

## MULTILOOP CALCULATIONS IN COVARIANT SUPERSTRING THEORY

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We give an explicit construction of the superstring multiloop amplitudes in terms of theta functions. We analyse the correlation functions of the space-time supersymmetry current and find that these contain unphysical poles. Using BRST-invariance we show that these poles have no physical effect for on-shell amplitudes, and that the partition function is given by a total derivative on moduli space.

*Introduction.* During the past year the understanding of string perturbation theory has grown rapidly. It is now realized that the analytic structure of the moduli space of Riemann surfaces plays a central role, and the powerful techniques of algebraic geometry have become available for studying the string integrand [1–4]. This has led to the explicit construction of the bosonic string partition function in terms of theta functions [4]. A somewhat different method for obtaining this explicit expression is based on Bose–Fermi equivalence [5–7]. Bosonization gives a representation of all chiral determinants relevant to string theory in terms of a bosonic functional integral, which can be calculated explicitly. Theta functions enter naturally in this construction.

Less explicit results are known for the fermionic string. The theory of super moduli spaces is just being developed [8], and a super-analogue of the theta function does not yet exist. Nevertheless, it is generally expected that superstring loop-amplitudes are well-behaved and finite. In particular, it has been argued that a number of so-called non-renormalization theorems hold, which state that the loop corrections to the vacuum amplitude and to the massless one-, two-, and three-point functions vanish to all orders [9,10]. These arguments are based on the existence of the conserved space-time supersymmetry current  $j_\alpha$ , which has been constructed in ref. [9] using the bosonized representation of the commuting ghosts  $\beta$  and  $\gamma$ .

In this paper we analyse the  $(\beta, \gamma)$  system on an arbitrary Riemann surface. Using the relation with an anti-commuting spin  $(\frac{3}{2}, \frac{1}{2})$  theory, we construct an explicit expression for the correlation functions, including those that contain spin fields. Applying these results to the correlation functions of the supersymmetry current  $j_\alpha$  of the superstring, we find that there is a supersymmetry anomaly:  $j_\alpha$  has other poles than those generated by operator products! These poles seem to bring superstring theory into great trouble. However, as we will show, the situation is saved by BRST-invariance. For physical on-shell amplitudes the residues of these poles are total derivatives on moduli space. On-shell supersymmetry remains unbroken, which is sufficient to ensure the validity of the non-renormalization theorems.

*The superstring partition function.* Covariant superstring perturbation theory can be formulated in terms of a two-dimensional local quantum field theory consisting of 10 scalar superfields  $X^\mu(z, \bar{z}, \theta, \bar{\theta})$ , with the action (in local complex coordinates)

$$S[X^\mu] = \frac{1}{2\pi} \int d^2z \int d^2\theta \frac{1}{2} \bar{D}X^\mu D X^\mu. \quad (1)$$

The  $g$ -loop contribution to a scattering amplitude is given by the functional integral over the  $X^\mu(z, \bar{z}, \theta, \bar{\theta})$ ,

living on a genus  $g$  super-Riemann surface  $\Sigma_g$ , and over all super-geometries of this surface. After gauge-fixing the expression becomes an integral over the super moduli space  $\mathbb{S}_g$ , the parameter space of all inequivalent super-conformal structures of  $\Sigma_g$ , described by  $3g-3$  complex commuting coordinates  $\{(m_i, \bar{m}_i) | i=1, \dots, 3g-3\}$  and  $2g-2$  complex Grassmann coordinates  $\{(\zeta_a, \bar{\zeta}_a) | a=1, \dots, 2g-2\}$  [8,11]. As a consequence of the splitting of the string modes into left and right movers, the integrand factorizes into a product of an analytic times an anti-analytic function on  $\mathbb{S}_g$  [1]. In particular, the vacuum amplitude, given by the integral of the string partition function, can be written as

$$\Gamma_g = \int_{\mathbb{S}_g} (\det \text{Im } \tau)^{-5} \bar{W} \wedge W, \tag{2}$$

where  $\tau$  is the period matrix of the surface  $\Sigma_g(m, \zeta)$  and

$$W = W_{\text{st}}(m; \zeta) W_{\text{gh}}(m; \zeta) \prod_i dm_i \prod_a d\zeta_a, \tag{3}$$

$$(\det \text{Im } \tau)^{-5} |W_{\text{st}}(m; \zeta)|^2 = \int DX^\mu \exp(-S[X^\mu]), \tag{4}$$

$$W_{\text{gh}}(m; \zeta) = \int DE DC \exp(-S[B, C]) \prod_{i=1}^{3g-3} \langle \eta_i | B \rangle \prod_{a=1}^{2g-2} \delta(\langle \alpha_a | B \rangle), \tag{5}$$

$$S[B, C] = \frac{1}{2\pi} \int d^2z \int d^2\theta B \bar{D}C. \tag{6}$$

Here we have suppressed the summation over all spin structures;  $W_{\text{gh}}$  is the chiral Faddeev-Popov super-determinant. The insertion of the operators  $\langle \eta_i | B \rangle$  and  $\delta(\langle \alpha_a | B \rangle)$  is necessary for the absorption of the  $B$  zero modes, corresponding to variations along the gauge slice. The  $\eta_i$  and  $\chi_a$  are a set of super-Beltrami differentials dual to the one-forms  $dm_i$  respectively  $d\zeta_a$  on  $\mathbb{S}_g$ , and have, as a consequence, vanishing Lie-brackets

$$\frac{\partial}{\partial m_i} \eta_j = \frac{\partial}{\partial m_j} \eta_i, \quad \frac{\partial}{\partial \zeta_a} \eta_i = \frac{\partial}{\partial m_i} \chi_a, \quad \frac{\partial}{\partial \zeta_a} \chi_b = -\frac{\partial}{\partial \zeta_b} \chi_a. \tag{7}$$

Different choices of the  $\eta_i$  and  $\chi_a$  correspond to different coordinate systems on  $\mathbb{S}_g$ . The partition function  $W_{\text{st}}W_{\text{gh}}$  transforms as a holomorphic half-density under these coordinate changes.

We take the  $\chi_a$  independent of all  $\zeta_a$ , so the total action can be expanded as

$$S[X^\mu, B, C] = S[X^\mu, B, C]|_{\zeta_a=0} + \sum_a \zeta_a \langle \chi_a | T[X^\mu, B, C] \rangle, \tag{8}$$

$$T[X^\mu, B, C] = -\frac{1}{2} D^2 X^\mu D X^\mu - CD^2 B + \frac{1}{2} DCDB - \frac{3}{2} (D^2 C) B \tag{9}$$

is the analytic super stress-energy tensor. We can now perform the integral over the anti-commuting moduli:

$$\begin{aligned} W(m_i, \chi_a) &= \int \prod_a d\zeta_a W_{\text{st}}(m; \zeta_a) W_{\text{gh}}(m_i; \zeta_a) \\ &= \int D[XBC] \exp(-S[XBC]) \prod_a \delta(\langle \chi_a | B \rangle) (\langle \chi_a | T \rangle + \partial/\partial \zeta_a) \prod_i \langle \eta_i | B \rangle. \end{aligned} \tag{10}$$

Here  $\partial/\partial \zeta$  acts on  $\prod \langle \eta_i | B \rangle$  according to (7). The new partition function  $W(m; \chi)$  is a holomorphic half-density on  $\mathbb{M}_g$ , the moduli space of ordinary genus  $g$  Riemann surfaces. Under variations of the anti-commuting Beltrami differentials  $\chi_a$  it changes with a total divergence on moduli space  $\mathbb{M}_g$ . To see this, note that

$$\delta_{\Delta\chi_b} (\delta(\langle \chi_b | B \rangle) \langle \chi_b | T \rangle) = \delta_{\text{BRST}} (\delta(\langle \chi_b | B \rangle) \langle \Delta\chi_b | B \rangle), \tag{11}$$

since  $\delta_{\text{BRST}}B(z, \theta) = T(z, \theta)$ . Next, using the BRST-invariance of  $W(m, \chi)$  and that  $\delta_{\Delta\chi_b}(\partial/\partial\zeta_b)\eta_i = (\partial/\partial m_i)\Delta\chi_b$ , a simple calculation shows that

$$\delta_{\Delta\chi_b}W(m_i; \chi_a) = \sum_i \frac{\partial}{\partial m_i} V_i,$$

$$V = \int D[XBC] \exp(-S[XBC]) \left( \delta(\langle\chi_b|B\rangle)\langle\Delta\chi_b|B\rangle \prod_{a \neq b} \delta(\langle\chi_a|B\rangle)(\langle\chi_a|T\rangle + \partial/\partial\zeta_a) \prod_{j \neq i} \langle\eta_j|B\rangle \right).$$
(12)

So the vacuum amplitude is indeed independent of the choice of the super-Beltrami differentials.

The expression (10) for  $W(m_i; \chi_a)$  is simplified if we choose the  $\chi_a$  to have no explicit dependence on the  $m_i$ . This means that we pick a set of  $\chi_a$  at some point on  $\mathbb{M}_g$  and define the  $\chi_a$  at other points by parallel transport, using the analytic stress energy tensor as connection [2]. The  $\zeta$ -derivatives in (10) then drop out.

The combination

$$X[\chi_a] = \delta(\langle\chi_a|B\rangle)\langle\chi_a|T\rangle$$
(13)

is BRST-invariant. Formally, it can be written as a BRST-variation:

$$X[\chi_a] = \delta_{\text{BRST}}\Theta(\langle\chi_a|B\rangle),$$
(14)

where  $\Theta$  is the Heavyside step function. The combined operation carried out by the  $X[\chi_a]$ , the absorption of a commuting  $B$  zero mode together with the integration over the corresponding super moduli parameter, is called ‘‘picture changing’’ [9], and  $X[\chi_a]$  is called a ‘‘picture changing operator’’. In ref. [8] the special class of  $X[\chi_a]$  corresponding to  $\chi_a(z) = \delta(z - z_a)$  was considered. For the original super-Riemann surface this corresponds to choosing conformal coordinate patches except for  $2g - 2$  points  $z_a$ , around which one defines transition functions with supersymmetry parameter  $\zeta_a$ . The picture changing operator in this limit is well defined if one takes eq. (14) as the definition of  $X[\chi_a]$ . With this special choice for the  $\chi_a$  the partition function  $W(m_i; z_a)$  finally becomes

$$W(m_i; z_a) = \prod_j \left( \int d^2w_j \eta_j(w_j) \right) W(w_i; z_a),$$

$$W(w_i; z_a) = \sum_{\delta} \varepsilon_{\delta} \int D[x\psi, bc, \beta\gamma]_{\delta} \exp(-S[x\psi, bc, \beta\gamma]) \prod_i b(w_i) \prod_a X(z_a),$$
(15)

$$X(z_a) = \oint dw j_{\text{BRST}}(w) \Theta(\beta(z_a)),$$
(16)

$$j_{\text{BRST}}(w) = c(T_{zz}[x, \psi, \beta, \gamma] + b\partial c) + \frac{1}{2}\gamma\partial x^{\mu}\psi^{\mu} + \frac{1}{4}\gamma^2 b.$$
(17)

Here we switched to component notation, and  $\sum_{\delta}\varepsilon_{\delta}$  denotes the sum over the spin structures with signs  $\varepsilon_{\delta}$  corresponding to the GSO projection.

The positions of the  $X(z_a)$  can be shifted, as we see from (12), by adding total derivatives on  $\mathbb{M}_g$ . Note, however, that if we make  $m_i$ -dependent shifts, we have to modify the expression (15) by including the  $\zeta$ -derivative terms. Modular invariance gives a restriction on the positions of the  $X(z_a)$ .  $\mathbb{M}_g$  contains orbifold points which correspond to surfaces having a discrete group of automorphisms. Modular invariance of  $W(m_i; z_a)$  requires that positions of the  $X(z_a)$  at the orbifold points must be left fixed (or be permuted) under the action of this automorphism group. Whether this requirement is consistent with imposing parallel transport of the  $z_a$  is not clear to us. If not,  $W(m_i; z_a)$  can not be globally of the form (15).

*Explicit construction of the partition function.* Our next goal is to express  $W(m_i; z_a)$  in terms of theta functions [12]. We will make use of the results obtained in ref. [7]. There the partition and correlation functions were studied of chiral fermionic ( $b, c$ ) theories with general spin  $\lambda$  respectively  $1-\lambda$ , and spin structure  $\delta$ . Using bosonization the following expression was found for the functional integral:

$$\int [Db Dc]_{\delta} \exp(-S[b, c]) \prod b(z_i) \prod c(w_j) = Z[\delta] \left( \sum z_i - \sum w_j - (2\lambda - 1)\Delta \right), \tag{18}$$

$$Z[\delta] \left( \sum q_i z_i - Q\Delta \right) = Z_1^{-1/2} \Theta[\delta] \left( \sum q_i z_i - Q\Delta \right) \prod_{i < j} E(z_i, z_j)^{q_i q_j} \prod_i \sigma(z_i)^{q_i Q}, \tag{19}$$

$$\sum_i q_i = Q(g-1), \quad Z_1 = Z \left( \sum_i^g z_i - w - \Delta \right) [\det_{ij} \omega_i(z_j)]^{-1}. \tag{20}$$

Here  $\Delta$  is the Riemann class,  $E(z, w)$  the prime-form [12], and  $\omega_i, i=1, \dots, g$ , are the canonical holomorphic one-forms on  $\Sigma_g$ . We use the notation

$$\sum_k z_k - \sum_k w_k = \sum_k \int_{z_k}^{w_k} \omega_i \in \mathbb{C}^g / \mathbb{Z}^g + \tau \mathbb{Z}^g. \tag{21}$$

The differential  $\sigma(z)$  represents the coupling to the background charge in the bosonized theory, and carries the conformal anomaly [7]. For us it is sufficient to know the quotient of two  $\sigma$ 's:

$$\frac{\sigma(z)}{\sigma(w)} = \frac{\Theta(z - \sum p_i + \Delta)}{\Theta(w - \sum p_i + \Delta)} \prod_i \frac{E(w, p_i)}{E(z, p_i)}. \tag{22}$$

It follows from the Riemann vanishing theorem [12], describing the zeroes of the theta function, that the RHS is independent of the  $p_i$  and furthermore that  $\sigma(z)$  has no zeroes or poles [7,12].

The reparametrization ghosts  $b$  and  $c$  correspond to the case  $\lambda=2, \alpha=0$ ; the  $\psi^\mu$  correlation functions follow from (18) with  $\lambda=\frac{1}{2}$ . The correlation functions containing spin fields are also of the form (19), with  $q=\pm\frac{1}{2}$ , since the spin fields are given by the vertex operators  $\exp(\pm i\frac{1}{2}\phi)$  in the bosonized theory.

The factor  $Z_1$  and the prime-form  $E(z, w)$  are related to the partition function, respectively the Green's function of a free scalar field. More precisely

$$\int Dx^\mu \exp(-S[x^\mu]) \prod_i \exp[iq_i^\mu x^\mu(z_i)] = (\det \text{Im } \tau)^{-d/2} |Z_1|^{-d} \prod_{i < j} F(z_i, z_j)^{q_i^\mu \cdot q_j^\mu} \delta \left( \sum q_i \right),$$

$$F(z, w) = \exp[-2\pi \text{Im}(z-w) (\text{Im } \tau)^{-1} \text{Im}(z-w)] |E(z, w)|^2. \tag{23}$$

For later use, we note that the RHS can be rewritten as

$$\int dp_i^\mu \left| Z_1^{-d/2} \exp \left[ i\pi p^\mu \cdot \tau \cdot p^\mu + 2\pi i p^\mu \cdot \left( \sum q_i^\mu z_i \right) \right] \prod_{i < j} E(z_i, z_j)^{q_i^\mu \cdot q_j^\mu} \right|^2 \delta \left( \sum q_i \right). \tag{24}$$

The interpretation is obvious:  $p_i^\mu$  ( $i=1, \dots, g$ ) is the loop momentum passing through the cycle  $a_i$ , and has to be integrated over, just as in ordinary Feynman diagrams. (The moduli correspond in this analogy to the Feynman parameters.) Note that the projection of (24) onto a given  $p^\mu$  is analytic, both on  $\Sigma_g$  and on  $\mathbb{M}_g$ .

The only missing ingredient are the correlation functions of the  $(\beta, \gamma)$  ghost system. These we will now analyse using the path integral. In order to avoid infinities associated with the integration over the zero modes

we have to introduce operators which restrict the zero-mode integration region. Examples of such operators are the  $\delta$ -function operators  $\delta(\beta(z))$  and the stepfunction operators  $\Theta(\beta(z))$ . Inserting  $\delta(\beta(z_0))$  into the path integral effectively means that we integrate over  $\beta$ -fields having a simple zero at  $z=z_0$  and thus each  $\delta$ -function eliminates one zero mode. For  $\Theta(\beta(z))$  the counting is different: to restrict  $n$  zero mode integrals one has to use  $n+1$  step functions. We can also introduce the operators  $\delta(\gamma(w))$  which force the  $\gamma$ -fields to have a zero at  $z=w$ . This is equivalent to allowing the fields  $\beta(z)$  to have a simple pole at  $z=w$  and in this way the operators  $\delta(\gamma(w))$  create a  $\beta$ -zero mode. So the reduction of the zero modes is given by

$$BV = \#\delta(\beta(z))'s - \#\delta(\gamma(w))'s + \#\Theta(\beta(z))'s - 1. \tag{25}$$

The requirement that all  $2(g-1)$  zero mode integrals extend over a finite region gives the condition

$$B = 2(g-1). \tag{26}$$

The conformal spin of  $\delta(\beta(z))$ ,  $\delta(\gamma(w))$  and  $\Theta(\beta(x))$  is respectively  $-\frac{3}{2}$ ,  $\frac{1}{2}$  and 0. Further,  $\delta(\beta(z))$  and  $\delta(\gamma(w))$  carry a ghost number charge  $+1$  respectively  $-1$ , i.e. opposite to  $\beta(z)$  respectively  $\gamma(w)$ . The total ghost number charge of all the operators in a correlation function must be equal to  $2(g-1)$ . So we get the additional relation

$$-\#\beta(z)'s + \#\gamma(w)'s + \#\delta(\beta(z))'s - \#\delta(\gamma(w))'s = 2(g-1). \tag{27}$$

The operators  $\beta$ ,  $\gamma$ ,  $\delta(\beta)$ ,  $\delta(\gamma)$ , and  $\Theta(\beta)$  are not all independent. One easily derives the following operator product relations:

$$\beta(z)\delta(\beta(w)) \sim (z-w)\partial_w\Theta(\beta(w)), \quad \delta(\beta(z))\delta(\gamma(w)) \sim (z-w)\cdot 1, \tag{28, 29}$$

$$\Theta(\beta(z))\gamma(w) \sim (z-w)^{-1}\delta(\beta(w)). \tag{30}$$

Combining (28) and (29) one finds the identification

$$\beta(z) = \partial_z\Theta(\beta(z))\delta(\gamma(z)). \tag{31}$$

To find all the correlation functions of the above fields it is sufficient to calculate the correlation functions of the type

$$G(y_i; w_j; x_k) = \int [D\beta D\gamma]_\delta \exp(-S[\beta, \gamma]) \prod_{i=1}^n \gamma(y_i) \prod_{j=1}^{n-2g+2} \delta(\gamma(w_j)) \prod_{k=1}^{n+1} \Theta(\beta(x_k)), \tag{32}$$

Since all others can be derived from these, using (30) and (31). For the computation we use the Fourier integral representation for the  $\delta$ - and stepfunction, and evaluate the path integral by expanding the fields  $\beta$  and  $\gamma$  in eigenfunctions of the laplacians  $A_{3/2}$  respectively  $A_{-1/2}$ . The resulting expression can be compared with the fermionic expressions of the correlation functions of the anti-commuting  $(b, c)$  system with  $\lambda = \frac{3}{2}$ , for which we have the result (19). In this way we find

$$G(y_i; w_j; x_k) = \frac{\prod_{i=1}^n Z[\delta](\sum x_k - \sum w_j - y_i - 2A)}{\prod_{l=1}^{n+1} Z[\delta](\sum_{k \neq l} x_k - \sum w_j - 2A)}, \tag{33}$$

and from this result we can express all  $(\beta, \gamma)$  correlation functions in terms of theta functions.

This analysis can be related to the bosonization prescription of ref. [9]. There it is shown that the commuting  $(\beta, \gamma)$  ghost system can be replaced by a scalar field  $\varphi$  coupled to a background charge  $Q = -2$  and two conjugate anti-commuting fields  $\xi$  and  $\eta$  with conformal spin 0 respectively 1. The prescription reads

$$\beta(z) = \partial_z \xi(z) \exp[-\varphi(z)], \quad \gamma(z) = \eta(z) \exp[\varphi(z)]. \tag{34}$$

Comparing this with (30) we are led to the following identifications:

$$\xi(z) = \Theta(\beta(z)), \quad \eta(z) = \partial_z \gamma(z) \delta(\gamma(z)), \quad \exp[\varphi(z)] = \delta(\beta(z)), \quad \exp[-\varphi(z)] = \delta(\gamma(z)). \tag{35}$$

Using (33) and (19) we find for the  $(\xi, \eta, \varphi)$  correlation functions

$$\left\langle \prod_{i=1}^{n+1} \xi(x_i) \prod_{j=1}^n \eta(y_j) \prod \exp[q_k \varphi(z_k)] \right\rangle_{\delta} = \frac{\prod_{j=1}^n \theta[\delta](-y_j + \sum x - \sum y + \sum qz - 2\Delta)}{\prod_{i=1}^{n+1} \theta[\delta](-x_i + \sum x - \sum y + \sum qz - 2\Delta)} \frac{\prod_{i < i'} E(x_i, x_{i'}) \prod_{j < j'} E(y_j, y_{j'})}{\prod_{i,j} E(x_i, y_j) \prod_{k < l} E(z_k, z_l)^{q_k q_l} \prod_k \sigma(z_k)^{2q_k}} \quad (36)$$

This equation has some surprising features. Firstly, there is one more  $\xi$ -field than  $\eta$ -fields, while the anomaly in the  $\xi\eta$ -current

$$j_{\xi\eta} = \lim_{z \rightarrow w} [\xi(z)\eta(w) - (z-w)^{-1}] \quad (37)$$

suggests that the number of  $\eta$ 's minus the number of  $\xi$ 's should be equal to  $(g-1)$ . The second strange feature of (36) is the occurrence of other poles than those generated by the operator products. The positions of these "unphysical" poles are given by the zeroes of the theta functions in the denominator. (The physical poles are represented by the prime-forms  $E(x, y)$ .) The unusual  $\xi-\eta$  counting is in fact a consequence of the presence of these extra poles:  $j_{\xi\eta}(w)$  has, in addition to those at the  $\xi$ 's and  $\eta$ 's,  $g$  other poles  $p_1$ , whose positions are determined by  $[\sum p_1] = [\sum x_i - \sum y_j + \sum q_k z_k - D_{\delta}]$  (where  $D_{\delta}$  is the divisor class of the spin-bundle  $L_{\delta}$ ).

As we will show in the next section, the additional poles are indeed unphysical in the sense that they have no physical effect for the superstring. For this we will use that the fields  $\beta$  and  $\gamma$  have no unphysical poles, as one can easily check, and secondly, that the residues of the poles coming from the theta function  $\theta[\delta](-x_i + \sum x - \sum y + \sum qz - 2\Delta)$  are independent of the position of  $\xi(x_i)$ . This follows directly from Riemann's vanishing theorem.

There are two spin fields  $\Sigma^+$  and  $\Sigma^-$  associated with the  $(\beta, \gamma)$  system, with conformal spin  $-\frac{5}{8}$  respectively  $\frac{3}{8}$ . In the bosonized theory  $\Sigma^+$  ( $\Sigma^-$ ) corresponds to the vertex-operator  $\exp(+\frac{1}{2}\varphi)$  ( $\exp(-\frac{1}{2}\varphi)$ ). Repeating the previous calculation for correlation functions containing spin fields, we find that (36) is also correct for half-integer  $q_i$ .

Combining the above results, we are now able to explicitly calculate the partition function (15) for any genus, as well as the expectation values of arbitrary vertex operators. For example for the two-loop partition function (with loop momenta  $p_k^{\mu}$ ) we find

$$W_{p_k^{\mu}}(w_j; z_a) = \sum_{\delta} \varepsilon_{\delta} \exp(i\pi p^{\mu} \cdot \tau \cdot p^{\mu}) \left[ \frac{Z[\delta](0)^4 Z[\delta](z_1 - z_2) Z(\sum w_j - 3\Delta)}{Z_1^5 Z[\delta](\sum z_a - 2\Delta)} \langle \partial x^{\mu}(z_1) \partial x^{\mu}(z_2) \rangle_{p_k^{\mu}} + \left( \frac{Z[\delta](0)^5 Z[\delta](2z_2 - 2\Delta) Z(z_1 - z_2 + \sum w_j - 3\Delta)}{Z_1^5 Z[\delta](\sum z_a - 2\Delta)^2} \langle J(z_1) \rangle + (z_1 \leftrightarrow z_2) \right) \right], \quad (38)$$

$$\langle \partial x^{\mu}(z_1) \partial x^{\nu}(z_2) \rangle_{p_k^{\mu}} = 2\pi p_i^{\mu} \cdot p_j^{\nu} \omega_i(z_1) \omega_j(z_2) + \eta^{\mu\nu} \partial_{z_1} \partial_{z_2} \log E(z_1, z_2), \quad (39)$$

$$\langle J(z_1) \rangle = \omega_i(z_1) \partial_i \log \left( \frac{\theta[\delta](2z_2 - 2\Delta)^2}{\theta(z_1 - z_2 + \sum w_j - 3\Delta)} \right) + \partial_{z_1} \log \left( \frac{\prod E(z_1, w_j)}{[E(z_1, z_2)]^5 \sigma(z_1)} \right).$$

The expressions for higher loops are unfortunately even more complicated and therefore not very enlightening. To gain more insight, let us study the implications of the space-time supersymmetry of the superstring.

*Space-time supersymmetry.* The space-time supersymmetry charges  $Q_{\alpha}^{+}$  and  $Q_{\alpha}^{-}$  are given by the contour integrals of the fermion emission vertices at zero momentum [9]:

$$Q_{\alpha}^{\pm} = \oint \frac{dz}{2\pi i} j_{\alpha}^{\pm}(z), \quad (40)$$

$$j_{\alpha}^{-}(z) = \exp(-if_{\alpha} \cdot h) \Sigma^{-}(z), \quad j_{\alpha}^{+}(z) = \exp(-if_{\beta} \cdot h) \Sigma^{+} \gamma_{\alpha\beta}^{\mu} \partial_z x^{\mu}(z) + \exp(-if_{\alpha} \cdot h) b \eta \exp[\frac{3}{2}\varphi(z)], \quad (41)$$

where  $\partial h^i = \psi^{2i-1} \psi^{2i}$  and  $f_{\alpha,\beta}$  are spinor weights of SO(10). The supersymmetry algebra is

$$\{Q_{\alpha}^{-}, Q_{\beta}^{+}\} = 2\gamma_{\alpha\beta}^{\mu} p^{\mu}, \quad p^{\mu} = \oint \frac{dz}{2\pi i} \partial_z x^{\mu}(z). \quad (42)$$

The correlation functions of the supersymmetry currents  $j_{\alpha}^{\pm}(z)$  are easily obtained, for example:

$$\left\langle j_{\alpha}^{-}(x) j_{\beta}^{+}(y) \partial x^{\mu}(z) \prod_{j=1}^{2g-2} \exp[2\varphi(w_j)] \right\rangle_{p_k^{\mu}} = \gamma_{\alpha\beta}^{\mu} \sum_{\delta} \varepsilon_{\delta} \frac{[Z[\delta](\frac{1}{2}x - \frac{1}{2}y)]^5}{Z[\delta](\frac{1}{2}x + \frac{1}{2}y + \sum 2w - 2A)} \langle \partial x^{\nu}(y) \partial x^{\nu}(z) \rangle_{p_k^{\nu}}. \quad (43)$$

The GSO projection ensures that this correlation function is single valued on  $\Sigma_g$  in  $x$  and  $y$ . Supersymmetry relations between different amplitudes are derived using contour deformations of the integral  $\oint dz j_{\alpha}^{-}(z)$ , [9,10]. However,  $j_{\alpha}^{-}$  has (in general) unphysical poles, which means that on a given Riemann surface space-time supersymmetry is broken! These unphysical poles seem to invalidate the previous proofs of the vanishing of the vacuum amplitude and of the dilaton tadpole [9,10]. Fortunately, as we will show below, for on-shell amplitudes it follows from BRST-invariance that the residues of the unphysical poles in  $j_{\alpha}^{-}(z)$  are given by total derivatives on  $\mathbb{M}_g$ , so that supersymmetry is restored after the integration over the moduli  $m_i$ . This result is sufficient to prove the vanishing of the zero-, one-, two- and three-point amplitudes for on-shell massless superstring states.

We now outline the proof for the vacuum amplitude. (For simplicity we take the  $z_a$  independent of the  $m_i$ .) By inserting the supersymmetry algebra (42) we can rewrite the partition function  $W_{p_k^{\mu}}(m_j; z_a)$  as follows:

$$W_{p_k^{\mu}}(m_j; z_a) = (p_i^{\mu} \gamma_{\alpha\beta}^{\mu})^{-1} \oint_{a_i} \frac{dy}{2\pi i} \oint_{C_y} \frac{dx}{2\pi i} A_{\alpha\beta}(x, y; z_a), \quad (44)$$

$$A_{\alpha\beta}(x, y; z_a) = \left( \int \exp(-S) \prod_j \langle \eta_j | b \rangle j_{\alpha}^{-}(x) j_{\beta}^{+}(y) \prod_a X(z_a) \right)_{p_k^{\mu}}. \quad (45)$$

where  $C_y$  is a contour on  $\Sigma_g$  surrounding the point  $y$ . As a function of  $x$ , the amplitude  $A_{\alpha\beta}(x, y; z_k)$  has a large number of unphysical poles. Their positions  $r_l$  are given by the zeroes of the function  $f(x) = \prod_{\delta} \theta[\delta](\frac{1}{2}x + \frac{1}{2}y + \sum z_a - 2A)$ , and their order is  $(g-1)$ . Since the only physical pole is at  $x=y$ , we have

$$\oint_{C_y} \frac{dx}{2\pi i} A_{\alpha\beta}(x, y; z_a) = - \sum_l \int_{C_l} \frac{dx}{2\pi i} A_{\alpha\beta}(x, y; z_a), \quad (46)$$

where  $C_l$  surrounds  $r_l$ . As already indicated, we can shift the position of, for example,  $X(z_l)$  by deforming its BRST contour. (Note that the BRST current has no unphysical poles.) The supersymmetry currents  $j_{\alpha}^{\pm}$  are BRST-invariant (upto total derivatives) so we get (cf. eq. (12))

$$A_{\alpha\beta}(x, y; z_1, z_{a'}) = A_{\alpha\beta}(x, y; z_1, z_{a'}) + \frac{\partial}{\partial m_j} B'_{\alpha\beta}(x, y; z_1, \tilde{z}_1, z_{a'}) + \text{total } x, y\text{-derivatives}, \quad (47)$$

$$B'_{\alpha\beta}(x, y; z_1, \tilde{z}_1, z_{a'}) = \left( \int \exp(-S) \prod_{i \neq j} \langle \eta_i | b \rangle j_{\alpha}^{-}(x) j_{\beta}^{+}(y) \xi(z_i) \xi(\tilde{z}_i) \prod_{a' \neq 1} X(z_{a'}) \right)_{p_k^{\mu}}.$$

Combining this result with (45) and (46) and using the fact that  $A(x, y; \tilde{z}_1, z_a)$  is regular at  $x=r_i$ , we deduce that the partition function  $W_{p_i}(m_j; z_a)$  is indeed a total derivative on  $\mathbb{M}_g$ :

$$W_{p_i}(m_j; z_a) = \frac{\partial}{\partial m_j} (p_i^\mu \gamma_{\alpha\beta}^\mu)^{-1} \oint_{C_i} \frac{dy}{2\pi i} \sum_T \oint_{C_i} \frac{dx}{2\pi i} B'_{\alpha\beta}(x, y; z_1, \tilde{z}_1, z_a). \quad (48)$$

(It is straightforward to check from (36) that the RHS is, as it should be, independent of  $\tilde{z}_1$ ). So we conclude that the vacuum amplitude (2), being the integral of a total derivative on  $\mathbb{M}_g$ , indeed vanishes to all orders<sup>#1</sup>.

*Conclusion.* In this paper we have given an explicit construction of the superstring partition function and amplitudes on an arbitrary Riemann surface. Our analysis has been local on moduli space, but the results can now serve as a basis for the study of the global properties (modular invariance, etc.) of string loop amplitudes. They are also useful for studying the factorization of the string integrand on the boundary of  $\mathbb{M}_g$  [7].

We have shown that the superstring partition function  $W$  is not identically zero, but a total derivative on  $\mathbb{M}_g$ . At first, this result seems to be in contradiction with that of ref [3], where it is argued that the string integrand vanishes identically. The hypothesis on which the proof of ref. [3] is based is that the string integrand factorizes on physical positive energy states. However, the factorization expansion of the partition function is not gauge invariant. As noted in ref. [2], and as can be seen in (12), gauge transformations in string theory change the integrand  $W$  by a total divergence. Eq. (48) states that for the superstring  $W$  itself is a total divergence, i.e.  $W$  is a pure gauge. For general gauge choices the spurious states of the string contribute to the partition function, and also to its factorization expansion. The coefficients of this expansion are total derivatives on the boundary of  $\mathbb{M}_g$ . The factorization hypothesis of ref. [3] corresponds to the special gauge choice for which the contribution of the spurious states, and therefore the total partition function, vanishes. It would of course be interesting to find an explicit geometric description of such a unitary gauge.

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<sup>#1</sup> One can argue that, because of BRST-invariance, there are no boundary terms.

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