}$  one chose to invoke. However, the terms called  $\mathbf{R}$  (the "resonant" terms) in general forbid our finding a solution of the type (11); from them we get terms  $\sim \psi_j \exp(i\psi_j)$  in  $\mathbf{E}^{(1)}$ , or "secular" terms. It simplifies the notation to define, for all  $j$ ,  $\psi_{-j} = -\psi_j$ ,  $\omega_{-j} = -\omega_j$ ,  $k_{-j} = -k_j$ .

It is the terms inside the double bracket ( $\llbracket \rrbracket$ ) which save us, terms which would not be present at all in a straightforward perturbation theory. In

the usual fashion, they may be chosen equal to  $4\pi e\mathbf{R}$ , so that the only inhomogeneous terms which survive on the right hand side of eq. (10) are those in  $-4\pi e\mathbf{N}$  (the "nonresonant" terms). For these, a solution of the type (11), with the  $\mathcal{E}_{jk}$  and  $\mathcal{J}_{jk}$  functions of the amplitudes only, may be found by trivial algebra.

Thus we have, as a condition for the existence of a wellbehaved solution to (10):

$$2i\omega_p \frac{\partial a_1}{\partial(\epsilon t)} + 2\Delta\omega_p a_1 = \frac{\omega_p^2 e k_1}{im\omega_2\omega_3} a_2^* a_3 e^{i\theta}, \quad (12a)$$

$$2ik_2 c^2 \frac{\partial a_2}{\partial(\epsilon x)} = - \frac{\omega_p^2 e k_1}{im\omega_p} \left( \frac{\omega_2}{\omega_p \omega_3} \right) a_1^* a_3 e^{i\theta}, \quad (12b)$$

$$2ik_3 c^2 \frac{\partial a_3}{\partial(\epsilon x)} = \frac{\omega_p^2 e k_1}{im\omega_p} \left( \frac{\omega_3}{\omega_p \omega_2} \right) a_1 a_2 e^{-i\theta}, \quad (12c)$$

and the three complex conjugate relations. These three relations, with the previous assumptions that  $a_2$  and  $a_3$  are independent of time, determine the solution for  $a_1$ ,  $a_2$ ,  $a_3$ , given  $a_2$ ,  $a_3$  at  $\epsilon x = 0$ . It is:

$$a_1 = \frac{\omega_p i e k_1}{2m\omega_2\omega_3} \left( \frac{a_2^* a_3}{\Delta} \right) e^{i\theta}, \quad (13a)$$

$$a_2 = a_2(0) e^{iK_2 \epsilon x}, \quad (13b)$$

$$a_3 = a_3(0) e^{iK_3 \epsilon x}, \quad (13c)$$

where the "wave number shifts",  $K_2$ ,  $K_3$  are:

$$K_2 = - \frac{\omega_p}{\omega_3^2} \frac{e^2}{4m^2 c^2} \frac{k_1^2}{k_2} \frac{|a_3(0)|^2}{\Delta}, \quad (13a)$$

$$K_3 = - \frac{\omega_p}{\omega_2^2} \frac{e^2}{4m^2 c^2} \frac{k_1^2}{k_3} \frac{|a_2(0)|^2}{\Delta}. \quad (14b)$$

The rest of the solution is straightforward. One can write

$$\begin{aligned} -4\pi e\mathbf{N} = & \sum_{i,j=-3}^3 [\eta_x(i, j) \hat{e}_x + \eta_y(i, j) \hat{e}_y] e^{i(\psi_i + \psi_j)} \\ & + [\text{complex conjugate}], \end{aligned} \quad (15)$$

where the coefficients  $\eta_x$ ,  $\eta_y$  (functions only of the  $a$ 's) can be read off from eq. (8). Then the  $\mathcal{E}_{jk}$  and  $\mathcal{J}_{jk}$  of eq. (11) are readily found as

$$\begin{aligned} \mathcal{E}_{jk} &= \frac{\eta_x(j, k)}{\omega_p^2 - \omega_j \omega_k}, \\ \mathcal{J}_{jk} &= \frac{\eta_y(j, k)}{\omega_p^2 - \omega_j \omega_k + c^2 k_k k_j}, \end{aligned} \quad (16)$$



where all the denominators in eq. (16) are finite and  $O(1)$  (those terms which *would* have led to  $O(\epsilon)$  denominators have been grouped into the resonant terms  $\mathbf{R}$ ).

To the solution just computed, we may add any solution of the associated homogeneous equation for  $\mathbf{E}^{(1)}$  (i.e., any solution of the  $\epsilon = 0$  system), but this is not of interest here. It is also easy (and not relevant here) to compute  $n^{(1)}$ ,  $\mathbf{v}^{(1)}$ , and  $\mathbf{B}^{(1)}$  in terms of the  $a$ 's and  $\psi$ 's by substituting eqs. (4) and (5) into eq. (1). The quantity of interest here is the amplitude of the plasma oscillation excited, eq. (13a). It is clear that by varying the detuning  $\Delta$ , we can make this as small as we like. We turn now to some numerical estimates to see how large we can make it.

3. *A typical case.* For comparison of the present results with those of ref. 1, we assume:

$$\omega_2, \omega_3 \approx 2.7 \times 10^{15} \text{ s}^{-1} \quad (\text{Ruby laser light})$$

$$\omega_p \approx 5.6 \times 10^{11} \text{ s}^{-1}$$

$$k_3 - k_2 = k_1 \approx (\omega_3 - \omega_2)/c \approx \omega_p/c \approx 19 \text{ cm}^{-1}$$

$$a_2, a_3 \approx 10^8 \text{ v/cm} = 10^6/3 \text{ e.s.u./cm}$$

$$\Delta \approx 6 \times 10^8 \text{ s}^{-1} \quad (\text{collision frequency} / 2\pi, \text{ corresponding to } \Gamma_e \omega_p \text{ of ref. 1}).$$

This leads to  $a_1/a_2 \sim 10^{-4}$ , which shows that the detuning is large enough to keep  $a_1$  small. However, if one keeps the collisional damping as a measure of the detuning, and increases the temperature by a factor of 300 (that is, if one considers a plasma of energies of the order of 3 keV instead of 10 eV, for the electrons), since  $\Gamma_e \omega_p \sim T^{-1/2} \ln A$ , where  $A/18$  is the number of particle inside a sphere with radius equal to the Debye length, one arrives at  $a_1/a_2 \sim 0.4$ , if the density is kept constant. This shows that now, it clearly is *not* permissible to treat the longitudinal field as "small" relative to the transverse field.

At still higher temperatures, the Landau damping becomes non-negligible and acts as the limiting factor in the detuning. This discussion also assumes, it is well to repeat, that the plasma density may be made arbitrarily uniform in space, which is perhaps questionable at sufficiently high temperatures. Whether the fractional departure from uniformity in the plasma frequency could be made smaller than  $\Gamma$  would have to be determined in the particular experimental situation.

Summarizing, it should be possible to excite electron plasma oscillations over a considerable range of amplitudes by shining properly-tuned laser beams through the plasma. The main limitations on the amplitude are results of dissipation and lack of control over the uniformity of the density.

The authors of ref. 1 were interested in this resonant excitation of plasma oscillation primarily in connection with possible use of the scattered light from a *third* beam as an optical density probe. We have taken the point of view here that the phenomenon is interesting simply as a controlled source of electron plasma oscillations to be studied experimentally. Such a source, which does not appreciably modify the plasma itself, has until now been largely lacking.

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We remark that if we had considered the pure *initial* value problem, more appropriate to a pulsed laser, instead of seeking steady state solutions as we have here, this reference would be directly relevant. Then,  $a_1$ ,  $a_2$ , and  $a_3$  would have all depended on  $et$  but not on  $ex$ , and eqs. (10) and (12) would have been correspondingly modified. With the simple replacement of the space variable  $x$  by  $et$  and a redefinition of the constants, sec. VI of the paper by Armstrong *e.a.* can in fact be taken over *in toto* for the present plasma situation. One amplitude must, of course, be interpreted as an electrostatic wave amplitude, whereas all three amplitudes in the paper by Armstrong *e.a.* are transverse. The results are so similar that it has seemed of little use to reproduce them here.
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