Shape from Diameter: Negative Results

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Abstract

Our objective is to automatically recognize parts in a structured environment (such as a factory) using inexpensive and widely-available hardware. We consider the planar problem of determining the convex shape of a polygonal part from a sequence of projections. Projecting the part onto an axis in the plane of the part produces a scalar measure, the *diameter*, which is a function of the angle of projection. The diameter of a part at a particular angle can be measured using an instrumented parallel-jaw gripper.

In this paper, we present the negative result that shape cannot be uniquely recovered: for a given set of diameter measurements, there is an (uncountably) infinite set of polygonal shapes consistent with these measurements. Since most of these shapes have parallel edges of varying lengths, we also consider the related problem of identifying a representative polygon with no parallel edges. We show that given a diameter function, deciding whether such a polygon exists is *NP*-Complete.

1 Introduction

In automated manufacturing it is often useful to sort parts according to shape. A common approach is to use machine vision, which can be sensitive to lighting conditions and requires coordination with a programmable manipulator. In this paper we explore an alternative approach that uses an inexpensive modification to the parallel-jaw gripper. For the class of convex parts with constant polygonal cross section (2.5 D parts), we consider the following problem: recover the shape of a part's cross section by grasping the part with a parallel-jaw gripper and measuring the distance between the jaws: the *diameter* of the part at some angle. Can a sequence of such measurements be used to determine part shape? Note that the answer is clearly negative if we admit curved parts, for example a circle cannot be distinguished from a Reuleaux triangle (cf. the Wankel rotary engine) merely by grasping and measuring its diameter.

We present the negative result that shape cannot be uniquely recovered even for polygonal parts: for a given set of diameter measurements: there is an (uncountably) infinite set of polygonal shapes consistent with these measurements. Since most of the shapes in this set have parallel edges of varying lengths, we consider the related problem of identifying a representative polygon with no parallel edges. We show that given a diameter function, deciding whether such a polygon exists is *NP*-Complete.

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These results motivate us to consider the problem of *recognizing* a part from a known (finite) set of parts. The results on this problem are detailed in [15]. The results of this technical report and those in [15] will appear in the International Journal of Robotics Research [16].

1.1 Preliminaries

Let S^1 denote the space of planar orientations $[0, 2\pi)$ and $\mathcal{R}_+$ the set of positive reals. Given a fixed part P in an $x-y$ coordinate frame, the diameter function $d : S^1 \to \mathcal{R}_+$ of P can be formally defined as follows. Imagine two (infinite) parallel lines l, h (supporting lines) both making angle ϕ with the x-axis, just touching P so that P lies entirely in the region between the two lines. In such a case we say that the supporting lines l, h are at *orientation* ϕ with respect to the (fixed) polygon P. The diameter of P at orientation ϕ, $d(\phi)$, is the distance between the two lines that are at orientation ϕ. The diameter function is continuous and has period π. See Fig. 1. Also, the diameter function of a part is the diameter function of its convex hull. Therefore we can only seek shape recovery of the *convex hull of a part* from its diameter function [7].

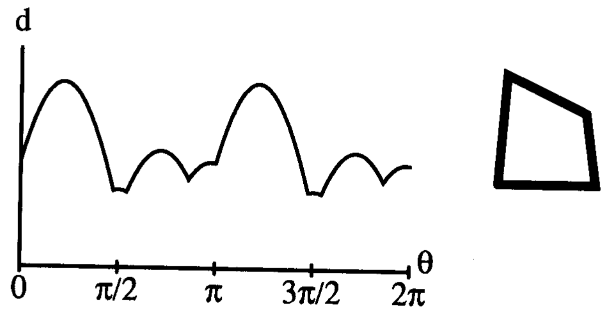


Figure 1: The diameter function for the four-sided part shown at the right.

Call a diameter function *valid* if it is the diameter function of a polygon. We begin, in Section 1.3 by characterizing valid diameter functions. In Section 2 we show that for every valid diameter function d there exist infinitely many polygons consistent with it. Thus, complete shape recovery of a polygon from its diameter function is impossible. However, we show that the orientation of every edge of the polygon and partial perimeters of the polygon along any orientation are recoverable from the diameter function. Looking at the proofs of these results, it becomes natural to consider shape recovery for the restricted class of polygons having no parallel edges (we call such polygons *Minimal polygons*). In section 2.1 we show that although the number of minimal polygons consistent with a given diameter function is finite, complete shape recovery still impossible. Also we show that deciding the question, "Given valid diameter function d, is there a minimal polygon consistent with d?" to be *NP*-complete.

1.2 Related work

The concept of *diameter* of a set of points, the maximizing distance over all pairs of points, is well studied in computational geometry [12, 4]. Diameter functions, termed width functions in [18], were applied by Jameson [10] to determine grasp stability for a part grasped in the jaws of a parallel-jaw gripper. Goldberg [8] used the diameter function to generate plans, in $O(n^2)$ time, to orient n-gonal parts. Rao and Goldberg [14, 13] extend these results to curved parts.

Our work has some relation to geometric probing which was introduced by Cole and Yap [3] and inspired by work in robotics and tactile sensing [9, 6]. Further related work is presented in [15].

1.3 Valid diameter functions

We call a diameter function *valid* if it is the diameter function of some polygon. Since the concept of a valid diameter function is so central to this paper, we now set about trying to characterize valid diameter functions.

A polygon P is specified by its n vertices in counter-clockwise order $v_0, v_1, \ldots, v_{n-1}$. Let $\oplus$ denote addition modulo n. The orientation of an face of P, $v_i, v_{i\oplus 1}$ is the angle made by it *bmodπ* *w.r.t* the positive x-axis. Let ϕ denotes an arbitrary orientation in S^1.

A continuous function $f : S^1 \rightarrow \mathcal{R}_+$ is said to be a "good" piecewise sinusoidal function (gpsf) if there exists a finite integer $Z \geq 4$, and a cyclic ordering of orientations

$$\phi_0 < \phi_1 < \ldots < \phi_{Z-1} < \phi_Z = \phi_0$$

such that $\forall j \in \{0, 1, \ldots, Z-1\}$, and $\forall \phi \in [\phi_j, \phi_{j+1}] : f(\phi) = l_j \cos(\phi + \alpha_j)$, for some $l_j \in \mathcal{R}_+$, and $\alpha_j \in S^1$.

Notice that a gpsf is continuous, single valued, and has a finite number of local maxima and local minima. A gpsf is differentiable at all but a finite number (at most Z) of orientations in S^1. For a gpsf f, let $MAX(f), MIN(f)$ denote the finite set of local maxima, local minima orientations, respectively. For a gpsf f, let $Z(f)$, the *size* of f, denote the finite integer Z, and $\Phi(f)$, the *set of transition orientations* of f, denote $\{\phi_0, \ldots, \phi_{Z-1}\}$ from the definition of the gpsf f. If $|S|$ refers to the cardinality of a finite set S, then for a gpsf f, $|MAX(f)| = |MIN(f)|$ and $|\Phi(f)| = Z(f)$.

From now on all sets of orientations, such as $\Phi(f), MIN(f), MAX(f)$, and others to be defined later, will be treated as ordered (circular) lists of orientations, *i.e.* their elements will be assumed sorted in a circular list. Two orientations ϕ_a, ϕ_b are said to be *adjacent* with respect to some property if they are adjacent in the (circular) list of all orientations having that property. For example, ϕ_a, ϕ_b are adjacent local maxima in a gpsf f if they are adjacent in $MAX(f)$. From the definitions of local maxima, local minima, notice that if we merge the circular lists $MAX(f), MIN(f)$, we get a new list $MAXMIN(f)$ in which elements of $MAX(f), MIN(f)$ alternate. That is, between every two adjacent local maxima, lies a unique local minima (and vice versa).

Theorem 1 *A function f is a valid diameter function if and only if*

1. *f is a gpsf,*
2. *f has period π, and*
3. $\mathrm{MIN}(f) \subseteq \Phi(f)$, $\mathrm{MAX}(f) \cap \Phi(f) = \emptyset$ *(that is, parameters of the sinusoid change at every local minima and never change at any local maxima).*

Proof: *"Only if:"* [7] states 1 and 2 as properties of valid diameter functions. Parameters of the sinusoid change at every local minima by the following argument. Let δ be an infinitesimal positive orientation. WLOG assume that x is an arbitrary local minima orientation. This implies that, at orientation x, an face $v_i v_{i\oplus 1}$ of P must be flush with one of l, h, say l. At orientation $x - \delta$, only one of v_i, v_j, say v_i, is in contact with l. Then, at orientation $x + \delta$ only v_j is in contact with l.

Let l_k, α_k (*resp.* l_{k+1}, α_{k+1}) be the parameters of the sinusoid in the region including $x - \delta$ (*resp.* $x + \delta$). Since these regions are closed, the diameter at x, $d(x) = l_k \cos(x + \alpha_k) = l_{k+1} \cos(x + \alpha_{k+1})$. x is local minima orientation of $d \Leftrightarrow d(x+\delta), d(x-\delta) > d(x)$. If $l_k = l_{k+1} > 0$ and $\alpha_k = \alpha_{k+1}$, then it must be the case that x is a minima of the sinusoidal function $\forall\theta, g(\theta) = l_k \cos(\alpha_k + \theta)$. Since $l_k > 0$, we have that all minima of g occur when g is negative. Therefore $g(x) < 0$. However, this is a contradiction since $g(x) = d(x)$ and $d(x) > 0$. Therefore, at least one of the parameters l, α have to change at local minima. Parameters never change at local maxima since a local maxima orientation x implies that only two vertices of P are in contact with l, h. These same vertices remain in contact at orientations $x \pm \delta$. Therefore, the same sinusoid "continues through" x.

"If:" Given function f satisfying 1., 2., 3. We show that the function f is a valid diameter function by constructing a polygon P that has f as its diameter function. Let 0 and $\phi*$, with $\phi* > 0$, be two adjacent local maxima in a gpsf f.[4] $\phi* \leq \pi$ (since π is a local maxima – due to 2). WLOG let there be 3 orientations, ϕ_1, ϕ_2, ϕ_3 from $\Phi(f)$, the set of transition orientations of f, in $(0, \phi*)$. The same proof can be easily extended whatever be the number of orientations from $\Phi(f)$ in $(0, \phi*)$. Exactly one of ϕ_1, ϕ_2, ϕ_3 is a local minima, WLOG ϕ_2.

Because of 2., Z (the size of f) is even. The polygon P that we construct will have $Z/2$ pairs of parallel faces (Z faces in all). Let P have centroid O. See Fig. 2. Let A, B be the vertices in contact with l, h, respectively, at orientation 0, assumed a local maxima. Since parameters do not change at local maxima (condition 3.) the entire polygon we construct should lie between perpendiculars to AB at A, B, *i.e.* wholly between l, h at orientation 0, and none of the 4 faces incident on A, B must be flush with l, h.

Now $|AO| = |OA| = f(0)/2$. Since we have assumed that there are three orientations, ϕ_1, ϕ_2, ϕ_3 from Φ, there will be three vertices v_1, v_2, v_3 encountered by h between orientations $(0, \phi*]$. Face Av_1 with orientation ϕ_1, v_1v_2 with orientation ϕ_2 (a local minima), and v_2v_3 with orientation ϕ_3.

Since h is in contact with v_3 for the local maxima at $\phi*$, v_3 can be specified as the point such that $\angle v_3OA = \phi*$ and $|v_3O| = f(\phi*)/2$. Specifying v_1 and v_2 is done by specifying the lengths of faces Av_1 and v_1v_2. These two lengths are, respectively, $\frac{t(\phi_1)}{2}, \frac{t(\phi_2)}{2}$, where

$$t(\phi_1) = \frac{f(\phi_2) - f(0)\cos(\phi_2)}{\sin(\phi_2 - \phi_1)}.$$

and

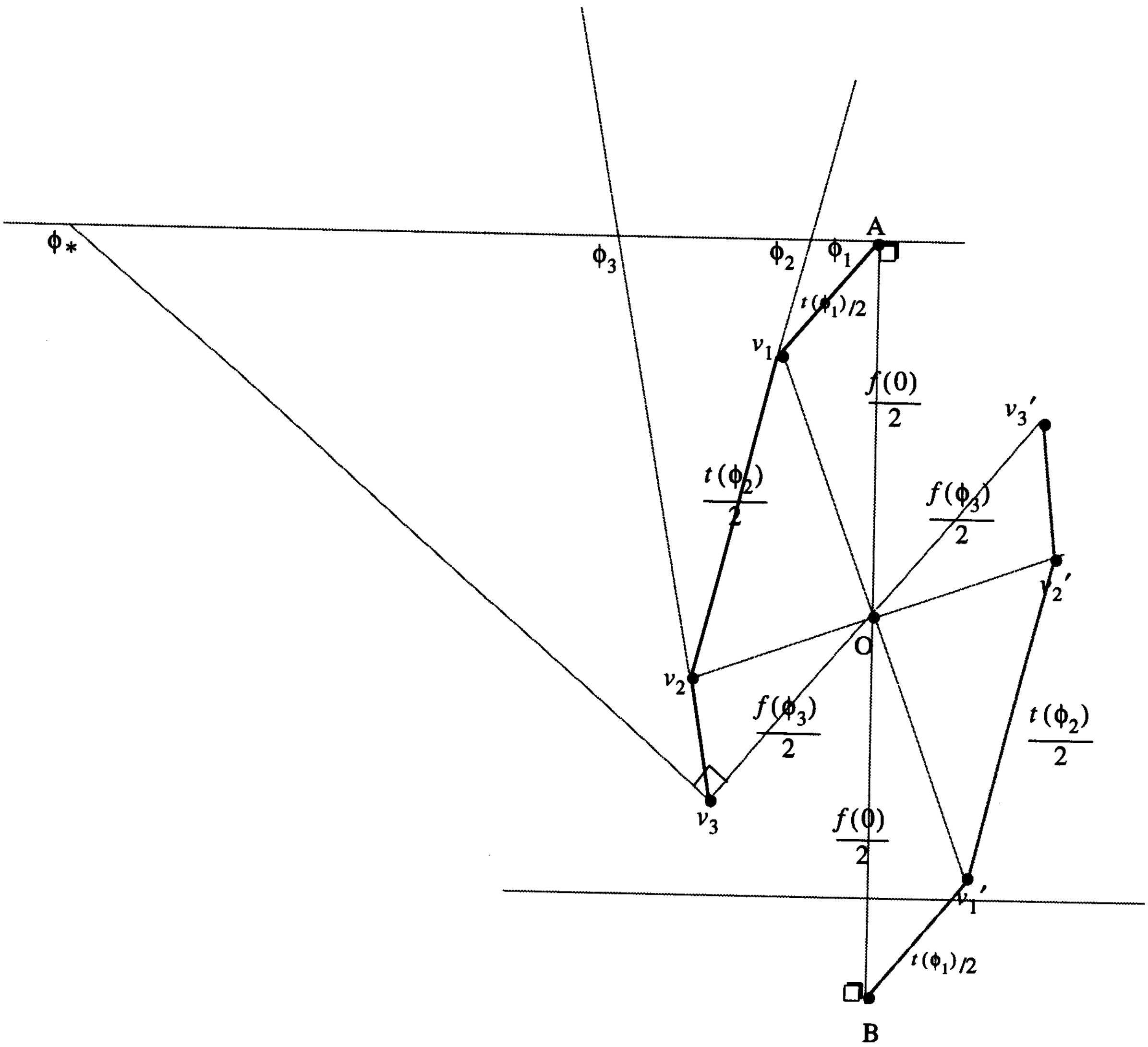
$$t(\phi_2) = \frac{f(\phi_3) - L\cos(\phi_3 + \alpha)}{\sin(\phi_3 - \phi_2)},$$

L, α being the parameters of the sinusoid between ϕ_1, ϕ_2.

It can be verified for now (this will be explained in Lemma 2) that if v_1, v_2, v_3 are specified in this way, and v_1', v_2', v_3' being their respective images through O, then the polygon $P(0, \phi*) \stackrel{\text{def}}{=} A, v_1, v_2, v_3, B, v_1', v_2', v_3', A$ has diameter function equal to f between orientations $[0, \phi*]$. Similarly we construct $P(\phi*, \phi_2*)$, where ϕ_2* is the first local maxima greater than $\phi*$, and so on until $P(\phi_k*, \pi)$, is constructed, where ϕ_k* is the largest local maxima less than π.[5] The polygon formed by taking the convex hulls of all these polygons is the desired polygon fully consistent with diameter function f. □

[4]WLOG zero may be chosen as one of the local maxima, because of invariance of d, P with respect to any constant rotation.

[5]π itself being a local maxima since 0 is assumed to be one.



Portion of polygon P (in thicker lines) that has f as its diameter function between orientations $[0, \phi *]$, both local maxima. The rest of P is similarly constructed. All angles and length of sides shown are obtained from the gpsf f.

Figure 2: Constructing polygon P consistent with gpsf f.

1.4 Further notation and an initial result

Let d denote a valid diameter function and P a polygon. Unless otherwise specified, d is the diameter function of P. Two circular lists of orientations are equal if they are equal after some fixed orientation (possibly zero) is added every element of one of them. From now on, maxima, minima stand for local maxima, local minima (in a diameter function), respectively.

Orientations in $k(d) = \Phi(d) - MIN(d)$ are called *kink orientations*, or more simply *kinks*. That is, kinks are the non-minima orientations at which the parameters of the sinusoid describing d change. Kinks and minima, *i.e.* orientations in $\Phi(d) = MIN(d) \cup k(d)$, are all and the only orientations at which an face of the polygon P is in contact with (at least) one of l, h. For example, an obtuse angled triangle has only one minima between $[0, \pi))$ when the largest side is in contact with one of l, h. When one of the other two sides is flush with the lines l, h, we get a kink orientation. Let $MINMAXKINK(d)$ denote a list of all maxima, minima, kinks in the diameter function d.

Let Φ_P denote the set of angles (module π) that the edges of polygon P make with the x-axis. Φ_P is $\Phi(d)$ restricted to the range $[0, \pi)$. Let m, k, respectively denote the number of minima, kinks in $[0, \pi)$, in the diameter function d of an n-gon P.

Consider orientations ϕ in which an face of P is flush with one of l, h. These are precisely the orientations in $\Phi(d)$. If the orientation is stable under jaw action (replacing l, h by a parallel jaw gripper nad squeezing, see [15]), it is a local minimum in d, and otherwise it is a kink orientation. Since there are n faces, we would expect $n = m + k$. This is true if P did not have any pairs of parallel faces. Let p be the number of pairs of parallel faces of P. Then a simple counting argument gives:

Lemma 1 $n - p = m + k$. □

Notice that the quantities on the left side n, p are the polygon's geometrical properties, while the quantities on the right m, k are properties of its diameter function. Let us refer to an n-gon having p pairs of parallel sides as an n,p-polygon. A diameter function having m minima and k kinks is an m,k-diameter function. Thus, it is $n - p$, rather than n alone that decides how "complex" the diameter function of the n,p-polygon is. Parallelograms (4,2-polygons) give the "simplest" diameter functions in the sense that they are the only polygons (among convex polygons) that have $n - p = 2$. Triangles (3,0-polygons), trapeziums (4,1-polygons), 5,2-pentagons, and 6,3-hexagons are the next simplest having $n - p = 3$. All other polygons have $n - p > 3$.

Corollary to Lemma 1 If an n_1, p_1-polygon and an n_2, p_2-polygon have the same diameter function, then $n_1 - p_1 = n_2 - p_2$. □

This is our first step towards shape recovery from diameter function.

2 Negative results

In this section we present our results implying that complete shape recovery of a planar part from its diameter function is impossible. We begin by investigating conditions for a polygon to be consistent with a given diameter function within a range of orientations (Lemma 2). Theorem 2 presents a necessary and sufficient condition for two polygons to have the same diameter function. Theorem 3 shows that there are infinitely many polygons, all satisfying the conditions of Theorem 2, and all having the same diameter function. From the proofs of these negative results for the general class of polygons, it becomes natural to seek shape recovery from diameter function for a special class of polygons: namely those without any pair of parallel faces. We call such polygons *minimal polygons* and consider the shape recovery problem restricted to these polygons in Section 2.1.

Let $t_P(\phi)$, $0 \leq \phi < \pi$, be zero if $\phi \notin \Phi_P$; otherwise it equals the sum of the lengths of all faces[6] of P that have orientation ϕ. $t_P(\phi)$ is called the *perimeter of P at orientation* ϕ. Let $t_P = \{t_P(\phi)|\phi \in \Phi_P\}$ sorted in the order of increasing ϕ. The subscripts P are dropped if we are discussing only one polygon.

A polygon P is said to be *consistent* with a valid diameter function d *between orientations* $[\phi_a, \phi_b]$ if the diameter function of P matches d between orientations $[\phi_a, \phi_b]$. This is written as $P \sim d[\phi_a, \phi_b]$.

Lemma 2 *Let P be a polygon and d some valid diameter function, not necessarily that of P. Further, let* $0 < \phi_1 < \phi_2$ *be an adjacent triplet of orientations in* $\Phi(d)$ *and also in* Φ_P. *Let P be consistent with d at orientations* $0, \phi_2$. *Then,*

$$P \sim d[0, \phi_2] \Leftrightarrow t_P(\phi_1) = \frac{d(\phi_2) - L\cos(\phi_2 + \alpha)}{\sin(\phi_2 - \phi_1)},$$

where L, α *are the parameters of the sinusoid in d between* $(0, \phi_1)$, i.e. $L\cos(\alpha) = d(0), L\cos(\alpha + \phi_1) = d(\phi_1)$.

Proof:

($\Leftarrow$) It is enough to show that P is consistent with d at ϕ_1 as well. That is, we need to show that $d(\phi_1)$ can be obtained from $t(\phi_1)$. Consider the equations $L\cos(\alpha) = d(0)$ and $L\cos(\phi_2 + \alpha) = d(\phi_2) - t(\phi_1) \cdot \sin(\phi_2 - \phi_1)$.

L, α can be determined (uniquely) from these equations (this is also shown in Appendix A) and $d(\phi_1)$ can be obtained as $L\cos(\alpha + \phi_1)$.

($\Rightarrow$) We first prove this portion assuming orientation 0 is a kink. Consider Fig. 3. In this figure, AA', BB' are faces of P at orientation 0. AC, BD are faces at orientation ϕ_1 and CE, DF are faces at orientation ϕ_2. Some of these faces could be of length 0. Let BX be a line perpendicular to the x-axis. Now $|AB| = L$ and $\angle ABX = \alpha$, the parameters of the sinusoid between $0, \phi_1$.

$|AC| + |BD| = t(\phi_1)$. Extend BD to Q so that $|DQ| = |AC|$. Draw lines parallel to CE (*i.e.* lines making angle ϕ_2 with the positive x-axis) through A, B. Draw a line perpendicular to these lines through Q intersecting them at S, T as shown. Extend BT to point R so that triangle ARB is right angled at R.

Now $ASTR$ is a rectangle and so $|ST| = |AR| = L\cos(\alpha + \phi_2)$. Also, $|TQ| = |BQ|\sin(\phi_2 - \phi_1)$. Finally notice that $|BQ| = t(\phi_1)$ and $|QS| = |TQ| + |ST| = d(\phi_2)$ to complete the proof.

Now let orientation 0 be a minima orientation. For simplicity, assume that there is only one face of P with orientations $0, \phi_1, \phi_2$ (the general case of there being two faces is very much like the proof above). See Fig. 4. AA' is the face of orientation 0. Faces AC, BD are of orientations ϕ_2, ϕ_1, respectively.

As before BX is perpendicular to AA' and $|AB| = L, \angle ABX = -\alpha$; L, α being the parameters of the sinusoid between $0, \phi_1$. Extend C (if necessary) to C' so that $C'D$ is perpendicular to AC.[7] T is a point on $C'D$ so that BT is perpendicular to $C'D$. Extend BT to R so that ARB is a right angled triangle. Now $ARTC'$ is a rectangle.

$|C'T| = |AR| = L\cos(\phi_2 + \alpha)$. $|BD| = t(\phi_1) = |TD|/\sin(\phi_2 - \phi_1)$. $|TD| + |C'T| = d(\phi_2)$. This completes the proof. □

[6]There can be at most two such faces due to the convexity assumption on P.
[7]The extension will be necessary iff ϕ_2 is a kink.

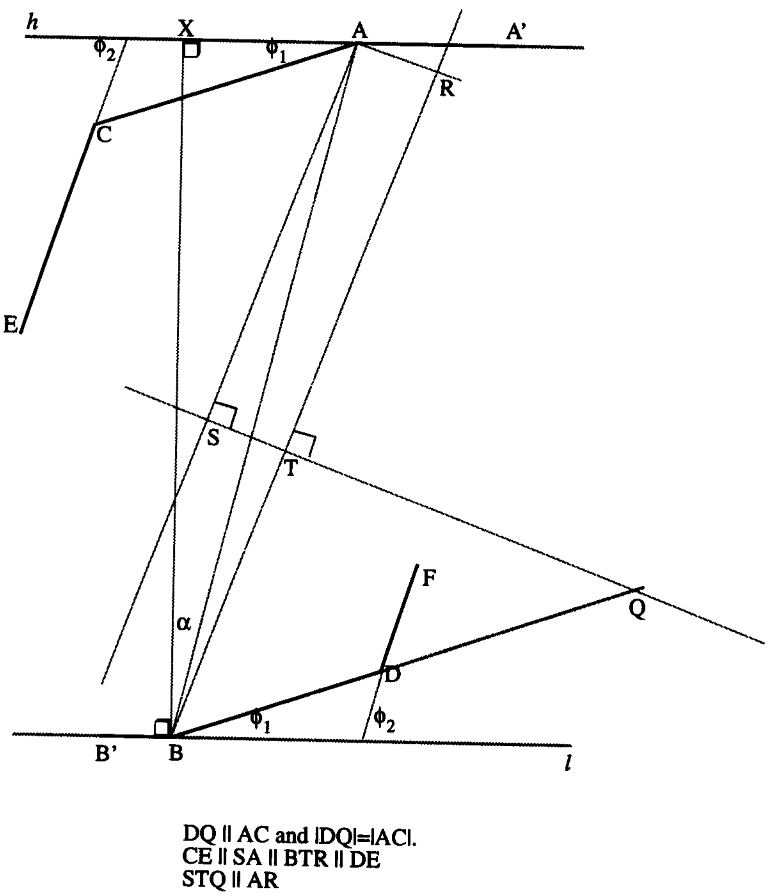
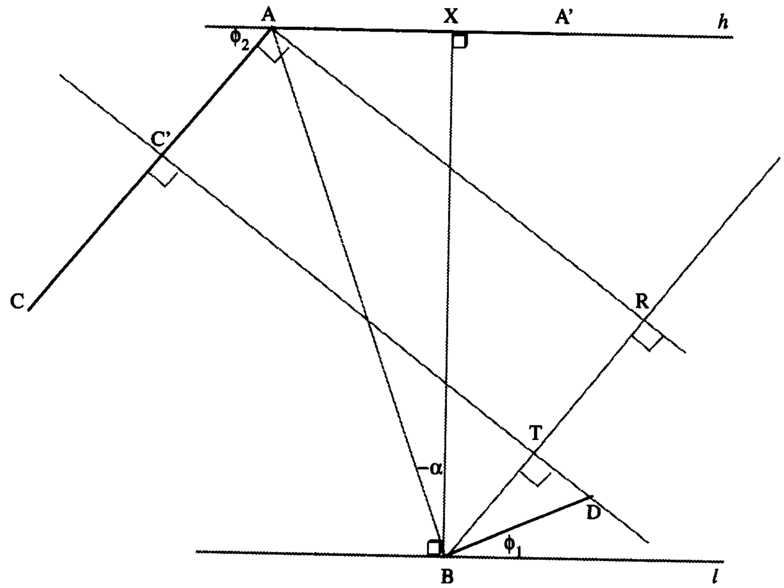


Figure 3: Orientation 0 is a kink.



AR || C'TD
AC'C || RTB

Figure 4: Orientation 0 is a minima.

Corollary to Lemma 2 Refer to the statement in Lemma 2.

There exists a unique triangle consistent with this diameter function if $\Phi_P = \{0, \phi_1, \phi_2\}$ *are in the order (kink,minima,kink) and* $\alpha + \phi_2 = \frac{\pi}{2}$.

There exist infinitely many pentagons (and other $n > 5$ *polygons) consistent with this diameter function between orientations* $[0, \phi_2]$.

Proof: To see the infinitely many pentagons, refer again to Fig. 3. Extend CE to meet BB' extended at Y. Now $ACYB'BA$ is a pentagon consistent with d between $[0, \phi_2]$. Only the sums of $|AC|$ and $|BD|$ are constrained.

The unique triangle ABC having this diameter function is obtuse-angled. Two of it's angles are $\phi_2 - \phi_1, \pi/2 + \alpha$ and a side has dimension $t(\phi_1)$. $\Box$

Now we are ready to prove the following theorem.

Theorem 2 *Two polygons* P, Q *have the same diameter function if and only if* $\Phi_P = \Phi_Q$ *and* $t_P = t_Q$.

Remark: From Cauchy's surface formula (See [1]), it follows that the integral of the diameter function of any planar part equals its perimeter. Thus, two parts having the same diameter function must have the same perimeter (but not necessarily vice versa). Our theorem above states something stronger for the class of polygonal parts – namely that the two parts must have the same set of partial perimeters.

Proof: *Only if:* Let P, Q both have diameter function d. Then $\Phi_P = \Phi_Q$ since each is equal to $\Phi(d)$ mod π. Let ϕ_1, ϕ_2, ϕ_3 be an adjacent triplet of orientations[8] in $\Phi(d)$.

[8]Recall from the definition of an gpsf that $\Phi(d)$ must have at least four transition orientations.

Assume WLOG that $\phi_2 \in [0, \pi)$. Then Lemma 2 gives a formula for $t(\phi_2)$ that must be satisfied by both polygons. A generalization of this shows that $t_P = t_Q$.

If: Let the two polygons have (valid) diameter functions d, d'. We prove $d = d'$ by showing that one can reconstruct a *unique* diameter function d, given these two lists Φ_P, t_P.

Let the given Φ_P have $Z = m + k$ orientations: $\Phi_P = \{\phi_0, \ldots, \phi_{Z-1}\}$. Let the diameter function between ϕ_j, ϕ_{j+1} (assume $j \pm 1$ are modulo Z) be the sinusoid $l_j \cos(\phi + \alpha_j)$. Thus we have the equations

$$l_j \cos(\phi_j + \alpha_j) = d(\phi_j), l_j \cos(\phi_{j+1} + \alpha_j) = d(\phi_{j+1}).$$

The unknowns are $d(\phi_j), \alpha_j, l_j$. If these are recovered, it is obvious that d is. This gives us $2Z$ equations in $3Z$ unknowns. The remaining Z equations are obtained from t information using Lemma 2:

$$t(\phi_j) = \frac{d(\phi_{j+1}) - l_{j-1}\cos(\phi_{j+1} + \alpha_{j-1})}{\sin(\phi_{j+1} - \phi_j)}.$$

We solve these $3Z$ equations for the $3Z$ unknowns $(\alpha_j, l_j, d(\phi_j))$. We know that *at least* one solution for the $3Z$ equations exists since d, d' are valid. However, there could exist more than one solution giving different d, d'. We show in Appendix A that at most one solution for these $3Z$ equations exists. Thus, the diameter function is uniquely constructed. □

In a sense, Φ_P, t_P is the *maximal non-redundant (and invertible) information* of the geometry of a polygon P obtainable from its diameter function, Φ_P giving the orientations of the faces of P, and t_P the perimeter along each orientation. However, the two lists do not completely determine P because there could be up to two faces along an orientation $t(\phi)$ and the $t(\phi)$ constraint is *only a constraint on the sum of the length of the two faces.* In fact, as Theorem 3 shows, there are infinitely many polygons P having the same valid diameter function (and hence the same Φ_P, t_P).

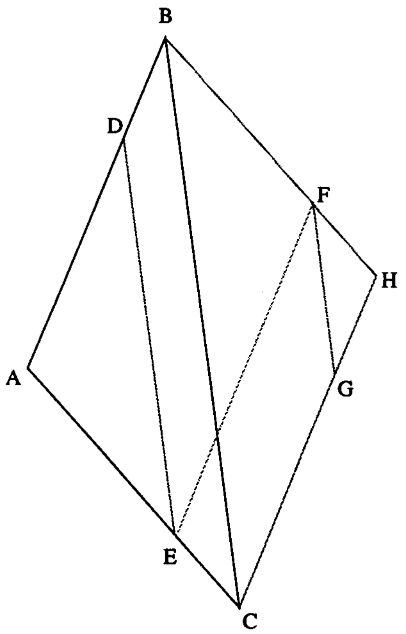
Diameter functions of parallelograms (4, 2-gons) are termed *trivial.*

Theorem 3 *For every non-trivial valid diameter function d there exist infinitely many polygons having diameter function d.*

Proof: Fig. 5 shows this is true for diameter functions of triangles and quadrilaterals. See also Fig. 6. Towards the generalization assume that P is a polygon having diameter function d. P exists since d is valid. Let A, C be two vertices of P touching l, h at a maxima orientation. Let this maxima orientation be the zero orientation, WLOG. Let D, B be the vertices adjacent to C in P (*i.e.* DC, BC are two faces of P). Likewise, let D^*, B^* be the vertices adjacent to A. D^* and D are on the same side of AC (as are B^* and B). D^* (*resp.* B^*) could be coincident with D (*resp.* B). For example, in Fig. 5: the quadrilateral case, $D = D^*, B = B^*$. Let $\phi_1, \pi - \phi_2, \phi_3, \pi - \phi_4$ be the orientations of faces CB, AB^*, AD^*, CD, respectively. Without loss of generality assume $\phi_2 < \phi_4, \phi_1 < \phi_3$. The other cases (including equality) are treated similarly. See Fig. 7. F', E' are can be arbitrarily chosen on CB, AB^*, respectively. G' is such that CG' is parallel and equal to B^*E'. Thus we have $t(\phi_2) = |AB^*| = |AE'| + |CG'|$. H' is determined similarly. It is defined so that AH' is parallel and equal to BF'. Now, $t(\phi_1) = |CB| = |CF'| + |AH'|$.

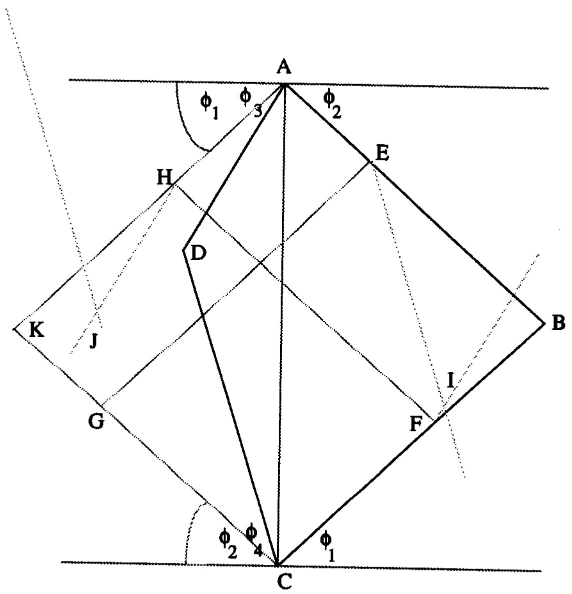
A line is drawn parallel to AD^* (*resp.* (DC)) through H' (*resp.* G'). Points $D^{*\prime}, D'$ are chosen on these two lines so that the distance between $D', D^{*\prime}$ is equal to that between D^*, D. Now the portion of the polygon P between D^*, D can be moved over to between $D^{*\prime}, D'$.

$B^{*\prime}, B'$ are defined in a similar manner. First a line is drawn parallel to AD^* (*resp.* DC) through F' (*resp.* E'). $B^{*\prime}, B'$ are chosen on these lines so that the distance between them equals



ABC is the given triangle.
Let x be any number such that $0<x<1$.
Pick points D,E on AB,AC , respectively,
such that $|DE| = x\,|BC|$ and DE $\parallel$ BC.
Flip the triangle about BC so that A falls on H.
CH $\parallel$ AB and BH $\parallel$ AC.
Pick points F,G on BH,CH, respectively,
such that $|FG| = (1-x)\,|BC|$ and FG $\parallel$ BC.

Now hexagon BDECGF has same diameter
function as triangle ABC, since
$|FG| + |DE| = |BC|$
$||BF| + |EC| = |AC|$
$||BD| + |GC| = |BA|$



$\phi_2 < \phi_4$, $\phi_1 < \phi_3$
(without loss of generality)

ABCD is the given quadrilateral.
Pick two numbers x,y such that $0 < x,y < 1$.
Pick points E (on AB) and F (on CD) so that
$|AE| = x\,|AB|$, $CF = y\,|CB|$.

K is a point such that AKCB is a parallelogram.
Pick points G, H, on CK, AK, respectively, so
that EG $\parallel$ BC and FH $\parallel$ BA.
Finally draw linesthrough G,H parallel to
CD, AD respectively intersecting at J.
Do the same for points E,F and let the
intersecting point be I.

Note that
1. $|CF| + |AH| = CB = t(\phi_1)$
2. $|AE| + CG| = |AB| = t(\phi_2)$
3. $|FI| + |HJ| = |AD| = t(\phi_3)$
4. $|EI| + GJ| = |CD| = t(\phi_4)$

Hence, quadrilateral ABCD and octagon AHJGCFIE
have the same diameter function.

Figure 5: Infinitely many hexagons/octagons having same diameter function as given triangle/quadrilateral.

Polygons with Identical Diameter Function

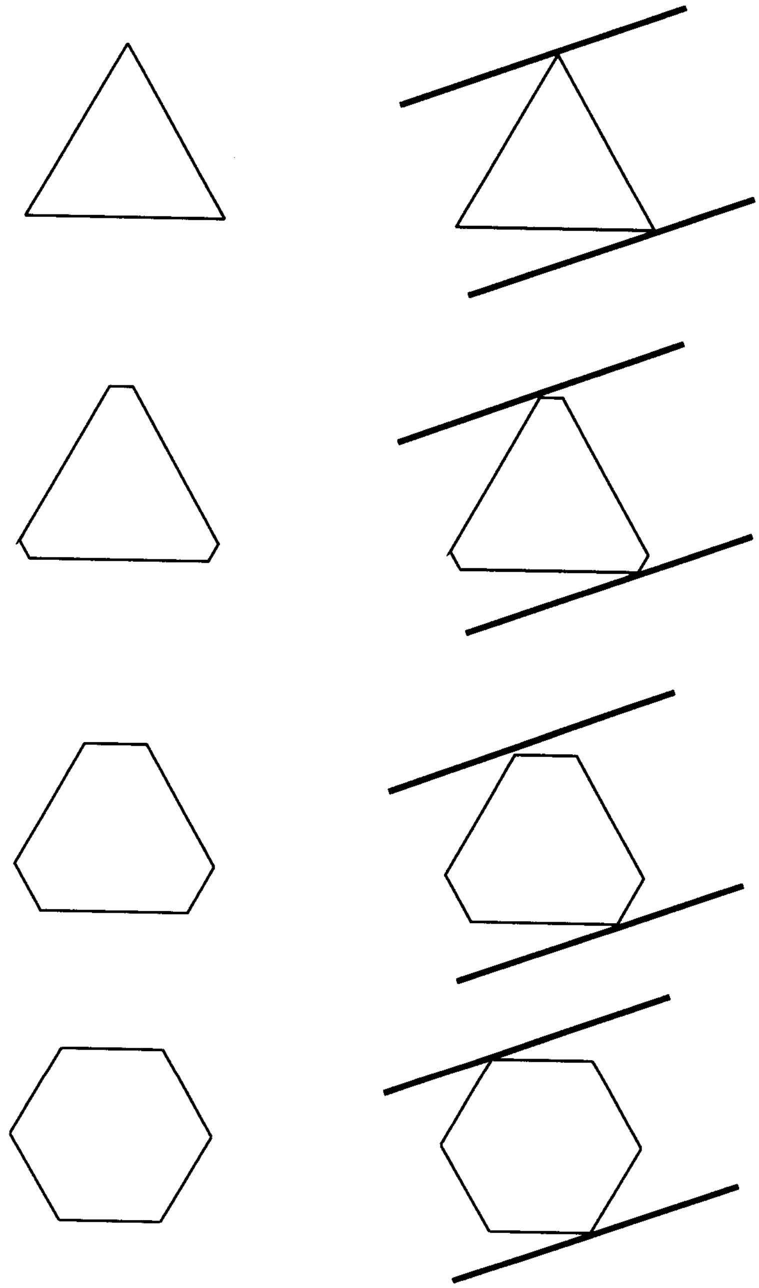
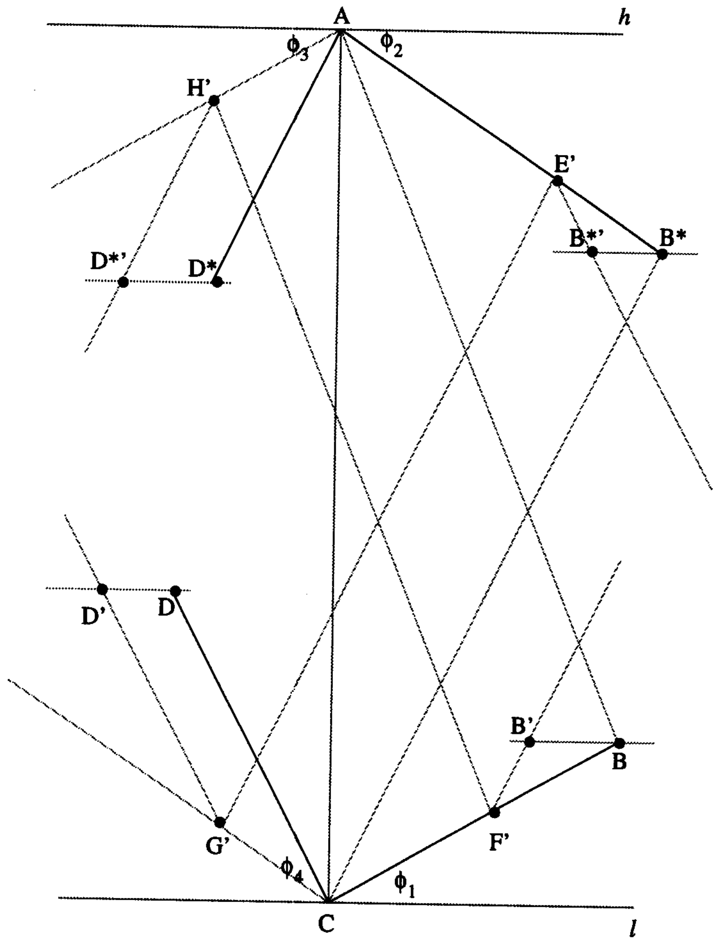


Figure 6: Triangles and hexagons having the same diameter function.



A, B*, B, C, D, D* are vertices of the original polygon P. Portions of P between B,B* and D,D* are not shown. In the new polygon P', D,D*,B,B* do not exist and are replaced by the 8 primed vertices shown. The method of construction of these eight points is shown in the proof of the theorem.

Both polygons P,P' have same perimeter along every orientation and therefore have the same diameter function.

The only parallel lines in the figure are :

$l \parallel h$
AD* ∥ H'D*' ∥ F'B'
DC ∥ D'G' ∥ E'B*'
AB* ∥ CG'
CB ∥ AH'
AB ∥ F'H'
CB* ∥ G'E'

Figure 7: Infinitely many polygons having the same diameter function as a given polygon.

that between B^*, B. The portion of P between B, B^* can be moved over to between $B', B^{*\prime}$. If this causes any problems of convexity, then take the faces $F'B'$, $E'B^{*\prime}$, and those originally between B, B^*, sort them by orientation, and arrange them between F' and E'.

Simple geometry can be applied to show that $|H'D^{*\prime}|+|B'F'| = |AD*|= t(\phi_3)$ and $|G'D'| + |E'B^{*\prime}|= |DC| = t(\phi_4)$. For example to show that $|H'D^{*\prime}|+|B'F'| = |AD*|$, draw a line through H' parallel to $D^{*\prime}D'$ intersecting AD^* at Z. Now note that triangle $F'B'B$ is congruent to triangle $H'ZA$ and so $|B'F'| = |AZ|$. Also note that $H'ZD^*D^{*\prime}$ is a parallelogram, and so $|H'D^{*\prime}|= |ZD^*|$.

Thus, the two polygons $P \stackrel{\text{def}}{=} A, D^*, \ldots, D, C, B, \ldots, B^*, A$ and $P' \stackrel{\text{def}}{=} A, H', D^{*\prime}, \ldots, D', G', C, F', B', \ldots, B^{*\prime}, E', A$ have the same diameter function by Theorem 2 since $t_P = t_{P'}$ and $\Phi_P = \Phi_{P'}$. Finally note that there are infinitely many P' since the choices of F', E' (along a line segment) were arbitrary. □

2.1 Minimal Polygons

Theorem 3 is a negative result for shape recovery from diameter: there exist infinitely many polygons consistent with a given measured diameter function. However, the proof of the theorem basically involved showing that a particular length $t(\phi)$ could be split, in infinitely many ways, into two segments (in the polygon), both of orientation ϕ whose lengths sum up to $t(\phi)$. Thus, most of these polygons would have parallel edges of varying lengths. This suggests that we might define a representative polygon as one without any parallel edges satisfying a given diameter function. Two obvious questions arise: does there always exist a representative polygon for a given diameter function; and if a representative polygon exists, is it always unique? We try and answer these questions in this section. The latter question is answered in the negative in Theorem 4 by constructing a counter-example, and the former question is shown *NP*-complete in Theorem 5. A major lemma in proving our *NP*-completeness result is showing that the problem of arranging a set of line segments, no two of which are parallel, into a convex polygon is *NP*-complete. This bears some resemblance to the result of [17] which shows that the problem of drawing (additional) line segments to connect a collection of given fixed line segments (by their end-points) into a simple circuit is *NP*-complete. In our case, we allow the segments to translate and we do not allow additional line connecting segments.

A *minimal polygon* is a (convex) polygon without any parallel edges, *i.e.* an n, p-polygon with $p = 0$.

Theorem 4 *Minimal polygons satisfying a given diameter function are not unique.*

Proof: See Fig. 8. □

The implication of this theorem is that even for minimal polygons, it is impossible to completely recover its shape from the diameter function.

Given an m, k-diameter function d, if there exists a minimal polygon P with n edges having d as its diameter function, then by Lemma 1, $n = m + k$. First notice that t_P, Φ_P are extractable from the diameter function d. We have seen above that there could be more than one minimal P. There can be at most a finite number (2^{n-1}) of possible minimal P since P is constrained to be convex. Now we tackle the question whether there always exists such a minimal P. We show that deciding this question is *NP*-complete in Theorem 5. Before we get there, we have to show some other problems *NP*-complete.

By *arranging* a set of n planar segments $S_0, \ldots, S_{n-1}$, we mean translating them in the plane so that two segments are either not intersecting, or if they intersect, they do so only at their

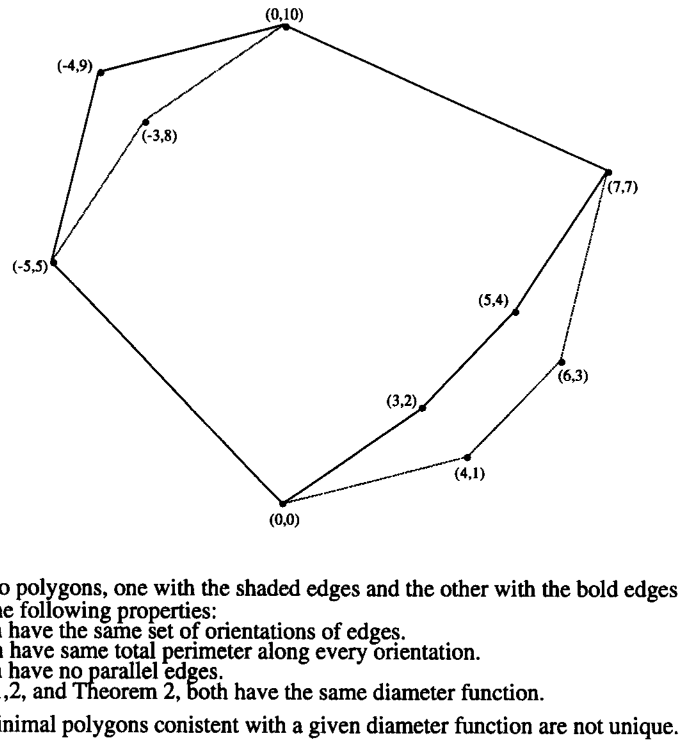


Figure 8: Two minimal polygons that have the same diameter function.

end-points. All sets in this section are multisets, *i.e.* they could contain more than one identical element. Consider the following problems.

POLYGON_FROM_SEGMENTS (PFS):

Given a set of n planar line segments, $S_0, \ldots, S_{n-1}$, each with a fixed length and orientation, can they be arranged so as to form a polygon (n-gon)?

Note that *forming a polygon* is equivalent to forming a simple polygon since the only intersections we allow between segments are at their end-points. Also note that we do not have any restriction on the orientations of the edges (any number of them could be parallel). Thus minimality of polygons is not addressed just as yet.

CONVEX_POLYGON_FROM_SEGMENTS (CPFS):

Given a set of n planar line segments, $S_0, \ldots, S_{n-1}$, each with a fixed length and orientation, can they be arranged so as to form a convex polygon (n-gon)?

MINIMAL_POLYGON_FROM_SEGMENTS (MPFS):

Given n segments, no two of which are parallel, does there exist an arrangement of them forming a convex polygon?

Lemma 3 *PFS is NP-complete.*

Proof: *PFS* is clearly in *NP* So it is sufficient to show that *PFS* is *NP*-hard. We do this by polynomial reduction to *PARTITION.*

PARTITION is the following problem and is well-known to be *NP*-complete [5].
Given a multiset of n numbers $a_0, \ldots, a_{n-1}$*, is there a set* $S \subset \{0, \ldots, n-1\}$ *such that*

$$\sum_{i \in S} a_i = \sum_{i \in \{0,\ldots,n-1\}-S} a_i.$$

We may assume that all the a_i are positive, WLOG. Given an instance I of *PARTITION* having n elements, consider the following set I' of $n+2$ segments (each segment is described as (l, ϕ), where $l \in \mathcal{R}_+$ is its length and $\phi \in [0, \pi)$ is angle made by it with x-axis):

$I' = \{(a_i, 0) | a_i \in \text{I}\} \cup \{(1, \pi/2), (1, \pi/2)\}$.

That is, I' contains n horizontal segments of widths $a_i \in I$, and two vertical segments of unit length. If the segments in I' can form a polygon, then it has to be a rectangle with the two segments $(1, \pi/2), (1, \pi/2)$ being opposite (and vertical) edges of the rectangle. Let $S = \{i | (a_i, 0)$ is a segment along the top edge of the rectangle$\}$. It is clear that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$ and hence I has a partition.

Conversely, if the numbers in I have a partition, then let S be the set of indices in one of the partitions. Then a rectangle can be formed as follows: two of the opposite vertical edges are $(1, \pi/2), (1, \pi/2)$. The segments in $\{(a_i, 0) | i \in S\}$ can form the top horizontal edge and the segments in $\{(a_i, 0) | i \in \{0, \ldots, n-1\} - S\}$ can form the lower horizontal edge. □

Corollary to Lemma 3 CPFS is NP-Complete.

Lemma 4 *MPFS is NP-complete.*

Proof: Again it is easy to see that *MPFS* is in *NP*. So it is sufficient to show that *MPFS* is *NP*-hard by transformation from *PARTITION*. Note that the proof of the *NP*-hardness of *PFS* does not apply here since the segments forming the horizontal edges of the rectangle are all parallel. The modification required from the proof of Lemma 3 is to make no two segments parallel.

Given an instance I of *PARTITION*, a multiset of n positive numbers, $I = \{a_i | 0 \leq i < n\}$, we produce in polynomial time an instance I' of *MPFS*, *i.e.* a set of $2n+3$ segments, $I' = \{s_0, \ldots, s_{2n-1}, s_{2n}, s_{2n+1}, s_{2n+2}\}$, no two of which are parallel, such that I has a partition if and only if we can form a convex polygon from I'.

Let $M = \sum_i a_i$. We may assume wlog that $M > 1$ and also that the set I is sorted in decreasing order, *i.e.* $a_i \geq a_{i+1}$.

Here we describe segments as (x, y) where x(*resp.*y) is the length of the projection of the segment on the x-axis (*resp.* y-axis). x, y are referred to as the *x-projection, y-projection*, respectively, of segment (x, y). Its length is $\sqrt{x^2 + y^2}$ and its orientation is Atan2(y, x) [9]. y is always taken ≥ 0 so that the orientation is always in $[0, \pi)$.

The exact description of the segments in I' is as follows. $\forall i \in \{0, 1, \ldots, n-1\}$ $s_{2i} = (a_i/2, M2^i + X_i)$, $s_{2i+1} = (-a_i/2, M2^i + X_i)$. And $s_{2n} = (0, M2^{n+2}), s_{2n+1} = (\epsilon, M2^{n+1}), s_{2n+2} = (-\epsilon, M2^{n+1})$.

The $X_i, 0 \leq X_i < 1/n, 0 \leq i < n-1$ can be chosen (in $O(n \log n)$ time) so as to make no two segments (among the first $2n$) parallel. $\epsilon > 0$ is so small that the segment $s_{2n+1} = (\epsilon, M2^{n+1})$ is the segment of largest orientation among all segments with orientation $< \pi/2$, and the segment $s_{2n+2} = (-\epsilon, M2^{n+1})$ is the segment of smallest orientation among all segments with orientation $> \pi/2$.

Notice that all segments have orientations $\phi \in (\pi/4, 3\pi/4)$ and exactly one (s_{2n}) has orientation exactly $\pi/2$. Also notice that no two segments have the same orientation (no two are parallel).

[9] Atan2(y, x) is the single valued inverse tangent function. For example, Atan2$(-\sqrt{3}, 1) = -\pi/3$ and Atan2$(\sqrt{3}, -1) = 2\pi/3$.

Suppose the segments in I' can be arranged to form a convex polygon P. Consider the position of s_{2n} with respect to s_{2n+1}, s_{2n+2}. Due to their orientations and the fact that P is convex the only possible configurations of these three segments[10] are shown in Fig. 9(b). Of these the first two configurations are impossible due to the y-projection considerations (since the sum of the y-projections of the adjacent segments (among $s_{2n}, s_{2n+1}, s_{2n+2}$) is greater than sum of y-projections of all other segments). Thus, only the third configuration is possible. In this configuration, let C (*resp.* C') be the upper (*resp.* lower) chain of segments between s_{2n} and s_{2n+2} (*resp.* s_{2n+1}).

For $i \in \{0, \ldots, n-1\}$, we say s_{2i}, s_{2i+1} are *twins* of each other (they have the equal y-projections and equal (but opposite) x-projections). The major claim is this proof is that every pair of twins must belong to the same chain of segments. Towards proving this claim for all twins, first consider the twins s_{2n-2}, s_{2n-1}. Let, if possible, these belong to different chains, say $s_{2n-2} \in C'$ and $s_{2n-1} \in C$ (the other case is treated symmetrically). Now since we assumed (WLOG) that the a_i were sorted, s_{2n-2} is the segment of largest orientation less than $\pi/2$, and s_{2n-1} is the segment of smallest orientation greater than $\pi/2$ among $s_0, \ldots, s_{2n-1}$, the set of twins. Thus, they have to be adjacent to s_{2n+1}, s_{2n+2} as shown[11] in Fig. 9(c). Now consider the y-projection of the four adjacent segments $s_{2n-2}, s_{2n-1}, s_{2n+1}, s_{2n+2}$. It adds up to $M(2^{n+2} + 2^n)$. The sum of y-projections of the remaining $2n-1$ segments (including s_{2n}) is strictly less than this quantity. This contradicts the assumption that the segments form a closed figure. Therefore, s_{2n-2}, s_{2n-1} have to be in the same chain C or C'. Now consider s_{2n-4}, s_{2n-3}. Essentially the same proof shows that both have to belong to the same chain since the sum of their y-projections is strictly greater than the sum of the y-projections of $s_0, \ldots, s_{2n-5}$. Continuing in this manner, it is not difficult to see that each pair of twins s_{2i}, s_{2i+1} have to belong to the same chain C or C'.

Now the x-projections of the chains C, C' have to be equal since the segments form a closed figure. Since both elements of a pair of twins exist in the same chain, the x-projection of C is equal to $\sum_{i \in S} a_i$, for some subset $S \subset \{0, \ldots, n-1\}$. Then, the x-projection of C' is equal to $\sum_{i \in S'} a_i$ where $S' = \{0, \ldots, n-1\} - S$. By the first statement of this paragraph, the two summations are equal and hence I is partitionable.

Now suppose I is partitionable, let S, S' be such that $S \cup S' = \{0, \ldots, n-1\}$, $S \cap S' = \emptyset$, and $\sum_{i \in S} a_i = \sum_{i \in S'} a_i$. We form a polygon P made up segments in I' as follows. First we form a convex chain of segments C from segments $\{s_{2i}, s_{2i+1} | i \in S\}$. This chain C is simply segments from $\{s_{2i}, s_{2i+1} | i \in S\}$ sorted in the following order. First consider all segments in this set with orientation between $(0, \pi/2)$ and sort them in decreasing order. To this sorted list, append the the segments with orientation between $(\pi/2, \pi)$ in decreasing order. The resulting list is the chain C. The first, last segments in this list will be twins, as will be the second, and second-last, and so on, till the middle two. In a similar fashion (sorting is in increasing order and the segments with orientations in $(\pi/2, \pi)$ precede the others), chain C' is formed. The polygon then is the following segments/chains in order: $s_{2n}, C, s_{2n+2}, s_{2n+1}, C'$. See Fig. 9(d). □

Now we define the following problem.

MINIMAL_POLYGON_FROM_DIAMETER_FUNCTION (MPFD):

Given an m, k-diameter function d, is there a minimal polygon P fully consistent with d?

Theorem 5 *MPFD is NP-Complete.*

[10]Remember that no segment has orientation between those of s_{2n} and s_{2n+1} and between those of s_{2n}, s_{2n+2}. Therefore, in a convex polygon, either the last three segments are adjacent and in the order $s_{2n+1}, s_{2n}, s_{2n+2}$ (Case (i) in Fig. 9(b)), or two of the three are adjacent, one of the two being s_{2n} (case (ii)), or s_{2n+1}, s_{2n+2} are adjacent (case (iii)). It cannot be the case that none of the three are adjacent.

[11]in the other case, *i.e.* $s_{2n-2} \in C, s_{2n-1} \in C'$, both s_{2n-1}, s_{2n-2} will be adjacent to s_{2n}.

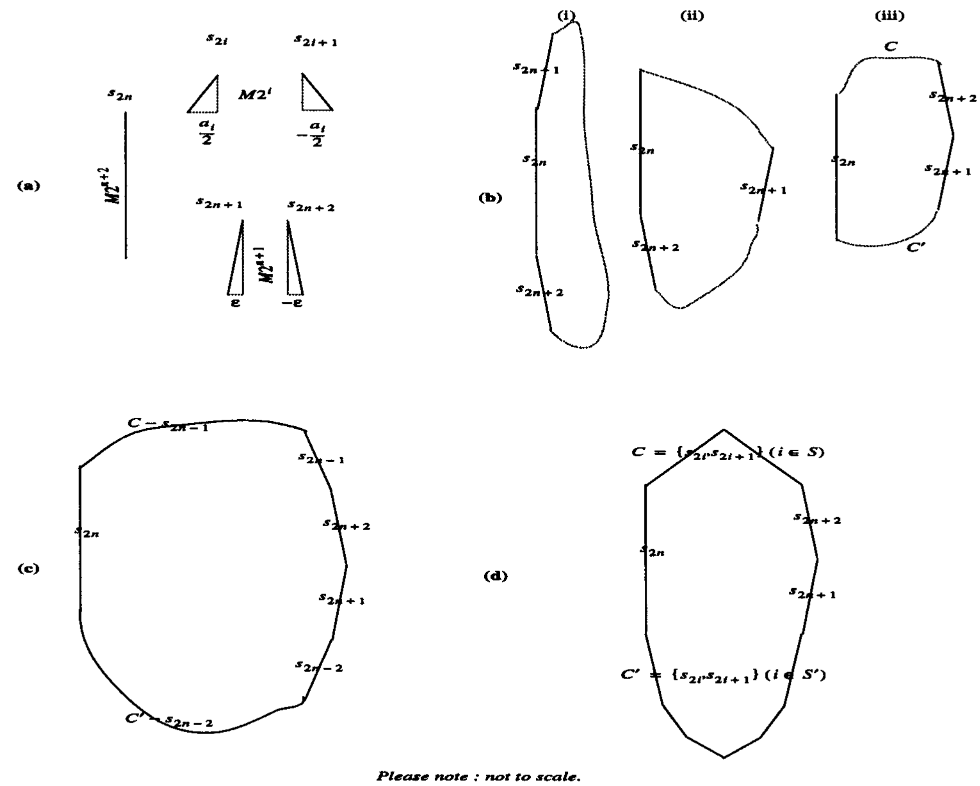


Figure 9: Various stages in the proof of the *NP*-completeness of *MPFS*.

Proof: We use the *NP*-completeness of *MPFS*. Let algorithm *MPFD(d)*, where d is an m,k-diameter function, return true (*resp.* false) according as whether there is (*resp.* is not) an $m+k,0$ polygon P fully consistent with d.

Then we solve the *MPFS* problem using the following algorithm:
INPUT: description (orientations, lengths) of n planar segments, no two of which are parallel.
OUTPUT: true/false whether or not they form a convex polygon.

1. Let Φ be a circular list of the orientations of the input segments in sorted order. Let t be a list of the lengths of the segments sorted according to the order in Φ.

2. Use equations 5 (in Appendix A) to determine whether there is some (valid) diameter function d that has $\Phi(d) = \Phi$ and $t_{P'} = t$, where P' is some polygon fully consistent with d.[12]

   If there does not exist such a d (equations 5 do not have a solution), return "FALSE" and exit.

3. We assume that there exists such a valid d. It is easy to compute d after solving equations 5. Now invoke *MPFD(d)*.

   Return "TRUE" if and only if *MPFD(d)* returned "true".

Complexity of the first step is clearly polynomial in n. The second steps involves forming and solving n linear equations in n unknowns, which basically involves inverting an $n \times n$ matrix which has polynomial complexity. Step 3 basically involves a call to *MPFD*. Therefore, if *MPFD* was polynomial time decidable, so would *MPFS*.

Correctness follows from the following. If the set of input segments forms a polygon, then the polygon would have to have a diameter function d which is easily computable given t, Φ, and d would have its minima and kinks exactly at orientations in Φ. If they did not form a polygon, then no diameter function would exist and this would be detected in step 2. above. However, it could happen that a valid diameter function d exists for a polygon P' that has $\Phi_{P'} = \Phi, t_{P'} = t$, but P' could have parallel edges. To check for parallel edges, step 3 calls *MPFD(d)*. If it returned true, then it implies the original set of segments form a convex polygon. And if the original segments form a polygon, step 3 would be executed and the call to *MPFD(d)* would return "true". □

3 Conclusion

We have considered the planar problem of determining the convex shape of a polygonal part from a sequence of measurements taken with a frictionless parallel-jaw gripper. In this report, we detail the negative results and in [15], we detail the positive results.

The negative results are related to the study of curves of constant width (*Gleichdicke*) in classical geometry, establishing a new link between computational geometry and robotics.

To extend these results to non-polygonal shapes, we note that one additional feature of non-polygonal diameter functions is the possible existence of flat regions in the diameter function. Also, the diameter function need not be piecewise sinusoidal anymore. It will be interesting to come up with an analog to Theorem 1 characterizing diameter functions of even a class of non-polygonal shapes, say the class of shapes made up of piecewise linear and piecewise circular segments (such parts are called *generalized polygonal* [11, 14]). Shape recovery is another question. The existence of curves of constant width (or "orbiforms" [2]) is known since Euler. Hence complete shape recovery

[12]Such a polygon exists since d is valid. However, remember that P' could have parallel edges.

is impossible. However, can we have analogs to Theorem 2 stating how much information about the geometry of the shape can be recovered? If so, we may be able to come up with grasp plans to recognize generalized polygonal parts from among a set.

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A Unique solution to $3Z$ equations

Here we show that the $3Z$ equations from the proof of Theorem 2,

$$l_j \cos(\phi_j + \alpha_j) = d(\phi_j), l_j \cos(\phi_{j+1} + \alpha_j) = d(\phi_{j+1}),$$

$$t(\phi_j) = \frac{d(\phi_{j+1}) - l_{j-1}\cos(\phi_{j+1} + \alpha_{j-1})}{\sin(\phi_{j+1} - \phi_j)},$$

$0 \le j < Z$, have at most one solution.

Let us abbreviate $d(\phi_j)$ as d_j, and $t(\phi_j)\cdot\sin(\phi_{j+1} - \phi_j)$ as C_j. Notice that C_j are strictly positive and are "known" quantities. Finally, we introduce Z new variables $x_j = \phi_j + \alpha_j$ to replace the α_j.

Thus, the $3Z$ unknowns are d_j, l_j, x_j, and the equivalent $3Z$ equations are:

$$l_j \cos(x_j) = d_j, l_j \cos(x_j + \phi_{j+1} - \phi_j) = d_{j+1}, \tag{1}$$

$$d_{j+1} - l_{j-1}\cos(x_{j-1} + \phi_{j+1} - \phi_{j-1}) = C_j. \tag{2}$$

A solution must comprise of strictly positive l_j, d_j, and therefore by equation 1 x_j can be restricted to the range $(-\pi/2, \pi/2)$.

Suppose we know all the d_j. Then, the x_j, l_j can be uniquely determined by the following argument. Consider equations 1. If the d_j are known, we can eliminate the l_j into equations of the form (from 1)

$$\cos(\phi_{j+1} - \phi_j) - \sin(\phi_{j+1} - \phi_j)\tan(x_j) = \frac{d_{j+1}}{d_j}.$$

From these equations, the x_j can be uniquely determined in the range $(-\pi/2, \pi/2)$. Once the x_j are uniquely determined, so are the l_j.

So the problem reduces to showing that there is at most one solution for the d_j. We do this by constructing Z linear equations in the d_j alone.

Out of the $3Z$ equations in (1),(2), consider three particular equations:

$$l_i \cos(x_i) = d_i, l_i \cos(x_i + \phi_{i+1} - \phi_i) = d_{i+1},$$

and

$$C_{i+1} = d_{i+2} - l_i \cos(x_i + \phi_{i+2} - \phi_i).$$

From the first two, we get

$$d_i \cos(\phi_{i+1} - \phi_i) - [l_i \sin(x_i)]\sin(\phi_{i+1} - \phi_i) = d_{i+1}. \tag{3}$$