How to survive in more than 3 dimensions?

An introduction to the Poincare Conjecture by Vincent v.d. Noort
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Introduction: Great Unsolved Problems in mathematics

The essence of all sciences is curiosity. Walking around in the world, one starts immediately posing questions to oneself. A huge advantage, in my view, of mathematics above almost all other sciences, is that at the moment you ask yourself a question, you know that, in principal, you are able to find the answer yourself by no other means than simple logical reasoning. This makes wondering in mathematics a very attractive and satisfactory way of spending your time. No dreadful, expensive experiments have to be undertaken to find your answer, nor do you have to spend hours in libraries, trying to find and understand the results of dreadful, expensive experiments carried out by others in the past. All you need is a healthy amount of common sense and knowledge of the meaning of the words used to phrase the question. And maybe a pencil and a piece of paper as an extension of your memory.

That's all very well, in theory! In practice some of the simplest looking questions turn out to be almost impossible to answer. This is however not only a source of frustration, but also or maybe mostly of a more romantic side of doing mathematics. In mathematics there are a number of questions, the socalled Great Unsolved Problems, which are world famous in the mathematical world, for being unanswered over hundreds or even thousands of years in spite of the attempts of some of the greatest minds in mathematics. They are surrounded by stories of people spending years on unsuccessful attempts to solve the problem, hardly eating or sleeping and eventually dying in loneliness, poverty and frustration over their failure. And still, these problems remain hugely attractive to both professional and amateur mathematicians. Although you know that many people who are probably smarter than you, have tried to solve the puzzle before you and failed, it is still possible that you by coincide will find a way of looking at it which hasn't been tried so far and which leads you to the answer and, as a consequence, immortal fame.

Apart from being unsolved, there are no clear criteria which a problem has to fulfill to get the status of a Great Unsolved Problem. It can help of course if the problem itself can be easily understood. This is the case, for example, with the Goldbach Conjecture. This conjecture states that every even number greater than 2 can be written as the sum of two prime numbers (40 = 23 + 17, 42 = 37 + 5, 44 = 31 + 13 etc). One can easily check that this holds indeed for every even number under 100 and with supercomputers it is checked for all even numbers up to some billions. But no-one has succeeded so far in proving that the conjecture holds true for *every* even number, which is quite a lot. Note that once we have either the knowledge that every even number can indeed be written as the sum of two prime numbers or the knowledge that there is some billion-digit even number which cannot be written that way, this would be of no use to anyone whatsoever. The fun is in getting the answer, not in having it.

Another Great Unsolved Problem, the Riemann Hypothesis (which is widely considered the Greatest Unsolved Problem since Andrew Wiles - after having locked himself up in his room for seven years - proved the Fermat's Last Theorem in 1992(?)) gained this status by an almost complete opposite strategy. For understanding the question one needs an awful lot of background knowledge in 'complex analysis' (not my favorite subfield of mathematics), which makes it impossible for me to state it here in understandable language. But over time, people have shown that a lot of other, less famous, questions can be answered once the Riemann Hypothesis turns out to be true and this has made a lot of mathematicians very eager to prove it. By now, the most important reason why mathematicians are eager to prove it is - of course - the fact it has become such a famous Great Unsolved Problem.

A third way of making an open problem into a Great Unsolved Problem, is a rather artificial one, and is carried out by the Clay Institute of Mathematics. This institute has selected 10 problems and for every problem promises a million dollars to the first person to solve it. All problems are like the Riemann Hypothesis (which is actually one of them) very hard to understand for someone with little background in the particular areas of mathematics they are about. Herefor they don't excite our imagination as much as the Goldbach Conjecture or Fermats Last Theorem, but they are still able to attract our attention when someone claims to have solved one of them.

This happened a few months ago when Martin Dunwoody, a mathematician at the university of Southampton, claimed he had proved the Poincare Conjecture. I intended to read and understand the question as well as the possible answer and explain it in more understandable language in the Scoop, the magazine of UvA mathematics and physics students. The possible proof (presented on Dunwoody's website) turned out to be very hard to understand and I'm not sure if my article in the scoop will ever appear. However, the Conjecture itself turned out to be reasonably understandable, and to make a first step in the right direction I will try to explain that here to an imaginary audience with no more than high school experience in mathematics.

The n-dimensional world.

The most remarkable things about the Poincare Conjecture is that over the past hundred years, the problem has been solved for 4 dimensions, 5 dimensions, 6 dimensions and so on up to infinity. The cases in which the dimension equals 1 or 2, have even been tackled at the end of the 19^{th} century, by Poincare himself. So by now, the entire open problem is reduced to finding a solution to the 3-dimensional version of the conjecture, which turns out to be the hardest of all.

This statement might sound strange for two reasons. Personally I was surprised that of all possible cases the 3-dimensional case turned out to be so hard, since of all spaces, the 3-dimensional space seems the most natural. We are actually living in one, I presume. To other people it might be the nonchalant talk about higher dimensions which makes them feel a little uncomfortable. Normal people and even physicists tend to consider objects in more than 3 dimensions as very hard to understand or even think about. To mathematicians however they are no problems at all. The reason is that in mathematics an object exists as soon as you give a precise definition to it, regardless as to whether it has any connection to the real world or not. Once you have the definition, you can reason with it and derive every other property the object should necessarily have. Sometimes people ask me: ‘But how do you picture a four dimensional space?’. The secret is: we don’t. Well, to be quite honest: when I picture a four dimensional space, I actually picture a three dimensional space and somehow succeed in making myself believe that it is a four dimensional space. But to do mathematics with it, you don’t really need to picture a mathematical object, since the definition contains all information you might possibly need.

Therefore, all we need to do is find a useful definition of an n-dimensional space (this is the usual mathematics slang: most of the time when a mathematician uses the word ‘n’, he means an arbitrary natural number.) which matches our ideas of what 1, 2- and 3-dimensional spaces look like. It is convenient to picture a one-dimensional space as a line, a two-dimensional space as a plane and a three dimensional space, well, like the space we’re living in. When you ask yourself why it is so obvious that a plane has two dimensions and a three-dimensional space has three, there are many possible answers. One of them is that there are at most 2 or 3 directions, in which you could draw lines that are all perpendicular to each other. Closely related to this, is the following brilliant idea that occurred to Descartes in the 17^{th} century and is now common knowledge. When you choose two or three of these perpendicular directions and a fixed point which you call the origin, you can give every point in the two (respectively three) dimensional space an unique name by giving two (respectively three) numbers indicating how many steps you have to take in the first direction, how many in the second (and how many in the third) to reach your point when starting at the origin. These are the so called Cartesian Coordinates (see figure 1). They enable us to identify the line with the set of all numbers, the plane with the set of all pairs of numbers and the three-dimensional space with the set of all combinations of three numbers.

This gives rise to an obvious way to define 4, 5 and n-dimensional spaces: the 4-dimensional space consists of all points of the form (a, b, c, d) where a, b, c and d are real numbers, and in general the points in the n-dimensional space are just all combinations of n numbers one can make. This definition also enable us to study lower dimensional objects floating around in higher dimensional spaces. Just like all points of the form (1, a, b) where a and b run over all real numbers while the 1 stays fixed, form a plane in 3-space (see picture 2) the set of all points of the form (2, a, b, c, 4) in the five-dimensional space is indistinguishable from the ordinary 3-space and therefore forms a three-dimensional ‘hyperplane’ in 5-space. Also smaller objects like cubes can simply be generalized to higher dimensions. A three-dimensional cube is for instance formed by all points in 3-space of which all three coordinates are in between 0 and 1 (see figure 3). A two-dimensional cube is usually called a square but can also be formed by taking all points in the plane with coordinates that are both between zero and one. Therefore it is easy to define an n-dimensional cube as the set of all points in n-space which have all their coordinates in between zero and one, or more generally in between two fixed numbers. It can be fun to determine the beautiful patterns in the number of edges, vertices, faces and other lower dimensional cubes occurring in the n-dimensional cube when n runs from 1 up to infinity.

However, just defining a space by giving every point in it a name, doesn’t provide much structure. There is at least one valuable concept we have in one-, two- and three-dimensional space which we want to keep in higher dimensions, and that is our ability to measure distances. Measuring the distance between two points on a line is easy: you just subtract the coordinates (see figure 4). In two dimensions you need Pythagoras’ Theorem. To compute the distance between the points (2, 3) and (10, 12) we note that the line from (2, 3) to (2, 12) has length 12 – 3 = 9, the line from (2, 12) to (10, 12) has length 10 – 2 = 8, both lines are perpendicular to each other and therefore the distance from (2, 3) to (10, 12) is $\sqrt{(9^2 + 8^2)}$, or in general the distance from (a_1, a_2) to (b_1, b_2) is $\sqrt{[(a_1 - b_1)^2 + (a_2 - b_2)^2]}$ (See figure 5).

Now when we want to compute the distance between two points in three-space, say (1, 3, 4) and (5, 10, 12), we can use quite the same trick. We know already the length of the line from (1, 2, 4) to (1, 10, 12): that is $\sqrt{(9^2 + 8^2)}$, as we computed in the 2-dimensional case. The line from (1, 10, 12) to (5, 10, 12) has length 5 – 1= 4 and is perpendicular to the previous line, since it is perpendicular to the whole plane with fixed first coordinate 1 (see figure 6). Therefore we can use Pythagoras again and find that the distance between (1, 3, 4) and (5, 10, 12) = $\sqrt{[(\sqrt{(9^2 + 8^2)})^2 + 4^2]} = \sqrt{(9^2 + 8^2 + 4^2)}$. In general we deduce that the distance between two points (a_1, a_2, a_3) and (b_1, b_2, b_3) is $\sqrt{[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2]}$.

Now we have found this, we can easily extend this to 4 and more dimensions. The distance between (2, 1, 3, 4) and (4, 5, 10, 12) in four-space can be found by looking at the line from (2, 1, 3, 4) to (2, 5, 10, 12) (which has length $\sqrt{(9^2 + 8^2 + 4^2)}$ as we saw), and the line from (2, 5, 10, 12) to (4, 5, 10, 12) which has length 2 and is perpendicular to the whole 3-dimensional hyperplane of points with first coordinate 2 in which the other line lies. So we can use Pythagoras to find that the distance between (2, 1, 3, 4) and (4, 5, 10, 12) is equal to $\sqrt{(9^2 + 8^2 + 4^2 + 2^2)}$. In general: the distance between points (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) in 4-space is $\sqrt{[(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 + (a_4 - b_4)^2]}$. And even more in general: the distance between two points $(a_1, a_2, ... a_n)$ and $(b_1, b_2, ... b_n)$ in n-space is $\sqrt{[(a_1 - b_1)^2 + (a_2 - b_2)^2 + ... + (a_n - b_n)^2]}$.

Now we have found a way to measure distances in our n-dimensional space, we can do practically whatever we want. That is: it is easy now to generalize concepts like area, volume, angle etcetera. It is also a lot easier to give the more-dimensional equivalent of some familiar geometric objects. I will give one example, which plays an important role in the Poincare Conjecture: the generalization of the circle.

An ordinary circle (that is: a one-dimensional circle in a two-dimensional plane) consists of all points in the plane at a fixed distance, the radius, from a fixed point in the plane, the center. Now we can measure distances in all dimensions, it is easy to define a more dimensional version of it. The n-dimensional sphere, S^n, is defined as consisting of all points in the n+1-dimensional space which lie on distance 1 from the origin. Note that S^1 is just an ordinary circle with radius one, while S^2 is the 2-dimensional surface of a 3-dimensional ball (also of radius one), an object which I use to picture as a soap bubble. Although S^3 is a three-dimensional object, there is no way to picture it, since it's living in four-dimensional space and you can't fold it into three-space using some sort of brute force.

Topology and Manifolds.

When you're studying mathematical objects floating around in n-dimensional spaces, it is useful to ask yourself the question: what properties of these objects am I interested in, and what properties do not interest me at all? For some purposes it is sufficient and even more natural to consider a circle of radius 1 centered at the origin as the same object as a circle of that radius centered at any other point in the plane. In some cases it is even more natural to consider all circles as examples of the same object, no matter what their radiuses are.

Topology, one of my favorite branches of mathematics, even goes a step further. In topology two object are considered the same if they can be deformed into each other by bending, stretching and folding. When doing topology, it helps to imagine that everything you think about is made of some super-elastic rubber. So a cube is considered the same object as a ball, a coffee-cup is the same object as a doughnut etcetera. The properties we are interested in are the ones that are unaffected by the stretching. These are called *topological properties*.

When you draw a circle on a balloon, the circle separates the surface of the balloon into a part that lies 'inside' the circle and a part that is on the 'outside' of it. And no matter how you stretch the balloon, (making the circle into a square, or making the area of the outside smaller than that of the inside, or the other way round) the circle will always separate the balloon into two areas. Apparently, this is a topological property. It is also clear that an object consisting of two non-intersecting lines can never be deformed into one line, by pushing and stretching only. The topological property which is relevant here, is *connectedness*. We say that an object is not connected when it falls apart in two or more separated parts. Any object which can't be separated without cutting or tearing, is called connected.

The challenge in topology is not to find what objects are topologically the same: most of the time it is relatively easy to write down a recipe how to fold and stretch one object into another. (Of course to do so, one needs a more precise definition of what is meant by 'stretching', but I won't give that here because we won't really need it.) However, showing that two objects are *not* topologically equivalent is sometimes a lot harder. The reason is that you simply don't have time to try the infinite number of ways of stretching the first object and check they do not turn the object into the second. So the only way to show that two objects are not equivalent is to find a topological property that the one has and the other has not. In the last example we used connectedness to show that two lines are not equivalent to one: the single line is connected, while the object consisting of two lines is not. Connectedness can also be used to show that a line is not equivalent to a plane: both are connected, but the plane is still connected when you remove one point from it, and the line is not. (Of course you still have to show that connectedness is indeed a topological property, but this is rather easy once you have a more precise definition of both connectedness and stretching.)

To show however that the two-dimensional surface of a ball is not equivalent to the two-dimensional plane, connectedness won't help us much. What we need here is another topological property, named *compactness*. Giving the general definition would require explanation of a lot of other concepts, but I can give a definition which is equivalent to the general definition on the particularly 'nice' kind of objects we consider here. An object is called compact if it satisfies two requirements:

1) For any sequence of points in the object, which has a limit in a bigger space surrounding the object, the limit should lie in the object itself.
2) If you put your object in a bigger, say n-dimensional space, you can put, in this space, an n-dimensional box around it in such a way that the volume of the box (that is: the product of the lengths of the vertices) is finite.

I guess this is not really clear, but an example can help. A plane of which the point (0, 0) is left out, doesn't satisfy 1) because the sequence (½ , ½), (1/3, 1/3), (¼, ¼), (1/5, 1/5), ... clearly has limit (0, 0) in the whole plane (the 'bigger space'). In the plane with the origin left out, it doesn't have a limit and therefore the plane with no origin is not compact. Note that both S^2 and the normal two-dimensional plane satisfy 1).

However, the plane doesn't satisfy 2). When you put both objects in 3-space, any box containing the entire plane will have infinite volume no matter how thin you make it. It is on the other hand rather easy to pack the ball surface into a finite box and therefore S^2 is compact and the plane is not. So they are indeed different objects, even topologically.

A special family of 'beautiful' objects which are studied from a topological point of view, are *manifolds*. Manifolds are n-dimensional objects that for n-dimensional people living on (or in) it (who are not able to travel far), are indistinguishable from an ordinary n-dimensional space. An example is the S^2, which is topologically equivalent to the surface of the earth. Even for us, three dimensional people it is very hard to see that the earth is indeed not (a piece of) a flat plane. For an ant, (which can be considered a two-dimensional creature) the only way to find out the earth is not a plane, would be to travel all around the world.

Other examples of manifolds are, of course, the n-dimensional spaces themselves, and all spheres S^n. Besides that we have - in two dimensions - the torus (surface of a doughnut) and the cylinder of infinite length. Locally these two look like a plane. When living on a torus or a cylinder, the only way to find out that you are not in a plane is to start walking in some direction and come back where you started after a long time. (Which will moreover happen only if you were lucky in choosing your direction to walk in.) Also an object consisting of two distinct tori is a manifold, in particular one that is not connected.

Since we are not able to travel very far outside the earth, it might as well be possible that the universe is not, as everybody believes, an ordinary 3-dimensional space, but some other three-dimensional manifold. In fact, some astronomers do believe the universe is an S^3 since this would match their Big Bang theories beautifully.

How many topologically different n-dimensional manifolds are possible is for most numbers n still an open question, as far as I know. At least it was at the time of Henry Poincare (1854-1912) and this is what his conjecture is about.

The Poincare Conjecture and the fundamental group

More precisely, the conjecture is about a special class of manifolds: compact connected manifolds. Using the definitions of compactness and connectedness I gave earlier in the text, the reader can check for himself which of the given examples of manifolds are compact and connected and which are not. For one thing, it is clear that not every two compact connected manifolds are topologically the same. For instance the ball surface S^2 and the doughnut surface named torus are both connected compact 2-dimensional manifolds which can't be stretched into each other. So we'll need a new topological property to see the difference between these two.

When looking at the two objects, it seems pretty clear what the difference is: the torus has a hole in it, which the sphere has not. The question is: how to describe this in a way that makes it clear this is a topological property. It was Poincare who came up with an elegant answer, which turned out to be a very powerful tool in making a distinction between a lot of other two-dimensional manifolds and even some objects of higher dimension. The key idea is to look at all possible ways to draw a closed loop on (or in, in case of more than 2 dimensions) the manifold you're considering. The question is now whether two loops can be deformed into each other by shifting them over the manifold and maybe stretch them a little bit. On a 2-dimensional sphere every two loops can be deformed into each other (which one can easily check for himself using a rubber band and an orange). The same is the case in a plane. On the other hand: in a plane with a hole in it, a loop which runs around the hole can never be shifted of stretched into a loop that doesn't, since the loop is not allowed to leave the plane. (You could picture the plane as a field, the hole as a very high tree standing in it and the loop as a rope with its ends tied together.) Therefore a plane with a hole in it is topologically different from both the sphere and the plane without the hole. Also on a torus, it is very easy to find two loops which can't be shifted onto each other. Therefore the torus too, is different from the sphere and the plane.

The set of all different ways of drawing a loop on a manifold (where 'different' means that the loops can't be deformed into each other by shifting and stretching), together with their interactions is called the *fundamental group* of the manifold. When this set consists of only one element, that is: if all loops can be deformed into each other as we see on the sphere, we say the fundamental group is trivial (the word mathematicians use to show that they are not impressed by a mathematical object).

The 2-dimensional sphere has a trivial fundamental group. So has every n-dimensional sphere with n = 3 or higher. To make this a little more credible I can say the following. The S^n is the very thin shell of a (n+1)-dimensional ball in (n+1)-dimensional space. When we make the shell a little thicker, this won't change the fundamental group (one can check that for the S^2). Making it thicker and thicker, one eventually finds that the n-dimensional sphere has the same fundamental group as a giant (n+1)-dimensional space, with a very small hole in it. For (n + 1) = 3 or higher this hole won't stand in your way when you try to shift one-dimensional loops into each other (in 3 dimensions this is perfectly clear). So the fundamental group of a 3- or higher dimensional space with a hole in it is trivial, and for that reason, the fundamental group of a 2- or higher dimensional sphere is trivial too.

The Poincare Conjecture now states that the S^n is the *only* compact connected n-dimensional manifold with trivial fundamental group. That is, when you are walking around in some high-dimensional space and suddenly bump into a weird looking n-dimensional compact connected object, you only have to check that its fundamental group is trivial to be sure that it is topological equivalent to the n-dimensional sphere we all know so well.

Progress solving the problem.

As I said, the conjecture is known to be true for all n except n = 3. For n = 1 and n = 2, the conjecture is pulverized by much stronger theorems. As far as n = 1 is concerned, it is known that *every* compact connected 1-dimensional manifold is equivalent to the circle no matter what it's fundamental group does look like. A theorem which doesn't leave much room for speculation. For n = 2 we have (mostly due to Poincare) the so-called "Full Classification of Compact Surfaces". A list consisting of all types of two-dimensional compact manifolds of which we know that every 2-dimensional manifold you can think of is equivalent to one on the list. Once you have a list like this, it is easy to check the conjecture and it turns out to be true. The cases n = 5, n = 6 and higher were done in one great sweep somewhere during the 60's of the 20th century, but it took until the 90's till someone could prove the conjecture for n = 4. The case n = 3 is still open. This illustrates a 'fact' which occurs in more often in mathematics but doesn't match one's expectations at all: it is often easier to prove things about spaces with more than 5 dimensions than about the more familiar 3- or 4-dimensional space.

If the proof by Martin Dunwoody turns out to be valid then the conjecture has finally reached the status of a theorem. But since I didn't hear anything about the million being given to him, it is very well possible that someone found a flaw in Dunwoody's proof.

Of course, I hope his proof is flawed and someday, let's say: tomorrow, I come up with some original way of looking at the problem which hasn't been tried before and leads me to a proof that *is* sound. What would be even better, though, is if I could find a counter-example. I.e.: a not yet known 3-dimensional manifold, which has trivial fundamental group but is nevertheless not equivalent to the 3-dimensional sphere (which therefore would prove the Poincare Conjecture to be false). And my joy would be complete if, after a few years, some brilliant physicist reads my article and realises that all problems in modern astronomy can be solved by assuming that the universe has this particular previously unknown shape discovered by me.

On the other hand, when it is about gathering immortal fame, I know that this is not the way to aim for it. Whoever proves the Poincare conjecture, either Martin Dunwoody or I or someone else, he will never be as famous as Poincare himself. Let alone as well-known as Fermat or Goldbach, the latter of who never did anything in mathematics worth memorizing, except for posing his conjecture as a question in a letter to Euler. If history teaches us one thing, it is that to get immortal fame in mathematics, it is far more efficient to ask a question that no-one can answer, than to answer a question that no-one will ask.