A Note on the Smyth Powerdomain Construction

Peter Knijnenburg

RUU-CS-93-02
January 1993

Utrecht University

Department of Computer Science

Padualaan 14, P.O. Box 80.089,
3508 TB Utrecht, The Netherlands,
Tel. : ... + 31 - 30 - 531454

A Note on the Smyth Powerdomain Construction

Peter Knijnenburg

Technical Report RUU-CS-93-02
January 1993

Department of Computer Science
Utrecht University
P.O.Box 80.089
3508 TB Utrecht
The Netherlands

ISSN: 0924-3275

A Note on the Smyth Powerdomain Construction

Peter Knijnenburg

Department of Computer Science
Utrecht University *

1 Introduction

Powerdomains are the order-theoretic analogue of powersets and are used to model non-determinism and concurrency [Plo81, Plo79, GS90, Smy78]. There exist three standard kinds of order-theoretic powerdomain: the Plotkin, Smyth and Hoare powerdomain. To obtain a powerdomain from a given ω-algebraic cpo D, one constructs the collection $\mathcal{F}(D)$ of finite, non-empty sets of finite elements from D (for necessary definitions, see below). The order on D can be lifted to a pre-order on $\mathcal{F}(D)$ is three different ways: the convex, upper and lower lifting, respectively. The powerdomain is then defined as the ideal completion of the resulting pre-ordered set [Plo81, GS90]. This 'abstract' description of the powerdomain is sufficient for theoretical purposes. But from the point of view of semantics, it would be convenient if it were given as a collection of subsets of D and an ordering relation. Hence we want to identify this collection and ordering relation and show that the resulting partially ordered set is isomorphic to the 'abstract' powerdomain. The usual representation for the Smyth powerdomain is the collection of non-empty, upwards closed, Scott-compact subsets of D ordered by reverse set inclusion $\supseteq$ [Plo81]. This description is somewhat tedious if one tries to construct a concrete semantics for an actual language. In this case one generally only encounters chains of finite sets of finite elements.

In this paper we define a different representation for the Smyth powerdomain. We use minimal representatives instead of maximal ones. The use of these minimal representatives is inspired by work of Meyer and De Vink [MdV88, Mey85, dV90]. It is argued in these works that the use of the Smyth powerdomain and minimal representatives facilitates proofs of various properties of denotational semantics for concurrent programming languages (see also [dBMO87]). Moreover, Libkin uses minimal representatives for the Smyth powerdomain in order to prove that the Smyth and Hoare powerdomain constrcutions commute [Lib92].

The powerdomain itself is defined using a different completion method, namely *chain completion* (see also [Plo81, Smy83]). It is not too difficult to show that the ideal completion and the chain completion of a *countable* pre-ordered set are isomorphic. However, at least in the realm of semantics, chain completion seems to be a more natural completion

*Dept. of Computer Science, Padualaan 14, 3584 CH Utrecht, the Netherlands. E-mail: peterk@cs.ruu.nl

method. Moreover, we found it more convenient to use in the powerdomain construction than ideal completion.

The paper is organized as follows. In section 2 we give a short recap on domain theory and the chain completion method. In section 3 we use this method to define the Smyth powerdomain. The construction immediately gives rise to a collection of subsets of the underlying domain which are again Smyth pre-ordered (this is not trivial: in the Plotkin powerdomain construction the Egli-Milner order on the finite elements 'becomes' the Plotkin order on the infinite ones, see [Plo81]). We identify minimal representatives for the equivalence classes for the pre-order. Finally we show how continuous functions on the underlying domain lift to continuous functions on the Smyth powerdomain.

Acknowledgements. The author wishes to thank Marcello Bonsangue, Jan van Leeuwen and Frank Nordemann for valuable comments on earlier versions of this paper.

2 Domains and Chain Completion

In this section we give a short review of notions relating to cpo's. For a fuller treatment, consult [GS90, Plo81]. A partially ordered set D is called a *cpo* iff every (countable) chain $(x_i)_i$ has a least upperbound, denoted by $\bigsqcup_i x_i$, in D. We assume that all partial orders considered here have a least element $\perp$. An element $d \in D$ is *finite* if for all chains $(d_i)_i$, if $d \sqsubseteq \bigsqcup_i d_i$, then $d \sqsubseteq d_k$ for some k. We denote the collection of all finite elements of D by $K(D)$. A partially ordered set D is *algebraic* iff for all $d \in D$ there exists a chain $(d_i)_i \subseteq K(D)$ such that $\bigsqcup_i d_i = d$. It is called *ω-algebraic* if moreover $K(D)$ is countable. If D is algebraic, then $K(D)$ is called the *basis* of D. An ω-algebraic cpo is also called a *domain*.

The ordering relation on a pre-order X induces an ordering relation on chains. Let $(x_n)_n$ and $(y_m)_m$ be chains in X. We say that $(x_n)_n$ *approximates* $(y_m)_m$, denoted as $(x_n)_n \precsim (y_m)_m$, iff $\forall n \exists m . x_n \sqsubseteq y_m$. Likewise we say that a chain $(x_n)_n$ is equivalent to a chain $(y_m)_m$, denoted as $(x_n)_n \sim (y_m)_m$, iff $(x_n)_n \precsim (y_m)_m$ and $(y_m)_m \precsim (x_n)_n$. We note the following simple but important result.

Lemma 2.1 *Let $(d_i)_i$ and $(e_j)_j$ be chains in a domain D such that $(d_i)_i, (e_j)_j \subseteq K(D)$. Then $(d_i)_i \precsim (e_j)_j$ iff $\bigsqcup_i d_i \sqsubseteq \bigsqcup_j e_j$.*

Next we consider functions that preserve (some of) the structure present in a domain. Let D and E be domains and $f : D \to E$. f is *monotonic* iff $d \sqsubseteq d'$ implies $f(d) \sqsubseteq f(d')$ for all $d, d' \in D$. f is *continuous* iff it is monotonic and $f(\bigsqcup_i d_i) = \bigsqcup_i f(d_i)$. In the sequel we denote the category of pre-ordered sets and monotone functions by **Ord**, and the category of domains and continuous functions by **Dom**. Let D and E be domains. Every monotone function $f : K(D) \to E$ yields a function $\uparrow f : D \to E$ given by

$$\uparrow f(x) = \bigsqcup f(x_i)$$

where $(x_i)_i \subseteq K(D)$ is some chain such that $x = \bigsqcup_i x_i$. For future reference we state the following lemma.

Lemma 2.2 *$\uparrow(\cdot)$ is an order preserving bijection between the collection of monotone functions $K(D) \rightarrow E$ and the collection of continuous functions $D \rightarrow E$.*

We now investigate how one can construct, in a uniform way, a domain out of a (countable) pre-ordered set. Intuitively, this construction adds (formal) limit points to the chains that exist in the pre-order. We call the construction *chain completion.* Chain completion also extends to mappings: every monotone function between two pre-ordered sets induces a *continuous* function between the completions of these pre-orders.

Since we want to complete a pre-ordered set to a domain, Proposition 2.1 suggests the following construction. Given a pre-order X, form the collection $C(X)$ of all chains in X. Then $C(X)$ with $\lesssim$ is a pre-ordered set. Let $\mathcal{C}X = C(X)/\!\sim$ be the corresponding partial order induced by the pre-order. The equivalence classes of $\mathcal{C}X$ are denoted by $[(x_i)_i]$. Usually we work with representatives of such an equivalence class.

Theorem 2.3 *Let X be a countable pre-order. Then $\mathcal{C}X$ is a domain and $K(\mathcal{C}X) \simeq X$.*

Proof All requirements are easily checked. We only show how to obtain lub's in $\mathcal{C}X$. Let

$$(x_i^1)_i \lesssim (x_i^2)_i \lesssim \cdots \lesssim (x_i^n)_i \lesssim \cdots \tag{1}$$

be a chain. Define the following collection $\{y_i : i < \omega\}$ inductively:

- $y_1 = x_1^1$;
- $y_{n+1} = x_k^{n+1}$ where k is the least index such that $y_n \sqsubseteq x_k^{n+1}$ and for all $m \leq n+1$ and $i \leq n+1$, we have that $x_i^m \sqsubseteq x_k^{n+1}$.

By assumption, this is a chain. Clearly, $(y_i)_i$ is a least upperbound for the chain (1) and we denote it by $\bigvee_{n<\omega}(x_i^n)_i$. It is straightforward to check that the construction of this diagonal is well-defined for equivalence classes of chains. That is, if $(z_i^n)_i \sim (x_i^n)_i$ for every n, then $\bigvee_{n<\omega}(x_i^n)_i \sim \bigvee_{n<\omega}(z_i^n)_i$. □

There is an interesting subcase in the proof of the preceding theorem, namely when for all n, $x_i^n \sqsubseteq x_i^{n+1}$ for all i. In this case the diagonal is indeed $(x_n^n)_n$. Moreover, it is not hard to see that any chain (with respect to $\lesssim$) in $\mathcal{C}X$ can be brought into this form.

Let X and Y be pre-ordered sets and let $f : X \rightarrow Y$ be monotonic. Then f can be extended to $\mathcal{C}f : \mathcal{C}X \rightarrow \mathcal{C}Y$ given by $\mathcal{C}f([(x_n)_n)]) = [(f(x_n))_n]$. Then $\mathcal{C}f$ is well-defined, monotonic and continuous. Moreover, with this definition, $\mathcal{C}$ is a functor from **Ord** to **Dom**.

3 The Smyth Powerdomain

This section concentrates on the interpretation of the (abstract) construction of the Smyth powerdomain as a collection of subsets of the underlying domain. The interpretation is inspired by the work of Meyer and De Vink [MdV88], who essentially described the interpretation for the special case where the underlying domain is the streamset on a (countable) alphabet.

For a domain D and $X, Y \in \mathcal{P}(D)$, the Smyth order is defined as $X \sqsubseteq_S Y$ iff $\forall y \in Y \exists x \in X . x \sqsubseteq y$. Let $\mathcal{F}(D)$ be the collection of finite, non-empty sets of finite elements in D.

Definition 3.1 *The Smyth powerdomain $\mathcal{P}^\sharp D$ on a domain D is the chain completion of the pre-ordered set $\mathcal{F}(D)$, pre-ordered by $\sqsubseteq_S$.*

We want to establish an isomorphism between this domain and a domain consisting of sets of elements from D. So given a chain $(X_i)_i \subseteq \mathcal{F}(D)$, what subset of D should we assign to it? A natural candidate is the set $Up(X_i)_i = \{\bigsqcup_i x_i : x_i \in X_i\}$. However, it is easy to see for equivalent chains $(X_i)_i \sim (Y_k)_k$ it may be the case that $Up(X_i)_i \neq Up(Y_k)_k$, although these sets are equivalent w.r.t. the Smyth order (see Proposition 3.5 below). We will use minimal representatives for the equivalence classes induced by the pre-order. We therefore introduce the following operator. For $X \subseteq D$,

$$\min(X) = \{x \in X : \forall y \in X . y \sqsubseteq x \rightarrow y = x\}$$

We expect that $X \equiv_S \min(X)$ for every $X \subseteq D$. We have to use a condition on X, however, for this to be true. Let D contain an infinite down-chain $x_0 \sqsupseteq x_1 \sqsupseteq x_2 \sqsupseteq \cdots$ Let $X = \{x_n : n < \omega\}$. Then $\min(X) = \emptyset$. Hence $X \not\equiv \min(X)$.

Definition 3.2 *Let D be a domain and let $X \in \mathcal{P}(D)$. X* has minimal elements *iff $\forall x \in X \exists x' \in \min(X) . x' \sqsubseteq x$.*

Lemma 3.3 *Let $X, Y \in \mathcal{P}(D)$. Assume X and Y have minimal elements.*

1. *$X \sqsubseteq_S Y$ iff $\min(X) \sqsubseteq_S \min(Y)$.*
2. *$X \equiv_S Y$ iff $\min(X) = \min(Y)$.*

In order to prove the next lemma we need a definition. A *down-chain* in a subset $X \subseteq D$ is an ordinal α and an injective function $f : \alpha \rightarrow X$ such that $\beta \leq \beta'$ implies $f(\beta') \sqsubseteq f(\beta)$. Note the reversal of the ordering. Usually we identify a down-chain with its image $f(\alpha)$.

Lemma 3.4 *Let $(X_i)_i \subseteq \mathcal{F}(D)$ be a chain. Then $X = Up(X_i)_i$ has minimal elements.*

Proof For $x \in X$, consider the collection $\downarrow x = \{y \in X : y \sqsubseteq x\}$. We have to show that this collection contains a minimal element. Let $f : \alpha \rightarrow \downarrow x$ be a downschain for some ordinal α. For all $y \in f(\alpha)$, $y = \bigsqcup_i y_i$ where $y_i \in X_i$. Fix an index i and define for all $\beta < \alpha$, $Y_\beta = \{y \in X_i : y \sqsubseteq f(\beta)\}$. Then $Y_\beta \neq \emptyset$ for all $\beta < \alpha$ and if $\beta < \gamma$ then $Y_\gamma \subseteq Y_\beta$. Since X_i is finite, $Y_i = \bigcap_{\beta < \alpha} Y_\beta \neq \emptyset$. Hence Y_i contains elements that are less than *every* element in the down-chain. Hence, by König's lemma, there exists a chain through these elements, say $(y_i')_i$. Now $\bigsqcup_i y_i' \in Up(X_i)_i$ and $\bigsqcup_i y_i' \sqsubseteq f(\beta)$ for all $\beta < \alpha$, that is, $\bigsqcup_i y_i'$ is a lowerbound for the down-chain. Hence every down-chain has a lowerbound in $\downarrow x$. By Zorn's Lemma, $\downarrow x$ contains a minimal element. □

Lemma 3.5 *For all chains $(X_i)_i, (Y_k)_k$,*

$$(X_i)_i \precsim (Y_k)_k \text{ iff } \min(Up(X_i)_i) \sqsubseteq_S \min(Up(Y_k)_k)$$

Proof If we denote $\hat{f}(X_i)$ by $(Y_n^i)_n$, then we may assume that $\bigvee(Y_n^i)_n$ is the chain $(Y_n^n)_n$. Furthermore, we denote the left hand side of the equation by L and the right hand side by R. We have the following equivalences.

$$f(\min(Up(X_i)_i)) \equiv_S \min(f(Up(X_i)_i)) \equiv_S \{\bigsqcup f(x_i) : x_i \in X_i \ \ \& \ \ (x_i)_i \text{ chain}\}$$

First, consider a chain $(y_n)_n$ with $y_n \in f(X_n)$. Then each y_n determines a finite non-empty set $Y_n' \subseteq Y_n^n$ given by $\{y \in Y_n^n : y \sqsubseteq y_n\}$. By König's Lemma, there exists a chain $(z_n)_n$ where $z_n \in Y_n^n$ such that $\bigsqcup z_n \sqsubseteq \bigsqcup y_n$. Hence $L \sqsubseteq_S R$.

Conversely, let $(z_n)_n$ be a chain with $z_n \in Y_n^n$. Then each z_n determines the subset $Z_n^i \subseteq Y_n^i$ for $i < n$ defined as $Z_n^i = \{y \in Y_n^i : y \sqsubseteq z_n\}$. Each Z_n^i is non-empty. Hence we can apply König's Lemma to find chains $(y_n^i)_n$ with $y_n^i \in Y_n^i$. By assumption, $\bigsqcup_n y_n^i \in f(X_i)$. By the way we chose the sets Z_n^i, each $\bigsqcup_n y_n^i \sqsubseteq \bigsqcup z_n$. Since all the sets $f(X_i)$ are finite we may again apply König's Lemma to choose a chain $(a_n)_n$ with $a_i \in f(X_i)$ and $a_i \sqsubseteq \bigsqcup_n y_n^k$ for some $k \geq i$. Hence $a_i \sqsubseteq \bigsqcup z_n$ for each i. Since every set $f(X_i)$ is finite, we can again apply König's lemma to find a chain $(x_i)_i$ with $x_i \in X_i$ such that $\bigsqcup f(x_i) \sqsubseteq \bigsqcup a_i$. From this it follows that $R \sqsubseteq_S L$. □

Lemmas 3.3 and 3.9 immediately yield the following proposition.

Proposition 3.10 *For each $f : D \to E$, $\min(Up(\mathcal{P}^\natural f([(X_i)_i]))) = \min(f(\min(Up(X_i)_i)))$.*

We can now state the main theorem of this paper.

Theorem 3.11 *$\mathcal{P}^\natural(\cdot) : \mathbf{Dom} \to \mathbf{Dom}$ is a functor.*

Proof The action of $\mathcal{P}^\natural$ on objects is given in Proposition 3.6, and its action on arrows in Proposition 3.10. It is easy to show that $\mathcal{P}^\natural(1_D) = 1_{\mathcal{P}^\natural D}$ and that $\mathcal{P}^\natural(g \circ f) = \mathcal{P}^\natural g \circ \mathcal{P}^\natural f$. □

We arrive at the following picture of the Smyth powerdomain on a domain D. Its elements are subsets of D of the form $\min(Up(X_i)_i)$ where $(X_i)_i \subseteq \mathcal{F}(D)$ is a chain, ordered by the Smyth order. A continious function $f : D \to E$ extends to a function $\mathcal{P}^\natural f : \mathcal{P}^\natural D \to \mathcal{P}^\natural E$ taking a set X to $\min\{f(x) : x \in X\}$. Most functions arising in the semantics of programming languages (using the Smyth powerdomain) are functions $\mathcal{P}^\natural D \to \mathcal{P}^\natural D$ that are lifted versions of functions $D \to D$ which have the additional property that they map $K(D)$ onto $K(D)$. Hence in practice it is often sufficient to consider only countable pre-ordered stets and monotone functions between them. For example, using the stream domain on an alphabet A as defined below Proposition 3.6, if we want to model processes as elements from $\mathcal{P}^\natural A^{str}$, we can define a sequential composition operator by stipulating that $v; w = vw$ and $u; w = u$ for $v \in A^+$, $u \in A^*\bot$ and $w \in A^+ \cup A^*\bot$. It is easy to see that this monotone map 'lifts' to the usual sequential composition operator, see [MdV88].

References

[dBMO87] J.W. de Bakker, J.-J.Ch. Meyer, and E.-R Olderog. Infinite streams and finite observations in the semantics of uniform concurrency. *Theoretical Computer Science*, 49(2,3):87–112, 1987.

[dV90] E.P. de Vink. *Designing stream based semantics for uniform concurrency and logic programming.* PhD thesis, Dept. of Computer Science, Free University, Amsterdam, 1990.

[GS90] C.A. Gunter and D.S. Scott. Semantic domains. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science, Vol. B*, chapter 12, pages 635–675. Elsevier, Amsterdam, 1990.

[Lib92] L. Libkin. An elementary proof that upper and lower powerdomain constructions commute. *Bulletin EATCS*, 48:175–177, 1992.

[MdV88] J.-J.Ch. Meyer and E.P. de Vink. Applications of compactness in the Smyth powerdomain of streams. *Theoretical Computer Science*, 57:251–282, 1988.

[Mey85] J.-J. Ch. Meyer. *Programming calculi based on fixed-point theory: Semantics and applications.* PhD thesis, Dept. of Computer Science, Free University, Amsterdam, 1985.

[Plo79] G.D. Plotkin. A powerdomain construction. *SIAM J. on Computing*, 5:452–487, 1979.

[Plo81] G.D. Plotkin. Post-graduate lecture notes in advanced domain theory (incorporating the Pisa lecture notes). Technical report, Dept. of Computer Science, Univ. of Edinburgh, 1981.

[Smy78] M.B. Smyth. Power domains. *J. of Comp. Syst. Sc.*, 16:23–36, 1978.

[Smy83] M.B. Smyth. Powerdomains and predicate transformers: a topological view. In J. Diaz, editor, *Proc. ICALP*, pages 662–676. Springer Verlag, Berlin, 1983.