Graded Model and Epistemic Logic

Drums 2 ESPRIT III (No.6156) Deliverable D6.2 (WP 6.3)

J.-J. Ch. Meyer and W. van der Meer

RUU-CS-93-44
December 1993

Utrecht University

Department of Computer Science

Padualaan 14, P.O. Box 80.089,
3508 TB Utrecht, The Netherlands,
Tel. : ... + 31 - 30 - 531454

ISSN: 0924-3275

Preface to the Report Version

This document is an official deliverable in the ESPRIT III Basic Research Action DRUMS II "Defeasible Reasoning and Uncertainty Management Systems", Project No. 6156. The DRUMS Project is concerned with the investigation of symbolic and quantitative (numerical) methods of representation and reasoning in order to deal with uncertain and defeasible knowledge. By symbolic methods mostly logical ones are meant, i.e. methods based on some logic(al formalism), whereas numerical methods are typically based on probabilities and generalisations of these. Typical problems addressed in the project are belief change, non-monotonic deduction, inconsistency in reasoning, abduction, algorithmic aspects and dynamic reasoning with partial models. Altogether, the DRUMS project covers a huge research area with many specialist sub-areas around the main topics of uncertainty and defeasibility, and many research institutes and researchers all over Europe working on these topics are involved in the project. One of the aims is to establish ways of integration and synthesis of the enormous number of diverse approaches already available in the literature as well as those currently in development.

The present deliverable is part of work done in the last area: (dynamic) reasoning with partial models, which includes the study of modal and epistemic logics to capture reasoning with uncertain and defeasible knowledge. In fact, this deliverable, entitled "Graded Modal and Epistemic Logics", is the first one of two planned within the project dealing with relations between multiple-valued, epistemic and dynamic logics to model non-monotonic reasoning. This first one deals with various ways of using modal logics for reasoning about knowledge and uncertainty, both qualitatively and quantatively. As such, part of this work should be interpreted as employing symbolic (logical) methods to reason quantitatively, thus providing a partial synthesis between symbolic and numerical methods for reasoning with uncertainty.

The deliverable is organised as follows: First the official body of the document is given, including a summary, the partners involved, the original description of the topic and aim, a global description of the results, and other information directed to the ESPRIT organisation, pertaining to this deliverable. Then in an Annex, the original papers in which the actual work is done, are provided. For the report version this annex is probably (hopefully) the most interesting part.

The editors, October 1993.

DRUMS 2 ESPRIT III (No.6156) Deliverable D6.2 (WP6.3)

Graded Modal and Epistemic Logics

J.-J. Ch. Meyer & W. van der Hoek
Utrecht University / Free University Amsterdam

Deliverable Summary
The deliverable contains a survey of various ways of using modal logics for reasoning about knowledge and uncertainty, including standard S5-based epistemic logic. Furthermore, a study is made of quantitative and qualitative modalities. In particular, a graded (or numerical) variant of the epistemic logic S5 is considered, and its logical properties are investigated, such as completeness, decidability and expressibility. In this logic we can express notions such as "knowing a formula φ modulo at most n exceptions to φ". The use of this graded logic for representing uncertain and defeasible knowledge is indicated. Finally, other quantitative and qualitative modalities related to probabilistic and possibilistic notions - such as "as least as likely / possible as" - together with their logics are studied. As will be shown below we can safely state the *no major milestones have been missed* in the execution of this part of WP6.3.

Partners Involved
As planned, this part of WP6.3 was performed by the F. U. Amsterdam - Theoretical Computer Science site (currently situated at Utrecht University). The intention is that the next part (deliverable D6.6, to be delivered at the end of the 3 years of the project) will also involve the FUA - AI Section, Blanes, Linköping and Warsaw sites, where the results of D6.2 will be used as some basic groundwork. In fact, there is already work in progress with the Blanes site on the modal interpretation of the knowledge meta-predicates in MILORD II, and separately with the FUA-AI-site on the temporal aspects of an epistemically based default logic.

Added Value of Participation in the DRUMS2 Project
Although the work of this part of WP6.3 has been performed by one site (FUA-Theory), as stated above, we can state that the participation in the context of the DRUMS2 project has influenced our work considerably and has helped us to complete the planned research in a satisfactory way. The work in progress was presented at several WP6 / WP3 - workshops, which provided very valuable feedback, which could be fruitfully incorporated in subsequent versions of the papers in this deliverable. Also, the many discussions we had with both fellow logicians and the quantitative-oriented colleagues in the project proved to be very useful for us.

Also, for us theorists, the DRUMS meetings gave a good view of the more practical issues involved in devising reasoning systems, which will surely influence further work in WP6.3.

Original Description of the Topic in the Project Proposal

"In graded modal logic it is possible to express that a formula is satisfied in a *number* of (accessible) worlds. This may give a tool to 'quantify' degrees of knowledge, whereas traditional (ungraded) epistemic modal logic only treats absolute knowledge: p is known iff p holds in *every* conceivable world. Thus, graded logic could contribute to the main objective of DRUMS 2:i.e. to bridge the gap between qualitative approaches to incomplete knowledge and quantitative (numerical) ones to uncertain knowledge. Moreover, discarding the exact number of worlds, one may introduce a qualitative comparison operator '$\varphi > \psi$', which means that there are more worlds / situations where φ holds than where ψ holds. Such a logic can be considered a qualitative abstraction of the graded modal logic. With a probabilistic interpretation in mind, '$>_p$' has already been studied in the literature. A link with epistemic logic is given by W. Lenzen's definition of belief: a formula φ is believed if $\varphi >_p \neg\varphi$ holds."

Original Description of the Aim in the Project Proposal

"Our plan concerning the study of these quantitative modalities is as follows. Firstly, we like to have a precise description of (the logic of) graded modalities; especially in terms of decidability, complexity, expressibility, especially for the graded variant of S5. Secondly, we want to investigate its use for epistemic logic. Especially, we like to make explicit in which situations it makes sense to base epistemic reasoning on democratic principles (when is it reasonable to 'believe' p if most of the words verify p?). Another topic we like to study is how this 'reasoning with exceptions' in graded epistemic logic is related to defeasible knowledge. Thirdly, we would like to relate the qualitative operator '>' with these graded modalities as well as with '$>_p$'. The output of this research will be used in the following research."

Global Description of Results

The results of the work reported in this deliverable are based on two papers, which are given in the Annex, viz.

1. W. van der Hoek & J.-J.Ch. Meyer, Graded Modalities in Epistemic Logic. This paper is submitted to the journal "Logique et Analyse" and contains work on so-called graded modalities as applied to epistemic logic.

2. W. van der Hoek & J.-J.Ch. Meyer, Modalities for Reasoning about Knowledge and Uncertainties. This paper was presented in a preliminary version as a tutorial at a forerunner of a DRUMS Workshop /Summerschool in Linköping, and will appear in a book edited by P.

Doherty and D. Driankov where the papers of the workshop are collected, revised and expanded.

The material in these two papers addresses the questions that we asked ourselves in the proposal. In the first paper we consider the use of graded modalities in epistemic logic. We then obtain the graded epistemic logic Gr(S5). In this logic we can express notions such as $K_n\varphi$: "knowing a formula φ modulo at most n exceptions to φ". An axiomatization of the logic is given. Besides foundational topics such as completeness and decidability (finite model property) the paper contains more practice-oriented issues pertaining to the use of the logic for plausible or defeasible reasoning. For instance, the following is a derivable theorem of the logic:

$$K_n(\varphi \rightarrow \psi) \rightarrow (K_m\varphi \rightarrow K_{n+m}\psi)$$

which expresses the intuitive idea that when modus ponens is used with uncertain knowledge the result is even weaker (more uncertain) than its premises. The modalities K_n together with their dualities M_n ($=\neg K_n \neg$) give rise to a spectrum of ever decreasing certainty : K_0 (absolute certainty), K_1, K_2, ..., M_2, M_1, M_0 (not an impossibility). Also generalised introspection properties are discussed such as $M_n\varphi \rightarrow K_0M_n\varphi$: if it is known that there are more than n situation in which φ holds, then it is known (for sure) that it is known that this is the case, or put differently, the agent is aware of the fact that he considers more than n φ-situations possible. By way of examples, it is shown how in the logic a democratic principle of belief (based on more than half of the situations are known to satisfy the assertion of concern) and the infamous lottery paradox can be treated. It is indicated how so-called numerical syllogisms can be represented and proved in the logic. Finally relations with other approaches such as concept languages and probabilistic logic are indicated.

The second paper is of a more survey-like nature. It collects well-known ways of using modal logics for reasoning about knowledge and uncertainty, as well as a survey of own results. Discussed are:

• standard modal (S5-like) epistemic logic including: the notions of both knowledge and belief and their interaction, the problem of logical omniscience of belief, epistemic states and honest formulas;

• modal approaches to reason about quantities, such as graded modalities and probabilistic modalities;

• qualitative modalities, that use quantities in an indirect, implicit way, such as e.g. "at least as many as" (based on counting), "as least as likely as" (probabilistic), "as least as possible as" (possibilistic).

In conclusion we may say that our most of the ingredients of our initial plan of research has been carried out and *no problems were encountered in the execution of this work part*: we have given a precise description of (the logic of) graded modalities; we have investigated the issues of completeness, decidability and expressibility for the logic Gr(S5). We have studied how Gr(S5) might be used as a logic for plausible / defeasible knowledge. In the logic it can be extended such that a belief of p if most of the worlds verify p, can be expressed. Finally, several relations of graded modalities with quantitative and qualitative modalities, as they appear in probabilistic and possibilistic logic, are established.

ANNEX

copies of papers for D6.2

Graded Modalities in Epistemic Logic*

W. van der Hoek
J.-J. Ch. Meyer#

Free University, Amsterdam
Department of Mathematics and Computers Science
De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands

ABSTRACT
We propose an epistemic logic with so-called graded modalities, in which certain types of knowledge are expressible that are less absolute than in traditional epistemic logic. Beside 'absolute knowledge' (which does not allow for any exception), we are also able to express 'accepting φ if there are at most n exceptions to φ'. This logic may be employed in decision support systems where there are different sources to judge the same proposition. We argue that the logic also provides a link between epistemic logic and the more quantitative (even probabilistic) methods used in AI systems. In this paper we investigate some properties of the logic as well as some applications.

1 Introduction

'Infallible' computers are computers that have multiple processors (usually each from a different company and programmed in a different way using different programming languages) to check and double-check on the results. Decisions are taken on a kind of democratic basis: the results that come up most often as the results of a certain calculation, are the ones that matter and are used to make a decision. The idea is just based on statistics: the chance that n independent processors are faulty at the same time is p^n for an already very small probability p. Typically, infallible computers are used in situations where the failure of a computer would have disastrous consequences, such as the stock exchange, certain security situations, and the so-called flying-by-wire (i.e. using a steering computer) of an airplane like the Airbus A320.

Decision Support Systems working on infallible computers, and devices with several input sensors in general, may have knowledge that is source-dependent. In this paper we will propose an epistemic logic that can deal with knowledge (or some may prefer the term *belief* here) that is not absolutely true in all worlds, but may have exceptions in the sense that there are worlds in which the assertion (that was believed) is nevertheless not true, such as in the case of a faulty processor or sensor in the situation described above.

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* This work is partially supported by ESPRIT III Basic Research Action No. 6156 (DRUMS II).
# Also at the University of Nijmegen, Dept. of Math & Comp. Sc., Toernooiveld, 6525 ED Nijmegen.

Consider an agent getting input from three different sources w_1, w_2 and w_3. Suppose furthermore, that two types of information are relevant for this agent, say p and q. All the sources agree on p: they mark p as true. Finally, in w_1 and w_2, q is true, whereas in w_3, it is false. When using 'standard' modal logic to model epistemic notions (cf. [MHV91] for an overview), one would consider the resources w_i ($i \leq 3$) to be worlds in an S5-Kripke model (cf. [HC68] for an introduction to modal logic), and observe that the agent knows p, i.e. Kp holds, but does not know q or $\neg q$, since he considers both alternatives to be possible: $Mq \wedge M\neg q$ holds.

This is about the limit of the expressibility of standard modal epistemic logic (we will formalize this claim in the next Section), where the only operators that are available are K and M, to express 'truth in all accessible worlds' and 'truth in some accessible world', respectively. Since the favourite system for knowledge (S5) may be interpreted on Kripke models in which the accessibility relation is universal (cf. [HM85, MHV91]), we may leave out the reference to this relation, leaving one with a system in which one can associate 'K' with 'all (worlds)' and M with 'some (world)' (cf. [GP90]).

However, in the above example, it might be desirable to be able to express that the agent has more confidence in q than in $\neg q$. (For instance, a robot which (who?) is searching for block A, may choose to first look for it on block B, if two of its sensors tell him it is there—while the third sensor tells him it is on block C.) One way to achieve this is to add a qualitative modality '>', enabling the agent to judge ($q > \neg q$), as is done in [Ho91a] (and, in a specifically epistemic context, see also [Le80]). Here, we will take an alternative approach, in which we add quantitative modalities to the language (M_n, $n \in \mathbb{N}$), with the intended meaning that more than n successors verify φ. Then, in the above example, we can describe the agent's point of view in a more precise manner, like for instance the fact that he considers exactly two q-alternatives possible ('input-sources'), and exactly one $\neg q$-alternative.

Actually, adding such 'graded modalities' to the modal language is not new. We refer to [Ho92a] for some history, and a first investigation into the expressibility, decidability and definability of this graded language. An application of those graded modalities, especially of the graded analogue of S5, has been studied in the area of *Generalized Quantifiers*, (cf. [HoR91]). In this paper, we try to explain how the greater expressive power of graded modalities may be used in *epistemic logic*. We already showed in [HoM92] how these new modalities may help to make some issues in the field of *implicit knowledge* explicit. However, there, the graded modalities are motivated to establish some properties on a 'meta-level'; adding them to the language enables one to more precisely define accurate models for implicit knowledge; in particular, we showed how one can employ graded modalities to define the intersection of accessibility relations. Here, though, we try to use the new operators directly in the object language in order to obtain a more fine-tuned epistemic logic. We think that, using the enriched language, one has an appropriate tool to deal with notions like 'uncertain', or 'almost certain' knowledge (or belief). The new operators may then be helpful to reason with *degrees of acceptance.* In fact, one may distinguish as many degrees of belief as there are graded modalities. To support our claim, we will sketch some directions in which such modalities might be employed.

The rest of this paper is organised as follows. In the following Section, we will introduce our main system, together with its natural semantics. In Section 3, we investigate how this system of Section 2 can be interpreted epistemically. Then we give some examples in Section 4 and conclude by indicating some further directions of research in Section 5.

2. The system Gr(S5)

Before plunging into the definitions of the graded language and the formal system, it may be useful to keep in mind how standard modal logic (together with its semantics) is used to model knowledge. There, $K\varphi$ (φ is known) is defined to be true in a Kripke model $(\mathcal{M},w)$ iff in all worlds v accessible from w, $(\mathcal{M},v)$ is a model for φ. Also, $M\varphi$ is defined to be $\neg K \neg\varphi$, which will be true in w iff φ is true in some accessible world v.

Now, consider the following Kripke model $\mathcal{M}_k = \langle W, R, \pi\rangle$, where $W = \{w_1, w_2, w_3, \ldots w_k\}$ $(k \geq 4)$, $R = W \times W$ and $\pi(p)(w_i)$ = true for all $i \leq k$; $\pi(q)(w_i)$ = true iff $i \notin \{2, 3\}$; $\pi(r)(w_i)$ = true iff $i \in \{1,3\}$ (cf. Figure 1 below).

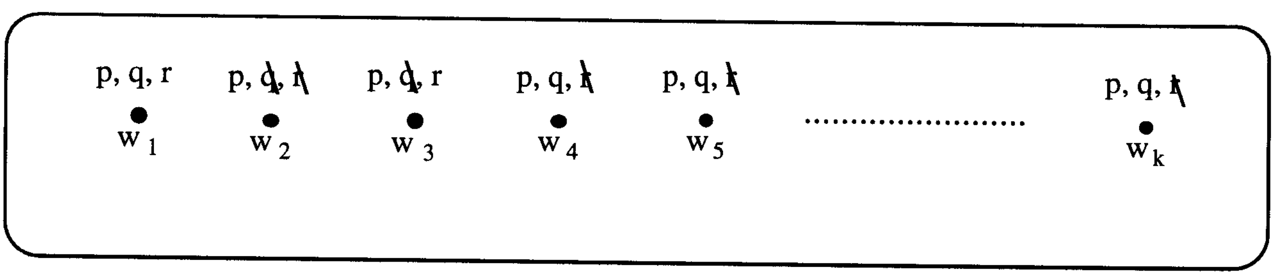


Figure 1

At the end of this Section, we will demonstrate, in a precise way, that in all *purely modal* formulas that are true in this model, one may freely interchange the role of q and r. In other words, despite the fact that q is true 'almost everywhere' in the model, and r is false 'almost everywhere', the modal language is too weak to express this difference between q and r. We claim that, both in the cases where worlds are interpreted in one-one correspondence to counterparts in the physical world (e.g. like sensors —Section 1) and where worlds correspond to possible (but made up) situations for some agent, a tool to distinguish 'q-statements' from 'r-statements' in the above model is highly desirable.

We provide such a tool by adding *graded modalities* M_0, M_1, ... to the modal language. The intended interpretation of $M_n\varphi$, ($n \in \mathbb{N}$) will be that there are more than n accessible worlds verifying φ. By defining $K_n\varphi \equiv \neg M\neg_n\varphi$, $K_n\varphi$ is true iff at most n accessible worlds refute φ. In terms of epistemic operators, note that $K_0\varphi$ boils down to $K\varphi$, so that we may interpret K_0 as our (certain) knowledge operator. Generally, $K_n\varphi$ means that the agent reckons with at most n exceptions for φ. Dually, $M_n\varphi$ then means that the agent considers more than n alternatives possible, in which φ is true. Now, what would be the appropriate properties of these 'defeasible' necessity operators? For instance, what kind of introspective properties are desirable? Many possibilities present themselves at this point, but for the time being we will remain on solid ground by considering the graded analogue of S5.

Our language L is built, in the usual way, from propositional atoms p, q, ..., $\in \mathsf{P}$, using the standard connectives $\bot$, $\top$, $\wedge$, $\vee$, $\neg$, $\rightarrow$ and $\leftrightarrow$. Moreover, if $\varphi \in \mathsf{L}$, then so is $M_n\varphi$ ($n \in \mathbb{N}$). From now on, we will assume that n, m, k $\in \mathbb{N}$. We use K_n as an abbreviation for $\neg M_n \neg$. Finally we introduce the abbreviation $M!_n\varphi$, where $M!_0\varphi \equiv K_0\neg\varphi$, $M!_n\varphi \equiv (M_{n-1}\varphi \wedge \neg M_n\varphi)$, if $n > 0$. From the definitions above, it is clear that $M!_n$ means 'exactly n'.

2.1 Definition. The system Gr(S5) is defined as follows (cf. [HoR91]). It has inference rules Modus Ponens and Necessitation:

R0 $\quad \vdash \varphi, \vdash \varphi \rightarrow \psi \Rightarrow \vdash \psi$

R1 $\quad \vdash \varphi \Rightarrow \vdash K_0\varphi$

It has also the following axioms (for each $n \in \mathbb{N}$):

A0 $\quad$ all propositional tautologies

A1 $\quad K_0(\varphi \rightarrow \psi) \rightarrow (K_n\varphi \rightarrow K_n\psi)$

A2 $\quad K_n\varphi \rightarrow K_{n+1}\varphi$

A3 $\quad K_0\neg(\varphi \wedge \psi) \rightarrow ((M!_n\varphi \wedge M!_m\psi) \rightarrow M!_{n+m}(\varphi \vee \psi))$

A4 $\quad \neg K_n\varphi \rightarrow K_0\neg K_n\varphi$

A5 $\quad K_0\varphi \rightarrow \varphi$

Before elaborating on the impact of the axioms on our intended epistemic reading of the operators, which we will do in the following Section, let us pause for a moment to sharpen our understanding of the postulates as such. The system with rules R0 and R1, axioms A0 - A3 is the graded modal analogue of K, the basic modal system—so let us refer to it with Gr(K). In Gr(K), A1 is a kind of 'generalized K-axiom' (cf. 2.3), A2 is a way to 'decrease grades' in the possibility operator (A2 is equivalent to $M_{n+1}\varphi \rightarrow M_n\varphi$) and using A3, one can go to 'higher grades'. To ensure that the definitions work out rightly, we take proposition 2.10 from [Ho92a]:

2.2 Proposition. The following are derivable in Gr(K) (and hence in Gr(S5)):

(i) $M_n(\varphi \wedge \psi) \rightarrow (M_n\varphi \wedge M_n\psi)$

(ii) $M!_n\varphi \wedge M!_m\varphi \rightarrow \bot$ $(n \neq m)$

(iii) $K_n\neg\varphi \leftrightarrow (M!_0\varphi \,\nabla\, M!_1\varphi \,\nabla\, \ldots \,\nabla\, M!_n\varphi)$ (∇ denotes 'exclusive or')

(iv) $\neg M_n(\varphi \vee \psi) \rightarrow \neg M_n\varphi$

(v) $M_{n+m}(\varphi \vee \psi) \rightarrow (M_n\varphi \vee M_m\psi)$

(vi) $M!_n\varphi \wedge M_m\varphi \rightarrow \bot$ $(m \geq n)$

(vii) $M_n(\varphi \wedge \psi) \wedge M_m(\varphi \wedge \neg\psi) \rightarrow M_{n+m+1}\varphi$

(viii) $(K_0\neg(\varphi \wedge \psi) \wedge (M_n\varphi \wedge M_m\psi)) \rightarrow M_{n+m+1}(\varphi \vee \psi)$

To see the system in action, we will give a derivation of a theorem which is a generalisation of the K-axiom from standard modal logic.

2.3 Proposition. The following is derivable in Gr(K) (and in Gr(S5)).

$K_n(\varphi \rightarrow \psi) \rightarrow (K_m\varphi \rightarrow K_{n+m}\psi)$.

Proof. We implicitly use the (Gr(K)-derivable) rule of substitution: $\vdash \alpha \leftrightarrow \beta \Rightarrow \vdash \varphi \leftrightarrow \varphi[^{\alpha}/_{\beta}]$. Then, observe that $\vdash$ A1 $\leftrightarrow (K_0(\varphi \rightarrow \psi) \rightarrow (M_n\varphi \rightarrow M_n\psi))$ (*). To see this, note that $\vdash K_0(\varphi \rightarrow \psi) \leftrightarrow K_0(\neg\psi \rightarrow \neg\varphi)$, and $\vdash (M_n\varphi \rightarrow M_n\psi) \leftrightarrow (K_n\neg\psi \rightarrow K_n\neg\varphi)$.

1	$\vdash \neg\psi \rightarrow (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi)$	A0
2	$\vdash K_0(\neg\psi \rightarrow (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi))$	R1,1
3	$\vdash M_{n+m}\neg\psi \rightarrow M_{n+m}((\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi))$	(*),2
4	$\vdash M_{n+m}((\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \neg\psi)) \rightarrow M_n(\varphi \wedge \neg\psi) \vee M_m(\neg\varphi \wedge \neg\psi)$	2.2(v)
5	$\vdash ((\varphi \wedge \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi)) \wedge ((\neg\varphi \wedge \neg\psi) \rightarrow \neg\varphi)$	A0
6	$\vdash (M_n(\varphi \wedge \neg\psi) \rightarrow M_n\neg(\varphi \rightarrow \psi)) \wedge (M_m(\neg\varphi \wedge \neg\psi) \rightarrow M_m\neg\varphi)$	R1,1, A1
7	$\vdash M_{n+m}\neg\psi \rightarrow (M_n\neg(\varphi \rightarrow \psi) \vee M_m\neg\varphi)$	A1, 3,4,6
8	$\vdash \neg M_m\neg(\varphi \rightarrow \psi) \rightarrow (\neg M_n\neg\varphi \rightarrow \neg M_{n+m}\neg\psi)$	A0,7
9	$\vdash K_n(\varphi \rightarrow \psi) \rightarrow (K_m\varphi \rightarrow K_{n+m}\psi)$	Def K_k,8

Note that, by taking n = m = 0 in 2.3, we get the K-axiom in Gr(S5). In the presence of the necessitation rule, this means that K_0 is a 'normal' modal operator. In fact, the axioms A4 and A5 are graded versions of Euclidicity and reflexivity, respectively. Before making this explicit, we give the definition of the models on which we want to interpret formulas of L.

2.4 Definition. A *Kripke structure* $\mathcal{M}$ is a tuple <W,π,R>, where W is a set (of 'worlds' or 'states'), π a truth assignment for each w ∈ W and R a binary relation on W. If R is both reflexive and Euclidean (i.e. $\forall xyz((Rxy \wedge Rxz) \rightarrow Ryz)$), we say that $\mathcal{M} \in \mathcal{S5}$. It is easily verified that the accessibility relations R of $\mathcal{M} \in \mathcal{S5}$ are equivalence relations. A model $\mathcal{M} \in \mathcal{S5}$ is known to be a model of (standard) S5 (cf. [MHV91]).

2.5 Definition. For a Kripke structure $\mathcal{M}$ we define the *truth of* φ at w inductively:

(i) $(\mathcal{M},w) \models p$ iff $\pi(s)(p) = true$, for all $p \in P$.

(ii) $(\mathcal{M},w) \models \neg\varphi$ iff not $(\mathcal{M},w) \models \varphi$.

(iii) $(\mathcal{M},w) \models \varphi \vee \psi$ iff $(\mathcal{M},w) \models \varphi$ or $(\mathcal{M},w) \models \psi$.

(iv) $(\mathcal{M},w) \models M_n\varphi$ iff $|\{w' \in W \mid Rww' \text{ and } (\mathcal{M},w') \models \varphi\}| > n$, $n \in N$.

2.6 Remark. Note that $(\mathcal{M},w) \models K_n\varphi$ iff $|\{w' \in W \mid Rww' \text{ and } (\mathcal{M},w') \models \neg\varphi\}| \leq n$. Also, note that the modal operators M and K (in the literature also written as M and L, or ◊ and □) are special cases of our indexed operators: $M\varphi \equiv M_0\varphi$ and $K\varphi \equiv K_0\varphi$.

2.7 Definition. We say that φ is *true* in $\mathcal{M}$ at w if $(\mathcal{M},w) \models \varphi$. If such an $\mathcal{M}$ and w exist for φ, we say that φ is *satisfiable*. Formula φ is *true* in $\mathcal{M}$ ($\mathcal{M} \models \varphi$) if $(\mathcal{M},w) \models \varphi$ for all w ∈ W, and φ is called *valid* ($\models \varphi$) if $\mathcal{M} \models \varphi$ for all $\mathcal{M}$. If $\mathcal{C}$ is a class of models (like $\mathcal{S5}$), $\mathcal{C} \models \varphi$ means that for all $\mathcal{M} \in \mathcal{C}$, $\mathcal{M} \models \varphi$.

With these semantic definitions, we can formalize our claim about the model of Figure 1. For two propositional variables x and y, let $[x \leftrightarrows y]\varphi$ be the formula obtained from φ by interchanging the x and y in φ. (This can be defined in terms of $[^u/_v]\,\varphi$, substitution of u for v in φ, as follows: $[x \leftrightarrows y]\varphi \equiv [^y/_z][^x/_y][^z/_x]\varphi$, where z is some atom not occurring in φ.) We suppose that the accessibility relation in the model of Figure 1 is *universal*, i.e. for all w and v we have Rwv.

2.8 Theorem. Let φ be a (non-graded) modal formula, and $\mathcal{M}$ the model of Figure 1. With the definition of $[x \leftrightarrows y]\varphi$ given above, we claim that $\mathcal{M} \models \varphi \Leftrightarrow \mathcal{M} \models [q \leftrightarrows r]\varphi$.

Proof. The theorem follows from the following observation. Let f: $W \to W$ be the following function: $f(w_1) = w_1$; $f(w_2) = w_2$; $f(w_3) = w_4$, $f(x) = w_3$ for all $x \in W \setminus \{w_1, w_2, w_3\}$. Then, we claim, that for all $w \in W$ and all non-graded modal formulas φ: $(\mathcal{M},w) \models \varphi \Leftrightarrow (\mathcal{M},f(w)) \models [q \leftrightarrows r]\varphi$. This claim is established using a simple induction, of which we demonstrate the modal case $\varphi = M\psi$: suppose that $(\mathcal{M},w) \models M\psi$. By the truth-definition of M, there must be some $v \in W$ such that $(\mathcal{M},v) \models \psi$. By the induction hypothesis, we obtain $(\mathcal{M},f(v)) \models [q \leftrightarrows r]\psi$. Since R is universal, we have $(\mathcal{M},f(w)) \models M[q \leftrightarrows r]\psi$ which is of course equivalent to $(\mathcal{M},f(w)) \models [q \leftrightarrows r]M\psi$, i.e. $(\mathcal{M},f(w)) \models \varphi$. The converse of this claim is proven similarly. The proof of the theorem then proceeds as follows: $\mathcal{M} \not\models \varphi \Leftrightarrow$ there is some w such that $(\mathcal{M},w) \models \neg\varphi \Leftrightarrow$ there is some w such that $(\mathcal{M},f(w)) \models [q \leftrightarrows r]\neg\varphi \Leftrightarrow$ there is some w such that $(\mathcal{M},f(w)) \models \neg[q \leftrightarrows r]\varphi \Leftrightarrow \mathcal{M} \not\models [q \leftrightarrows r]\varphi$.

We end this introduction to Gr(S5) by recalling the following results:
2.9 Theorem. (Completeness: [Fi72], [FC88]). For all $\varphi \in L$, Gr(S5) $\vdash \varphi$ iff $\mathcal{S5} \models \varphi$.
Thus $\mathcal{S5}$ is also a class of models characterizing Gr(S5).

2.10 Theorem. (Finite models: [Ho92a]). Any $\varphi \in L$ is satisfiable iff it is so on a finite model.

2.11 Theorem. (Freedom of nestings: cf. [HoR91]). In Gr(S5), each formula is equivalent to a formula in which no nestings of (graded) modal operators occur.

Related to the last theorem, a popular slogan in modal logic is that in S5, 'the inner modality always wins', we have e.g. $KM\varphi \equiv M\varphi$, $MK\varphi \equiv K\varphi$ and $MM\varphi \equiv M\varphi$ in S5. However, in the case of Gr(S5) this is not always sufficient: we *do* have $M_3M_5\varphi \equiv M_5\varphi$, but instead of $M_5M_3\varphi \equiv M_3\varphi$ we now have $M_5M_3\varphi \equiv M_5\top \wedge M_3\varphi$, accounting for the fact that $M_5M_3\varphi$ implies that, so to speak, 5 worlds are around.

3 Epistemic Reading

Returning to the main point of this paper: how can Gr(S5) serve as an appropriate starting point to study epistemic phenomena? To start with, R0 and A0 express that we are dealing with an (extension of) classical propositional logic: we may use Modus Ponens and reason 'classically' (A0). By taking S5 as a 'standard' system for knowledge, the observations in the preceding Section suggest that we interpret $K_0\varphi$ as 'φ is known' (by the agent: for the moment, we focus on one-agent systems, although graded modalities do not prevent us from studying multi-agent systems—on the contrary, cf. [HoM92]).

Then, R0, R1, A0 and A5 find their motivation in the same fashion as the corresponding properties in S5, i.e., we may use Modus Ponens, the agent knows all (Gr(S5))-derivable facts, we are dealing with an extension of propositional logic (A0) and moreover the agent cannot know facts that are not true (A5).

In order to interpret the other axioms, we need to have some intuition about the meaning of $K_n\varphi$. The semantics suggest, that it should be something like 'the agent reckons with at most n exceptional situations for φ', or 'the agent "knows-modulo-n-exceptions" φ'. Thus, the greater n is in $K_n\varphi$, the less confidence in φ is uttered by that sentence. The latter observation immediately hints at A2, $K_n\varphi \rightarrow K_{n+1}\varphi$: if the agent foresees at most n exceptions to φ, he also does so with at most n+1 exceptions. Of course, the generalisation of A3, for $n > 0$; $K_n\varphi \rightarrow \varphi$ is not valid: if the agent does not know φ for sure, i.e., if he allows for exceptions regarding φ, he cannot conclude that φ is the case. Thus $K_n\varphi$ expresses a form of "uncertain knowledge".

In standard S5, we have the axiom $\neg K\varphi \rightarrow K\neg K\varphi$, expressing the agent's negative introspection: if he does not know a given fact, he knows that he does not (this is of course an 'over-idealised' property of knowledge, especially if we have in mind capturing human knowledge; see [MHV91] for a short discussion and further references). We may write this introspection axiom equivalently as

(1) $M\varphi \rightarrow KM\varphi$,

saying that the agent has awareness (see [FH88] or [HoM89] for a discussion on this 'awareness'—defined in a technical sense) of what he considers to be possible. Now that we have at hand a more fine-tuned mechanism to distinguish between 'grades' of possibility, it seems straightforward to strengthen the bare introspection formula (1) to

(2) $M_n\varphi \rightarrow K_0M_n\varphi$,

saying that the agent is aware of the fact that he considers more than n φ-situations possible. (2) is equivalent with our axiom A4. Note that (2) is at the same time the 'most general' way to generalise (1): it implies, (using A2 m-1 times) for instance $M_n\varphi \rightarrow K_mM_n\varphi$.

In the same spirit, we can interpret A1: if the agent knows that φ implies ψ, then, if he believes that there can be at most n exceptions to φ, he will not imagine more than n exceptions to ψ, since every exception to ψ will be an exception to φ as well, i.e. $K_0(\varphi \rightarrow \psi) \rightarrow (K_n\varphi \rightarrow K_n\psi)$, or equivalently (cf. 2.3), $K_0(\varphi \rightarrow \psi) \rightarrow (M_n\varphi \rightarrow M_n\psi)$ (A1'). In epistemic logic, the K-axiom, $K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$, has been considered a source of *logical omniscience* ([FH88] or

[Ho92c]), which yields too idealistic a notion of knowledge (and certainly of belief). It would mean that the agent is capable to close his knowledge (belief) under logical implication. However, now that we allow for weaker notions of knowledge, it appears that the K-axiom is only valid for K_0, which we may consider as a kind of 'ideal' knowledge. Instead of a K-axiom for each K_n, we have the much more realistic (cf. 2.3)

(3) $K_n(\varphi \to \psi) \to (K_m\varphi \to K_{n+m}\psi)$.

This seems very reasonable (suppose n, m > 0): if the agent has some confidence that φ implies ψ, and also has some confidence in φ, his conclusion that ψ holds should be stated with even less certitude than that of the two assertions separately. This is reminiscent of *plausible* ([Re76]) or *defeasible reasoning*, where reasoning under uncertainties is also the topic of investigation. Note that (3) guarantees that, the longer the chain of reasoning with uncertain arguments, the less certain the conclusion can be stated by the agent. Moreover, note that, although (3) holds, if n > 0 we do not have $K_n(\varphi \to \psi) \to (K_n\varphi \to K_n\psi)$: this makes it questionable to call K_n a *modal operator* (if n > 0). However, here we do so because of the interpretation of such operators in Kripke models.

Finally, to understand A3, we must recall that $M!_n\varphi$ means that the agent is aware of exactly n possible situations in which φ is true. But then, A3 simply states this property of additivity: if the agent knows that φ and ψ are mutual exclusive events, and he is thinking of exactly n situations in which φ is true and, at the same time, m situations in which ψ is true, altogether he has to reckon with (n + m) situations in which one of these two alternatives is the case.

Up to now, we have been deliberately slightly vague about what $M_n\varphi$ and $M!_n\varphi$ exactly should mean. For instance, is this index n within the scope of the agent's knowledge? That is, does the agent know himself of (exactly) n concrete situations in which φ holds, and if so, is it possible that there are still other situations he does not know about where φ holds as well? This makes sense in situations in which the agent has to make decisions that depend on rules that allow for *exceptions*. The alternative interpretation is, that these n situations are only known to the reasoner *using* the system at a meta-level, interpreting K_n as some abstract n-degree of knowledge (or perhaps belief, if n is greater than some threshold)? We believe that the logic can be used in both these cases, and will not fix the interpretation in this paper.

It is argued (cf. [HM85]) that the axiom which distinguishes knowledge (K) from belief (B) is $(K\varphi \to \varphi)$. Instead of $(B\varphi \to \varphi)$, for belief, the weaker axiom $\neg B\bot$ is added. Now that we have (infinitely) many operators around, we might see how they behave in this respect. In

Gr(S5), $\neg K_n\bot$ (meaning that more than n possibilities are reckoned with) is derivable only for n = 0. If we would have (add) $\neg K_n\bot$, implying that he allows for more than n possibilities and hence he 'does not know too much' (if n is big). And indeed, as long as the agent considers at least one possible world, it means that he does not know contradictions ($\neg K\bot$). In case he has no epistemic alternative left his knowledge is all encompassing but inconsistent ($K\varphi$, for any φ). This is of course excluded in S5 (and hence in Gr(S5)), but so far, there was no way to exclude the extreme case of an 'omniscient knower', i.e., one for which ($K\varphi \leftrightarrow \varphi$) holds. Semantically speaking, there was no way to define the class of Kripke models in which each world had more than one successor. Using graded modalities this can be enforced by adding $M_1\top$ to any system.

4 Examples

When interpreting K_n as an 'n-degree of knowledge', we recall that the higher the degree, the less certain the knowledge. The picture is denoted in the following chain:

$$K_0\varphi \rightarrow K_1\varphi \rightarrow \ldots K_n\varphi \rightarrow K_{n+1}\varphi \ldots \Rightarrow \ldots M_{n+1}\varphi \rightarrow M_n\varphi \rightarrow \ldots \rightarrow M_1\varphi \rightarrow M_0\varphi.$$

Here, the '$\rightarrow$' denotes logical implication. If, semantically speaking, the number of alternatives is infinite, the sequence is an infinite one, and '$\Rightarrow$' denotes implication, in the sense that all M_i-formulas are logically weaker than all the K_j-formulas. We could, as argued above, interpret the strongest formula in this chain ('$K_0\varphi$') as "φ is known", and the weakest ('$M_0\varphi$') as "φ is not impossible"—but even as "φ is believed", cf. [HoM89].

If, however, the number of alternatives is *finite*, say N, we get the sequence

$$K_0\varphi\ (\equiv M_{N-1}\varphi) \rightarrow K_1\varphi\ (\equiv M_{N-2}\varphi) \rightarrow \ldots K_n\varphi\ (\equiv M_{N-n-1}\varphi) \rightarrow \ldots K_{N-1}\varphi\ (\equiv M_0\varphi) \rightarrow K_N\varphi\ (\equiv\top)$$

In fact, this is the case in the situation of the introduction, where the agent is capable to sum up a complete description of the model by listing a (finite) number of possible situations determined by some finite set of propositional atoms.

The property that each formula of L is equivalent to one in which no nestings of the operator occur (2.11), supports to consider an *S5*-model to be a collection of 'points' (worlds) that can have certain properties (summarized by the atomic formulas that are true in each world), the language L being sufficiently expressive to sum up the quantitative distribution of those properties over the model. Alternatively, identifying worlds with truth assignments to primitive propositions, as is usual in standard *S5*-models, we can view a Gr(S5)-model as a multi-set of truth assignments rather than a set of these as in standard, ungraded modal logic. A special case, of course, is that situations (= truth assignments) occur only once in a description. We shall refer to these models as *simple* (referring to the original Latin meaning of this word). Note that in simple models it is still sensible to use graded modalities, since an assertion (even a primitive proposition) may nevertheless hold in more than one situation, as e.g. p in the situations {p is true, q is false} and {p is true, q is true}.

To be more specific, let us consider a simple example. Suppose we are given that the agent knows $(p \vee q)$ and also $(p \vee r)$. Since q and r are 'independent' propositional atoms, we try to formalise our intuition that the agent has more confidence in p than in q (or r). Given the three propositional atoms, the agent will consider five of the eight (a priori) possible worlds: the worlds in which $(\neg p \wedge (\neg q \vee \neg r))$ is true, left out. Thus, assuming that we have a simple model in the sense above, we get $(M!_5T \wedge M!_4p \wedge M!_3q \wedge M!_3r)$, indicating that indeed, p is the 'most frequent' atom. (This is perhaps more appealing when interpreting the premises as $(\neg q \rightarrow p)$ and $(\neg r \rightarrow p)$, expressing that there are two (independent) reasons for p.)

Michael Freund has proposed a formal system for defaults, in which the number of worlds refuting some default is important when imposing an order on such defaults. In Freund's words: "… if we have to choose between two assertions of Δ that are in conflict, our natural move is to drop the one that is violated by the greatest number of worlds…" (Cf. Fr93]). Graded modalities provide a tool to explicitly reason with such numerical values. However, in Freund's general approach, the worlds themselves may have attached weights to them, so that a full treatment seems to be out of scope here; in our set up, all worlds would have the same weight (although generalizing this to arbitrary weights seems to be feasible).

The following example is well known in the literature on probabilistic reasoning ([Pe88]) and on non-monotonic reasoning ([Gi87]) where it is called the lottery paradox. It deals with the situation of a lottery with n tickets, numbered 1 … n. Let w_i denote 'ticket i will be the winning ticket' $(1 \leq i \leq n)$. Many default theories (cf. [Gi87]) allow one to obtain the defeasible conclusion $\neg w_i$ for each $i \leq n$, using a default rule expressing "if you can assume that $\neg w_i$, conclude $\neg w_i$". In particular, one derives $(\neg w_1 \wedge \ldots \neg w_n)$, raising the question why we call the happening a lottery, if we can derive on forehand that no ticket will (probably) win.

In our graded language, we would model the situation as follows, using the premises P1-P2:

P1 $K_0\neg(w_i \wedge w_j)$ $(i \neq j)$ — no two tickets will win simultaneously

P2 $M!_nT \wedge M!_1w_i$ $(i \leq n)$ — of all n possibilities, there is one in which ticket i wins

From these premises, one safely deduces that $K_0(w_1 \vee w_2 \vee \ldots \vee w_n)$, and even $K_0(w_1 \nabla w_2 \nabla \ldots \nabla w_n)$ (with ∇ standing for exclusive or) expressing that exactly one of the tickets will win. Moreover, one deduces $K_1\neg w_i$, expressing, that, except for (at most one) possibility, ticket i will not win. One should compare this with ungraded modal approaches, in which it is possible to express P1 together with the fact that there are *at least* n possibilities (the latter is done by adding $[M(w_1 \wedge \neg w_2 \wedge \neg w_3 \wedge \ldots \wedge \neg w_n) \wedge M(\neg w_1 \wedge w_2 \wedge \neg w_3 \wedge \ldots \wedge \neg w_n) \wedge \ldots \wedge M(\neg w_1 \wedge \neg w_2 \wedge \neg w_3 \wedge \ldots \wedge w_n)]$ but in which there is no way to guarantee that there are *at most* n of such worlds (adding copies to the 'intended model' is never excluded).

In the example of the introduction, the number of worlds (sources) was fixed. This gives rise to considering $Gr_k(S5)$, with fixed $k \in \mathbb{N}$, which is obtained from Gr(S5) by adding $M!_k\top$ to it. Let

$k^\wedge = \min\{m \in \mathbb{N} \mid m > \frac{1}{2}k\}$.

Using a preference modality (*use belief* in the sense of Perlis [Pe86]) expressed by operator P as in [MH91], we may express the democratic principle of infallible computers in $Gr_k(S5)$, with k denoting the number of computers, as $P\varphi \leftrightarrow K_{k^\wedge}\varphi$, that is, φ is preferred (is a practical/working/use belief) iff it is true in more than the half of all sources. Note that there is no logical omniscience in this respect, in a way resembling the local reasoning approach of [FH88].

However, note that here, P is not a normal modality as it *is* in [MH91], since, as follows from our discussion about the K-axiom, $P(\varphi \rightarrow \psi) \rightarrow (P\varphi \rightarrow P\psi)$ is not valid. To illustrate this, consider the case of an airplane with three sensors w_1, w_2 and w_3 in which "it is foggy" (φ) is true according to w_1 and w_2 (and not according to w_3), and "permission to take off" (ψ) according to sensor w_1 only. Then we have that both $P(\varphi \rightarrow \psi)$ (since $\varphi \rightarrow \psi$ is true in w_1 and w_3) and $P\varphi$ (since φ is true in w_1 and w_2), thus both φ and $(\varphi \rightarrow \psi)$ are working beliefs, without the conclusion "permission to take off" (ψ) being one.

One might contrast this with the situation where *rules* are added to the system (in the form of (certain) knowledge: cf. [MH91]). For instance, in the above example, $K_0(\varphi \rightarrow \neg\psi)$ might be a rule (it is *known* by the decision support system, independently of the information supplied by the sources, that fog is sufficient to deny permission to leave). If in addition, $P\varphi$ would be the case (the systems supposes φ based on the information of its sources), it would take as a working belief $\neg\psi$, i.e. there is no permission to fly! (This follows directly from axiom A1: $K_0(\varphi \rightarrow \neg\psi) \rightarrow (K_{k^\wedge}\varphi \rightarrow K_{k^\wedge}\neg\psi)$, i.e., $K_0(\varphi \rightarrow \neg\psi) \rightarrow (P\varphi \rightarrow P\neg\psi)$.)

Recall that $Gr_k(S5) = Gr(S5) + M!_k\top$. Using Proposition 2.2 we see that for any φ, $M!_0\varphi \nabla M!_1\varphi \nabla \ldots \nabla M!_k\varphi$ is derivable in $Gr_k(S5)$. Here, a formula of the form $M!_m\varphi$ is rather informative, since we know the *relative number* of occurrences of φ. This is close to adding *probabilities* to (modal) logic. In the literature, there have been several attempts to do so (Cf. [FH89, HR87]). In order to avoid the problem of losing *compactness* (cf. [Ho92b]) it was suggested to allow only for a *finite* number of probabilities ([FA89, Ho92b]). We will give the main idea now.

Let us denote the language with graded modalities with L_C (a language for counting). The language L_P (a language for probabilities) is like L_C, but instead the operators M_n, we now add operators $P^{>}_{r}$, for each $r \in [0,1]$. The intended meaning of $P^{>}_{r}\varphi$ is, that the probability of φ is greater than r. Two interpret this language, we assume to have a finite set F, such that $\{0, 1\} \subseteq F \subseteq [0,1]$ and $\forall r,s(r \in F \wedge s \in F \wedge r + s \leq 1) \Rightarrow (r + s \in F)$. Now, a Probability Kripke model $\mathcal{M}$ over F is a tuple $<W, \pi, R, P_F>$, where W is a set of worlds, π as before, R is a serial relation on W ($\forall w \exists v Rwv$) and P_F: $W \times \mathcal{P}(W) \rightarrow F$ is a function from the powerset of W to F, for each $w \in W$, satisfying:

- $X \cap Y = \varnothing \Rightarrow P_F(w, X \cup Y) = P_F(w,X) + P_F(w,Y)$ $\qquad X, Y \in \mathcal{P}(W)$
- $P_F(w,\{v \mid Rwv\}) = 1$

The truth definition for L_P formulas is obtained straightforwardly, with the modal case

$$(\mathcal{M},w) \models P^{>}_{r}\varphi \text{ iff } P_F(w,\{v \mid Rwv \text{ and } (\mathcal{M},v) \models \varphi\}) > r$$

We denote the class of all these models by $\mathcal{PK}_F$. In [Ho92b] a logic P_FD is given, such that we have $P_FD \vdash \varphi \Leftrightarrow \mathcal{PK}_F \models \varphi$.

Let $Gr_k(K)$ be $Gr(K) + M!_kT$. This class is obviously sound and complete with respect to $\mathcal{K}_k$, the class of Kripke models in which each world has exactly k successors. Moreover, let

$$Fk = \{0 = \frac{0}{k}, \frac{1}{k}, \frac{2}{k}, \ldots, \frac{k}{k} = 1\}$$

Given a model $\mathcal{M}_c = <W, \pi, R> \in \mathcal{K}_k$, we straightforwardly associate a model $\mathcal{M}_p = <W, \pi, R, P_{Fk}> \in \mathcal{PK}_{Fk}$ with it by stipulating that $P_{Fk}(w,v) = \frac{1}{k}$ if Rwv holds, and $P_{Fk}(w,v) = 0$ otherwise. The relation between the classes of valid formulas of those two models is as follows. Let τ: $L_C \rightarrow L_P$ be a translation from graded to probabilistic formulas, distributing over the logical connectives and such that $\tau(M_n\varphi) = \bot$ if $n \geq k$ and $\tau(M_n\varphi) = P^{>}_{r}\tau(\varphi)$, with $r = \frac{n}{k}$, if $n < k$. We claim that for all $\varphi \in L_C$, $(\mathcal{M}_c,w) \models \varphi$ iff $(\mathcal{M}_p,w) \models \tau(\varphi)$. Conversely, we define σ: $L_P \rightarrow L_C$ as a translation that distributes over the connectives and for which moreover $\sigma(P^{>}_{r}\varphi) = M_n\sigma(\varphi)$, where $n = \max\{m \mid \frac{m}{k} \leq r\}$. In this case, we have for all $\varphi \in L_P$, $(\mathcal{M}_p,w) \models \varphi$ iff $(\mathcal{M}_c,w) \models \sigma(\varphi)$. Note that, although in general $\tau(\sigma(\varphi)) \neq \varphi$ and $\sigma(\tau(\varphi)) \neq \varphi$, we do have $\mathcal{M}_p \models \varphi \leftrightarrow \tau(\sigma(\varphi))$ and $\mathcal{M}_c \models \varphi \leftrightarrow \sigma(\tau(\varphi))$.

The above observation immediately ties up the ways to reason about relative occurrences with ways to reason about probabilities, at least for the minimal graded modal logic $Gr_k(K)$. However, we argued that the natural graded system to reason epistemically with (or about) numbers, is Gr(S5). So what is the counterpart, in the sense of the paragraph above, of $\mathcal{S5}_k$, the semantic class of $Gr_k(S5)$? Models for $Gr_k(S5)$ are $\mathcal{M} = \langle W, \pi, R \rangle$ in which R is an equivalence relation, and for which $\forall w(|\{v \mid Rwv\}| = k)$. But, by an argument using generated models (which still holds for the graded cases, cf. [Ho92a]), we can also conceive them as models in which R is *universal* ($\forall wvRwv$) and for which $\forall w(|\{v \mid Rwv\}| = k)$. In such a model, there is no need to explicitly refer to the relation R. So let $\mathcal{U}_k = \{\mathcal{M} = \langle W, \pi \rangle \mid |W| = k\}$, where $(\mathcal{M},w) \models M_n\varphi$ iff $|\{v \in W \mid (\mathcal{M},v) \models \varphi\}| > n$. Then we have that $Gr_k(S5) \vdash \varphi \Leftrightarrow \mathcal{U}_k \models \varphi$.[1]

On the side of probabilistic models , we add the following constraint in order to compare them with $\mathcal{U}_k$. The class $\mathcal{PKU}_k$ is a subclass of probabilistic $\mathcal{PK}$-models $\mathcal{M} = \langle W, \pi, R, P_{Fk} \rangle$ in which P_{Fk} is such that $\forall wv(P_{Fk}(w,v) = \frac{1}{k})$. In particular, we have a kind of reflexivity: $P_{Fk}(w,w) = \frac{1}{k}$. We claim that we have, for all $\varphi \in L_C$: $\mathcal{U}_k \models \varphi \Leftrightarrow \mathcal{PK}_k \models \tau(\varphi)$ and for all $\varphi \in L_P$: $\mathcal{U}_k \models \sigma(\varphi) \Leftrightarrow \mathcal{PK}_k \models \varphi$.

We now proceed by mentioning the use of graded operators to express the 'numerical syllogisms' as introduced in [AP88]. In the following, the left hand side is our translation of the numerical syllogisms on the right hand side.

1	$M!_7d$	exactly 7 days of the week are known
2	$M!_5(w \wedge d)$	I know 5 of them to be working days
3	$\underline{M_3(s \wedge d)}$	<u>at least 4 days are shopping days</u>
4	$\therefore M_1(w \wedge s)$	∴ I know at least 2 days to go working and shopping.

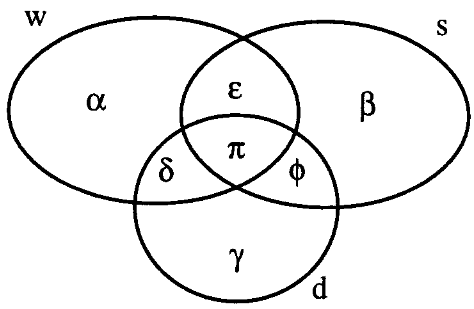
To prove such a conclusion formally, it turns out to be worthwhile to split up the set of formulas (the formulas in d, w and s), in to a set of partitions (cf. Figure 2).

The formal derivation now reads as follows (note $\Gamma = \{\alpha, \beta, \gamma, \delta, \varepsilon, \phi, \pi\}$):

(1)	$M!_7(\gamma \vee \delta \vee \phi \vee \pi)$	translation of 1 (cf. Figure 2)
(2)	$M!_5(\delta \vee \pi)$	translation of 2
(3)	$M_3(\phi \vee \pi)$	translation of 3
(4)	$K_0\neg((\phi \vee \pi) \wedge (\gamma \vee \delta))$	definition of Γ
(5)	$\neg M_7(\gamma \vee \delta \vee \phi \vee \pi)$	(1), def. M!

[1]We will not prove this here, but the result is easily obtained by combining results of [Fi72] and [HR91].

(6)	$\neg M_3(\gamma \vee \delta)$	2.2.(viii), (3), (4), (5)
(7)	$(K_0(\delta \rightarrow (\gamma \vee \delta)) \rightarrow (M_3\delta \rightarrow M_3(\gamma \vee \delta))) \wedge K_0(\delta \rightarrow (\gamma \vee \delta))$	A1' and A0, R1
(8)	$\neg M_3\delta$	A0, (6), (7)
(9)	$M!_0\delta \vee M!_1\delta \vee M!_2\delta$	(8), 2.2.(iii)
(10)	$\neg M_1\pi \rightarrow (M!_0\pi \vee M!_1\pi)$	2.2.(iii)
(11)	$K_0\neg(\delta \wedge \pi) \rightarrow [((M!_0\delta \vee M!_1\delta \vee M!_2\delta) \wedge (M!_0\pi \vee M!_1\pi)) \rightarrow$ $(M!_0(\delta \vee \pi) \vee M!_1(\delta \vee \pi) \vee M!_2(\delta \vee \pi)]$	A0, A3
(12)	$\neg M_1\pi \rightarrow (M!_0(\delta \vee \pi) \vee M!_1(\delta \vee \pi) \vee M!_2(\delta \vee \pi)$	(9)(10)(11), $\vdash K_0\neg(\delta \wedge \pi)$
(13)	$[(M!_0(\delta \vee \pi) \vee M!_1(\delta \vee \pi) \vee M!_2(\delta \vee \pi)] \rightarrow \neg M_2(\delta \vee \pi)$	A0,def. M!
(14)	$(M!_5(\delta \vee \pi) \rightarrow M_4(\delta \vee \pi)) \wedge (M_4(\delta \vee \pi) \rightarrow M_2(\delta \vee \pi))$	def. M!, A2 twice
(15)	$(\neg M_2(\delta \vee \pi) \wedge M!_5(\delta \vee \pi)) \rightarrow \bot$	A0, (14)
(16)	$(\neg M_1\pi \wedge M!_5(\delta \vee \pi)) \rightarrow \bot$	A0, (12), (13), (15)
(17)	$M_1\pi$	(2), (16)
(18)	$M_1(\pi \vee \varepsilon)$	(17), A1'



For example, it is understood that $w \leftrightarrow (\alpha \vee \delta \vee \varepsilon \vee \pi)$. If $\Gamma = \{\alpha, \beta, \gamma, \delta, \varepsilon, \phi, \pi\}$, then, for all different x1, ..., xk$\in \Gamma$ $(2 \leq k \leq 7)$:
(*) $\vdash \neg (x1 \wedge \ldots \wedge xk)$

Figure 2

We round off this Section by mentioning a link between the graded formalisms presented here and so called *terminological* or *concept languages*, used for knowledge representation (cf. [DLN91] for these languages, and [HoR92] for a deeper analysis of the connection with graded modal systems). Such languages provide a means for expressing knowledge about hierarchies of sets of objects with common properties. Expressions in such languages are built up using *concepts* and *roles*. Compound expressions are then made using a number of constructs.

Typical examples of such constructs are intersection, complement and restricted quantification, yielding examples like the concept 'mathematicians whose pupils are all clever' ($m \sqcap$ ALL p c, with the modal counterpart $(m \wedge [R_p]c)$, where $[R_p]$ is the necessity operator for the relation R_p). Many of such concept languages also allow for number restriction, as in the concept 'mathematicians who have at least 4 clever pupils ($m \sqcap \geq 4$ p c). Obviously, here the graded modalities come into play: the latter concept would translate into the formula $(m \wedge \langle R_p \rangle_3 c)$, where $<R_p>$ is the dual of $[R_p]$. In [HoR92], one can find several results on relating several concept languages in the hierarchy of concept languages to (some fragment of) some graded modal logic.

5 Conclusion

We have argued that extending the modal language with graded modalities (taking into account the number of accessible worlds) gives some interesting options for epistemic logic. We provided some examples of how this new language can be used in an epistemic context. Particularly, we indicated how these operators can be used in the context of a fixed number of sources. It thus provides us with a framework for *reasoning with exceptions.*

We think the graded modalities are especially useful in 'laboratory-like situations', where explicit bounds are prescribed. Areas of application that may be worthwhile may therefore typically be found in situations where numbers of counter-examples have a clear evidence and meaning. Typical examples (that have not been worked out by us, yet) may thus be found in 'laboratory situations' like (reasoning about) *a voting* or in a *legal context* (where for instance a petition is granted when at least *n* requirements are met) or more generally, intelligent databases of which the quantities of the data matters (cf. [Ho92c], for several examples).

We see two lines of future work. Firstly, we may transfer some standard questions from 'standard' epistemic logic to the graded language. For instance, it might be interesting to study the introspection properties more systematically, like was done e.g. in [Ho91b]. Secondly, we think that several of our proposals have natural generalisations. For instance, where the P-operator models the notion of 'more than-a-half', we could have such operators P_n for 'more than-an-n-th'.

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Modalities for Reasoning about Knowledge and Uncertainties‡

Wiebe van der Hoek
John - Jules Meyer #

Department of Mathematics and Computer Science
University of Utrecht
P.O. BOX 80.089, 3508 TB Utrecht, The Netherlands

0 Introduction

In this chapter we will discuss other ways (than using partial logic explicitly) to deal with the 'gap' in truth values. Recall from the chapter by T.Langholm that, apart from the possibility to represent this gap in a partial model, an alternative way to deal with it is by considering a *class* of *complete* models (or, as T.Langholm call it, *complete possible scenarios*) It is important to note, that if we like to think about proposition r as unknown, this does not mean that one thinks about r as having no truth value; it rather happens to be the case that this value is *unknown*. However, until now we had no way in our object language to distinguish between those possibilities; until now it was impossible to express that r is unknown, although of course it is either true or false.

In this chapter, we will deal with ways to express explicitly that something is known or not known. Here, we do this by adding simply a (modal) operator K to the language, with intended meaning of $K\varphi$ that φ is known. We are then allowed to re-embrace classical tautologies like $(Kp \vee \neg Kp)$, without trivialising knowledge: i.e. we do not have $(Kp \vee K\neg p)$.

With *modalities*, we refer to their formal definition and use in the area of (philosophical) logic. Modal logic is the logic of 'must be' and 'may be' (according to one of the standard introductions ([HC68]) to this field); the logic of *necessity* and *possibility*, (as another standard introduction ([Ch80]) puts it). The 'standard', or 'neutral' symbols for necessity and possibility are '$\Box$' ("box") and '$\Diamond$' ("diamond"), respectively. They are considered 'dual' in the following sense; $\Box p \equiv \neg\Diamond\neg p$, i.e. "necessarily p" is equivalent to "not possibly not p".

This is indeed the standard, or 'philosophical' (cf. [Ga82]) notion of modality, which gives us a way to distinguish between facts that are 'accidentally' true—like "it is raining" (r)—and those that are necessarily true—like "necessarily, all events have a cause" ($\Box c$). This standard modality has been studied since Aristotle, and was fully recognised by Kant in his 'Kritik der

‡ This work was partially supported by the project DRUMS which is funded by a grant from the Commission of the European Communities under ESPRIT III-program, Basic Research Project 6156, and also by the Free University of Amsterdam.
# Also at the University of Nijmegen, Dept. of Math. & Comp. Sc., Toernooiveld, 6525 ED Nijmegen.

Reinen Vernunft'; however, the subject of modalities was rigorously expelled from the field of formal logic by Frege, using the argument that when we say that a proposition is necessarily true, we only give an impression of the *reasons* for our judgement. And, according to Frege, it is only the *content* of a judgement that is logically relevant.

As is clearly exposed in [Ga82], the 'philosophical' modalities were re-introduced through the backdoor to cope with the problems that several logicians had with the fact that, around 1925, the material implication was declared to be *the* implication for formal logic. Logicians (but also logicians-to-be, as may be confirmed by everyone who has taught an introductory course in logic) have had problems with the paradoxes of this material implication which come to surface in tautologies like $(p \rightarrow q) \vee (q \rightarrow p)$, ("of any two assertions, at least one implies the other"); $p \rightarrow (q \rightarrow p)$ ("true facts are implied by anything"); $\neg q \rightarrow (q \rightarrow p)$ ("false facts imply anything"). As we know, the material implication $(p \rightarrow q)$ is nothing else but $\neg(p \wedge \neg q)$: it is not the case that p and $\neg q$ happen to be true simultaneously. Many attempts to strengthen the material implication '$\rightarrow$' to a 'strong implication' (say '$\twoheadrightarrow$') follow the idea that $(p \twoheadrightarrow q)$ should not just mean that $(p \wedge \neg q)$ *happens* to be untrue, but rather that it is *impossible* for p and $\neg q$ to be true at the same time: i.e. $(p \twoheadrightarrow q) \equiv \neg\Diamond(p \wedge \neg q)$ or, equivalently: $(p \twoheadrightarrow q) \equiv \Box(p \rightarrow q)$. Thus, we return to our classical necessity operator to model "p (strongly) implies q" by "necessarily, p implies q".

One of the most appealing features of modal logic is, we think, its very natural semantics. In previous chapters, hints were already made at modelling the 'truth gap' as follows. Suppose we know p, we know also $\neg q$, and we are not sure about r. One way to model this is by 'splitting' our model of the world into two alternatives: one in which $(p \wedge \neg q \wedge r)$ is true, and one in which $(p \wedge \neg q \wedge \neg r)$ holds. If we are also uncertain about s, we again split our two alternatives; however, if we *do* know that $(r \rightarrow s)$ holds, of the four a priori alternatives, we discard $(p \wedge \neg q \wedge r \wedge \neg s)$. In modal logics, such alternatives for the world are called *possible worlds* (and a collection of such worlds is a *Kripke structure*), and, as our example suggests, to know φ then boils down to φ being true in all possible worlds! So, what happens is, that we simply collect all the 'possible scenarios' for a formula in one structure, and enrich our language to allow us to explicitly reason about them.

Recall the observation of T. Langholm that '... the interest in models was influenced strongly by a wish to capture precisely a notion of *entailment*: φ entails ψ if ψ is never false whenever φ is true'. This should be compared with the argument to re-introduce modalities in order to bypass the problems with implication, and the suggestion to replace it by $\Box(p \rightarrow q)$. Recalling that the latter formula is true in a Kripke structure exactly iff $(p \rightarrow q)$ is true in all (possible) worlds, we recognise two rather similar attempts to capture a notion of entailment by taking into account a collection of models (or worlds).

When the modal operator '$\Box$' is interpreted in the standard way, i.e. as 'necessity', we say that we are dealing with *alethic* modalities (the Greek word *alètheia* means *truth)*. It is interesting to observe that in such diverse areas as linguistics, philosophy, computer science and artificial intelligence, it was recognised that there exists a great variety of modalities (differing from the alethic ones, each in its own way). To give some examples, apart from the alethic interpretation of '$\Box$' *(must be)* we may give it others, like temporal *(always);* deontic *(ought to be);* dynamic *(true after every execution of an action);* arithmetical *(provably);* default *(normally);* epistemic *(known)* or doxastic *(believed).*

In this chapter, we show that modal logic is useful when studying notions like knowledge, belief, assuming by default, supposing, considering ... (for simplicity, we will call all such notions 'epistemic', here). In Section 1, we will give the formal definitions for the basic modal systems, and enrich them with operators to model knowledge and belief. Then, in Section 2, we will deviate from those classical systems, and introduce modal operators to deal with uncertainty. The ways we deal with those uncertainties can be characterised as 'quantitative'. In Section 3 we then pay attention to some 'qualitative' modal operators; in Section 4 we round off with some conclusions.

1 Modal Epistemic Logic

1.1. Modal Systems

We start out by defining our modal language. In general, given a set P of propositional atoms and O of operators, such a language is the smallest set $\mathsf{L}(\mathsf{P},\mathsf{O}) \supseteq \mathsf{P}$ which is closed both under infix attachment of $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$, and prefix placing of $\neg$ and operators $O \in \mathsf{O}$. If $|\mathsf{O}| > 1$, we say to have a multi-modal logic. To present the minimal modal logic K, we will assume that $\mathsf{O} = \{\Box\}$. In the introductory Section, we already mentioned the variety in possible interpretations for this operator '$\Box$'. We now present a uniform mathematical interpretation for it.

A Kripke model $\mathcal{M}$ for a modal language L with one modal operator $\Box$ is a tuple $\langle W,R,\pi\rangle$, where W is a non-empty set of worlds, $R \subseteq W \times W$ a binary relation, and π: $W \rightarrow \mathsf{P} \rightarrow \{\text{true}, \text{false}\}$ a truth assignment to the propositional atoms for each world $w \in W$. The truth definition (TD) for $\varphi \in \mathsf{L}$ at w, written $(\mathcal{M},w) \models \varphi$, is defined inductively as follows:

$(\mathcal{M},w) \models p$ iff $\pi(w)(p) = \text{true}$
$(\mathcal{M},w) \models \psi \wedge \chi$ iff $(\mathcal{M},w) \models \psi$ and $(\mathcal{M},w) \models \chi$
$(\mathcal{M},w) \models \neg\psi$ iff not $(\mathcal{M},w) \models \psi$
TD($\Box$) $(\mathcal{M},w) \models \Box\psi$ iff for all v for which Rwv, $(\mathcal{M},v) \models \psi$.

We say that an operator that is defined like $\Box$ for R, is a *necessity operator for R.* For any modal operator $\Box$, we define $\Diamond = \neg\Box\neg$. $\Diamond$ is called the *dual of* $\Box$ or, alternatively, the *possibility operator for R.* Furthermore, we say that φ is *satisfiable* if φ is true at some world w in some model $\mathcal{M}$, φ is *true in model* $\mathcal{M}$ ($\mathcal{M} \models \varphi$) if $(\mathcal{M},w) \models \varphi$ for all worlds of $\mathcal{M}$, and, finally, φ is *valid* ($\models \varphi$) if it is true in all models. For any class $\mathcal{C}$ of models, we write $\models_{\mathcal{C}} \varphi$ if φ is true in all models in $\mathcal{C}$. The class of all Kripke models $\mathcal{M}$ is denoted by $\mathcal{K}$.

As is easily verified, we have ($\models \varphi \Rightarrow\ \models \Box\varphi$) (but not $\models \varphi \rightarrow \Box\varphi$!) and ($\models \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$). The latter property formalises a kind of Modus Ponens over all possible worlds. Moreover, we observe that, for any propositional tautology φ (this may be read in a broad sense, including formulas like $(\Box p \vee \neg\Box p)$) we have $\models \varphi$.

The above notion of validity can be axiomatised in the logic K as follows. Since modal logic is just (another) logic, we first add a propositional basis to our system:

A1) Any axiomatisation of the propositional calculus
R1) $\vdash \varphi, \vdash \varphi \rightarrow \psi \Rightarrow \vdash \psi$.

On top of that, the following modal axioms A2 and rule R2 for '$\Box$' are added:

A2) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ Distribution
R2) $\vdash \varphi \Rightarrow \vdash \Box\varphi$ Necessitation

An operator that satisfies A2 and R2 is called a *modal* operator. If a modal logic L satisfies A1 and R1, together with A2 and R2 for all operators in some set O, we say that L is *normal over* O. The following rule is derivable in K and all its extensions we will deal with in this chapter:

Lemma 1.1. Let $[^{\alpha}/_{\beta}]\varphi$ be any formula, which arises from φ by substituting any occurrence(s) of β in φ by α. Then the following rule of substitution Sub is derivable in K and its extensions:

Sub) $\vdash \alpha \leftrightarrow \beta \Rightarrow \vdash \varphi \leftrightarrow [^{\alpha}/_{\beta}]\varphi$

This substitution rule should be clearly distinguished from $\vdash (\alpha \leftrightarrow \beta) \rightarrow (\varphi \leftrightarrow [^{\alpha}/_{\beta}]\varphi)$, which is not true! We are tempted to say that modal logic is so to speak *invented* to avoid the latter property: it is exactly what extricates modal logic from being *extensional*: indeed, if propositions p and q have the same truth value, we do not want to conclude that *knowing* p is equivalent to *knowing* q, or that the claim that *always* p is the same as *always* q (take for p: Clinton is president; q: it is raining). Modal logic replaces this principle of extensionality by that of *intensionality*, expressed by the lemma above: it guarantees that we are allowed to substitute formulas that are equivalent in *all contexts*. (Note that $\vdash (\alpha \leftrightarrow \beta)$ implies $\vdash \Box(\alpha \leftrightarrow \beta)$.)

Comparing our observations following the truth-definition with the axioms and rules of K, we immediately see that K is sound for $\mathcal{K}$ but in fact we even have completeness:

Proposition 1.2. For all $\varphi \in$ L, $\vdash_K \varphi \Leftrightarrow \models_{\mathcal{K}} \varphi$.

1.2. Knowledge and Belief

It is customary to use the symbols 'K' and 'B' if the intended interpretation of the box is epistemic or doxastic, respectively. As will be clear by now, the modal language by itself already provides us with a powerful mechanism to reason about knowledge and belief. For instance, there is no restriction on the *scope* of the operators, enabling us to write down arbitrary *nested* formulas—a feature that is provided by the bare modal language, but the epistemic use of the operators also gives a natural justification to allow such nestings. Theoretically, this feature naturally invites one to analyse (the suitability of) a variety of properties (like $Kp \rightarrow KBp$, or $\neg Bp \rightarrow K\neg Bp$—an exciting venture, for which some buoys are set out in [Ho92a]).

Formally, in this subsection we consider languages L(P,O), where $O \subseteq \{K, B\}$. Since here we are not interested in the (explicit or distributed) knowledge in a group of agents, this suffices for our purposes. Semantically, the operators K and B receive the same truth definition as □: they are both interpreted as a necessity operator with respect to two accessibility relations R and S, respectively. Thus, the relation R gets the following interpretation when it has a knowledge operator K as its necessity operator: Rxy if the agent considers world y as an epistemic alternative for world x. Obviously, this relation should then be an *equivalence*. This time, we will present the formal semantics once we have given the logical systems for knowledge and belief.

As far as axiomatisations are concerned, we define the following systems:

Definition 1.3. The modal system S5 is normal over {K} and has, on top of that, the following axioms:

A3) $K\varphi \rightarrow \varphi$

A4) $K\varphi \rightarrow KK\varphi$

A5) $\neg K\varphi \rightarrow K\neg K\varphi$

Next, KD45 is normal over {B}, has the axioms A4, and A5 (with 'B' instead of 'K') and the axiom D:

D) $\neg B\bot$

Finally, KB is the logic one obtains by combining the S5-axioms for K, the KD45-axioms for B and the following interaction axioms:

A6) $K\varphi \rightarrow B\varphi$

A7) $B\varphi \rightarrow KB\varphi$.

In words: knowledge is understood to be *veridical* (A3), and satisfies *positive* as well as *negative introspection* (A4 and A5, respectively). Belief, on its turn, is not assumed to be veridical, but satisfies the weaker property that beliefs are consistent (D). Finally, knowledge is to be understood stronger than belief (A6) and one is *conscious* about one's beliefs (A7).

As far as completeness is concerned, the crucial thing to notice is, that when we add properties for K or B (on top of A2), it has consequences for the accessibility relations R and S, respectively. For instance, let us consider the axiom A3, $K\varphi \rightarrow \varphi$. If this formula has to be valid on our Kripke models, it says that for all φ, if φ is true in all accessible worlds y (for which Rxy), it should be true in world x itself. This is guaranteed if we know that R is reflexive, since then we have Rxx. We say that axiom A3 *corresponds* to reflexivity. We leave it to the reader to check that in the same way, A4 corresponds to transitivity ($\forall xyz(Rxy \wedge Ryz) \rightarrow Rxz)$), and A5 to Euclidicity ($\forall xyz(Rxy \wedge Rxz) \rightarrow Ryz$). Furthermore, the scheme D corresponds to seriality ($\forall x\exists y Sxy$). Finally, A6 corresponds with $S \subseteq R$ and A7 with ($\forall xyz(Rxy \wedge Syz) \rightarrow Sxz$)). For a systematic treatment of such correspondences, the reader is referred to [Ho92a].

Definition 1.4. Let $\mathcal{KD}45$ be the class of Kripke models $\mathcal{M} = \langle W,R,\pi\rangle$ such that R is serial, transitive and euclidean. $\mathcal{S}5$ is the class of models in which R is an equivalence. Finally, $\mathcal{KB}$ is the class of models $\langle\mathcal{M},R,S,\pi\rangle$ such that $\langle\mathcal{M},R,\pi\rangle \in \mathcal{S}5$; $\langle\mathcal{M},S,\pi\rangle \in \mathcal{KD}45$; $S \subseteq R$ and, for all x, y and z $\in$ W, ((Rxy $\wedge$ Syz) $\rightarrow$ Sxz)

Proposition 1.5. For all $\varphi \in L(P,\{K\})$, $\vdash_{S5} \varphi \Leftrightarrow \models_{\mathcal{S}5} \varphi$; for all $\varphi \in L(P,\{B\}) \vdash_{KD45} \varphi \Leftrightarrow \models_{\mathcal{KD}45} \varphi$; for all $\varphi \in L(P,\{K, B\}) \vdash_{KB} \varphi \Leftrightarrow \models_{\mathcal{KB}} \varphi$.

Remark 1.6. The proof of Proposition 1.5 is by now folklore in the modal literature (Cf. [Ch80, HC68]), and most often done using a *Henkin-style* procedure. For future reference, we sketch the idea behind it. In such a proof, a (so-called *canonical*) model $\mathcal{M}^c$ is build for a consistent formula φ as follows. The worlds of such a model are maximal consistent sets Γ. Propositional formulas are governed by the condition $\pi^c(\Gamma)(p) = p \in \Gamma$. Then, a relation R^c is defined using the clause ($R^c\Gamma\Delta \Leftrightarrow$ for all $\Box\delta \in \Gamma$, we have $\sigma \in \Delta$). Then, to round off the completeness proof, one has to establish the following *coincidence* or *truth lemma:*

$$\text{For all formulas } \gamma,\ (\mathcal{M}^c,\Gamma) \models \gamma \Leftrightarrow \gamma \in \Gamma.$$

This is sufficient to prove satisfiability of our given consistent φ: its consistency implies that $\varphi \in \Phi$ for some maximal consistent set, and thus φ is satisfied in $\mathcal{M}^c$ at world Φ. It is important to note that the definition of R^c is the same for several modal logics: the axioms of the logic (that determine the behaviour of maximal consistent sets) steer the properties of this accessibility relation. For instance, when A3 is an axiom of the logic, R^c will be reflexive: if $\Box\gamma \in \Gamma$ then (because of A3) also Γ, so $R^c\Gamma\Gamma$.

It turns out that these systems are rather well-behaved. For instance, from [Ho92a] we learn that in fact nesting of the modal operators are, logically speaking, superfluous:

Theorem 1.7. [Ho92a]. Let X, Y $\in$ {K, B, $\neg$} and $\overline{X}$ be a sequence of X's. Let φ be any KB-formula. Then $\vdash_{KB} \overline{X}Y\varphi \leftrightarrow (\neg)Y\varphi$, where the '$\neg$' is present if the number of '$\neg$' in $\overline{X}$ is odd.

Moreover, as far as the system S5 is concerned, we have the following.
Proposition 1.8. Let $\mathcal{U}$ be the class of Kripke models $\mathcal{M} = \langle W,R,\pi\rangle$ in which R is universal (i.e., for all x, y $\in$ W, Rxy). Then $\vdash_{S5} \varphi \Leftrightarrow\ \models_{\mathcal{U}} \varphi$.

The latter proposition guarantees that we can think of S5-models as just a non-empty collection of complete propositional models. In particular, in such a collection $\mathcal{M}$ we then have $\mathcal{M} \models \varphi \Leftrightarrow \forall w\text{: } (\mathcal{M},w) \models \varphi \Leftrightarrow \forall w\text{: } (\mathcal{M},w) \models K\varphi \Leftrightarrow \exists w\text{: } (\mathcal{M},w) \models K\varphi$. As a consequence, we may assume that for each w and w' $\in$ W, $\pi(w) \neq \pi(w')$, allowing us to identify a world w with its truth-assignment $\pi(w)$. In such a case (i.e., when we may identify worlds with their truth-assignment), we call the model *normal.* In such a model, a universal relation can be conceived as characterising exactly what an agent knows, which then, on its turn, is generated by the propositional kernel of his knowledge. (We will briefly address this further in 1.3.) Since in a model $\mathcal{M} = \langle W,R,\pi\rangle \in \mathcal{U}$ the relation R is always W $\times$ W, we also write $\mathcal{M} = \langle W,\pi\rangle$ in such a case. In that case the truth definition for $K\varphi$ in w boils down to the fact that φ is true everywhere. Summarising, if $\mathcal{NORMAL} = \{\mathcal{M} = \langle W,\pi\rangle \mid W \text{ is normal}\}$ then we have $\vdash_{S5} \varphi \Leftrightarrow\ \models_{\mathcal{NORMAL}} \varphi$.

We end this subsection by mentioning some draw-backs of the epistemic notions developed so far. Recall from 1.1 that A2 and R2 together are equal to saying that '$\Box$' is a modal operator. Having an epistemic interpretation of '$\Box$' in mind, the question of whether these notions are captured adequately and realistically breaks to surface. There is indeed a problem with the notions so far, known as the problem of "*logical omniscience*". This problem pertains to a kind of too idealistic notion of knowledge and belief. Consider for instance the property that beliefs are closed under logical consequences. Especially for a notion of belief, which should be more fallible if human everyday beliefs are to be captured, this property is obviously not true. On the other hand, this property holds for the S5-notion of knowledge and the KD45-notion of belief. But there are also other (related) properties that are not very realistic, which nevertheless hold in S5 / KD45. We list some of them (for KD45-belief) below, including the one mentioned already:

(LO1)	$B\varphi \wedge B(\varphi \rightarrow \psi) \rightarrow B\psi$	(Closure under implication)
(LO2)	$\models \varphi \Rightarrow \models B\varphi$	(Belief of valid formulas)
(LO3)	$\models \varphi \rightarrow \psi \Rightarrow \models B\varphi \rightarrow B\psi$	(Closure under valid implication)
(LO4)	$\models \varphi \leftrightarrow \psi \Rightarrow \models B\varphi \leftrightarrow B\psi$	(Belief of equivalent formulas)
(LO5)	$(B\varphi \wedge B\psi) \rightarrow B(\varphi \wedge \psi)$	(Closure under conjunction)
(LO6)	$B\varphi \rightarrow B(\varphi \vee \psi)$	(Weakening of belief)
(LO7)	$B\varphi \rightarrow \neg B \neg \varphi$	(Consistency of beliefs)
(LO8)	$B(B\varphi \rightarrow \varphi)$	(Belief of having no false beliefs)
(LO9)	B **true**	(Believing truth)

We briefly mention two ways to get around this problem. One approach that certainly deserves mentioning in this volume, is by allowing the worlds to be partial *themselves* (cf. [Th92, Le84, Wa89]! In some sense, one thus obtains two ways of modelling ignorance: the modal and partial approach (for some motivation for combining them in one system, cf. the chapter of Jaspars and Thijsse in this volume). By choosing a partial logic without any tautologies (cf. [Th92]) one obviously avoids, for example, the properties LO2 and LO6. Moreover, by choosing a multi-valued logic in which p, $(p \rightarrow q) \not\models q$, one can avoid most of the problems mentioned above.

In the second approach, a notion of *awareness* is introduced. This is related to the denial of axiom A5: it is claimed that not knowing φ (for example when one is not 'aware' of φ) is not sufficient to conclude that one knows that φ is not known. A (syntactically defined) notion of awareness then can limit this kind of inferences. It also solves LO6 and LO2: if the agent is not aware of p, he will not believe $(p \vee \neg p)$. For more on this, see [Th92, FH88a, HM89].

1.3. Circumscribing Epistemic States

In this section we shall be concerned with the characterisation of the *epistemic state* of an agent, that is to say, a description of what the agent knows *and what he does not*. So we are interested in describing both the knowledge and ignorance of the agent. This is very much related to autoepistemic logic ([Mo85]) and the logic of "All I know" of [Le90], but here we follow a treatment of this subject by Halpern & Moses [HM84]. They proposed an elegant mathematical treatment of this phenomenon, using the notions of a *stable set* of epistemic formulas and **S5** Kripke models. In this section we shall briefly discuss some ingredients of their approach.

Definition 1.9. A set Σ of epistemic formulas is *stable* if it satisfies the following:

(St 1) all instances of propositional tautologies are elements of Σ;

(St 2) if $\varphi \in \Sigma$ and $\varphi \rightarrow \psi \in \Sigma$ then $\psi \in \Sigma$;

(St 3) $\varphi \in \Sigma \Leftrightarrow K\varphi \in \Sigma$
(St 4) $\varphi \notin \Sigma \Leftrightarrow \neg K\varphi \in \Sigma$
(St 5) Σ is propositionally consistent.

Here (St 5) means that from Σ one cannot derive a contradiction ($\perp$) by means of Propositional Logic (A1, R1) only. Keeping in mind that a stable set has to model an agent's knowledge set $\Sigma = \{\sigma \mid K\sigma \text{ holds}\}$, St 1 is related to the Necessitation rule of S5; St 2 to S5-axiom A2; St 3 to A3 and A4; St 4 to A3 and A5 and, finally St 5 is related to A3 again.

We now give some well-known properties of stable sets (cf. e.g. [Mo85]). By an objective formula we mean a formula in which there are no occurrences of the K-operator.

Proposition 1.10. A stable set of epistemic formulas is uniquely determined by the objective formulas it contains.

Proposition 1.11. Let Σ, Σ' be stable sets such that $\Sigma \subseteq \Sigma'$. Then $\Sigma = \Sigma'$.
PROOF. Suppose that Σ, Σ' are stable such that $\Sigma \subseteq \Sigma'$ and $\Sigma \neq \Sigma'$. Then there is a formula φ such that $\varphi \in \Sigma'$ and $\varphi \notin \Sigma$. By (St 3) and (St 4) it then holds that $K\varphi \in \Sigma'$ and $\neg K\varphi \in \Sigma$. Since $\Sigma \subseteq \Sigma'$ and $\Sigma \neq \Sigma'$, also $\neg K\varphi \in \Sigma'$, so that both $K\varphi$ and $\neg K\varphi$ are elements of Σ'. This contradicts (St 5).

Now, suppose that φ is a formula that describes all the facts that have been learnt by the agent. What then is the epistemic state of the agent if she "only knows φ"? Clearly, this epistemic state has to contain φ, but what else? The state must be *minimal* in some sense. However, it is not the minimal stable set with respect to *set-inclusion* that contains φ, since different stable sets are incomparable with respect to $\subseteq$, due to Proposition 1.11 above. However, by Proposition 1.10 a stable set Σ of epistemic formulas (which we identify with an epistemic state) is uniquely determined by the purely propositional formulas that it contains. We denote by Prop(Σ) the subset of Σ that exactly contains all purely propositional formulas. Now we may try as the "least" epistemic state containing φ the stable set that contains φ and for which the purely propositional part is the least (with respect to $\subseteq$).

Such a least stable set is not defined for every formula φ: consider for example $\varphi = Kp \vee Kq$. Every stable set Σ that contains φ, has to contain either p or q, for suppose $p \notin \Sigma$ and $q \notin \Sigma$. By St 4, we conclude that $\neg Kp$ and $\neg Kq \in \Sigma$; which contradicts St 5 and the fact that $\varphi \in \Sigma$. However, there are stable sets Σ_1, Σ_2 such that $\varphi, p \in \Sigma_1$, $q \notin \Sigma_1$, and $\varphi, q \in \Sigma_2$, $p \notin \Sigma_1$. Thus $\text{Prop}(\Sigma_1) \cap \text{Prop}(\Sigma_2)$ contains neither p nor q. So there is no stable set Σ that contains φ such that $\text{Prop}(\Sigma) \subseteq \text{Prop}(\Sigma_1)$ and $\text{Prop}(\Sigma) \subseteq \text{Prop}(\Sigma_2)$.

The example above is justified by our intuition that an agent who says that she *only knows* Kp $\vee$ Kq is, in some sense, not *honest:* the only reason to know that you know p or that you know q seems to be that you know at least one of p and q, which is stronger than knowing only Kp $\vee$ Kq. We call a formula φ for which it makes sense to say that you only know φ an *honest* formula:

Definition 1.12. A formula φ is *honest*$_S$ if there is a stable set Σ^φ that contains φ and such that for all stable sets Σ containing φ it holds that $Prop(\Sigma^{\varphi}) \subseteq Prop(\Sigma)$.

The intention is that Σ^{φ} denotes the stable set representing the epistemic state of the agent who "knows only φ" (if φ is honest). We can characterise this notion by means of *NORMAL S5*-Kripke models. If $\mathcal{M} = \langle S, \pi \rangle$ is a normal **S5**-Kripke model, then $K(\mathcal{M})$ is the set of facts that are known in $\mathcal{M}$: $K(\mathcal{M}) = \{\varphi \mid \mathcal{M} \models \varphi\} = \{\varphi \mid \mathcal{M} \models K\varphi\}$. $K(\mathcal{M})$ is called the *theory of* $\mathcal{M}$ or *knowledge in* $\mathcal{M}$. Moreover, if $\mathcal{M}_1 = \langle S_1, \pi_1 \rangle$ and $\mathcal{M}_2 = \langle S_2, \pi_2 \rangle$ are two such models, their union is defined as: $\mathcal{M}_1 \cup \mathcal{M}_2 = \langle W_1 \cup W_2, \pi_1 \cup \pi_2 \rangle$, where $(\pi_1 \cup \pi_2)(w) = \pi_1(w)$ if $w \in W_1$, and $= \pi_2(w)$, if $w \in W_2$ (since we identified worlds with their truth assignments, this is always possible). We define the subset relation by: $\mathcal{M}_1 \subseteq \mathcal{M}_2$ iff $W_1 \subseteq W_2$.

Lemma 1.13. $K(\mathcal{M})$ is a stable set.

Proposition 1.14 ([HM84]). Every stable set Σ of epistemic formulas determines an *S5* Kripke model $\mathcal{M}_\Sigma$ for which it holds that $\Sigma = K(\mathcal{M}_\Sigma)$. Moreover, if P is a finite set, then $\mathcal{M}_\Sigma$ is the unique *S5*-Kripke model with this property.

Corollary 1.15. Stable sets are closed under **S5**-consequences.

Which Kripke model is associated with the epistemic state of an agent that "only knows φ"? The bigger the model the less is known. Therefore we take the union of all (normal) **S5**-models $\mathcal{M} = \langle S, \pi \rangle$ in which it holds that $K\varphi$: $\mathcal{M}_\varphi = \bigcup \{\mathcal{M} \mid \mathcal{M} \models K\varphi\}$. It appears that this 'big model' is not always a model of $K\varphi$, i.e. not always $\mathcal{M}_\varphi \models K\varphi$! This is seen as follows: let us consider again $\varphi = Kp \vee Kq$. Let $\mathcal{M}_1 = \{s \mid \pi(s)(p) = true\}$ and $\mathcal{M}_2 = \{s \mid \pi(s)(q) = true\}$. Clearly, $\mathcal{M}_1 \models K\varphi$ and $\mathcal{M}_2 \models K\varphi$. Obviously, there is some $s_1 \in \mathcal{M}_1$ such that $(\mathcal{M}_1, s_1) \models \neg q$ and some $s_2 \in \mathcal{M}_2$ for which $(\mathcal{M}_2, s_2) \models \neg p$. Let us now consider $\mathcal{M}_\varphi$, with s some state in $\mathcal{M}_\varphi$. Since s_1 and s_2 are states in $\mathcal{M}_\varphi$, we have $(\mathcal{M}_\varphi, s) \models \neg Kp \wedge \neg Kq$, i.e. $\mathcal{M}_\varphi \not\models Kp \vee Kq$.

Definition 1.16. φ is $honest_M \Leftrightarrow \varphi \in K(\mathcal{M}_\varphi)$.

Theorem 1.17 (Halpern & Moses [HM84]). For any epistemic formula φ it holds that:
(i) φ *is* $honest_S \Leftrightarrow \varphi$ *is* $honest_M$
(ii) φ is $honest_S \Rightarrow K(\mathcal{M}_\varphi) = \Sigma^{\varphi}$.

On the basis of epistemic states we can associate an entailment relation $\mathrel{|\!\sim}$ between honest formulas and arbitrary formulas as follows. The relation $\varphi \mathrel{|\!\sim} \psi$ indicates which consequences can be derived by the agent if he only knows the honest formula φ.

Definition 1.18. Let φ be honest. Then we define: $\varphi \mathrel{|\!\sim} \psi \Leftrightarrow \psi \in \Sigma^{\varphi}$.

It is interesting to study the behaviour of this entailment relation. For instance, one immediately sees, that it is *reflexive*: for any honest formula φ, $\varphi \mathrel{|\!\sim} \varphi$. We end this Section by giving some examples of $\mathrel{|\!\sim}$, implying that this relation is moreover a *non-monotonic* one: this follows from items iii and v below. In [MH93] we moreover demonstrate that $\mathrel{|\!\sim}$ is *cumulative*. We return to this in chapter [MH93] of this volume.

Example 1.19 Let p and q be two distinct primitive propositions. Then:

(i) $p \vdash p$
(ii) $p \vdash Kp$
(iii) $p \vdash \neg Kq$
(iv) $p \wedge q \vdash Kq$
(v) $p \wedge q \not\vdash \neg Kq$
(vi) $p \vdash (p \vee q) \wedge K(p \vee q)$
(vii) $p \vee q \vdash K(p \vee q)$
(viii) $p \vee q \vdash \neg Kp \wedge \neg Kq$
(ix) $p \wedge q \vdash Kp \wedge Kq$
(x) $p \wedge (p \rightarrow q) \vdash Kq$

2 Modalities for Quantities

Consider an agent getting input from three different sources w_1, w_2 and w_3. Suppose furthermore, that two types of information are relevant for this agent, say p and q. All the sources agree on p: they mark p as true. Finally, in w_1 and w_2, q is true, whereas in w_3, it is false. Using standard modal logic, one would consider the resources w_i $(i \leq 3)$ to be worlds in an S5-Kripke model, and observe that the agent knows p, i.e. Kp holds, but does not know q or $\neg q$, since he considers both alternatives to be possible: $Mq \wedge M\neg q$ holds.

This is about the limit of the expressibility of standard modal epistemic logic. To be more precise, consider the following Kripke model $\mathcal{M}_k = \langle W, R, \pi \rangle$, where $W = \{w_1, w_2, w_3, \ldots w_k\}$ $(k \geq 4)$, $R = W \times W$ and $\pi(p)(w_i) = \text{true}$ for all $i \leq k$; $\pi(q)(w_i) = \text{true}$ iff $i \notin \{2, 3\}$; $\pi(r)(w_i) = \text{true}$ iff $i \in \{1,3\}$ (cf. Figure 1 below).

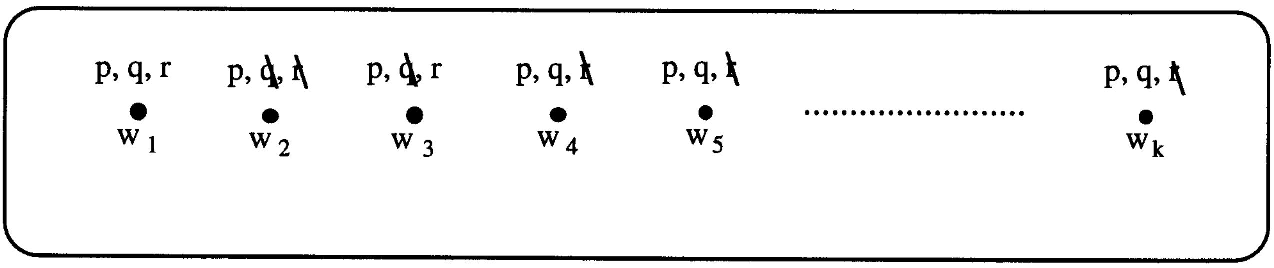

Figure 1

In [HM93], we demonstrate that in all *purely modal* formulas that are true in this model, one may freely interchange the role of q and r. In other words, despite the fact that q is true 'almost everywhere' in the model, and r is false 'almost everywhere', the modal language is too weak to express this difference between q and r. We claim that, both in the cases where worlds are interpreted in one-one correspondence to counterparts in the physical world (e.g. like sensors) and the case in which worlds correspond to possible (but made up) scenarios for some agent, a tool to distinguish 'q-statements' from 'r-statements' in the above model is highly desirable.

In this Section we will treat two such tools; the first is obtained by adding *numerical* (or *graded*) *modalities* to the language, in the second approach we will focus on *probabilistic modalities.*

2.1 Graded Modalities

First, we will discuss a way to add quantitative modalities M_n, $(n \in \mathbb{N})$, to the language, with the intended meaning of $M_n\varphi$ that more than n successors verify φ. We refer to [Ho92b] for some history, and a first investigation into the expressibility, decidability and definability of this graded language. An application of those graded modalities, especially of the graded analogue of S5, has been studied in the area of *Generalized Quantifiers*, (cf. [HR93a]). In [HR93b], a link was provided between the graded formalisms presented here and so called *terminological* or *concept languages*, used for knowledge representation. Such languages provide a means for expressing knowledge about hierarchies of sets of objects with common properties. Here, we try to explain how the greater expressive power of graded modalities may be used in *epistemic logic*.

We showed in [HM92] how these new modalities may help to make some issues in the field of *implicit knowledge* explicit. However, there, the graded modalities are motivated to establish some properties on a 'meta-level'; adding them to the language enables one to define more accurate models for implicit knowledge. In particular, they enable one to modally define the intersection of accessibility relations. Here, though, we try to use the new operators directly in the object language in order to obtain a more fine-tuned epistemic logic. We think that, using the enriched language, one has an appropriate tool to deal with notions like 'uncertain', or 'almost certain' knowledge (or belief). The new operators may then be helpful to reason with *degrees of acceptance.*

As already mentioned, the intended interpretation of $M_n\varphi$, ($n \in \mathbb{N}$) will be that there are more than n accessible worlds verifying φ. By defining $K_n\varphi \equiv \neg M\neg_n\varphi$, $K_n\varphi$ is true iff at most n accessible worlds refute φ. In terms of epistemic operators, note that $K_0\varphi$ boils down to $K\varphi$, so that we may interpret K_0 as our (certain) knowledge operator. Generally, $K_n\varphi$ means that the agent reckons with at most n exceptions for φ. Dually, $M_n\varphi$ then means that the agent considers more than n alternatives possible, in which φ is true. Since we are mainly interested in epistemic systems here, for the time being we will remain on solid ground by considering the graded analogue of S5. In this Section, our language L(Gr) = L(P,O) is built from a set of atoms P and operators $O = \{M_n \mid n \in \mathbb{N}\}$. Apart from K_n, we introduce the abbreviation $M!_n\varphi$, where $M!_0\varphi \equiv K_0\neg\varphi$, $M!_n\varphi \equiv (M_{n-1}\varphi \wedge \neg M_n\varphi)$, if $n > 0$. From the definitions above, it is clear that $M!_n$ means 'exactly n'.

Definition 2.1. The system Gr(S5) is defined as follows (cf. [HR93a]). It has inference rules Modus Ponens and Necessitation:

R0 $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$
R1 $\vdash \varphi \Rightarrow \vdash K_0\varphi$
It has also the following axioms (for each $n \in \mathbb{N}$):

A1 all propositional tautologies
A8 $K_0(\varphi \to \psi) \to (K_n\varphi \to K_n\psi)$
A9 $K_n\varphi \to K_{n+1}\varphi$
A10 $K_0\neg(\varphi \wedge \psi) \to ((M!_n\varphi \wedge M!_m\psi) \to M!_{n+m}(\varphi \vee \psi))$
A11 $\neg K_n\varphi \to K_0\neg K_n\varphi$
A12 $K_0\varphi \to \varphi$

The system with rules R0 and R1, axioms A1, A8 - A10 is the graded modal analogue of K, the basic modal system—so let us refer to it with Gr(K). In Gr(K), A8 is a kind of 'generalized K-axiom' (cf. 2.3), A9 is a way to 'decrease grades' in the possibility operator (A9 is equivalent to $M_{n+1}\varphi \rightarrow M_n\varphi$) and using A10, one can go to 'higher grades'. To ensure that the definitions work out rightly, we take proposition 2.10 from [Ho92b]:

Proposition 2.2. The following are derivable in Gr(K) (and hence in Gr(S5)):

(i) $M_n(\varphi \wedge \psi) \rightarrow (M_n\varphi \wedge M_n\psi)$

(ii) $M!_n\varphi \wedge M!_m\varphi \rightarrow \bot$ $(n \neq m)$

(iii) $K_n\neg\varphi \leftrightarrow (M!_0\varphi \,\nabla\, M!_1\varphi \,\nabla \dots \nabla\, M!_n\varphi)$ (∇ denotes 'exclusive or')

(iv) $\neg M_n(\varphi \vee \psi) \rightarrow \neg M_n\varphi$

(v) $M_{n+m}(\varphi \vee \psi) \rightarrow (M_n\varphi \vee M_m\psi)$

(vi) $M!_n\varphi \wedge M_m\varphi \rightarrow \bot$ $(m \geq n)$

(vii) $M_n(\varphi \wedge \psi) \wedge M_m(\varphi \wedge \neg\psi) \rightarrow M_{n+m+1}\varphi$

(viii) $(K_0\neg(\varphi \wedge \psi) \wedge (M_n\varphi \wedge M_m\psi)) \rightarrow M_{n+m+1}(\varphi \vee \psi)$

For a Kripke model $\mathcal{M}$ we extend the *truth definition of* φ at w for our new operator as follows:
(TD M_n) $(\mathcal{M},w) \models M_n\varphi$ iff $|\{w' \in W \mid Rww' \text{ and } (\mathcal{M},w') \models \varphi\}| > n$, $n \in N$.

Remark 2.3. Note that $(\mathcal{M},w) \models K_n\varphi$ iff $|\{w' \in W \mid Rww' \text{ and } (\mathcal{M},w') \models \neg\varphi\}| \leq n$. Also, note that the modal operators M and K (or $\Diamond$ and $\Box$) are special cases of our indexed operators: $M\varphi \equiv M_0\varphi$ and $K\varphi \equiv K_0\varphi$.

We end this introduction to Gr(S5) by recalling the following results:

Theorem 2.4. (Completeness: [Fi72], [FC88]). For all $\varphi \in$ L(Gr), Gr(S5) $\vdash \varphi$ iff $\mathcal{S5} \models \varphi$. Thus $\mathcal{S5}$ is also a class of models characterizing Gr(S5).

Theorem 2.5. (Finite models: [Ho92b]). Any $\varphi \in$ L(Gr) is satisfiable iff it is so on a finite model.

Theorem 2.6. (Freedom of nestings: cf. [HR93a]). In Gr(S5), each formula is equivalent to a formula in which no nestings of (graded) modal operators occur.

Related to the last theorem, a popular slogan in modal logic is that in S5, 'the inner modality always wins', we have e.g. $KM\varphi \equiv M\varphi$, $MK\varphi \equiv K\varphi$ and $MM\varphi \equiv M\varphi$ in S5. However, in the case of Gr(S5) this is not always sufficient: we *do* have $M_3M_5\varphi \equiv M_5\varphi$, but instead of $M_5M_3\varphi \equiv M_3\varphi$ we now have $M_5M_3\varphi \equiv M_5\top \wedge M_3\varphi$, accounting for the fact that $M_5M_3\varphi$ implies that, so to speak, 5 worlds are around.

How can Gr(S5) serve as an appropriate starting point to study epistemic phenomena? To start with, R0 and A1 express that we are dealing with an (extension of) classical propositional logic: we may use Modus Ponens and reason 'classically' (A1). By taking S5 as a 'standard' system for knowledge, the observations in the preceding Section suggest that we interpret $K_0\varphi$ as 'φ is known'. Then, R0, R1, A1 and A12 find their motivation in the same fashion as the corresponding properties in S5, i.e., we may use Modus Ponens, the agent knows all (Gr(S5))-derivable facts, we are dealing with an extension of propositional logic (A1) and moreover the agent cannot know facts that are not true (A12).

The semantics tells us, that $K_n\varphi$ should mean something like 'the agent reckons with at most n exceptional situations for φ', or 'the agent "knows-modulo-n-exceptions" φ'. Thus, the greater n is in $K_n\varphi$, the less confidence in φ is uttered by that sentence. The latter observation immediately hints at A9, $K_n\varphi \to K_{n+1}\varphi$: if the agent foresees at most n exceptions to φ, he also does so with at most n+1 exceptions. Of course, the generalisation of A10, for $n > 0$; $K_n\varphi \to \varphi$ is not valid: if the agent does not know φ for sure, i.e., if he allows for exceptions on φ, he cannot conclude that φ is the case. Thus $K_n\varphi$ expresses a form of "uncertain knowledge".

Axiom A8 says that if the agent knows that φ implies ψ, then, if he believes that there can be at most n exceptions to φ, he will not imagine more than n exceptions to ψ, since every exception to ψ will be an exception to φ as well, i.e. $K_0(\varphi \to \psi) \to (K_n\varphi \to K_n\psi)$. Recall from Section 2.1 that the K-axiom, $K(\varphi \to \psi) \to (K\varphi \to K\psi)$, has been considered a source of *logical omniscience*. However, now that we allow for weaker notions of knowledge, it appears that the K-axiom is only valid for K_0, which we may consider as a kind of 'ideal' knowledge. Instead of a K-axiom for each K_n, we have the much more realistic

(*) $$K_n(\varphi \to \psi) \to (K_m\varphi \to K_{n+m}\psi).$$

This seems very reasonable (suppose n, m > 0): if the agent has some confidence that φ implies ψ, and also has some confidence in φ, his conclusion that ψ holds should be stated with even less certitude than that of the two assertions separately. This is reminiscent of *plausible* ([Re76]) or *defeasible reasoning*, where reasoning under uncertainties is also the topic of investigation. Note that (*) guarantees that, the longer the chain of reasoning with uncertain arguments, the less certain the conclusion can be stated by the agent. Moreover, note that, although (*) holds, if $n > 0$ we do not have $K_n(\varphi \to \psi) \to (K_n\varphi \to K_n\psi)$: this makes it questionable to call K_n a *modal operator* (if $n > 0$). However, here we do so because all the operators receive some natural interpretation in Kripke models.

When interpreting K_n as an 'n-degree of knowledge', we recall that the higher the degree, the less certain the knowledge. The picture is denoted in the following chain:

$$K_0\varphi \to K_1\varphi \to \ldots K_n\varphi \to K_{n+1}\varphi \ldots \Rightarrow \ldots M_{n+1}\varphi \to M_n\varphi \to \ldots \to M_1\varphi \to M_0\varphi.$$

Here, the '$\to$' denotes logical implication. If, semantically speaking, the number of alternatives is infinite, the sequence is an infinite one, and '$\Rightarrow$' denotes implication, in the sense that all M_i-formulas are logically weaker than all the K_j-formulas. We could, as argued above, interpret the strongest formula in this chain ('$K_0\varphi$') as "φ is known", and the weakest ('$M_0\varphi$') as "φ is not impossible"—but even as "φ is believed", cf. [HM89].

If, however, the number of alternatives is *finite*, say N, we get the sequence

$$K_0\varphi\ (\equiv M_{N-1}\varphi) \to K_1\varphi\ (\equiv M_{N-2}\varphi) \to \ldots K_n\varphi\ (\equiv M_{N-n-1}\varphi) \to \ldots K_{N-1}\varphi\ (\equiv M_0\varphi) \to K_N\varphi\ (\equiv \top)$$

In fact, this is the case in the situation of the introduction to this subsection, where the agent is capable to sum up a complete description of the model by listing a (finite) number of possible situations determined by some finite set of propositional atoms.

Identifying worlds with truth assignments to primitive propositions, as is usual in standard *S5*-models, we can view a Gr(S5)-model as a multi-set of truth assignments rather than a set of these as in standard, ungraded modal logic. A special case, of course, is that situations (= truth assignments) occur only once in a description. We shall refer to these models as *simple* (referring to the original Latin meaning of this word). Note that in simple models it is still sensible to use graded modalities, since an assertion (even a primitive proposition) may nevertheless hold in more than one situation, as e.g. p in the situations {p is true, q is false} and {p is true, q is true}. For a deeper discussion on this and some specific examples, we refer to [HM93].

In several examples, the number of worlds (sources) may be assumed to be fixed. This gives rise to considering $Gr_k(S5)$, with fixed $k \in \mathbb{N}$, which is obtained from Gr(S5) by adding $M!_k T$ to it. Let $k^\wedge = \min\{m \in \mathbb{N} \mid m > 0.5\, k\}$. Using a preference modality (*use belief* in the sense of Perlis [Pe86]) expressed by operator P as in [MH91], we may express a democratic principle in $Gr_k(S5)$, as $P\varphi \leftrightarrow K_{k^\wedge}\varphi$, that is, φ is preferred (is a practical/working/use belief) iff it is true in more than the half of all sources. Note that this operator P does not obey the Distribution Axiom (A2): in this sense it contributes to solving some aspects of logical omniscience. Of course, other tresholds (than 0.5) might as well be taken.

2.2 Probabilistic Modalities

Recall that $Gr_k(S5) = Gr(S5) + M!_k T$. Using Proposition 2.2 we see that for any φ, $M!_0\varphi \nabla M!_1\varphi \nabla \dots \nabla M!_k\varphi$ is derivable in $Gr_k(S5)$. Here, asserting $M!_m\varphi$ is informative, since we know the *relative number* of occurrences of φ. This is less the case when the number of worlds is not finite, fixed, or known. In such a case, it seems more appropriate to add *probabilities* to (modal) logic. In the literature, there have been several attempts to do so (Cf. [FH89a, HR87]); we will compare them after we have informally introduced a language for a system that we will explore here.

Whereas probability theory on itself is a quite well-understood area in mathematics, its applications in AI and computer science in general justify the study and analysis of our reasoning about probabilities. In spite of the existing (and fastly growing) literature on probabilistic reasoning, (cf. [Ni86] allowing truth *values* to range between 0 and 1) we know of few endeavours of defining a logic enabling explicit *reasoning about* probabilities.

The logic P_FD, as introduced in [FA89], is essentially modal. Instead of interpreting the modal (possibility) operator as a 'likelihood operator' (as is done in [HR87]), it is a logic designed for reasoning with (exact) probabilities. Formally, this logic contains operators $P_r^>$ (in our notation) for each $r \in [0,1]$. The intended meaning of $P_r^>\varphi$ becomes "the probability of φ is strictly greater than r". In terms of this operator, operators $P_r^<$, $P_r^\geq$, $P_r^\leq$ and $P_r^=$ can be defined, with self explanatory meanings. $P_0^>$ is identified with the classical possibility operator M (or '◊'). This is the bridge between probability and modal logic.

We think P_FD is a good starting point for studying the interrelations between 'probabilities' and 'modalities'. With our semantics, we are able to interpret the above logics that deal with both modalities and probabilities, in one and the same (formal) semantics. In order to evaluate $P_r^>\varphi$ at world w we assume to have a function P_F for w which assigns a probability to each formula. A peculiar property of P_FD is that it only allows probability measures (for each world, on the set of formulas) that have a *finite range* (F). This assumption restores compactness for our logic of probabilities, as we will explain in the sequel.

Although restricting ourselves to a finite set F of granted measures (not eliminating the possibility to *reason* about and *express* properties of arbitrary probabilities) is a serious logical restriction, we think it allows for interesting applications. Taking F = {0,1} gives us ordinary modal logic. If $0.5 \in F$, we might represent Lentzen's logic of Belief ([Le80]). Letting F have 9 elements might be suitable to model Driankov's linguistic estimates *impossible, extremely unlikely, very low chance, small chance, it may, meaningful chance, most likely, extremely likely, certain* (cf. [Dr87]. Also more general applications to fuzzy reasoning seem interesting. We think that in many occasions, to let an agent reason about probabilistic events, the granularity of F can be chosen conveniently.

We now proceed with providing some formal details of the logic P_FD. The (uncountable) language L(Prob) of P_FD is $L(P,\{P_r^{>} \mid r \in 0,1]\}$. We also write $(\varphi \nabla \psi)$ for $((\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi))$.

2.7. Definition. A set F is a base (for a logic P_FD) if it satisfies:

i. F is finite
ii. $\{0,1\} \subseteq F \subseteq [0,1]$
iii. F is 'quasi-closed under addition': $r, s \in F \;\&\; (r + s) \leq 1 \Rightarrow (r + s) \in F$.
iv. F is 'closed under taking complements': $r \in F \Rightarrow (1 - r) \in F$.

P_FD is given relative to a fixed base[1] $F = \{r_0, \ldots, r_n\} \subseteq [0,1]$. We assume that $r_i < r_{i+1}$, if $i < n$ (implying $0 = r_0$, $r_n = 1$). We will now give the axioms for P_FD, as they are given in [FA89]. However, we use a slightly adapted notation. Basic operator in [FA89] is M_r, with intended meaning of $M_r\varphi$: 'φ has a probability strictly greater than r'. We denote the latter with $P_r^{>}\varphi$. Moreover, we will use the following abbreviations (for all $s, r \in [0,1]$):

def1 $P_r^{\geq}\varphi \equiv \neg P_{1-r}^{>}\neg\varphi$
def2 $P_r^{<}\varphi \equiv P_{1-r}^{>}\neg\varphi$
def3 $P_r^{\leq}\varphi \equiv \neg P_{1-r}^{<}\neg\varphi$
def4 $P_r^{=}\varphi \equiv \neg P_r^{>}\varphi \wedge \neg P_r^{<}\varphi$

P_FD has inference rules R1 and R2 and axioms A1, A13 - A18:

R1 Modus ponens: From φ and $\varphi \to \psi$ derive ψ.
R2 Necessitation: From φ derive $P_1^{\geq}\varphi$.

A1 Propositional tautologies.
A13 $P_1^{\geq}(\varphi \to \psi) \to [(P_r^{>}\varphi \to P_r^{>}\psi) \wedge (P_r^{>}\varphi \to P_r^{\geq}\psi) \wedge (P_r^{\geq}\varphi \to P_r^{\geq}\psi)]$ $\qquad r \in [0,1]$
A14 $P_1^{\geq}(\varphi \to \psi) \to (P_r^{\geq}\varphi \to P_s^{>}\psi)$ $\qquad r, s \in [0,1]$ and $s < r$
A15 $P_0^{\geq}\varphi$
A16 $P_{r+s}^{>}(\varphi \vee \psi) \to (P_r^{>}\varphi \vee P_s^{>}\psi)$ $\qquad r, s \in [0,1]$ such that $r + s \in [0,1]$

[1]Although properties iii and iv are not explicitly mentioned in [FA], in the following it will be obvious that they are necessary.

A17	$P^{\geq}_{1}\neg(\varphi \wedge \psi) \rightarrow ((P^{>}_{r}\varphi \wedge P^{\geq}_{s}\psi) \rightarrow P^{>}_{r+s}(\varphi \vee \psi))$	$r, s \in [0,1]$ such that $r + s \in [0,1]$
A18	$P^{>}_{r_i}\varphi \rightarrow P^{\geq}_{r_{i+1}}\varphi$	$i < n$

Before reflecting on the definitions and axioms of this logic, we prove a small lemma:

2.8. Lemma. The following are derivable in $P_F D$:

a. $P^{\leq}_{1}\varphi$

b. $P^{\geq}_{1}\varphi \leftrightarrow P^{=}_{1}\varphi$

2.9. Remark. Although the definitions def1 - def4 all are very straightforward in our notation, it is not made explicit until 2.12 that they have the desired properties; for instance, that $P^{\geq}_{r}\varphi$ equals $(P^{>}_{r}\varphi \vee P^{=}_{r}\varphi)$. By 2.8.b, R2 is equivalent to $(\vdash \varphi \Rightarrow \vdash P^{=}_{1}\varphi)$, which explains the name 'Necessitation'. Axiom 7 is the axiom where our finite set F comes into play. The axiom forbids any formula to have a probability which is not in F: if the probability for φ is greater than some $r_i \in F$, it is at least the next value of F, viz. r_{i+1}. Axiom A17 resembles additivity: cf. Axiom A10 of Section 2.1. Axiom A16 explains how the probability of a disjunction is 'distributed' over its disjuncts. A15, together with lemma 2.8 and 2.12, where we show that $P^{<}_{r}$, $P^{\leq}_{r}$, $P^{\geq}_{r}$ and $P^{=}_{r}$ are well-defined) guarantees that probabilities are in a proper range: $\vdash (P^{\geq}_{0}\varphi \wedge P^{\leq}_{1}\varphi)$. Axiom A13 implies that the probability of the consequent of any necessary implication is always at least the probability of the antecedent. It also records that 'greater than' implies 'at least'. Finally, A14 is the only axiom with which one can change from one probability to a smaller one. Although A14 is, like A13, stated for necessary implications, in this case we can weaken it to:

A14' $\quad P^{\geq}_{r}\varphi \rightarrow P^{>}_{s}\varphi$, $r, s \in [0,1]$ and $s < r$.

To see that this implies A14, we reason as follows: $P^{\geq}_{1}(\varphi \rightarrow \psi) \wedge P^{\geq}_{r}\varphi \Rightarrow_{A13} P^{\geq}_{r}\psi \Rightarrow_{A14'} P^{>}_{s}\psi$

The name 'Necessitation rule' for R2 suggests a connection with classical modal logic. We can make this relation explicit as follows; for a proof, we refer to [Ho92c].

2.10 Proposition. Let, by definition, $\Box\varphi = P^{\geq}_{1}\varphi$[2]. Then:

a. (1) $\vdash \varphi \Rightarrow \vdash \Box\varphi$

   (2) $\vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

   (*) $\vdash \neg\Box\bot$

b. We say that a formula $\in$ L is *modal* if it is built from atomic propositions, using only the logical connectives and the modal operator $\Box$. KD is the modal logic K + axiom D (Cf. Definition 1.3). We claim, that for all modal formulas φ: $P_F D \vdash \varphi$ iff $KD \vdash \varphi$.

2.11. Definition. We say that a logic $\mathcal{L} \supseteq (P_F D \setminus A18)$ is *permitting a probability-assignment* (is ppa) if for each $\mathcal{L}$-consistent set Γ and each formula φ there is some $r \in [0,1]$, such that $\Gamma \cup \{P^{=}_{r}\varphi\}$ is $\mathcal{L}$-consistent.

It is obvious that the property of permitting probabilities is necessary to allow models that assign probabilities to formulas. It can be shown that $P_F D$ is ppa (Cf. [Ho92c]). It is this property that will guarantee completeness with respect to our semantics. In [Ho92c] it is demonstrated, that for $P_F D$, this completeness is obtained in a very straightforward way; this completeness proof should be compared with the one given in [FA89], in which these

[2] by lemma 2.8, this is equivalent to stipulating $\Box\varphi = P^{=}_{1}\varphi$.

probabilities are obtained in a rather indirect manner. However, where the benefit of axiom A18 is that each formula has a probability, its drawback is that each formula has a probability *in* the finite set *F*, as we shall see. This is, of course, a serious restriction.

2.12. Theorem. The following are P_FD-theorems for all φ, ψ in the language and all r, s $\in$ [0,1].

a. $P_r^{\geq}\varphi \leftrightarrow (P_r^{>}\varphi \vee P_r^{=}\varphi) \wedge P_r^{\leq}\varphi \leftrightarrow (P_r^{<}\varphi \vee P_r^{=}\varphi)$

b. $P_r^{>}\varphi \; \nabla \; P_r^{=}\varphi \; \nabla \; P_r^{<}\varphi$

c. $\neg(P_r^{=}\varphi \wedge P_s^{=}\varphi) \qquad (r \neq s)$

d. $(\neg P_r^{<}\varphi \leftrightarrow P_r^{\geq}\varphi) \wedge (\neg P_r^{>}\varphi \leftrightarrow P_r^{\leq}\varphi)$

e. $P_r^{=}\varphi \leftrightarrow (P_r^{\geq}\varphi \wedge P_r^{\leq}\varphi)$

f. $(P_r^{>}\varphi \rightarrow P_s^{>}\varphi) \;\; (s \leq r)$

g. $P_r^{=}\varphi \leftrightarrow P_{1-r}^{=}\neg\varphi$

Proof. See [Ho92c].

Theorem 2.12 does not state remarkable properties: it just says that the defined operators have their intended properties.

2.13. Definition. A *Probabilistic Kripke model* $\mathcal{M}$ is a tuple <W, R, P, V> where W, R and V are as in standard Kripke models but now P: W $\times$ $\mathcal{P}$(W) $\rightarrow$ [0,1] is a function from the powerset of W to F, for each w $\in$ W, called a *probability measure over* W.

i $\quad P(w,\emptyset) = 0$

ii $\quad P(w,\{v \mid Rwv\}) = 1$

iii $\quad$ Let for i $\in$ countable I, $X_i \in \mathcal{P}(W)$. Then $(i \neq j \Rightarrow X_i \cap X_j = \emptyset) \Rightarrow P(\cup_{i\in I} X_i) = \Sigma_{i\in I}P(X_i)$

If condition iii holds only for finite index sets I, we say that P is finitely additive, else countably additive. We will also write $P_w\{v\}$ and $P_w[\varphi]$ instead of $P(w)(\{v\})$ or $P(w)([\varphi])$, respectively. The truth definition for P_FD formulas is obtained straightforwardly, with the modal case

$TD(P_r^{>}) \qquad (\mathcal{M},w) \models P_r^{>}\varphi$ iff $P_F(w,\{v \mid Rwv \text{ and } (\mathcal{M},v) \models \varphi\}) > r$

We denote this class of models with $\mathcal{PK}$ (for probabilistic Kripke models). If the range of P is some base F $\subseteq$ [0,1], then we also write <W, R, P_F, V> and say that $\mathcal{M}$ is a *Relativized Probabilistic Kripke model*. $\mathcal{PK}_F$ is the set of probabilistic models with a finite base, in which the measures P_F are moreover finitely additive.

Note that we have a truly recursive truth-definition, and that we no longer need a separate clause for $P_1^{\geq}\varphi$ as was needed in [FA89]. We might allow some more generality in the definition of Kripke models. For instance, we might leave out the accessibility relation R as a primitive and define Rwv $\Leftrightarrow$ $P_F(w,\{v\}) > 0$ (cf. [Ho91]). Also, instead of forcing the range of P_F to be F, we might weaken this to

$P_F(w,X) \in F$ for each X that is the denotation of a formula φ

In [Ho92c], a Henkin-style completeness proof is given for this logic P_FD with respect to the semantics of Definition 2.13. Note that, in the canonical model (Cf. Remark 1.6) one now also has to specify $P_F(\Gamma,\Omega)$, for any maximal consistent set Γ and set of maximal consistent sets Ω. In fact, P_F is determined by specifying it for all singletons Ω, and to do so, the fact P_FD is permitting probabilities is crucial. This property, on its turn, is proven using the fact that the formula

$$P^{=}_{r_0}\varphi \nabla P^{=}_{r_1}\varphi \nabla \ldots \nabla P^{=}_{r_n}\varphi$$

(recall that $F = \{0 = r_0, r_1, \ldots, r_n = 1\}$) is derivable in $P_F D$. As an immediate consequence, we obtain:

2.14. Corollary. $P_F D$ is ppa; if Γ is $P_F D$-consistent, then for each $\varphi \in L$ there is a $r \in [0,1]$ such that $\Gamma \cup \{P^{=}_r\varphi\}$ is also $P_F D$-consistent.

It appears, that to really exploit the formula above (it says that any formula φ must have some probability in F) we have to manipulate *formulas* in the canonical model rather than arbitrary *maximal consistent sets.* This is achieved by doing the Henkin construction using only *finite* maximal consistent sets: then, one can identify Δ with the conjunction of its members $\underline{\Delta}$, so that, in order to decide on $P_F(\Gamma,\{\Delta\})$, one can use the fact that $P^{=}_{r_0}\underline{\Delta} \nabla P^{=}_{r_1}\underline{\Delta} \nabla \ldots \nabla P^{=}_{r_n}\underline{\Delta}$ is derivable, and hence, consistent with Γ; this means that we can assign a probability to $\underline{\Delta}$ (and hence to Δ). This procedure can be consistently extended to all Γ and Δ (Cf. [Ho92c] for more details).

Given this definition of Probability model, we elaborate a bit on the problem of compactness for $P_F D$-like systems, and the role of A18 in this. Axiom A18 is a logical compromise. On one hand, it restricts us to probability measures with finite range, on the other hand, it guards us against some serious logical complications. To be more precise, consider $P_F D \setminus A18$, and let

$$\Gamma = \{\neg P^{=}_r q \mid r \in [0,1]\}.$$

Our first claim is that Γ is consistent (see [Ho92c] for a justification). Our next claim is that we can use Γ to show that $P_F D \setminus A18$ is not compact. For, although Γ is not satisfiable if we interpret $P^{>}_r\varphi$ as "the possibility of φ is greater than r", (i.e., not satisfiable on Probabilistic Kripke Models ($\mathcal{PK}$): models similar to those of $\mathcal{PK}_F$, but allowing F to be $\mathbb{R}$) each finite subset of Γ is (in fact each proper subset Γ' of Γ is). Stated equivalently: although (for each $s \in [0,1]$), $\Gamma\setminus\{P^{=}_s\varphi\} \models P^{=}_s\varphi$, (in which we mean by '$\models$' $\mathcal{PK}$-validity) there is no finite subset $\Gamma' \subseteq \Gamma\setminus\{P^{=}_s\varphi\}$ such that $\Gamma' \models P^{=}_s\varphi$.

It seems that the only way to resolve this (at least to avoid consistency of Γ) is to admit an infinitary logic, for which $\sum_{(r\in[0,1])} P^{=}_r\varphi$ holds, i.e., guaranteeing that each assertion has a probability. We only know of one attempt of allowing an infinite rule that guarantees such a property, viz. in a logic for an operator '$P^{=}_s$', as done in [Al91].

We conclude this Section be briefly mentioning some other approaches to probabilistic logics. Two recent papers in the area of AI, "A Logic for Reasoning about Probabilities" ([FHM90]) and "Uncertainty, Belief and Probability" ([FH89b]) present a profound investigation and a framework to systematically reason with probabilities. In their formalism, a formula is typically a Boolean combination of expressions of the form $a_1 w(\varphi_1) + \ldots a_k w(\varphi_k) \geq c$, where $a_1, \ldots a_k$, c are integers. $P_F D$ is, in some sense, at least as expressive, as is shown in [Ho92c].

However, in the formalism of [FH89b] and [FHM90], the φ's must be purely propositional, thus omitting 'higher order weight formulas' ([FHM90]). Moreover, they do not allow 'w-free formulas' disabling expressions like "p is true, although its probability is less than 0.1". These restrictions do not apply to P_FD. Moreover, as is also explained in [Ho92c], the logics of [FH89b] and [FHM90] are not compact.

Also, in the two approaches mentioned, the formal system explicitly postulates as axioms all valid formulas of linear inequalities, and the completeness proofs heavily rely on results in the area of linear programming. Finally, in both contributions it is claimed that, if (the interpretations of) primitive propositions is measurable, their semantics is equivalent to that of [Ni86], in which a probability is given to each $2^{|\Phi|}$ different assignments (worlds) of the primitive propositions in Φ, implying that they use a finite set of primitive propositions[3], and, more importantly, thus discarding 'features of worlds that are not captured by primitive propositions' ([FH89b]).

3 Qualitative Modalities

A common objection against the approaches of the previous Section is summarised in the question: *Where do the numbers come from?* People (and, more specifically, experts, like doctors or engineers), are often reluctant in expressing their opinions in exact probabilities. They rather give their judgement in a qualitative manner: instead of a doctor saying "Given the symptoms of patient x, I think he has disease A with probability 0.716", a more realistic utterance is "Given his symptoms, I rather think patient x has disease A than B". Such qualitative judgements are often sufficient in every day life. An engineer confronted with a failing car will not bother about exact estimates for the possible causes of this unpleasant situation. He merely reasons as follows: "In most situations like this, it is the battery. If that is o.k., I'll better check the wires. If they are fine too, there is the—a priori small possibility—that the tank is empty. If that is untrue too, I'll better get my check-list". This list, on its turn, is probably organised along the same principle: from often occurring causes to very rare ones.

What we do in this Section is augment the language of modal logic with a binary operator '$\geq$'. The formula $\varphi \geq \psi$ can be read as 'φ is at least as likely, probable, or trustworthy as ψ is'. We present a logic for this operator $\geq$. Then, we provide it with two semantics, both based on Kripke structures. The first semantics is based on the idea of *counting* worlds, as in Section 2.1. In the second semantics, one has to compare the *measures* that are assigned to the φ-worlds and the ψ-worlds. In fact, historically, the second semantics preceded the first one, but to be in line with the presentation in Section 2, we first present the—conceptual more easy—semantics based on counting.

The rest of this Section is organised as follows. In Subsection 3.1 we introduce the logic QM for $\leq$, which has an infinite axiom scheme $B = \{B(m) \mid m \in \mathbb{N}\}$ on top of a set of simple axioms. We derive some of its properties and introduce its first semantics: Kripke models with finitely many successors. In 3.2, we recall the notion of probability model and state that the $\geq$-theory of those models is exactly the same as that of the models from Section 3.1. We shortly discuss the scheme B as well as some other qualitative operators in 3.3.

3.1 At Least as Many as

The language $L(QM) = L(P,\{\geq\})$ for our logic QM, (for Qualitative Modalities), has a two place

[3]Although this assumption is made explicit in [FH], in [FHM] the starting point of their system seems to be an infinite set of propositional atoms!

modal operator '≥'. Using that, we can define the operators '>', '<', '≤', '~' and '□' straightforwardly: the formula $\psi \leq \varphi$ is defined as $\varphi \geq \psi$, $\varphi > \psi$ as $((\varphi \geq \psi) \wedge \neg(\psi \geq \varphi))$. $(\psi < \varphi)$ means $(\varphi > \psi)$ and we use $\varphi \sim \psi$ for $(\psi \leq \varphi) \wedge (\varphi \geq \psi)$. The one place operator □ is defined as $\Box\varphi = (\varphi \geq \mathit{true}) \wedge (\mathit{true} \geq \varphi)$. $\Diamond = \neg\Box\neg$. We assume that the binding power of the connectives is weaker than that of the operators: $\psi \leq \varphi \wedge \chi$ means $(\psi \leq \varphi) \wedge \chi$.

3.1 Definition. The class of models $\mathcal{FKD}$ is the set of Kripke structures $\mathcal{M} = \langle W,R,\pi\rangle$, for which R(w) is non-empty but finite for all $w \in W$. We write R(w) for {v| Rwv} and $[\varphi]$ for {w| $(\mathcal{M},w) \models \varphi$}. The truth definition for ≥ is

TD(≥) $(\mathcal{M},w) \models \varphi \geq \psi$ iff $|Rw \cap [\varphi]| \geq |Rw \cap [\psi]|$

$\mathcal{FKD}$ stands for Finite Kripke models verifying the scheme D, guaranteeing that each world has at least one successor.

3.2 Definition. AX consists of the following rules and axioms.

R1 modus ponens, i.e. $\vdash \varphi \;\&\; \vdash \varphi \to \psi \Rightarrow\; \vdash \psi$.

R2 necessitation, i.e. $\vdash \varphi \Rightarrow\; \vdash \Box\varphi$.

A1 all propositional tautologies.

A19 $\Box(\varphi \leftrightarrow \varphi') \wedge \Box(\psi \leftrightarrow \psi') \to (\varphi \geq \psi \to \varphi' \geq \psi')$

A20 $\varphi \geq \psi \vee \psi \geq \varphi$

A21 $\varphi \geq \mathit{false}$

A22 $\mathit{true} > \mathit{false}$

A23 $\varphi \geq \psi \wedge \psi \geq \chi \to \varphi \geq \chi$

A24 $[\Box\neg(\varphi \wedge \chi) \wedge \Box\neg(\psi \wedge \chi)] \to [(\varphi \geq \psi) \leftrightarrow ((\varphi \vee \chi) \geq (\psi \vee \chi))]$.

The axioms of AX all look rather reasonable, keeping in mind the meaning of '≥'. We first give a result in the spirit of Proposition 2.10: AX can be considered to be (an extension of) normal modal logic: it has necessitation (R2) as a rule of inference, and the K-axiom as a theorem (3.4.v). Recall from Definition 1.3 that KD is the modal logic K, with the axiom D: $\Diamond\top$. We say that $\varphi \in$ L(QM) is *modal* if it does not contain ≤, <, >, ≥, or ~.

3.3 Proposition. For all modal φ, $\mathcal{FKD} \models \varphi$ iff KD $\vdash \varphi$.

The following proposition not only shows that the K-axiom is AX-derivable (3.4.v), but also that '≥' is a kind of intermediate for this property (3.4.iii and 3.4.iv).

3.4 Proposition.

i. AX $\vdash \varphi \geq \psi \leftrightarrow (\varphi \wedge \neg\psi) \geq (\neg\varphi \wedge \psi)$

ii. AX $\vdash \varphi \geq \psi \leftrightarrow \neg\psi \geq \neg\varphi$

iii AX $\vdash \Box(\varphi \to \psi) \to \psi \geq \varphi$

iv. AX $\vdash \psi \geq \varphi \to (\Box\varphi \to \Box\psi)$

v. AX $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

Proof. We prove i, as an example. For complete proofs, see [Ho91].

By necessitation, we have $\Box[\neg[(\varphi \wedge \neg\psi) \wedge (\varphi \wedge \psi)] \wedge \neg[(\neg\varphi \wedge \psi) \wedge (\varphi \wedge \psi)]]$. Applying A24 to this yields $(\varphi \wedge \neg\psi) \geq (\neg\varphi \wedge \psi) \leftrightarrow [(\varphi \wedge \neg\psi) \vee (\varphi \wedge \psi)] \geq [(\neg\varphi \wedge \psi) \vee (\varphi \wedge \psi)]$. The left hand side of the latter inequality is equivalent to φ; its right hand side is equivalent to ψ. Using A19, we get the equivalence $(\varphi \wedge \neg\psi) \geq (\neg\varphi \wedge \psi) \leftrightarrow \varphi \geq \psi$.

The properties expressed in proposition 3.4 are perhaps more appealing in the semantics of our operator. Propositions iii - v relate set inclusion with set magnitudes. Moreover, if $\#\chi$ is the numbers of worlds verifying χ, then 3.4.i amounts to $\#\varphi \geq \#\psi$ iff $\#\varphi - \#(\varphi \wedge \psi) \geq \#\psi - \#(\varphi \wedge \psi)$ and ii to $\#\varphi \geq \#\psi$ iff $\#\top - \#\psi \geq \#\top - \#\varphi$. However, until now, we have stated no completeness result. In fact, the system AX is still too weak for such.

3.5 Definition. Let Γ (of length m) = $\gamma_1,\ldots,\gamma_m$ be a sequence of m formulas. let $N_i(\Gamma)$ express that exactly i elements of Γ are true, for each $i \leq m$. So $N_0(\Gamma) = \neg\gamma_1 \wedge \neg\gamma_2 \wedge \ldots \wedge \neg\gamma_m$, $N_1(\Gamma) = (\gamma_1 \wedge \neg\gamma_2 \wedge \ldots \wedge \neg\gamma_m) \vee (\neg\gamma_1 \wedge \gamma_2 \wedge \ldots \wedge \neg\gamma_m) \vee \ldots \vee (\neg\gamma_1 \wedge \neg\gamma_2 \wedge \ldots \wedge \neg\gamma_{m-1} \wedge \gamma_m)$, and so on. We say that two sequences Γ and Δ of length m are *balanced*, $\Gamma \mathrm{E} \Delta$, iff

$$\Gamma \mathrm{E} \Delta \qquad \bigvee_{i=0}^{m}[N_i(\Gamma) \wedge N_i(\Delta)],$$

that is, $\Gamma \mathrm{E} \Delta$ iff the number of formulas of Γ that are true is exactly the same as the number of true Δ-formulas. Finally, we say that Γ and Δ are *balanced everywhere,* $\Gamma \mathbb{E} \Delta$, iff $\Box(\Gamma \mathrm{E} \Delta)$.

So, E and $\mathbb{E}$ are a kind of *generalized equivalence.* For instance, $\varphi \mathbb{E} \psi = \Box(\varphi \leftrightarrow \psi)$. Now we are able to formulate the additional scheme that we will need:

3.6 Definition. Given the operator $\mathbb{E}$ and $m \in \mathbb{N}$, we define

$$\text{B(m)} \qquad \varphi_1,\ldots,\varphi_m \mathbb{E} \psi_1,\ldots,\psi_m \rightarrow [(\varphi_1 \geq \psi_1) \wedge \ldots \wedge (\varphi_{m-1} \geq \psi_{m-1})] \rightarrow \psi_m \geq \varphi_m$$

Read semantically, B(m) expresses that if each of the accessible worlds verifies equally many φ's as it verifies ψ's (if they are 'balanced' in the φ's and the ψ's), and if the amount of successors that verifies φ_i is at least the amount of successors that verifies ψ for the first $i \in \{1, \ldots, m-1\}$, then (to 'keep the balance') there cannot be *more* φ_m-successors than ψ_m-successors, hence there should be at least as many ψ_m-successors as φ_m-successors.

3.7 Definition QM is the logic AX together with the scheme B(m), for all m.

3.8 Proposition. The axioms A23 and A24 follow from the other QM-axioms.
Proof.
A23: Observe that $\varphi,\psi,\chi \mathbb{E} \psi,\chi,\varphi$ is a (propositional) tautology. Next, apply B(3) to this.

A24:

$$\Box\neg(\varphi \wedge \chi) \wedge \Box\neg(\psi \wedge \chi) \Rightarrow_{A1} \Box(\bigvee_{i=0}^{2}[N_i(\varphi,(\psi \vee \chi)) \wedge N_i(\psi,(\varphi \vee \chi))])$$
$$\Rightarrow_{3.5} \varphi,(\psi \vee \chi) \mathbb{E} \psi,(\varphi \vee \chi) \Rightarrow_{B(2)} (\varphi \geq \psi) \rightarrow (\varphi \vee \chi) \geq (\psi \vee \chi).$$
$$\Box\neg(\varphi \wedge \chi) \wedge \Box\neg(\psi \wedge \chi) \Rightarrow_{A1} \Box(\bigvee_{i=0}^{2}[N_i((\varphi \vee \chi),\psi) \wedge N_i((\psi \vee \chi),\varphi)])$$
$$\Rightarrow_{3.5} (\varphi \vee \chi),\psi \mathbb{E} (\psi \vee \chi),\varphi \Rightarrow_{B(2)} (\varphi \vee \chi) \geq (\psi \vee \chi) \rightarrow (\varphi \geq \psi).$$

That B(m) does not follow from the other axioms, is shown in [Ho91].

From proposition 3.3 we know that the modal theory of $\mathcal{FKD}$ is exactly that of $\mathcal{KD}$, the theory of K + seriality. This was because the truth definition we gave for $\Box$ and the 'usual' definition for the necessity operator coincide on finite models. Let us conclude this Section by observing that the operator $\geq$ is not modally definable, like we did for the graded modalities using Figure 1. To do this for $\geq$, let $W = \{u,v_1,v_2\}$, $W' = \{u',v'\}$ and consider the frames $\mathcal{F} = \langle W, W \times W\rangle$ and $\mathcal{F}' = \langle W', W' \times W'\rangle$, denoted in figure 2.

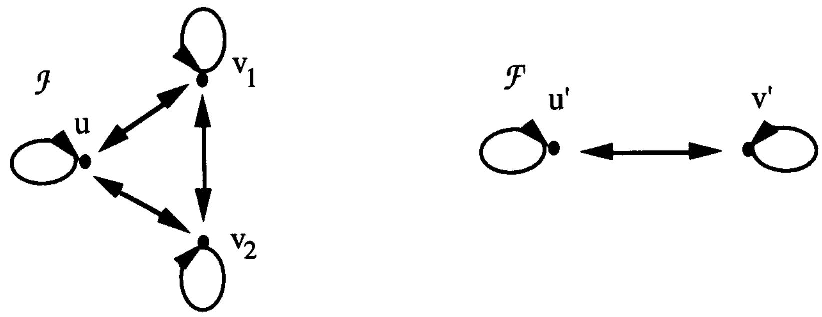

figure 2

It is easily verified that the mapping f: $\mathcal{G} \rightarrow \mathcal{F}$, with f(u) = u' and f($v_1$) = f($v_2$) = v' is a function that can be used to demonstrate that, for all *modal* formulas φ: $(\mathcal{G},u) \models \varphi \Rightarrow (\mathcal{F},u') \models \varphi$ (Cf. [Ho91]). However, this is not true for arbitrary L(QM)-formulas. For instance, let ψ be $(p > \neg p) \vee (\neg p > p)$. Then although $(\mathcal{G},u) \models \psi$, we have $(\mathcal{F},u') \not\models \psi$.

3.2 At Least as Likely as

Segerberg ([Se71]) and Gärdenfors ([Gä75]) interpreted the logic QM on countably additive Kripke models $\in \mathcal{PK}$(cf. Definition 2.13). The truth definition then is

TD($\geq$) $(\mathcal{M},w) \models \varphi \geq \psi$ iff $P_w[\varphi] \geq P_w[\psi]$.

Note that we immediately have $(\mathcal{M},w) \models \Box\varphi$ iff $P_w[\varphi] \geq P_w[\top]$ (iff $P_w[\varphi] = 1$).

Without proof, we state that QM is sound with respect to $\mathcal{PK}$, i.e. QM $\vdash \varphi \Rightarrow \mathcal{PK} \models \varphi$. For the axiom B(m), we refer to [Se71b]. Note that, on such probability models, although sets (of worlds) may be infinite, their *measure* now is always finite, even bounded. A probability model need not have an accessibility relation, but note that each world has a non-empty set with measure > 0. That is the reason that we will be able to make a connection between $\mathcal{FKD}$ and $\mathcal{PK}$: in $\mathcal{FKD}$ each world has at least one successor.

3.9 Proposition.

i. For each $\mathcal{M} = \langle W,R,\pi \rangle \in \mathcal{FKD}$ there is a $U(\mathcal{M}) = \langle W,F,\pi \rangle \in \mathcal{PK}$, *the uniform probability model over* $\mathcal{M}$, such that $W \subseteq W'$ and for all $w \in W$, $(\mathcal{M},w) \models \varphi$ iff $(U(\mathcal{M}),w) \models \varphi$.

ii. For all $\varphi \in$ L(QM), $\mathcal{PK} \models \varphi \Rightarrow \mathcal{FKD} \models \varphi$.

Proof. In [Ho91].

Both Segerberg and Gärdenfors prove the following:

3.10 Theorem. If QM $\nvdash \varphi$ then $(\mathcal{M},w) \models \neg\varphi$ for some finite probability model $\mathcal{M}$.

Together with the remark about QM's soundness for $\mathcal{PK}$, this theorem implies that we have that $\mathcal{PK}$ is determined by QM: QM $\vdash \varphi \Leftrightarrow \mathcal{PK} \models \varphi$. But it says more: each QM-consistent formula has a *finite* $\mathcal{PK}$ model. This implies that in the definition of probability measure, we might as well have required finite additivity instead of countable additivity. It is this property of having finite models for each satisfiable formula that was exploited in [Ho91] to get a completeness result for $\mathcal{FKD}$. The main pillar of that proof is the following lemma, which was referred to as the rationalising lemma. The lemma guarantees that if there is a probability measure P: $\mathcal{P}(A) \to [0,1]$ where A is a finite set, there will also be a *rational* probability measure P': $\mathcal{P}(A) \to \cap$ Q that agrees with P for as far as ≥ is concerned, i.e. for which for all X, Y ⊆ A: $P(X) \geq P(Y) \Leftrightarrow P'(X) \geq P'(Y)$.

3.11 Lemma (Rationalising). Let A be finite, and P: $\mathcal{P}(A) \to [0,1]$ a probability measure. Then there is a function P': $\mathcal{P}(A) \to [0,1] \cap Q$ such that

i. P' is a probability measure.
ii $\forall X,Y \subseteq A$: $P(X) = P(Y) \Leftrightarrow P'(X) = P'(Y)$.
iii $\forall X,Y \subseteq A$: $P(X) < P(Y) \Leftrightarrow P'(X) < P'(Y)$.

Proof. in [Ho91]

3.12 Theorem.

i. For all $\varphi \in L(QM)$, $\mathcal{PK} \models \varphi \Leftrightarrow \mathcal{FKD} \models \varphi$.
ii. $QM \vdash \varphi \Leftrightarrow \mathcal{FKD} \models \varphi$.

As was pointed out before, the axiom scheme B(m) is necessary to guarantee completeness with respect to the class of probability models. De Finetti conjectured that the conditions expressed by the axioms of AX (section 2) were necessary and sufficient for the existence of a probability measure (on a powerset $\mathcal{P}(A)$) that agreed with ≥. However, it was shown by Kraft, Pratt and Seidenberg ([KPS59]), by giving a ingenious counterexample, that they were not sufficient. They also showed that adding the scheme *B* did the job. Segerberg ([Se71]) and Gärdenfors ([Gä75]) put this result of [KPS59] to work to obtain their completeness proof of QM with respect to $\mathcal{PK}$.

There are, in general, two ways to increase the expressive power of a system. One way to do so is to shift to a richer language, as was done in the approach of [FHM90] for probabilistic modal logic. Another way to bypass the scheme *B* is to add additional inference rules to the system. We briefly speculated on this in [Ho91]. We will not digress on this here, but instead mention that there has been a great variety of proposals (each with its own specific rules) for systems with qualitative axioms that guarantee the existence of a probability measure (that agrees with a qualitative ordering) (cf. [Do69] or [SZ76]). This has also been done for *conditional probabilities* ([Do69, SZ82]. In [SZ89], also conditions on *upper and lower probabilities* are given that are necessary and sufficient for a probability measure.

3.3 At Least as Possible as.

We conclude this Section by mentioning a binary modal operator '≥' in the field of *possibilistic logic* ([DP88, CH91]). In this Qualitative Possibility Logic (QPL), the interpretation of $\varphi \geq \psi$ is: 'φ is at least as *possible* as ψ'. QPL consists of our axioms A19-A23 and moreover replaces our additivity axiom (A24) by a property called *disjunctive stability*:

A24' $(\varphi \geq \psi) \to ((\varphi \vee \chi) \geq (\psi \vee \chi))$

By choosing $\chi = \neg\psi$, we obtain that $(\varphi \geq \psi) \to ((\varphi \vee \neg\psi) \geq \top)$ is a tautology. If we now take $\varphi = \neg\psi$, we get $(\neg\psi \geq \psi) \to (\neg\psi \geq \top)$ as a tautology. So we see, (since A20 is also valid for this 'possibilistic >', yielding $(\neg\psi \geq \psi) \vee (\psi \geq \neg\psi)$), that for each formula ψ, either ψ itself or its negation is at least as possible as any tautology.

We conclude by mentioning some analogies between probabilistic (qualitative, modal) logics and possibilistic (qualitative modal) logics. Like Kraft, Pratt and Seidenberg ([KPS59]) gave sufficient conditions for a relation '≥' on $\mathcal{P}(W)$ in order to correspond with a probability measure on W, Dubois ([Du86]) gave such conditions for '≥' on $\mathcal{P}(W)$ in order to correspond with a *possibility* relation on W. And, like Segerberg and Gärdenfors ([Se71, Gä75]) used the result of [KPS59] to obtain a qualitative probabilistic modal logic, Dubois and Prade ([DP88]) used Dubois' result to obtain a qualitative possibilistic modal logic. Farinâs del Cerro and Herzig ([FC91]) then showed that the latter result is equivalent to a conditional logic introduced by Lewis [Le73]). Moreover, in [CH91], some natural alternative semantics are given for qualitative possibilistic logic (*sphere* semantics), and possibilistic modal logic (multi-relational models). Since we think that given the two interpretations for '≥', possibilistic modal logic is the natural counterpart of graded modal logic, it seems interesting to find the graded analogue of this multi-relational semantics.

4 Conclusion

We discussed several means to enrich the modal language with quantitative and qualitative operators. We showed, in a technical sense, that they indeed extend the modal language, and we also gave several examples to use this greater expressiveness. Those examples all had an epistemic flavour: the modalities were used to reason about knowledge and uncertainties. Other applications of the graded modal language are to be found in the field of generalized qauntifier theory ([HR93a]), concept languages ([HR93b]), or to obtain results on expressibility of trees ([Sc92]) or in computational semantics ([Be87]).

Although probability theory is a well developed discipline in mathematics, not many approaches have been given to employ a logic that enables one to explicitly reason about probabilities. We discussed some way to embed this in a modal framework, and we mentioned some problems. The qualitative modal operator '≥' can be defined on kripke models in which the primitive underlying feature is either 'counting' or 'measuring': for the logic of '≥', this makes no difference. If we relate it to a notion of 'possibility', however, the behaviour of '≥' changes.

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