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ON THE THEORY OF STATIONARY WAVES IN PLASMAS

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Synopsis

Existing theories of stationary plasma oscillations lead to a dispersion equation (2), involving an integration across a pole. It is here shown that this difficulty is of purely mathematical origin, and can be overcome by a proper treatment. This treatment leads to a complete set of stationary solutions, which are much more numerous than the usual plasma oscillations. In particular, their wave lengths and frequencies are not connected by a dispersion equation, but independently assume all real values. Special superpositions of these stationary solutions correspond to the usual plasma oscillations. They constitute slightly damped plane waves, which do obey the dispersion equation (2), the integral being interpreted as a Cauchy principal value. An arbitrary initial distribution behaves (after a short transient time) like a superposition of such waves, as far as the density is concerned.

1. *Introduction.* The theory of plasmas deals with gases consisting of particles with long-range interaction. Such interaction gives rise to collective motions which cannot be adequately treated by means of the familiar picture of collisions. The best known example is a gas of electrically charged particles, for instance electrons. The total charge is often supposed to be neutralised by a constant charge density of opposite sign. In the following we confine ourselves to this case; moreover we only take into account the Coulomb force and neglect the magnetic field.

Usually there are also short-range forces, which give rise to collisions, in addition to the collective motion. If these collisions are frequent enough to insure that local equilibrium is established, the theory becomes quite elementary. Only five macroscopic parameters are needed to describe the state of the gas at each point, *viz.*, the density, the three components of the centre of gravity motion, and the temperature. Their dependence on space and time can be found by applying the laws of conservation of mass, momentum, and energy [1]).

This paper is concerned with the opposite extreme, namely the case in which individual collisions may be neglected altogether. A complete description of the behaviour of the gas then requires the determination of a velocity distribution $f(\mathbf{r}, \mathbf{v}, t)$ for each space-time point. This function obeys a nonlinear transport equation, which for small deviations from the equilibrium state may be linearised in the usual way.

At first sight the most promising approach to solve this linearised transport equation seems to consist in finding stationary plane wave solutions. Tonks and Langmuir [2]) showed, indeed, that when the individual motion of the particles is neglected, standing density waves are possible with a fixed frequency

$$\omega_P = (4\pi e^2 n_0/m)^{\frac{1}{2}}. \tag{1}$$

(e is the charge, m the mass of the particles, and n_0 is the average number of particles per unit volume.) Vlasov [3]) allowed for a random motion with a velocity distribution $f_0(\mathbf{v})$ for the equilibrium state, and found that waves with wave vector $\mathbf{k}$ and frequency ω are possible, if ω and $\mathbf{k}$ are related to each other by the dispersion equation

$$\frac{4\pi e^2}{m} n_0 \frac{\mathbf{k}}{k^2} \int \frac{\partial f_0}{\partial \mathbf{v}} \frac{\mathrm{d}\mathbf{v}}{\mathbf{kv} - \omega} = 1. \tag{2}$$

Other derivations have been given by Bohm and Gross [4]) and Pines and Bohm [5]), who arrived at the slightly different dispersion equation (33) discussed in section 9.

Unfortunately the above equation suffers from a serious flaw: the denominator in the integrand vanishes for certain $\mathbf{v}$, so that the integration cannot really be carried out. Vlasov decreed that the Cauchy principal value has to be taken, without trying to justify this choice. The present article will show that this choice was indeed correct.

Bohm and collaborators avoided the difficulty by assuming that $f_0(\mathbf{v})$ is zero in some range of $\mathbf{v}$ containing the zeros of the denominator. The simplest is to assume $f_0(\mathbf{v}) = 0$ for $|\mathbf{v}| > v_{\max}$ and to consider only plane wave solutions for which $\omega > |\mathbf{k}|\, v_{\max}$. They found that for each $\mathbf{k}$ satisfying

$$k^2 \langle \mathbf{v}^2 \rangle_0 \ll \omega_P^2 \tag{3}$$

there is just one ω (or rather one pair of values $\pm\, \omega$), approximately given by

$$\omega^2 \approx \omega_P^2 + k^2 \langle \mathbf{v}^2 \rangle_0.$$

$\langle \mathbf{v}^2 \rangle_0$ is the mean square velocity in equilibrium:

$$\langle \mathbf{v}^2 \rangle_0 = \int \mathbf{v}^2 f_0(\mathbf{v}) \mathrm{d}\mathbf{v}.$$

Hence to every wave vector $\mathbf{k}$ corresponds one stationary solution, called 'plasma oscillation'.

However, the assumption of a rigorously cut off f_0 is rather artificial, and it is unsatisfactory that the calculation should fail for an f_0 that decreases rapidly without actually vanishing. Another objection is, that there is no indication whether these stationary plasma oscillations are complete, in the sense that every other solution of the equation of motion is a superposition of them.

An alternative device to avoid the singularity in (2) is to consider only complex **k** and ω. These complex values give rise to exponentially decreasing or increasing waves, so that it is necessary to take the boundary conditions (in space and time) into account. One is thus led to the Laplace transform method, which was used by Landau [6]) and Twiss [7]). This method gives a complete solution without divergence difficulties, but it does not answer the question why these difficulties do appear in the stationary wave method.

Bohm and Gross explained the divergence in the integral (2) as due to the occurrence of 'trapped electrons', which amounts to a breakdown of the linear approximation. However, although nonlinear effects may be important in applications to actual physical systems, it is hard to believe that the linear theory should not be mathematically consistent in itself. Another explanation was recently forwarded by Ecker [8]), who ascribed the divergence in (2) to a breakdown of the underlying physical model. He carefully discussed the validity of the concept of distribution function, and argued that the continuous function f_0 should be replaced by a series of δ functions. Then the integral in (2) reduces to a sum, in which case there are no divergence difficulties.

We summarise the state of affairs. The linearised transport equation can be solved without difficulties in terms of time dependent solutions by means of a Laplace transformation. On the other hand, the search for stationary solutions leads to absurdities, if the particle velocities are distributed continuously in **v** space. If, however, the velocities are confined to a discrete set of points in **v** space, there are no difficulties, *no matter how densely* **v** *space is filled by these discrete points*. This makes apparent that the divergence in (2) has nothing to do with physics, but is due to an inability of the mathematical procedure used to deal with the continuous case.

The object of this paper is to furnish a mathematical treatment which permits to use the stationary wave approach in the case of a continuous non-vanishing velocity distribution f_0. It will be shown that a complete set of stationary plane wave solutions can be constructed. There is no dispersion equation, because for given **k** a continuous range of values for ω is possible. The plasma oscillations envisaged by Bohm *et al.* do not belong to these stationary solutions; they are special wave packets, which, owing to a sort of resonance, have an exceptionally long life time.

2. *Derivation of the basic equation*. In this section the usual results are derived and reduced to one dimension. The calculation is essentially the one given by Vlasov [3]).

Let the number of particles in a given unit volume with given velocity be

$$f(\mathbf{r}, \mathbf{v}, t) = n_0 f_0(\mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t),$$

where n_0 and f_0 refer to the equilibrium state. (It happens to be convenient to add n_0 explicitly as a separate factor, so that f_0 is normalised to unity, although f and f_1 are not.) The evolution of f is given by Boltzmann's transport equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \frac{e}{m}\,\mathbf{E}\,\frac{\partial f}{\partial \mathbf{v}} = \left[\frac{\partial f}{\partial t}\right]_{\text{collisions}}. \tag{4}$$

$\mathbf{E}$ is the averaged electric field strength,

$$\mathbf{E}(\mathbf{r}, t) = e\int \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}\mathbf{r}' \int f_1(\mathbf{r}', \mathbf{v}', t)\mathrm{d}\mathbf{v}'. \tag{5}$$

The right-hand side of (4) accounts for short-range forces, and also for local deviations of the actual field strength from its averaged value (5). According to the program announced in the introduction, these collisions are neglected, so that the right-hand side of (4) is put equal to zero. The remaining transport equation is to first order in f_1

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\frac{\partial f_1}{\partial \mathbf{r}} + \frac{e}{m}\,n_0\,\mathbf{E}\,\frac{\partial f_0}{\partial \mathbf{v}} = 0. \tag{6}$$

We now look for solutions of (6) that have the form

$$f_1(\mathbf{r}, \mathbf{v}, t) = g(\mathbf{v})\, \mathrm{e}^{i(\mathbf{kr}-\omega t)},$$

with constants $\mathbf{k}$ and ω. Substitution of this 'Ansatz' in (6) gives with the aid of (5)

$$(-\,i\omega + i\mathbf{kv})\, g(\mathbf{v}) + \frac{e^2}{m}\, n_0 \frac{-\,4\pi i\mathbf{k}}{k^2}\, \frac{\partial f_0}{\partial \mathbf{v}} \int g(\mathbf{v}')\,\mathrm{d}\mathbf{v}' = 0.$$

It is convenient, though not essential, to suppose that f_0 only depends on the magnitude of $\mathbf{v}$, so that $f_0(\mathbf{v}) = f_0(v)$. Then the equation for $g(\mathbf{v})$ becomes

$$(\omega - \mathbf{kv})\, g(\mathbf{v}) = -\,\frac{4\pi e^2}{m}\, n_0 \frac{\mathbf{kv}}{k^2 v}\, f_0'(v) \int g(\mathbf{v}')\,\mathrm{d}\mathbf{v}'. \tag{7}$$

From this equation one readily finds (2), but we prefer to carry out some of the integrations first. For this purpose, let the direction of $\mathbf{k}$ be taken as z axis, and put

$$\iint_{-\infty}^{+\infty} g(\mathbf{v})\, \mathrm{d}v_x\, \mathrm{d}v_y = \bar{g}(v_z).$$

Then the whole equation (7) may be integrated over v_x and v_y with the aid of the identity

$$\iint_{-\infty}^{+\infty} v^{-1}\, f_0'(v)\, \mathrm{d}v_x\, \mathrm{d}v_y = -\,2\pi f_0(v_z), \tag{8}$$

yielding

$$(\omega - kv_z)\,\bar{g}(v_z) = \frac{8\pi^2 e^2}{m}\, n_0 \frac{v_z}{k} f_0(v_z) \int_{-\infty}^{+\infty} \bar{g}(v_z')\mathrm{d}v_z'. \tag{9}$$

This is a homogeneous linear equation for the function $\bar{g}(v_z)$. Clearly the integral $\int \bar{g}(v_z')\,\mathrm{d}v_z'$ must not be zero, so that it may be normalised to 1. The solution is then

$$\bar{g}(v_z) = \frac{8\pi^2 e^2 n_0}{mk} \frac{v_z f_0(v_z)}{\omega - kv_z}, \tag{10}$$

provided that ω and k are so chosen that

$$\frac{8\pi^2 e^2 n_0}{mk} \int_{-\infty}^{+\infty} \frac{v_z f_0(v_z)}{\omega - kv_z}\, \mathrm{d}v_z = 1.$$

This is identical to (2) on account of (8).

The function (10) is a satisfactory solution if the numerator $v_z f_0(v_z)$ vanishes for the value $v_z = \omega/k$, for which the denominator is zero. This is certainly realised when f_0 is supposed to be cut off above some fixed value $v_{\max}$, provided that the ratio ω/k that follows from the dispersion equation turns out greater than $v_{\max}$ — which is roughly the contents of the condition (3) of Pines and Bohm.

3. *Stationary plane wave solutions.* The present treatment is based on the observation that it is not necessary for (10) to be free from singularities. Since $\bar{g}(v_z)$ is a distribution function, it is sufficient to know how to use it for calculating averages, *i.e.*, how to integrate it after multiplication with other functions *). Hence (10) is a good distribution function as soon as a prescription is given how to integrate across the pole. This prescription cannot be determined *a priori*, for instance by decreeing that the Cauchy principal value is to be taken. On the contrary, every different way of dealing with the pole gives rise to a different distribution function $\bar{g}(v_z)$. All these different functions are comprised in

$$\bar{g}(v_z) = \frac{8\pi^2 e^2}{mk} n_0 v_z f_0(v_z) \left[\mathcal{P} \frac{1}{\omega - kv_z} + \lambda \delta(\omega - kv_z) \right], \tag{11}$$

where $\mathcal{P}$ indicates that the first term in [ ] is to be interpreted as a principal value, and λ is an arbitrary function of ω and k.

The idea that the expression [ ] is the appropriate way of writing the reciprocal of the factor $\omega - kv_z$ is due to Dirac [9]). He only used $\lambda = \pm \pi i$, which is equivalent with the familiar procedure of integrating above or below the pole. The rigorous justification of the δ term requires the mathematical theory of operators with continuous spectra. A more intuitive

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*) This amounts to saying that $\bar{g}(v_z)$ must be a 'distribution' in the sense of L. Schwartz.

way of arriving at the same result consists in first breaking up the continuous velocity distribution f_0 into a large number of discrete velocities, as advocated by E c k e r, and subsequently making the transition to the limit of a continuous distribution. It has been previously shown in a different connection, that this transition indeed gives birth to the δ term [10]).

Another heuristic way of arriving at (11) is as follows. Sometimes a damping term is added in the equation of motion [11]), *e.g.*, by replacing the collision term in (4) with $-\beta f_1$. This has the effect of changing in the ensuing equations ω into $\omega' = \omega + i\beta$. If one subsequently lets β go to zero, one formally gets the result (11), with $\lambda = -\pi i$. This would suggest that it is incorrect to introduce λ in (11) as an undetermined parameter. However, by regarding the absence of damping as the limit of a positive damping which tends to zero, one has introduced a boundary condition, namely, that at infinity there may only be outgoing waves (retarded solution). There are, of course, no stationary solutions obeying this condition, nor does it lead to a complete set of solutions, as it is not self-adjoint. To obtain the complete set of stationary solutions, one must also consider the limiting solutions for negative damping tending to zero. These give solutions of the form (11) with $\lambda = \pi i$. A linear combination of both yields again the stationary solution (11) with arbitrary λ.

The expression (11) satisfies (9), if it is consistent with the chosen normalisation

$$1 = \int_{-\infty}^{+\infty} \bar{g}(v_z)\, \mathrm{d}v_z = \frac{8\pi^2 e^2}{mk} n_0 \left[\mathcal{P} \int_{-\infty}^{+\infty} \frac{v_z f_0(v_z) \mathrm{d}v_z}{\omega - kv_z} + \frac{\lambda}{k} \frac{\omega}{k} f_0\left(\frac{\omega}{k}\right) \right]. \quad (12)$$

This condition can be fulfilled, for every real ω and k, by choosing λ appropriately. Thus the appearance of the new parameter λ makes it possible to satisfy the consistency equation without having to relate ω to k.

The only exceptions are the values of ω and k for which the coefficient of λ happens to be zero, *i.e.*, firstly $\omega = 0$ and, secondly, those values for which $f_0(\omega/k) = 0$.

The first exception is trivial: it is readily checked that for $\omega=0$, $k\neq 0$, eq. (9) has no solution at all. If both ω and k are zero, f_1 is constant in space and time and should therefore be incorporated in $n_0 f_0$.

The second exception, the case that f_0 vanishes for certain velocities, will be excluded in the present paper. We shall assume that f_0 never vanishes, although it may become very small, for instance at high velocities. Obviously this assumption cannot lead to physically incorrect results.

4. *The general solution.* To simplify the notation we introduce
(i) the phase velocity $u = \omega/k$;
(ii) the plasma frequency ω_P defined by (1);
(iii) the plasma phase velocity $u_P = \omega_P/k$;
(iv) the function $F(v_z) = 2\pi v_z f_0(v_z)$.

Then the (normalised) solution associated with given k and u is

$$\bar{g}_{k,u}(v_z) = \frac{\omega_{\rm P}^2}{k^2} F(v_z) \left[\mathcal{P} \frac{1}{u - v_z} + \lambda(k, u)\, \delta(u - v_z) \right], \qquad (13)$$

where $\lambda(k, u)$ is determined by (12), or

$$\omega_{\rm P}^2 \left[\mathcal{P} \int_{-\infty}^{+\infty} \frac{F(v_z) {\rm d}v_z}{u - v_z} + \lambda(k, u)\, F(u) \right] = k^2. \qquad (14)$$

The most general solution for f_1 that can be obtained in this way is a superposition of these stationary solutions. To avoid irrelevant complications we confine ourselves to functions f_1 that do not depend on the space coordinates x and y, so that only vectors $\mathbf{k}$ in the z direction need be considered. Moreover, since we are not interested in the distribution of v_x and v_y, we may integrate over both, so that we are only concerned with the integrated distribution function

$$\bar{f}_1(z, v_z, t) = \iint_{-\infty}^{+\infty} f_1(\mathbf{r}, \mathbf{v}, t)\, {\rm d}v_x\, {\rm d}v_y.$$

For this we have found the general form

$$\bar{f}_1(z, v_z, t) = \iint_{-\infty}^{+\infty} C(k, u)\, \bar{g}_{k,u}(v_z)\, {\rm e}^{ik(z-ut)}\, {\rm d}k\, {\rm d}u, \qquad (15)$$

where $C(k, u)$ is an arbitrary function.

The problem is now to show that this solution is complete, in the sense that any given initial distribution $\bar{f}_1(z, v_z, 0)$ can be obtained by constructing a suitable $C(k, u)$. In other words, the equation

$$\iint_{-\infty}^{+\infty} C(k, u)\, \bar{g}_{k,u}(v_z)\, {\rm e}^{ikz}\, {\rm d}k\, {\rm d}u = \bar{f}_1(z, v_z, 0)$$

has to be solved for $C(k, u)$, supposing that $\bar{f}_1(z, v_z, 0)$ is an arbitrary given function of z and v_z.

The first step is trivial; by Fourier transformation one finds

$$\int_{-\infty}^{+\infty} C(k, u)\, \bar{g}_{k,u}(v_z)\, {\rm d}u = \mathfrak{f}(k, v_z), \qquad (16)$$

where $\mathfrak{f}(k, v_z)$ is the Fourier transform of $\bar{f}(z, v_z, 0)$ with respect to z, and hence also an arbitrary function of k and v_z. Using (13) one finds for (16)

$$\omega_{\rm P}^2 \left[\mathcal{P} \int_{-\infty}^{+\infty} \frac{C(u)}{u - v_z} {\rm d}u + \lambda(v_z)\, C(v_z) \right] = \frac{k^2 \mathfrak{f}(v_z)}{F(v_z)}. \qquad (17)$$

k has no longer been written as a variable, because it only enters as a parameter and may therefore be treated as a constant.

5. *Solution of equation* (17). The singular integral equation (17) for $C(v_z)$ is of a type that has been extensively studied in recent years by Russian mathematicians [12]). At first sight one might think that our problem

suffers from the additional complication that $\lambda(v_z)$ is not an explicitly given function, but is determined implicitly by (14). However, the remarkable similarity between (17) and (14) will turn out to be a great help, which permits us to sidestep the general mathematical theory. The only mathematics needed are some results of the theory of Hilbert transforms [13], which are summarised in the next paragraph.

Every square integrable function $F(u)$ has a Fourier transform $\Phi(p)$ such that

$$F(u) = \int_{-\infty}^{+\infty} \Phi(p) \, \mathrm{e}^{ipu} \, \mathrm{d}p.$$

It can be uniquely decomposed into a 'positive-frequency part' $F_+(u)$ and a 'negative-frequency part' $F_-(u)$, by putting

$$F_+(u) = \int_0^{\infty} \Phi(p) \, \mathrm{e}^{ipu} \, \mathrm{d}p, \quad F(u) = \int_{-\infty}^{0} \Phi(p) \, \mathrm{e}^{ipu} \, \mathrm{d}p.$$

$F_+(u)$ has an analytic continuation without singularities in the upper half of the complex u plane, and $F_-(u)$ in the lower half. We shall write (for real u)

$$F_+(u) - F_-(u) = F_*(u);$$

it can then be shown that

$$F_*(u) = \frac{1}{\pi i} \mathcal{P} \int_{-\infty}^{+\infty} \frac{F(u')}{u' - u} \, \mathrm{d}u'. \tag{18}$$

If F is a real function, F_+ and F_- are complex conjugate to each other, and F_* is purely imaginary.

With the use of (18), eq. (14) for λ takes the form

$$-\pi i \, F_*(u) + \lambda(u) \, F(u) = u_{\mathrm{P}}^{-2},$$

while the integral equation (17) may be written

$$\{\lambda(u) + \pi i\} \, C_+(u) + \{\lambda(u) - \pi i\} \, C_-(u) = u_{\mathrm{P}}^{-2} \, \mathfrak{f}(u)/F(u). \tag{19}$$

Elimination of λ yields

$$\{1 + 2\pi i \, u_{\mathrm{P}}^2 \, F_+(u)\} \, C_+(u) + \{1 - 2\pi i \, u_{\mathrm{P}}^2 \, F_-(u)\} \, C_-(u) = \mathfrak{f}(u). \tag{20}$$

Here the given function $\mathfrak{f}(u)$ appears as the sum of two terms, one of which is holomorphic in the upper half plane, the other being holomorphic in the lower half plane. Consequently, if $\mathfrak{f}(u)$ itself is decomposed into a positive-frequency part $\mathfrak{f}_+(u)$ and a negative-frequency part $\mathfrak{f}_-(u)$ one may put

$$C_+(u) = \frac{\mathfrak{f}_+(u)}{1 + 2\pi i \, u_{\mathrm{P}}^2 \, F_+(u)}, \quad C_-(u) = \frac{\mathfrak{f}_-(u)}{1 - 2\pi i \, u_{\mathrm{P}}^2 \, F_-(u)}, \tag{21}$$

provided that the denominators do not vanish in their respective half planes of regularity.

To prove that this proviso is satisfied, first note that it is sufficient to consider the first denominator, because the second one is its complex conjugate. We shall show that the argument of the complex quantity

$$Z(u) = 1 + 2\pi i\, u_P^2\, F_+(u) = X(u) + iY(u) \tag{22}$$

returns to its original value when u describes a large contour consisting of the real axis from $-R$ to $+R$ and a semicircle with radius R in the upper half plane. If R is large enough, $F_+(u)$ is very small on the semicircle, so that $Z(u)$ remains in the vicinity of 1. Along the real axis one finds for the imaginary part

$$Y(u) = 2\pi u_P^2 \operatorname{Re} F_+(u) = \pi u_P^2\, F(u) = 2\pi^2 u_P^2\, u f_0(u). \tag{23}$$

On account of our assumption that f_0 never vanishes, (23) is positive for $u > 0$ and negative for $u < 0$, so that $Z(u)$ crosses the real axis only once, namely for $u = 0$. This point of intersection, however, lies on the right of the origin, because

$$X(0) = 1 + \pi i\, u_P^2\, F_*(0) = 1 + 2\pi\, u_P^2 \int_{-\infty}^{+\infty} f_0(u')\, du' > 0.$$

Hence the closed path described by $Z(u)$ does not enclose the origin.

This completes the proof that in (19) one may indeed take for C_+ and C_- the expressions (21), so that

$$C(u) = \frac{\mathfrak{f}_+(u)}{1 + 2\pi i\, u_P^2\, F_+(u)} + \frac{\mathfrak{f}_-(u)}{1 - 2\pi i\, u_P^2\, F_-(u)} \tag{24}$$

is a solution of the integral equation (17). That the solution is unique can be shown by taking $\mathfrak{f}(u) \equiv 0$. Then (19) becomes

$$\{1 + 2\pi i\, u_P^2\, F_+(u)\}\, C_+(u) = -\{1 - 2\pi i\, u_P^2\, F_-(u)\}\, C_-(u);$$

this states that a positive-frequency function is identical to a negative-frequency one, which can only be true if both are zero.

6. *Some mathematical remarks.* The above considerations would be quite rigorous if there was a reason to suppose that all functions concerned were indeed square integrable, and if, moreover F satisfies a Lipschitz condition. However, these are certainly not the natural conditions to impose on distribution functions like $F(v_z)$ and $\mathfrak{f}(v_z)$ (or f_0 and f_1 respectively), although they are fulfilled in most of the specific cases to which one would like to apply the theory. Yet there are exceptions, such as the case in which there are separate 'beams' of particles, each with a given velocity, so that f_0 contains one or more delta functions. From a physical point of view these delta functions may be regarded as the somewhat artificial limit of distributions with very high, but finite peaks. It is therefore clear that the limit-

ation to square integrable distribution functions cannot give rise to any physically incorrect results. It would be more satisfactory, however, to adapt the mathematics in such a way that no other properties of f_0 and f_1 are required than those which follow from their very nature as velocity distributions. The above philosophy suggests that this ought to be possible, but we shall not try to do it here.

This objection against assuming square integrability does not hold with respect to the z dependence of $\bar{f}_1(z, v_z, 0)$. For this dependence obeys the additional physical condition that the total electrostatic energy must be finite, which, in terms of the density excess

$$n_1(\mathbf{r}, t) = \int f_1(\mathbf{r}, \mathbf{v}, t)\, \mathrm{d}\mathbf{v},$$

reads

$$\tfrac{1}{2} e^2 \iint \frac{n_1(\mathbf{r}, t)\, n_1(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}\, \mathrm{d}\mathbf{r}\, \mathrm{d}\mathbf{r}' < \infty.$$

Although this condition is not quite the same as square integrability, it is sufficiently similar to expect that it can likewise be employed as a justification for the Fourier transformation in (16).

7. *Explicit form of the solution.* Collecting the results (15), (24), (13), and using the identity

$$\mathfrak{f}_\pm(u) = (2\pi)^{-1} \int_{-\infty}^{+\infty} \mathrm{e}^{-ikz'}\, \mathrm{d}z' \int_{-\infty}^{+\infty} \delta_\pm(u - v_z')\, \bar{f}_1(z', v_z', 0)\, \mathrm{d}v_z'$$

and the abbreviation (22), one finds

$$\bar{f}_1(z, v_z, t) = \iint_{-\infty}^{+\infty} G_t(z, v_z \mid z', v_z')\, \bar{f}_1(z', v_z', 0)\, \mathrm{d}z'\, \mathrm{d}v_z', \tag{25}$$

where the 'propagator' G_t is

$$G_t(z, v_z \mid z', v_z') = (2\pi)^{-1} \iint_{-\infty}^{+\infty} \mathrm{e}^{ik(z-z'-ut)} \times [Z(k, v_z)\, \delta_+(v_z - u) + Z^*(k, v_z)\, \delta_-(v_z - u)] \times \left[\frac{\delta_+(u - v_z')}{Z(k, u)} + \frac{\delta_-(u - v_z')}{Z^*(k, u)}\right] \mathrm{d}k\, \mathrm{d}u.$$

If one is not interested in the velocity distribution at t, but only in the density, then (25) must be integrated over v_z. The relevant formula is easily found to be

$$\int_{-\infty}^{+\infty} G_t(z, v_z \mid z', v_z')\, \mathrm{d}v_z = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \mathrm{e}^{ik(z-z'-ut)} \left[\frac{\delta_+(u - v_z')}{Z(k, u)} + \frac{\delta_-(u - v_z')}{Z^*(k, u)}\right] \mathrm{d}k\, \mathrm{d}u. \tag{26}$$

8. *Evolution of a plane density wave.* The general result will now be applied to an initial disturbance in which the density excess $n_1(z, 0)$ varies trigonometrically, while the velocity distribution is described by some function $g_0(v_z)$ independent of z

$$f_1(z, v_z, 0) = g_0(v_z)\, \mathrm{e}^{ik_0 z}. \tag{27}$$

The density at time t is then found from (26) to be

$$n_1(z, t) = \mathrm{e}^{ik_0 z} \int_{-\infty}^{+\infty} \mathrm{e}^{-ik_0 ut} \left[\frac{g_{0+}(u)}{Z(u)} + \frac{g_{0-}(u)}{Z^*(u)} \right] \mathrm{d}u.$$

If k_0 is taken positive, the second term in [ ] does not contribute for $t > 0$, as is seen by shifting the integration path in the $-i$ direction. The remaining expression

$$n_1(z, t) = \mathrm{e}^{ik_0 z} \int_{-\infty}^{+\infty} \frac{g_{0+}(u)}{Z(u)}\, \mathrm{e}^{-ik_0 ut}\, \mathrm{d}u \tag{28}$$

can be interpreted as follows.

According to (28), $n_1(z, t)$ consists of a superposition of plane waves having *the same wave vector k_0 but different phase velocities u.* The amount to which each wave contributes to the density is given by the complex amplitude

$$A(u) = \frac{g_{0+}(u)}{Z(u)} = \frac{k_0^2}{k_0^2 + 2\pi i\, \omega_{\mathrm{P}}^2\, F_{+}(u)}\, g_{0+}(u). \tag{29}$$

This amplitude is the product of two factors. The factor $g_{0+}(u)$ indicates how much of each wave with a particular u was present in the initial distribution (27). The factor $1/Z(u)$ indicates how effective this particular wave is in contributing to the density. For large $|u|$ the latter factor tends to 1.

We now proceed to investigate the general aspect of the function $n_1(z, t)$. The dependence on z is trivial; hence it suffices to keep z fixed and to study the dependence on t. Equation (28) determines $n_1(t)$ $(t > 0)$ as the Fourier transform of $A(u)$. The following is therefore merely an exercise in the theory of Fourier integrals. The only complication is that $A(u)$ is given in the form of a product of the two functions $g_{0+}(u)$ and $1/Z(u)$, so that a number of cases have to be distinguished, according to the various possible ways in which these functions can behave.

It is convenient to characterise the smoothness of $A(u)$ by a 'variation length' U_A roughly defined by

$$\left[\frac{\mathrm{d}A}{\mathrm{d}u}\right] \sim \frac{[A]}{\mathrm{U}_A}.$$

Here the brackets indicate the order of magnitude of $\mathrm{d}A/\mathrm{d}u$ and A. An estimate of the time τ after which the Fourier integral (28) becomes small is then obtained by putting

$$k_0 \mathrm{u}_A \tau \sim 2\pi. \tag{30}$$

Similarly u_f, u_g, and u_Z will be used to characterise the smoothness of f_0, g_{0+}, and $1/Z$ respectively. For example, if f_0 is a Maxwell-Boltzmann distribution one has $\mathrm{u}_f \sim (\varkappa T/m)^{\frac{1}{2}}$ (in obvious notation). Moreover, the variation of f_0 is roughly equal to that of F, and therefore of Z. The latter is equal to the variation u_Z of $1/Z$, *unless there are values of u for which $Z(u)$ becomes very small.* This can only occur if values of u come into play for which $f_0(u)$ is small; this case is treated in the next section.

We are now in a position to consider various possible cases separately.

FIRST CASE ($\mathrm{u}_g \sim \mathrm{u}_Z \sim \mathrm{u}_f$). Let the initial disturbance be distributed over the velocities in roughly the same way as the undisturbed velocities are distributed. More precisely, it is supposed, on the one hand, that $\mathrm{u}_g \sim \mathrm{u}_f$; and, on the other hand, that $g_0(v_z)$ is negligible for those velocities for which f_0 is small, so that $\mathrm{u}_Z \sim \mathrm{u}_f$. Then clearly $\mathrm{u}_A \sim \mathrm{u}_f$, so that (30) becomes

$$\tau \sim 2\pi(k_0 \mathrm{u}_f)^{-1} = \lambda_0/\mathrm{u}_f.$$

The physical interpretation is simply that it takes this time τ for the particles that had originally the same position to become dispersed over a region of the size of a wavelength λ_0, through the effect of their different individual velocities. It is clear that then the initial variation of the density is no longer recognisable.

SECOND CASE ($\mathrm{u}_g \ll \mathrm{u}_Z \sim \mathrm{u}_f$). Suppose that $g_0(v_z)$ is a sharp peak about v_z^0 with width δv_z. That means that the initial disturbance is confined to an almost mono-energetic beam of particles. Moreover, it must be supposed that v_z^0 lies in a region where f_0 is not small. Then $\mathrm{u}_Z \sim \mathrm{u}_f \ll \mathrm{u}_g$, so that $\mathrm{u}_A \sim \mathrm{u}_g \sim \delta v_z$. Consequently

$$\tau \sim \lambda_0/\delta v_z.$$

In the limit $g_0(v_z) \to \delta(v_z - v_z^0)$ the lifetime τ becomes infinite. In this limit, the expression (28) for n_1 may be written

$$n_1(z, t) = \frac{e^{ik_0(z - v_z^0 t)}}{2Z(k_0, v_z^0)} - \mathcal{P}\int_{-\infty}^{+\infty} \frac{e^{ik_0(z-ut)}\,\mathrm{d}u}{2\pi i\, Z(k_0, u)\,(u - v_z^0)}.$$

The first term has the form of a disturbance that is bodily carried along with the beam, while the second term describes the correlated motion of particles with nearby velocities.

9. *Plasma oscillations*. The oscillation discussed by Bohm *et al.* was subject to the condition (3), which states that the phase velocity must be much higher than the velocities for which f_0 is appreciable. In fact, their solution is only exact if f_0 is rigorously zero in the vicinity of the phase

velocity. In this section it is shown that the present treatment leads to a similar oscillation if f_0 is small without actually vanishing. This will be our

THIRD CASE ($\mathrm{U}_Z \ll \mathrm{U}_f \sim \mathrm{U}_g$). It is supposed that $g_0(v_z)$ is about as smooth as $f_0(v_z)$, but extends to high values of v_z for which f_0 is very small. For these v_z the imaginary part of $Z(v_z)$ is small according to (23). On the other hand, the real part of $Z(u)$ is

$$X(u) = 1 + 2\pi \frac{\omega_{\mathrm{P}}^2}{k_0^2} \mathcal{P} \int_{-\infty}^{+\infty} \frac{u' f_0(u')}{u' - u} \, \mathrm{d}u'.$$

This may vanish for a suitable value u_{B} of u. This equation for u_{B}, namely $X(u_{\mathrm{B}}) = 0$, is identical to (2), *provided one supplements the latter by prescribing that the principal value must be taken.*

In the neighbourhood of u_{B} one has approximately

$$Z(u) = iY(u_{\mathrm{B}}) + (u - u_{\mathrm{B}})X'(u_{\mathrm{B}})$$

which shows that $Z(u)$ has a zero at

$$u_0 = u_{\mathrm{B}} - iY(u_{\mathrm{B}})/X'(u_{\mathrm{B}}).$$

Consequently $A(u)$ has a pole *near* the real axis, which corresponds to a peak *on* the real axis at u_{B}. How to deal with such peaks in the Fourier transform is well known from the theory of resonance scattering. One writes

$$1/Z(u) = c/(u - u_0) + \zeta(u),$$

where c is the residue $1/Z'(u_0)$, and $\zeta(u)$ has no pole at u_0 and may therefore be considered smooth. Now the first term contributes to (28)

$$C\, \mathrm{e}^{ik_0(z - u_{\mathrm{B}} t)}\, \mathrm{e}^{-t/\tau_{\mathrm{B}}}, \tag{31}$$

where
$$1/\tau_{\mathrm{B}} = k_0 Y(u_{\mathrm{B}})/X'(u_{\mathrm{B}}) \tag{32}$$

and
$$C = c g_{0+}(u_0) \approx g_{0+}(u_{\mathrm{B}})/X'(u_{\mathrm{B}}).$$

These are good approximations if τ_{B} is very large, *i.e.*, if $f_0(u_{\mathrm{B}})$ is very small. If, moreover, $\zeta(u)$ is really a smooth function, it will give a contribution to (28) that dies out rapidly, so that after a short transient time (of order $(k_0 \mathrm{U}_\zeta)^{-1}$) the density $n_1(z, t)$ is practically given by (31). This is a damped wave, whose phase velocity U_{B} and damping constant τ_{B} depend on the wave length $2\pi/k_0$ alone, while the particular form of the initial disturbance only influences the amplitude C.

Thus we have found collective motions that correspond to the usual plasma oscillations. Their frequency is given by the same equation (2), but they are damped with a decrement (32). This reduces to the value that L a n d a u [6]) found by means of Laplace transformation, if one takes for f_0 a Maxwell-Boltzmann distribution, and evaluates u_{B}, $Y(u_{\mathrm{B}})$ and $X'(u_{\mathrm{B}})$ with the same approximations that he used.

It may be surprising to find a damped wave as the solution of the equation (6), which is invariant for time reversal. This appearance of irreversibility, however, is the same phenomenon that is known from statistical mechanics. It is due to the fact that out of the ∞^2 numbers $\bar{f}_1(z, v_z, t)$ that are needed to specify the state at time t, we only pay attention to the ∞^1 numbers $n_1(z, t)$. The initial state was so chosen that the $n_1(z, 0)$ had certain non-equilibrium values. From there the $n_1(z, t)$ tend to their equilibrium value, namely zero, both in the forward and in the backward time direction. Of course, the function $\bar{f}_1(z, v_z, t)$, which specifies the state completely, does not tend to an equilibrium, because there is no mechanism through which the information involved in the complete specification of the initial state can get lost.

It follows that, if one knows the density at one time, $t = 0$ say, it is not possible to compute it at other times, unless the complete distribution function $\bar{f}_1(z, v_z, 0)$ is known. This fact was emphasized by Landau, and is also clearly demonstrated by (28). Indeed, any functional dependence of $n_1(z, t)$ on t (for $t > 0$) can be obtained, by choosing for its Fourier transform $A(u)$ an appropriate function, *i.e.*, by choosing an appropriate velocity distribution $g_0(u)$ for the initial disturbance (27). On the other hand, it can be asserted that, for all those $\bar{f}_1(z, v_z, 0)$ that satisfy the conditions of this section (*viz.*, that they are of the form (27), where g_0 is smooth and extends to high velocities), there is some period $(\lambda_0/\mathrm{U}_\zeta < t < \tau_\mathrm{B})$ during which the density is approximately equal to (31).

When the value of f_0 at the velocity u_B is allowed to go to zero (for instance by considering low temperatures), the lifetime τ_B tends to infinity, so that the limit is just the usual stationary plasma oscillation. Our treatment is only rigorous if f_0 does not actually vanish, but it can be extended to that case. That case, however, is already covered by the treatment of Bohm and Gross.

A final remark should be made on the slightly different form that Bohm and Gross found for the dispersion equation, *viz.*,

$$\frac{4\pi e^2}{m} n_0 \int \frac{f_0(\mathbf{v})}{(\mathbf{kv} - \omega)^2} \, d\mathbf{v}. \tag{33}$$

If $f_0(\mathbf{v})$ is zero, for all $\mathbf{v}$ for which the denominator vanishes, the equation is equivalent to (2) (by partial integration). If $f_0(\mathbf{v})$ is not zero for those values of $\mathbf{v}$, (33) has no meaning, unless a prescription is added how to deal with the singularity. This prescription is uniquely determined by requiring that (33) must be equivalent to (2). It amounts to defining the value of the singular integral in (33) as half the sum of the values obtained by integrating above and below the pole.

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