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1955

Physica XXI
• 355–359

COEFFICIENTS OF VISCOSITY FOR A FLUID IN A MAGNETIC FIELD OR IN A ROTATING SYSTEM

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Synopsis

The linear equations between the elements of the viscous pressure tensor and the rates of deformation are investigated for the case of an isotropic fluid in an external magnetic field or for the equivalent case of a rotating fluid. Since these equations can be incorporated within the thermodynamics of irreversible processes, the Onsager reciprocity relations hold for the scheme of phenomenological coefficients.

For the present case the viscous behaviour is seen to be described by 8 coefficients between which one Onsager relation exists. The remaining 7 independent coefficients can be combined in a linear way so as to yield 5 coefficients of ordinary viscosity, the other 2 coefficients then describing the volume viscosity and a cross-effect between the ordinary and the volume viscosity, respectively. For the special case of vanishing volume viscosity the equations are compared with those derived from kinetic theory by Chapman and Cowling for an ionized gas in a magnetic field.

The macroscopic description of viscosity can be developed from the viewpoint of the thermodynamics of irreversible processes [1] [2] [3]. In this theory an expression for the entropy production σ (per unit time and volume) due to the irreversible phenomena occurring within a system is derived by means of the conservation laws and the second law of thermodynamics. For the contribution σ_v of viscous flow one then finds

$$T\sigma_v = -\mathbf{\Pi} : \text{Grad}\ \mathbf{v}, \tag{1}$$

where T is the temperature, $\mathbf{\Pi}$ the viscous pressure tensor, $\mathbf{v}$ the barycentric velocity and : denotes the interior product of two tensors, contracted twice. We shall restrict ourselves to the case usually met with that $\mathbf{\Pi}$ is a symmetric tensor and denote the six independent cartesian components as

$$\begin{aligned} \pi_{xx} &= \pi_1, & \pi_{yz} &= \pi_4, \\ \pi_{yy} &= \pi_2, & \pi_{xz} &= \pi_5, \\ \pi_{zz} &= \pi_3, & \pi_{xy} &= \pi_6. \end{aligned} \tag{2}$$

In (1) the tensor Grad **v** then can be replaced by its symmetric part which we shall denote by $\boldsymbol{\epsilon}$ with components

$$\begin{aligned} \varepsilon_{xx} &= \varepsilon_1, & \varepsilon_{yz} &= \tfrac{1}{2}\varepsilon_4, \\ \varepsilon_{yy} &= \varepsilon_2, & \varepsilon_{xz} &= \tfrac{1}{2}\varepsilon_5, \\ \varepsilon_{zz} &= \varepsilon_3, & \varepsilon_{xy} &= \tfrac{1}{2}\varepsilon_6, \end{aligned} \tag{3}$$

such that we have

$$T\sigma_v = -\boldsymbol{\Pi} : \boldsymbol{\epsilon} = -\Sigma_{i=1}^{6} \pi_i \varepsilon_i. \tag{4}$$

According to the thermodynamics of irreversible processes we next assume linear relationships between the elements of $\boldsymbol{\Pi}$ and $\boldsymbol{\epsilon}$ which occur in (4) as 'fluxes' and 'forces' in the thermodynamic sense. These 'phenomenological equations' can be written as

$$-\pi_i = \Sigma_{k=1}^{6} L_{ik}\, \varepsilon_k, \qquad (i = 1, \ldots, 6), \tag{5}$$

(we shall not consider cross-effects between viscosity and other irreversible phenomena although such effects might exist).

We now suppose the fluid (*e.g.*, an ionized gas) to be placed in a homogeneous external magnetic field or to rotate with a constant angular velocity. The magnetic field strength or the angular velocity will be denoted by the comprehensive symbol H. Supposing that the fluid itself is isotropic we then want to investigate the symmetry properties of the phenomenological equations (5).

a. *Spatial symmetry.* If we choose the x-axis in the direction of H it follows from the isotropy of the fluid that the relations (5) are invariant with respect to rotations about the x-axis. By straightforward calculation (*e.g.*, introducing an infinitesimal rotation) one then finds for the scheme of phenomenological coefficients

	ε_1	ε_2	ε_3	ε_4	ε_5	ε_6
$-\pi_1$	L_{11}	L_{12}	L_{12}	0	0	0
$-\pi_2$	L_{21}	L_{22}	L_{23}	L_{24}	0	0
$-\pi_3$	L_{21}	L_{23}	L_{22}	$-L_{24}$	0	0
$-\pi_4$	0	$-L_{24}$	L_{24}	$\tfrac{1}{2}(L_{22}-L_{23})$	0	0
$-\pi_5$	0	0	0	0	L_{55}	L_{56}
$-\pi_6$	0	0	0	0	$-L_{56}$	L_{55}

(6)

in which only 8 coefficients are left.

b. *Parity.* Since any axis perpendicular to the x-direction is a 2-fold axis of rotation the relations (5) are invariant for a rotation of the coordinate system by an angle π about the z-axis. This leads to

$$L_{ik}(\mathrm{H}) = (-1)^n L_{ik}(-\mathrm{H}), \tag{7}$$

if the index z figures n times in π_i and ε_k together (*cf.* (2) and (3)). Hence we find that

$$\left.\begin{array}{l} L_{11}, L_{12}, L_{21}, L_{22}, L_{23} \text{ and } L_{55} \text{ are even functions of H,} \\ L_{24} \text{ and } L_{56} \qquad\qquad ,, \quad \text{odd} \quad ,, \quad ,, \text{ H.} \end{array}\right\} \tag{8}$$

c. *The Onsager relations.* For the phenomenological coefficients the Onsager reciprocal relations [3] [4]

$$L_{ik}(\text{H}) = L_{ki}(-\text{H}) \tag{9}$$

hold. In view of (6) and the parity relations (8) we are left with only one true Onsager relation, *viz.*,

$$L_{12} = L_{21}, \tag{10}$$

by which the number of independent coefficients is further reduced to 7 (5 of them being even functions of H and 2 odd).

d. *Ordinary viscosity, volume viscosity and their cross-effect.* Each of the tensors $\mathbf{\Pi}$ and $\boldsymbol{\epsilon}$ can be split up into a tensor with zero trace and a scalar multiple of the unit tensor $\boldsymbol{\delta}$

$$\left.\begin{array}{l} \mathbf{\Pi} = \mathring{\mathbf{\Pi}} + \tfrac{1}{3}\pi\boldsymbol{\delta}, \\ \boldsymbol{\epsilon} = \mathring{\boldsymbol{\epsilon}} + \tfrac{1}{3}\vartheta\boldsymbol{\delta}, \end{array}\right\} \tag{11}$$

where π and ϑ are the traces

$$\left.\begin{array}{l} \pi = \Sigma_{i=1}^{3}\, \pi_i, \\ \vartheta = \Sigma_{i=1}^{3}\, \varepsilon_i = \operatorname{div} \mathbf{v}. \end{array}\right\} \tag{12}$$

The expression (4) for the entropy production then can be rewritten as

$$-\, T\sigma_v = \mathring{\mathbf{\Pi}} : \mathring{\boldsymbol{\epsilon}} + \tfrac{1}{3}\pi\vartheta. \tag{13}$$

We now can write the phenomenological equations in a form corresponding to (13). From (5) and (12) we find, using also (6) and (10),

$$-\pi = (L_{11}+2L_{12})\,\mathring{\varepsilon}_1 + (L_{12}+L_{22}+L_{23})(\mathring{\varepsilon}_2+\mathring{\varepsilon}_3) + \tfrac{1}{3}(L_{11}+4L_{12}+2L_{22}+2L_{23})\vartheta, \tag{14}$$

and therefore

$$\left.\begin{array}{l} -\mathring{\pi}_1 = \tfrac{2}{3}(L_{11}-L_{12})\,\mathring{\varepsilon}_1 + \tfrac{1}{3}(2L_{12}-L_{22}-L_{23})(\mathring{\varepsilon}_2+\mathring{\varepsilon}_3) + \tfrac{2}{9}(L_{11}+L_{12}-L_{22}-L_{23})\vartheta, \\ -\mathring{\pi}_2 = \tfrac{1}{3}(-L_{11}+L_{12})\,\mathring{\varepsilon}_1 + \tfrac{1}{3}(-L_{12}+2L_{22}-L_{23})\,\mathring{\varepsilon}_2 + \tfrac{1}{3}(-L_{12}-L_{22}+2L_{23})\,\mathring{\varepsilon}_3 + \\ \qquad\qquad + \tfrac{1}{9}(-L_{11}-L_{12}+L_{22}+L_{23})\,\vartheta + L_{24}\,\varepsilon_4, \\ -\mathring{\pi}_3 = \tfrac{1}{3}(-L_{11}+L_{12})\,\mathring{\varepsilon}_1 + \tfrac{1}{3}(-L_{12}-L_{22}+2L_{23})\,\mathring{\varepsilon}_2 + \\ \qquad + \tfrac{1}{3}(-L_{12}+2L_{22}-L_{23})\,\mathring{\varepsilon}_3 + \tfrac{1}{9}(-L_{11}-L_{12}+L_{22}+L_{23})\,\vartheta - L_{24}\,\varepsilon_4. \end{array}\right\} \tag{15}$$

Since $\overset{\circ}{\varepsilon}_1 + \overset{\circ}{\varepsilon}_2 + \overset{\circ}{\varepsilon}_3 = 0$ these equations can be given a more symmetrical form. Writing

$$\left.\begin{array}{ll} 2L_{11} - 4L_{12} + L_{22} + L_{23} \equiv 6\mu_1, & L_{55} \equiv \mu_3, \\ L_{11} - 2L_{12} + 2L_{22} - L_{23} \equiv 6\mu_2, & L_{24} \equiv \eta_1, \\ L_{11} + 4L_{12} + 2L_{22} + 2L_{23} \equiv 9\mu_v, & L_{56} \equiv \eta_2, \\ L_{11} + L_{12} - L_{22} - L_{23} \equiv 3\zeta, & \end{array}\right\} \quad (16)$$

we find the following scheme of phenomenological coefficients connecting the two sets of quantities which occur in (13):

	$\overset{\circ}{\varepsilon}_1$	$\overset{\circ}{\varepsilon}_2$	$\overset{\circ}{\varepsilon}_3$	ε_4	ε_5	ε_6	$\frac{1}{3}\vartheta$
$-\overset{\circ}{\pi}_1$	$2\mu_1$	0	0	0	0	0	2ζ
$-\overset{\circ}{\pi}_2$	0	$2\mu_2$	$2(\mu_1-\mu_2)$	η_1	0	0	$-\zeta$
$-\overset{\circ}{\pi}_3$	0	$2(\mu_1-\mu_2)$	$2\mu_2$	$-\eta_1$	0	0	$-\zeta$
$-\pi_4$	0	$-\eta_1$	η_1	$2\mu_2-\mu_1$	0	0	0
$-\pi_5$	0	0	0	0	μ_3	η_2	0
$-\pi_6$	0	0	0	0	$-\eta_2$	μ_3	0
$-\pi$	2ζ	$-\zeta$	$-\zeta$	0	0	0	$9\mu_v$

(17)

μ_1, μ_2, μ_3, μ_v and ζ are even functions of H, η_1 and η_2 are odd.

The coefficients μ_1, μ_2, μ_3, η_1 and η_2 describe ordinary viscosity, μ_v is the coefficient of volume (or bulk) viscosity and ζ describes a cross-effect between ordinary and volume viscosity.

With regard to symmetry and parity the scheme (17) can be compared with the equations given by Chapman and Cowling [5]) for the stress tensor of a simple gas in a magnetic field, derived from kinetic theory. The scheme is in agreement with these equations apart from an apparent error of sign in the latter (the coefficients of $2\overset{\circ}{e}_{yz}$ in p_{yy} and p_{zz}, resp., should be the opposite of the coefficients of $\overset{\circ}{e}_{yy}$ and $\overset{\circ}{e}_{zz}$, resp., in p_{yz}; this follows from the spatial symmetry and is confirmed by the Onsager relations (9)). It may be noted that in Chapman and Cowling's approximation μ_v and ζ vanish.

e. *The case of isotropy.* If H = 0 the above equations reduce to the well-known linear relationships between the stresses and rates of deformation in an isotropic system. As a matter of fact, for complete isotropy we have in addition to (6)

$$\left.\begin{array}{ll} L_{12} = L_{21} = L_{23}, & L_{24} = L_{56} = 0, \\ L_{11} = L_{22}, & L_{55} = L_{44}, \end{array}\right\} \quad (18)$$

so that only two independent coefficients are left (the Onsager relations become trivial for this case). By (16) this means

$$\mu_1 = \mu_2 = \mu_3, \qquad \zeta = \eta_1 = \eta_2 = 0, \tag{19}$$

and (17) reduces to a diagonal scheme, pertaining to the equations

$$\left.\begin{aligned} -\mathring{\mathbf{\Pi}} &= 2\mu_1 \mathring{\boldsymbol{\epsilon}}, \\ -\pi &= 3\mu_v \vartheta, \end{aligned}\right\} \tag{20}$$

or

$$-\mathbf{\Pi} = 2\mu_1 \mathring{\boldsymbol{\epsilon}} + \mu_v \vartheta\, \boldsymbol{\delta} = 2\mu_1 \boldsymbol{\epsilon} + \lambda\vartheta\, \boldsymbol{\delta}, \tag{21}$$

where λ is the 'second coefficient of viscosity' defined by

$$\mu_v = \lambda + \tfrac{2}{3}\mu_1. \tag{22}$$

The authors wish to thank Professor I. Prigogine for a remark which led to this note.

Received 14-1-55.

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