Spin Correlation and Discrete Symmetry in Spinor Bose-Einstein Condensates

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We study spin correlations in Bose-Einstein condensates of spin 1 bosons with scatterings dominated by a total spin equal 2 channel. We show that the low energy spin dynamics in the system can be mapped into an $o(n)$ nonlinear sigma model. $n = 3$ at the zero magnetic field limit and $n = 2$ in the presence of weak magnetic fields. In an ordered phase, the ground state has a discrete Z_2 symmetry and is degenerate under the group $[U(1) \times S^{n-1}]/Z_2$. We explore consequences of the discrete symmetry and propose some measurements to probe it.

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Recently, there has been a burst of theoretical and experimental activities on spinor Bose-Einstein condensates (BEC) [1–6]. An optical trap confines alkali atoms independent of spins and liberates spin degrees of freedom [2,3]. For sodium (^{23}Na) or rubidium (^{87}Rb) atoms with nuclear spin $I = 3/2$ and electrons at s orbits, the energy splitting between hyperfine multiplets is of order 100 mK. At temperatures as low as 100 nK, ^{23}Na and ^{87}Rb atoms can be considered as simple bosons with a hyperfine spin $F = 1$. The ground state of N spin 1 noninteracting bosons has $(N + 1)N/2$ folds spin degeneracy, by contrast to its magnetically trapped cousins. Presumably, hyperfine spin-dependent scattering lifts the degeneracy and leads to a spin correlated state. Optically trapped BEC therefore sets up a platform for studying quantum magnetism in many boson systems and adds a new dimension to already extremely rich physics in these systems.

The spin-dependent two-body interaction in BEC is characterized by $U_2(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2)[c_0 + c_2\mathbf{F}_1\dot{\mathbf{F}}_2]$, as suggested in an early paper [4]. Here $c_0 = (g_0 + 2g_2)/3$ and $c_2 = (g_2 - g_0)/3$; $g_F = 4\pi\hbar^2 a_F/M$, M is the mass of the atom, and a_F is the s wave scattering length in the total spin F channel. Thus, the spin correlation in a BEC is determined by c_2. For ^{87}Rb, $g_2 < g_0$ or $c_2 < 0$ and the scattering is dominated by the total spin $F = 0$ channel. In the ground state, all spins of atoms prefer to align in a certain direction and have a maxima magnetization [4].

For ^{23}Na studied experimentally [2], the scattering between ^{23}Na atoms is dominated by the total spin $F = 2$ channel, i.e., $g_2 > g_0$. The scattering between ^{23}Na atoms thus leads to an "antiferromagnetic" spin correlation. Efforts have been made to understand the ground state properties, exact excitation spectra, and collective modes [4–6]. Many interesting predictions, such as spin waves and spin mixing dynamics, were made for ^{23}Na BEC where $c_2 > 0$.

In this paper, we show the spin dynamics in BEC with $c_2 > 0$ is characterized by an $o(n)$ nonlinear sigma model (NLσM) of n components. $n = 3$ at zero magnetic field limit and $n = 2$ in the presence of a weak magnetic field. Spin correlations in spinor BEC can be studied in the context of the NLσM. We identify that the internal order parameter space for BEC as $[S^1 \times S^2]/Z_2$ (zero field limit) and the ground state is nematically ordered. We explore consequences of the discrete Z_2 symmetry.

To describe the spin correlated BEC, it is most convenient to introduce Weyl representation of SU(2) involving polynomials of a unit vector (u, v) [7]. Each unit vector is represented by a point Ω on a sphere with polar coordinates (θ, ϕ); namely, $u = \exp(i\phi/2)\cos(\theta/2)$ and $v = \exp(-i\phi/2)\sin\theta/2$. The corresponding hyperfine spin operators are $F^+ = u\partial/\partial v$, $F^- = v\partial/\partial u$, and $F_z = (u\partial/\partial u - v\partial/\partial v)/2$. The scalar product between two wave functions g and f is defined as $\int g^*(u, v) \times f(u, v)\, d\Omega/4\pi$. (We reserve $\mathbf{\Omega}$ for the spin rotations discussed below.)

Under spin rotations $\mathcal{R} = \exp(iF_z\chi_1/2)\exp(iF_y\theta_1/2)\exp(iF_z\phi_1/2)$, u and v transform into

$$\begin{aligned} u(\mathbf{\Omega_1}, \chi_1) =\ & \exp(i\chi_1/2)[\cos\theta_1/2\exp(-i\phi_1/2)u \\ & + \sin\theta_1/2\exp(i\phi_1/2)v], \\ v(\mathbf{\Omega_2}, \chi_2) =\ & \exp(-i\chi_2/2)[-\sin\theta_2/2\exp(-i\phi_2/2)u \\ & + \cos\theta_2/2\exp(i\phi_2/2)v], \end{aligned} \tag{1}$$

where $\mathbf{\Omega}_{1,2} = (\theta_{1,2}, \phi_{1,2})$. Spin-1 wave functions are polynomials of degree 2 in u and v. $\sqrt{3}\,u^2$, $\sqrt{6}\,uv$, and $\sqrt{3}\,v^2$ correspond to $m = 1$, 0, and -1 states. All $F = 1$ states can also be expressed in terms of $\sqrt{6}\,u(\mathbf{\Omega}_1)v(\mathbf{\Omega}_2)$ with $\mathbf{\Omega}_{1,2}$ properly chosen.

The Hamiltonian for spin-1 bosons can be written as

$$\mathcal{H} = -\frac{1}{2M}\sum_\alpha \nabla_\alpha^2 + \sum_{\alpha,\beta}\left[\frac{c_0}{2} + \frac{c_2}{2}\mathbf{F}_\alpha \cdot \mathbf{F}_\beta\right]\delta(\mathbf{r} - \mathbf{r}') - \sum_\alpha \mathbf{F}_{z\alpha} g\mu_B H\,. \tag{2}$$

The second term is hyperfine spin-dependent interaction with $c_2 > 0$, and the last term is the coupling with an external magnetic field $\mathbf{H} = H\mathbf{e}_z$; g is a Landé factor of an atom and μ_B is the Bohr magneton.

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The wave function of N spin-1 bosons generally can be written as [8]

$$\Psi(\{\mathbf{r}_\alpha\}) = \mathcal{P}\Pi_{\alpha=1\cdots N}\Phi_{N_\alpha}(\mathbf{r}_\alpha)\sqrt{6}\,u_\alpha(\mathbf{\Omega}_{1\alpha}(\mathbf{r}_\alpha)) \times v_\alpha(\mathbf{\Omega}_{2\alpha}(\mathbf{r}_\alpha)). \quad (3)$$

$\mathcal{P}$ is the permutation of $\{N_\alpha\}$, $\{\mathbf{\Omega}_{1\alpha}, \mathbf{\Omega}_{2\alpha}\}$. N_α labels a one-particle orbital state. Phases $\chi_{1,2}$ in Eq. (1) are absorbed by a gauge transformation of the wave function Φ_{N_α} introduced above. For BEC under consideration, we take $\mathbf{\Omega}_{1\alpha,2\alpha} = \mathbf{\Omega}_{1,2}(\mathbf{r})$ and $\Phi_{N_\alpha}(\mathbf{r}) = \Phi(\mathbf{r})$ [$\Phi(\mathbf{r})$ is a complex scalar field]. By introducing $\mathbf{n}(\mathbf{r}) = (\mathbf{\Omega}_1 + \mathbf{\Omega}_2)/2$, $\mathbf{L}(\mathbf{r}) = (\mathbf{\Omega}_1 - \mathbf{\Omega}_2)/2$, and $\Phi(\mathbf{r}) = \sqrt{\rho(\mathbf{r})}\exp(i\chi(\mathbf{r}))$, we derive from Eqs. (2) and (3) an effective Hamiltonian as a function of two pairs of variables: $\{\mathbf{n}(\mathbf{r}), \mathbf{L}(\mathbf{r})\}$ and $\{\rho(\mathbf{r}), \chi(\mathbf{r})\}$. These are collective variables of the N interacting spin-1 bosons, which describe the spin dynamics and phase dynamics, respectively. In this representation, the hyperfine spin-dependent interaction in Eq. (2) is mapped into $\mathcal{H}_s = \int d\mathbf{r}\, c_2 \mathbf{L}^2(\mathbf{r})\rho^2(\mathbf{r})$, which depends only on collective variable $\mathbf{L}$ when $\rho(\mathbf{r})$ is taken as a constant. This indicates that each atom acquires an inertial $I_0 = 1/2c_2\rho$ in the presence of hyperfine spin-dependent scatterings with $c_2 > 0$; the rotation energy in the presence of a finite spin moment $\mathbf{L}$ is thus $\mathbf{L}^2/2I_0$.

In the most interesting limit, we can introduce the local spin density as $\mathbf{l}(\mathbf{r}) = \mathbf{L}(\mathbf{r})\rho(\mathbf{r})$. $\mathbf{n}$ and $\mathbf{l}$ satisfy the constraint $\mathbf{n}(\mathbf{r}) \cdot \mathbf{l}(\mathbf{r}) = 0$. Commutation relations between ρ and χ, $\mathbf{n}(\mathbf{r})$ and $\mathbf{l}(\mathbf{r})$, are given as $[\rho(\mathbf{r}), \chi(\mathbf{r}')] = i\hbar\delta(\mathbf{r} - \mathbf{r}')$, $[\mathbf{n}_\alpha(\mathbf{r}), \mathbf{n}_\beta(\mathbf{r}')] = 0$, $[\mathbf{l}_\alpha(\mathbf{r}), \mathbf{n}_\beta(\mathbf{r}')] = i\hbar\epsilon^{\alpha\beta\gamma}\mathbf{n}_\gamma\delta(\mathbf{r} - \mathbf{r}')$, and $[\mathbf{l}_\alpha(\mathbf{r}), \mathbf{l}_\beta(\mathbf{r}')] = i\hbar\epsilon^{\alpha\beta\gamma}\mathbf{l}_\gamma\delta(\mathbf{r} - \mathbf{r}')$. $\epsilon^{\alpha\beta\gamma}$ is an antisymmetric tensor. These identities are valid when $\mathbf{L}$ per atom is much less than unity and $\mathbf{n}(\mathbf{r})$ can be considered as a classical "vector," components of which commutate with each other. The corresponding Lagrangian density can be derived as $\mathcal{L} = \mathcal{L}_s + \mathcal{L}_c + \mathcal{L}_{sc}$ [8], with

$$\mathcal{L}_c \approx \frac{\rho}{2M}\left[(\nabla\chi(\mathbf{r})) + \frac{1}{v_c^2}(\partial_\tau\chi)^2\right], 2$$

$$\mathcal{L}_s = \frac{\rho}{2M}\left[(\nabla\mathbf{n}(\mathbf{r}))^2 + \frac{1}{v_s^2}(\partial_\tau\mathbf{n})^2\right], \quad (4)$$

$$\mathcal{L}_{cs} = \frac{\rho}{M}\frac{1}{v_s^2}[\mathbf{n} \times \partial_\tau\mathbf{n} \cdot \mathbf{n} \times (\nabla\chi \cdot \nabla)\mathbf{n}].$$

Here $\rho = \rho(0)$, $v_c = \sqrt{2\rho c_0/M}$, and $v_s = \sqrt{2\rho c_2/M}$. We introduce $\tau = it$ as the imaginary time. Nonlinearity is imposed via a constraint $|\mathbf{n}^2| = 1$ at a low frequency limit. Equation (4) is the main result of the mapping, and we have kept contributions which are lowest order in terms of ∂_τ and ∇. $\mathcal{L}_c$ is taken in a Gaussian approximation and should be replaced by a full Gross-Pitaevskii Lagrangian in general. We will be mostly interested in the spin sector, and simplification in $\mathcal{L}_c$ does not affect conclusions here. $\mathcal{L}_s$ in Eq. (4) represents an $o(3)$ NLσM.

The last term $\mathcal{L}_{sc}$ characterizes a coupling between a spin rotation and the superflow in BEC due to Berry's phase. This term, however, is linear in ∂_τ and quadratic in spatial gradient and is negligible compared with $\mathcal{L}_s$, $\mathcal{L}_c$ at the long wavelength limit. Particularly, such a coupling vanishes in a configuration where $\mathbf{L}$ is zero.

At the zero field limit, there exists a saddle point solution for the spin sector $\mathbf{n}(\mathbf{r}) = \mathbf{n}_0$, $\mathbf{l}(\mathbf{r}) = 0$. $\mathbf{n}_0$ lives on a unit sphere. By expanding Eq. (4) around the saddle point solution, we obtain spin waves with soundlike spectrum $\omega = \sqrt{4c_2\rho/M}\,k$, which can also be obtained in the Gross-Pitaevskii approach [4]. However, Eq. (4) here is valid for any point $\mathbf{\Omega}_1 \sim \mathbf{\Omega}_2$ on the unit sphere. The effective NLσM derived here allows us to describe spin correlated states well beyond the Gross-Pitaevskii approach.

For a symmetry broken state, the ground state wave function in Eq. (3) with $\mathbf{\Omega}_1 = \mathbf{\Omega}_2 = \mathbf{n}$ is invariant under a global transformation $\mathbf{n}, \chi \to -\mathbf{n}, \chi + \pi$;

$$\Psi(\mathbf{n}, \chi) = \Psi(-\mathbf{n}, \chi + \pi), \quad \Psi(\mathbf{n}) = (-1)^N\Psi(-\mathbf{n}), \quad (5)$$

where χ is the phase of the scalar field $\Phi(x)$ introduced before. In obtaining this symmetry, we notice $u(\mathbf{n}) = \exp(i\pi/2)v(-\mathbf{n})$, with $\pi/2$ from a phase of a spin-1/2 particle under a 180° rotation. Equation (5) shows that the many-body wave functions characterized by $(\mathbf{n}, \chi)$ and $(-\mathbf{n}, \chi + \pi)$ are indistinguishable. Thus, the spinor BEC under consideration has a discrete Z_2 symmetry and the order parameter space is $\mathcal{R} = [S^1 \times S^2]/Z_2$, with Z_2 as a two-element group of integers modulo 2.

As in a classical uniaxial nematic liquid crystal where diatomic molecules are indistinguishable upon an inversion of their directors $\mathbf{n}$ and the internal order parameter space is S^2/Z_2, the Z_2 symmetry identified here also indicates that there exists a tensor order parameter [9]. For the purpose of demonstration, let us introduce a "director" $\mathbf{d}_x = [uv^* + vu^*]/2$, $\mathbf{d}_y = [uv^* - vu^*]/2i$, $\mathbf{d}_z = [uu^* - vv^*]/2$. One then can show for the ground state wave function in Eq. (3) $\langle\mathbf{d}\rangle = 0$, but nematic order parameter $Q_{\alpha\beta} = 1/\rho[\langle\mathbf{d}_\alpha\mathbf{d}_\beta\rangle - 1/3\,\mathrm{Tr}\langle\mathbf{d}_\alpha\mathbf{d}_\beta\rangle]$ is nonzero. Here $\langle\,\rangle$ stands for an average taken over the ground state wave function Ψ. In fact,

$$Q_{\alpha\beta} = \tfrac{2}{5}[-3\mathbf{n}_{0\alpha}\mathbf{n}_{0\beta} + \delta_{\alpha\beta}]. \quad (6)$$

According to Eq. (6), the director $\mathbf{d}$ aligns in a plane perpendicular to $\mathbf{n}_0$.

Topological defects in this case are of particular interest because of the discrete symmetry in Eq. (5). Following the general theory for the classification of defects in a symmetry broken state, the Z_2 symmetry leads to π spin disclinations superimposed with half-vortices, which otherwise do not exist. The corresponding wave function of composite linear singularities (Z_2 strings) is

$$\lim_{\xi\to\infty}\mathbf{n}(\xi) = \mathrm{Re}\left(\frac{\xi-\xi_0}{|\xi-\xi_0|}\right)^{m+1/2}\mathbf{e}_x + \mathrm{Im}\left(\frac{\xi-\xi_0}{|\xi-\xi_0|}\right)^{m+1/2}\mathbf{e}_y\,,$$
$$\lim_{\xi\to\infty}\mathbf{v}_s(\xi) = \frac{1/2+n}{M|\xi-\xi_0|}\left[\mathrm{Im}\left(\frac{\xi-\xi_0}{|\xi-\xi_0|}\right)\mathbf{e}_x - \mathrm{Re}\left(\frac{\xi-\xi_0}{|\xi-\xi_0|}\right)\mathbf{e}_y\right]. \tag{7}$$

Here $\xi = x + iy$ and lines are located at $\xi_0 = x_0 + iy_0$; n, m are integers and $\mathbf{v}_s$ is superfluid velocity [10]. Each string is characterized by (m, n). However, $(-1, n)$, $(\pm 3, n)$, and $(\pm 5, n)$ strings can be obtained by deforming string $(1, n)$ and are homotopically identical to $(1, n)$.

In a composite string given in Eq. (7), $\mathbf{n}$ evolves into $-\mathbf{n}$ along a loop enclosing ξ_0. The corresponding spin wave function changes its sign under an inversion $\mathbf{n} \to -\mathbf{n}$, following the identity $u(\mathbf{n})v(\mathbf{n}) = -u(-\mathbf{n})v(-\mathbf{n})$. A superflow $\mathbf{v}_s$ of a half-vortex is present to compensate the π phase under $\mathbf{n} \to -\mathbf{n}$ rotation and ensure the single valuedness of the wave function. This composite structure is different from a linear defect in a classical nematic liquid where π disclinations are free topological excitations. In fact, in a coherent spinor BEC, a bare π-spin disclination carries a cut along which phase changes abruptly from π to 2π or 0. This cut starting at the disclination ends only at the boundary of the system and costs an energy linear in terms of the size of BEC. For a similar reason, the energy cost to have a π disclination and a half-vortex separated at a distance L is linearly proportional to L. Composite strings in Eq. (7) should be considered as results of confinement of π spin disclinations and half-vortices in spinor BEC [11].

In the presence of an external magnetic field along the z direction, $\mathbf{\Omega}_{1,2} = \mathbf{n} \pm \mathbf{e}_z g\mu_B H/4c_2\rho$ and $\mathbf{n}$ satisfies constraint

$$\mathbf{n}\cdot\frac{\mu_B H}{4c_2\rho}\,\mathbf{e}_z = 0\,. \tag{8}$$

Obviously, an external magnetic field breaks the S^2 symmetry and confines the low frequency sector of $\mathbf{n}$ in a plane perpendicular to $\mathbf{H}$ itself. The Lagrangian in the presence of a magnetic field is that of an $O(2)$ NLσM; it has S^1 symmetry at the frequency $\omega \ll \mu_B H$. At the high frequency limit, $\mathbf{n}$ precesses in a field $4c_2\mathbf{l}(x)$, much larger than the external field, and S^2 symmetry is restored.

As a consequence, the order parameter space for the quantum spin nematic state in an external magnetic field is $\mathcal{R} = [S^1 \times S^1]/Z_2$. The nematic order parameter $Q_{\alpha\beta}$ is still given by Eq. (7), with the easy plane of $\mathbf{d}$ parallel to the external field. Wave functions for linear defects are of the same forms as those in Eq. (7), but with all strings (m, n) homotopically distinguishable.

We have restricted ourselves to the weak magnetic field limit and neglected the possible quadratic Zeeman shift $\mathcal{H}_{QZ} = \sum_\alpha QH^2F_{z\alpha}^2$ (the external field H is along the $\mathbf{e}_z$ direction). Inclusion of the quadratic Zeeman shift yields an additional term $\mathcal{L}_{QZ} = \int \rho QH^2(\mathbf{n}_x^2 + \mathbf{n}_y^2)\,dx$ to the NLσM derived. The main effect of this contribution is to align $\mathbf{n}$ along the external field. When this shift dominates, the ground state is left with a double degeneracy: $\mathbf{n} = \mathbf{e}_z$ and $\mathbf{n} = -\mathbf{e}_z$. The spin wave develops an energy gap of order QH^2. We will focus on the zero magnetic field case in the rest of the discussions.

In general, following Eq. (4), spin correlated BEC can be studied by considering a NLσM,

$$\mathcal{L}_s = \frac{1}{2f}(\partial_\mu \mathbf{n})^2, \qquad \mathbf{n}^2 = 1\,;$$
$$f = (16\pi)^{1/2}(\rho\Delta a^3)^{1/6}, \qquad \Delta a = \frac{a_2 - a_0}{3}\,. \tag{9}$$

We introduce dimensionless length and time: $\tilde{\mathbf{r}} = \mathbf{r}\rho^{1/3}$, $\tilde{\tau} = \tau v_s \rho^{1/3}$. Derivatives ∂_μ are defined as $(\partial_{\tilde{\tau}}, \partial_{\tilde{x}}, \partial_{\tilde{y}}, \partial_{\tilde{z}})$. f^{-1} is a square root of the ratio between potential energy at an interatomic scale $\hbar^2\rho^{2/3}/2m$ and zero point kinetic (rotation) energy $c_2\rho/2$ of an individual atom. The zero point motion is absent for noninteracting bosons but gets stronger when c_2 increases, or the effective inertial I_0 gets smaller. The $o(3)$ NLσM has order and disordered phases at $d > 1$, depending on the parameter f. Most of the qualitative results about spin correlations in BEC can be obtained in a renormalization group (RG) approach [8]. For $\rho\Delta a^3 = 10^{-6}$ as is in experiments, a long range nematic order should be observed.

In 1D case, at zero temperature and zero field, one should expect there will be no long range order and the state is nematically disordered, following the RG results of NLσM [12]. These nematic disordered states mimic the quantum spin liquid states proposed in the literature of Heisenberg antiferromagnetic systems (HAFS). However, we notice that the NLσM derived from the microscopic Hamiltonian in this paper does not have a θ term $\mathcal{L}_\theta = \theta/4\pi \int d\tau\,dx\,\mathbf{n}\cdot\partial\mathbf{n}/\partial\tau \times \partial\mathbf{n}/\partial x$, which is generally present in HAFS studied before [13]. Absence of a θ term, which ensures an energy gap in the excitation spectrum of nematic state, follows a fact that the Berry's phase under rotations of $\mathbf{n}$ vanishes identically [8].

Two remarks are in order. (1) At a high density limit, one should also take into three-body, four-body elastic scatterings. This can further modify the short distance dynamics but will not affect conclusions arrived above in a qualitative way. (2) One should be cautious about the definition of "phase" since the alkali atoms under investigation are in a long lived metal stable gaseous state. The lifetime of alkali atomic gas is limited by three-body inelastic collisions [14]. The collision rate is proportional to the square of the number density of atoms and increases dramatically as the density increases. This sets a practical limit in order for the quantum disordered nematic liquid to be probed in BEC.

Since the experiment [2] was done in a BEC cloud with a few million sodium atoms, it is also particularly interesting to consider the symmetry restoring due to a finite size effect. We take a weakly interacting limit where quantum fluctuations of finite wavelength are negligible. In a zero mode approximation, the Hamiltonian becomes

$$\mathcal{H}_{z.m} = \rho c_2 \frac{\mathbf{L}^2}{2N}. \tag{10}$$

$\mathbf{L}$ is the total spin of the N-bosons system and can be considered as an angular momentum operator defined on a unit sphere of $\mathbf{n}$ [8]. And $[\mathbf{n}, \mathbf{n}] = 0$, $[\mathbf{L}_\alpha, \mathbf{n}_\beta] = i\hbar\epsilon^{\alpha\beta\gamma}\mathbf{n}_\gamma$, and $[\mathbf{L}_\alpha, \mathbf{L}_\beta] = i\hbar\epsilon^{\alpha\beta\gamma}\mathbf{L}_\gamma$. The energy spectrum is given by $\mathbf{L}^2 = l(l+1)$, which is identical to that obtained in [6]. $l = 0, 2, 4, \ldots, N$, if N is an even number; and $l = 1, 3, 5, \ldots$, otherwise. The energy gap between lowest lying excitations is inversely proportional of N and vanishes at large N limit.

Now consider a wave packet with $\mathbf{n}$ confined within a region centered at $\mathbf{n}_0 = \mathbf{e}_z$ of size $\sqrt{\langle \delta^2 \mathbf{n} \rangle_0} \ll 1$ on the unit sphere at $t = 0$. Spread $\langle \delta^2 \mathbf{n} \rangle$ is defined as the expectation value of $\mathbf{n}_x^2 + \mathbf{n}_y^2$. In spherical polar coordinates (θ, ϕ) where $\mathbf{n}$ is a vector represented by $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, the corresponding wave packet (for an even N) can be constructed as

$$\Psi(\theta, \phi, t) = \frac{1}{B} \sum_{l=2n} \exp\left(-\frac{l^2}{4\sigma} - it\frac{l(l+1)c_2\rho}{2N}\right) \times Y_{l,0}(\theta, \phi). \tag{11}$$

Here $B = \sum_l \exp(-l^2/2\sigma)$; $\langle \delta^2 \mathbf{n} \rangle_0 = A_0/\sigma$ with constant A_0 estimated to be a constant of unity. $Y_{l,0}(\theta, \phi)$ are spheric harmonics. $\sigma \gg 1$ but σ/N is vanishingly small. The energy of this wave packet is $\Delta E = 2A_0\sigma c_2\rho/N$. In the limit $N \to \infty$, an infinitesimal external field will stabilize this wave packet with respect to the rotation invariant state.

The spread of $\mathbf{n}$ at a time t is

$$\langle (\delta\mathbf{n})^2 \rangle_t = \langle (\delta\mathbf{n})^2 \rangle_0 + 4A_0\sigma\left(t\frac{c_2\rho}{N}\right)^2, \tag{12}$$

which is valid at $t < \tau_c = \sigma/A_0\Delta E$. At $t > \tau_c$, the spread is of order unity. Therefore, $\sqrt{\langle \delta^2 \mathbf{n} \rangle_t}$ exceeds the initial spread $\sqrt{\langle \delta^2 \mathbf{n} \rangle_0}$ at a time of order $1/\Delta E$. At a longer time τ_c, $\mathbf{n}$ starts to explore the whole unit sphere S^2 and the rotation symmetry is restored due to spin-dependent scatterings in BEC.

Equation (12) also imposes restrictions on measurements. A measurement of $\mathbf{n}$ in ^{23}Na BEC discussed here excites the ground state to an excited state of energy ΔE, where $\mathbf{n}$ has a finite spread on S^2 and ΔE is infinitesimally small in the thermal dynamical limit. The time scale to restore the broken symmetry is determined by the two-body spin dependent scattering lengths and the number of atoms in BEC. A measurement taken at a time scale longer than $\sigma/A_0\Delta E$ should reveal the symmetry restoring because of zero point motions of $\mathbf{n}$. With $N \sim 10^7$ and $c_2\rho \sim 100$ Hz(500 nK), the symmetry restoring time is about 1 day which is much longer than the lifetime of the trap. However, for a smaller cloud with 10^{3-4} atoms, the symmetry restoring can take place within 10–100 sec, before recombination processes take place.

Nematically ordered BEC has very fascinating optical properties [8]. In the presence of spin-orbital couplings, the dielectric constant is a tensor expressed in terms of $Q_{\alpha\beta}$. This suggests that birefringence takes place in the system as a direct evidence of the hidden Z_2 symmetry. Another experimental consequence associated with the broken symmetry is the enhanced small angle light scattering due to thermal fluctuations of $\mathbf{n}$. For ^{23}Na, the light scattering amplitude can be 4 orders of magnitude higher than that in an isotropic BEC and the nematic BEC is optically turbid. Finally, the Z_2 symmetry also implies that there exists a local Z_2 gauge field in BEC. This was recently considered and will be published elsewhere [15].

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