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IMPROVED DETERMINATION OF THE PLASMA POTENTIAL AND THE SHEATH RADIUS OF POSITIVE CYLINDRICAL AND SPHERICAL LANGMUIR-PROBES

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Synopsis

The probe current can be described by a formula from the orbital motion theory given by Langmuir. The probe current depends on the voltage difference between the probe and the plasma, the electron temperature, the probe radius and the sheath radius. On the other hand, an extension of Langmuirs formula for the space charge limited current is presented. In this extended formula the velocities of the electrons entering the sheath is accounted for. For several probe characteristics – with well-defined plasma potentials – the orbital motion formula together with the corrected space charge formula have been checked. A good agreement exists. On the other hand, in the case of a probe characteristic with a badly known plasma potential, the two formulas can be used together to determine the plasma potential and the sheath radius.

Introduction. It is a well known fact that the electron density in a low pressure plasma can be derived with Langmuir probes from a determination of the probe current at plasma potential.

A second method is the calculation of the density from the slope of the $I^2 - V$ plot (cylindrical case) or the $I - V$ plot (spherical case); however, application of this method is limited to a range of radii of the space charge sheath, below which the formulas are not valid and above which electron collisions in the sheath contradict the assumptions. The commonest method for the determination of the density is the first one, but it requires an accurate knowledge of the plasma potential. In many experiments it is not possible to determine the plasma potential accurately. (The "knee" in the semilogarithmic $I - V$ plot does not stand out clearly). In the next, formulas will be derived for the determination of the plasma potential from the $I - V$ probe characteristics.

The situation of a probe in a plasma and of the collector in an inverted vacuum diode are in many aspects the same. Therefore first formulas are derived for the concentric cylindrical or spherical diode with the emitter outside. These formulas are used to describe the behaviour of a probe in a plasma.

1. *Orbital motion of electrons between an emitter and a collector (when space charge is negligible).* *) The current I_c on the collector (radius r_c) from a concentric external emitter (radius $r_e > r_c$) is influenced by the fact that the electrons have initial velocities with tangential components. As a consequence a part of the electrons follow comet-like orbits and do not reach the collector (fig. 1). Assuming no influence of space charge, the current to the collector, in the case of Maxwellian velocity distribution, is given by[1][2]:

$$I_c = S_c J_e f(r_e/r_c, \eta) \qquad (1a) \text{ cylinder}$$

$$I_c = S_c J_e F(r_e/r_c, \eta) \qquad (1b) \text{ sphere.}$$

The formulas ($1a$) and ($1b$) can be taken together as

$$I_c = J_e S_e \theta_1 \ (\equiv I_e \theta_1) \tag{1}$$

in which θ_1 is equal to $(r_c/r_e)\, f$ for the cylindrical case and equal to $(r_c/r_e)^2 F$ for the spherical case.

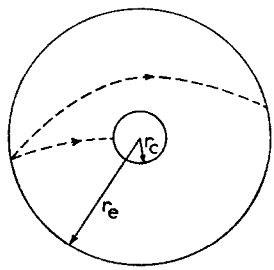

Fig. 1. Orbital motion of electrons between an emitter (radius r_e) and a collector (radius r_c).

S_c is the surface of the collector, S_e is the surface of the emitter. J_e is the electron current density at the surface of the emitter directed inwards. $\eta = eV/KT_e$, V is the voltage between the emitter and the collector ($V_{\text{collector}} > V_{\text{emitter}}$), T_e is the electron temperature (assuming Maxwellian energy distribution). A derivation and calculation of the functions f and F for the cylindrical and spherical diode respectively are given by Langmuir[2]).

2. *Space charge limited current between an emitter and a collector (in a diode without orbital motion).* The electron current I_c cannot increase indefinitely,

*) As the derivation of the formulas for the cylindrical case is quite the same as for the spherical case, the treatment will be simultaneously.

but will become space charge limited. The space charge limitation follows from the Poisson equation and the condition $\mathrm{d}V/\mathrm{d}r = 0$ for some r value. We suppose the choice of the emission current from the emitter such, that $\mathrm{d}V/\mathrm{d}r$ is zero near the surface of the emitter, because this situation has a great similarity with the surroundings of a probe in a plasma. For electrons starting with negligible initial velocities from the emitter, and thus with negligible orbital motion, the space charge limited current between the emitter and collector will be (in esu):

$$I' = \frac{2}{9}\left(\frac{2e}{m}\right)^{\frac{1}{2}} \frac{LV^{\frac{3}{2}}}{(-\beta)^2\, r_c} \qquad (2a) \text{ cylinder}$$

$$I' = \frac{4}{9}\left(\frac{2e}{m}\right)^{\frac{1}{2}} \frac{V^{\frac{3}{2}}}{(-\alpha)^2} \qquad (2b) \text{ sphere.}$$

L is the length of the cylinder.

The factors $(-\beta)^2$ and $(-\alpha)^2$ are functions of the parameter r_e/r_c only and are tabulated by Langmuir and Blodgett[3]) and by Ollendorff[8]).

The formulas (2a) and (2b) can be taken together as

$$I' = \theta_2 V^{\frac{3}{2}} \qquad (2)$$

in which θ_2 contains the geometrical factors of the cylinder and the sphere. The current I' is the space charge producing current, and in this case also the current to the collector.

3. *Space charge limited current between an emitter and a collector, with electrons leaving the emitter with initial velocities.* Formula (2) will now be changed in such a way that it can be used for the case where the electrons from the emitter do have *initial velocities* and consequently orbiting electrons do occur. An exact solution of the Poisson equation for this case is mathematically too involved.

In formula (2) is I' equal to I_c, for in that (hypothetical) case every electron leaving the emitter region reaches the collector. On the other hand, electrons with initial velocities can pass the collector. They do not contribute to I_c, but, nevertheless, contribute to the space charge, and even two times because of the relative high importance of the space charge region near the emitter[3]). The orbiting electrons contribute to the space charge on their way *to* and *from* the collector (fig. 3) whereas the electrons striking the collector contribute only once. This means that the number of electrons passing the space charge region each second is equal to $2J_eS_e - I_c$, which is the current that has to be substituted for I' in equation (2).*).

*) The collector current I_c from eq. (1) has been derived for the case when the space charge is negligible. However, according to Langmuir[9]) the equation is also applicable, when the current is so large that the space charge is nearly sufficient to bring $\mathrm{d}V/\mathrm{d}r$ to zero at the surface of the emitter. We shall describe the orbital motion in this case also with eq. (1).

Electrons which have randomly directed initial velocities will have higher velocities in the space charge region than electrons which start at rest. This results in a decrease in space charge, approximately given by a factor $(1 + 2.50\,\eta^{-\frac{1}{2}})$ by which θ_2 in equation (2) has to be multiplied.*)

The final formula for electrons emitted with randomly directed initial velocities is determined by the equality between the electrons contributing to the space charge and the space charge current of equation (2) corrected for the extra velocity:

$$2J_eS_e - I_c = \theta_2(1 + 2.50\,\eta^{-\frac{1}{2}})\,V^{\frac{3}{2}}. \tag{3}$$

The orbital motion formula (1) is correct between the emitter and collector. This means that I_c and also J_e in formula (1) and (3) may be indentified, when (1) is also used in the case where $\mathrm{d}V/\mathrm{d}r = 0$ at the surface of the emitter.

Elimination of J_eS_e from (1) and (3) leads to

$$I_c = \frac{(1 + 2.50\,\eta^{-\frac{1}{2}})\,\theta_2}{(2/\theta_1) - 1}\,V^{\frac{3}{2}} \tag{4}$$

Substituting the values of θ_1 and θ_2 gives for the collector current on a cylinder:

$$I_c = \frac{2}{9}\left(\frac{2e}{m}\right)^{\frac{1}{2}}\frac{LV^{\frac{3}{2}}}{r_cg}$$

with

$$g \equiv (-\beta)^2\left(\frac{2r_e}{fr_c} - 1\right)(1 + 2.50\,\eta^{-\frac{1}{2}})^{-1} \tag{5a}$$

and on a sphere:

$$I_c = \frac{4}{9}\left(\frac{2e}{m}\right)^{\frac{1}{2}}\frac{V^{\frac{3}{2}}}{G}$$

with

$$G \equiv (-\alpha)^2\left(\frac{2}{F}\left(\frac{r_e}{r_c}\right)^2 - 1\right)(1 + 2.50\,\eta^{-\frac{1}{2}})^{-1}. \tag{5b}$$

Or numerically (with I_c in amperes and V in volts):

cylinder:
$$I_c = 1.47 \times 10^{-5}\,\frac{LV^{\frac{3}{2}}}{r_cg} \tag{6a}$$

sphere:
$$I_c = 2.94 \times 10^{-5}\,\frac{V^{\frac{3}{2}}}{G}. \tag{6b}$$

*) The derivation of the correction factor is rather complicated for the cylindrical and spherical case. For the plane electrode geometry and Maxwellian initial velocities this factor is given by Langmuir [3]), namely $(1 + 2.66\,\eta^{-\frac{1}{2}})$. This is a useful approximation for large η. For $\eta < 10$, $(1 + 2.50\,\eta^{-\frac{1}{2}})$ is a more adequate approximation according to our calculations.

The factors g and G are functions of r_e/r_c and η. We have calculated these functions and they are given in figure 2 and figure 3.

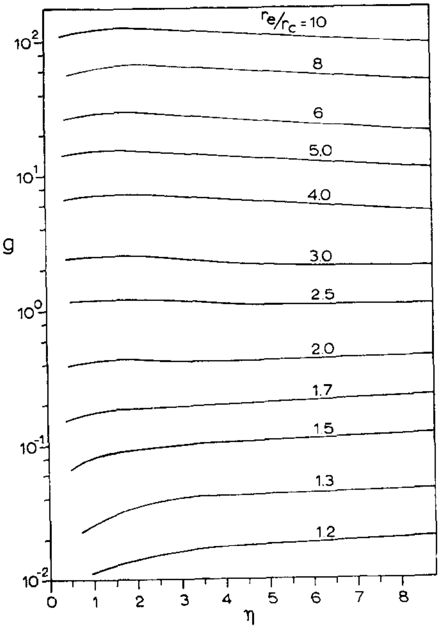

Fig. 2. g as a function of η for different values of r_e/r_c. (cylindrical case, formula 5*a*).

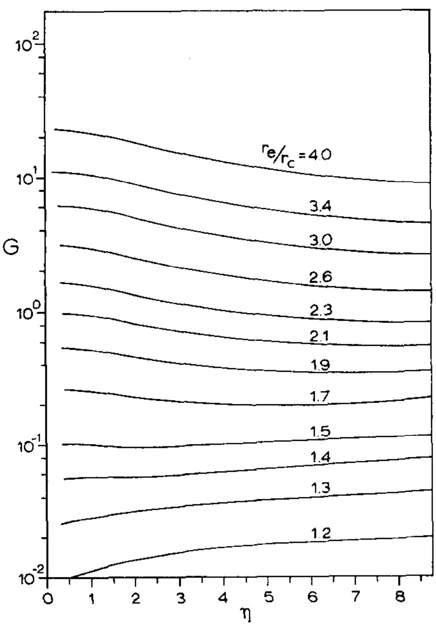

Fig. 3. G as a function of η for different values of r_e/r_c. (spherical case, formula 5*b*).

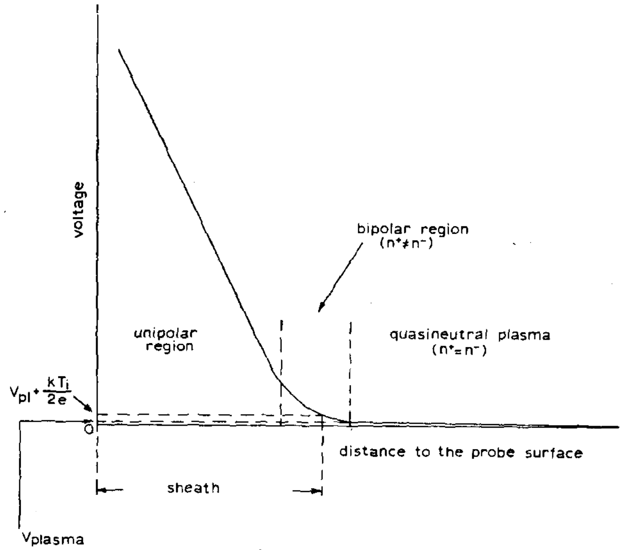

Fig. 4. The voltage in a plasma near a positive probe as a function of the distance to the probe surface.

4. *The probe in a plasma.* We want to apply the formulas (1*a*), (1*b*), (5*a*) and (5*b*) to the cylindrical and spherical probe in a plasma. The probe is comparable with the collector, the plasma with the emitter and the space-charge sheath with the region between the emitter and the collector. A description of the sheath will show the appropriateness of this method.

In the neighbourhood of the probe (fig. 4) there is generally a large potential gradient. For a positive probe, electrons and only a few ions are present in this region ("unipolar region"). Around the former region exists a "bipolar" region in which $n^+ \neq n^-$. The positive ions penetrate into this region as a result of their thermal velocities. The ion density distribution is a Boltzmann distribution. An analogous description can be given for a negative probe. Outside the bipolar region with $n^+ \neq n^-$ there is a quasi-neutral plasma with $n^+ \approx n^-$ which shades off into the undisturbed plasma. The most important part of the potential difference between the plasma and the probe is in the unipolar region. In the cases under consideration we shall assume that the mean free path of the electrons is large compared to the thickness of the unipolar region $(r_e - r_c)$. Then the influence of the gas atoms will be negligible. These facts permit us to compare the unipolar region around a probe with the region between an emitter and a collector in a vacuum diode. The question arises where one must locate the pseudo emitter at the outside of the sheath, because it shades off asymptotically. Some authors use an artificial conception "boundary of the sheath". Bohm[5][6][7] gives the following criterion for this boundary for the negative probe: the boundary of the sheath is the surface where $V = kT_e/2e$. We shall take the boundary for the positive probe at $V = kT_i/2e$. This assumption is arbitrary. The influence of the choice of V on the location of such a boundary may be estimated from the dependence of V on r. Chen[10]) has calculated $V(r)$ for a negative probe, but it holds for a positive probe too. Taking $V = kT_i/4e$ instead of $V = kT_i/2e$ a difference of 8% in r appears.

Bohm assumes $\mathrm{d}V/\mathrm{d}r = 0$ at the sheath boundary for the negative probe. This is in conflict with the assumptions, but it will be a good approximation. We shall use the same approximation for the description of the positive probe.

The description of the sheath justifies the application of the formulas for the diode to interpret the experimental results obtained with cylindrical and spherical probes.

5. *Experimental results*. A check of the formulas (1) and (5) can be made with the aid of experimental probe characteristics with well-determined plasma potentials.

When in an experimental probe characteristic the plasma potential and the electron temperature are well known, η is fixed for every I_c. With the formulas (5) and fig. 2 or fig. 3 the function r_e/r_c (η) may be calculated. On the other hand, one can calculate the probe current I_c from formula (1), when S_cJ_e and $r_e/r_c(\eta)$ are known. For this purpose we will substitute in formula (1) the current measured at plasma potential and for r_e/r_c (η) the values derived with formula (5). In this way we can recalculate the probe characteristic. When formula (1) and formula (5) are correct, the calculated probe characteristic must be the same as the experimental probe characteristic.

Probe characteristics were measured with cylindrical and spherical probes under several circumstances (variation of gas pressure, type of gas, tube current and probe diameter). The agreement between formula (1) and (5) appears to be satisfactory. The differences in I_c at a given probe voltage are a few percents only. An illustration of the results is given in table I.

Remark: the agreement between I_c (calculated) and I_c (measured) is not trivial, although we use the formula for the orbital motion in the space charge formula too. This is illustrated by the following facts. In the first place, a wrong choice of the plasma potential will cause a difference between I_c (calculated) and I_c (measured). For example, when we locate the plasma potential in the cylindrical case 0.6 volt negative to the real plasma potential, for $\eta = 2.23$ the ratio between I_c (calculated) and I_c (measured) will be 0.71; for the plasma potential 0.3 volt negative, the ratio will be 0.86. With small deviations for the plasma potential, the ratio will change linearly. If we make the comparison between the measured and calculated I_c with formula (1) and the uncorrected space charge formula (2) instead of formula (1) and (5) a serious disagreement will appear.

TABLE I

Experimental check of the orbital motion formula (1) and the space charge formula (6). r_e/r_c is determined with formula (6); I_c (meas.) is the experimentally determined probe current; I_c (calc.) the probe current calculated with formula (1); p_0 is the reduced pressure (0°C); I is the tube current; I_{pl} is the probe current at plasma potential.

Cylindrical probe, radius 9.85×10^{-4} cm, mixture of neon and argon (1% argon), reduced pressure 4.28 torr, tube current 0.61 A; tube radius 1.2 cm; electron temperature is 1.5 volt and the probe current at plasma potential I_{pl} is 4.3 mA.

V (volt)	η	r_e/r_c	I_c (mA) meas.	I_c (mA) calc.
0.35	0.23	1.7	3.8	3.8
1.35	0.90	2.5	5.0	5.1
2.35	1.57	3.0	5.8	5.9
3.35	2.23	3.4	6.5	6.5
4.35	2.90	3.8	7.3	7.2

Spherical probe, radius 2.79×10^{-2} cm, gas filling with helium, tube radius 3.2 cm.

V (volt)	T_e (volt)	I (mA)	p_0 (10^{-2} torr)	I_{pl} (μA)	η	r_e/r_c	I_c (mA)	I_c (mA)
14.1	7	257	6	258	2.02	2.5	0.71	0.69
7.4	7	257	6	258	1.03	2.2	0.47	0.49
16.1	5.8	65	12	115	2.78	3.1	0.38	0.39
9.1	5.8	65	12	115	1.57	2.6	0.25	0.26
18.5	10.2	2570	11	4510	1.82	1.7	9.70	9.85

6. *Determination of the plasma potential.* In the foregoing it appears that the calculated I_c agrees well with the measured I_c if the plasma potential is chosen correctly. When we assume the correctness of the formulas (1) and (5), one can use them together to determine the plasma potential. This procedure is especially useful when the "knee" in the probe characteristic does not stand out clearly. For this purpose we take a provisional plasma potential and an arbitrary I_c. In the way described above we may calculate the I_c too. The calculated I_c must be the same as the measured I_c when the plasma potential is chosen correctly. The correct plasma potential may be found by changing the provisional plasma potential to make the difference between the calculated I_c and the measured I_c sufficiently small. In practice, two or three steps are sufficient to reach an accuracy of about 0.2 volt.

7. *Determination of the sheath radius.* The plasma density can be determined also with the use of the slope of the I^2 - V plot for the cylindrical probe and the slope of the I - V plot for the spherical probe. For a correct application of the formulas for the plasma density it is necessary to have a sufficiently large r_e/r_c (about 5 to 10), but a small sheath radius r_e in order to avoid electron collisions in the sheath. The calculation (e.g. with formula

6) of r_e/r_c (η) is very useful to test these conditions. This makes it possible to determine the maximum tolerable probe diameters and probe voltages beforehand.

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