

BOUND STATES OF ^4He WITH LOCAL INTERACTIONS

J.A. TJON

Institute for Theoretical Physics, University of Utrecht, The Netherlands

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The Yakubovsky four-body equations for the 0^+ -state are solved in the boundstate region for the case of local central forces of the Yukawa type, using the Hilbert–Schmidt expansion. As a result the groundstate energy of ^4He is found to be -29.6 MeV.

In the past few years considerable attention has been paid to the three-nucleon problem within the framework of the Faddeev equations. There exists by now various computational techniques to determine for a given arbitrary two-body interaction the actual solution of the three-body equations [e.g. 1]. On the other hand the four-nucleon system has been investigated quantitatively in the integral equation formulation only for the simplest case of separable Yamaguchi potentials [2, 3]. In this note we explore the possibility of solving the four-body equations for the case that the two-nucleon interactions are given both in the spin-singlet and triplet channels by pure s-wave local potentials of the Yukawa type.

$$V(r) = -\lambda_A \frac{\exp(-\mu_A r)}{r} + \lambda_R \frac{\exp(-\mu_R r)}{r}. \quad (1)$$

The interaction parameters employed in eq. (1) are those found in ref. [4] and denoted by set (I–III). They have been determined from fitting the overall s-wave phase shifts.

The four-body equations which have been examined are of the Yakubovsky–Faddeev type [5] and are reduced to a coupled set of integral equations in one continuous variable by introducing the Hilbert–Schmidt (H.S.) expansion in the two-particle, $(3+1)$ - and $(2+2)$ -particle amplitudes. Clearly, the usefulness of this method depends strongly on the number of terms needed in the expansion. As is known [6] the two-particle T -matrix can be represented as

$$t(k, k'; z) = \sum_m g_m(k, z) d_m(z) g_m(k', z), \quad (2)$$

with $d_m(z) = -(1/4\pi) \lambda_m(z)/(1 - \lambda_m(z))$. Here $\lambda_m(z)$ and $g_m(k; z)$ are the two-particle H.S. Eigenvalues and eigenfunctions defined through the equation

$$\lambda_m(z) g_m(k, z) = 4\pi \int_0^\infty V(k, k') (z - k'^2)^{-1} g_m(k', z) k'^2 dk'. \quad (3)$$

Using the separable representation (2) and following the same procedure of Narodetsky et al. [2] the Yakubovsky equations are reduced to a set of integral equations. For the case of spinless particles and total angular momentum zero they are of the form

$$\begin{aligned} A_{nn'}(q, q'; z) &= X_{nn'}(q, q'; z) \\ &- \sum_{n''} \int_0^\infty q''^2 dq'' \left\{ X_{nn''}(q, q''; z) \frac{\eta_{n''}(z - q''^2)}{1 - \eta_{n''}(z - q''^2)} A_{n''n'}(q'', q'; z) + Y_{nn''}(q, q''; z) \frac{\xi_{n''}(z - q''^2)}{1 - \xi_{n''}(z - q''^2)} B_{n''n'}(q'', q'; z) \right\}, \\ B_{nn'}(q, q'; z) &= -2 \sum_{n''} \int_0^\infty q''^2 dq'' Y_{n''n}(q'', q; z) \frac{\eta_{n''}(z - q''^2)}{1 - \eta_{n''}(z - q''^2)} A_{n''n'}(q'', q'; z), \end{aligned} \quad (4)$$

Table 1

The binding energies of the ground state of the tri-nucleon system (E_t) and the four-nucleon system (E_α) as a function of $N_\lambda^{(\pm)}$.

$N_\lambda^{(+)}$	$N_\lambda^{(-)}$	E_t	E_α
1	0	9.47	36.6
1	1	7.96	26.4
1	2	7.76	24.7
2	0	10.13	40.8
2	1	8.60	30.2
2	2	8.38	28.5
3	2	8.53	29.5
4	2	8.56	29.6

Table 2

The boundstate energies of the groundstate and excited state of ^4He for various N_η and N_ζ with $N_\lambda^{(+)} = 1$, $N_\lambda^{(-)} = 0$.

$N_\eta^{(+)}$	$N_\eta^{(-)}$	$N_\zeta^{(+)}$	$N_\zeta^{(-)}$	E_0	E_1
1	0	1	0	-36.6	-
1	1	1	0	-36.5	-
2	0	1	0	-37.1	-9.52
2	0	1	1	-37.1	-9.52
2	0	2	0	-37.3	-9.66
2	1	1	0	-36.9	-9.52
2	1	2	0	-37.2	-9.65
2	1	2	1	-37.2	-9.65
2	2	2	0	-37.2	-9.65
3	1	2	0	-37.2	-9.65

where the generalized potentials are defined as

$$X_{nn'}(q, q'; z) = \frac{1}{2} \left(\frac{3}{2\sqrt{2}} \right)^3 \int_{-1}^1 dx \sum_m w_{nm}(q_2; z - q^2) d_m(z_1) w_{n'm}(q_1; z - q'^2),$$

$$Y_{nn'}(q, q'; z) = \frac{1}{2} \left(\frac{3}{2} \right)^3 \int_{-1}^1 dx \sum_m w_{nm}(s_2; z - q^2) d_m(z_2) v_{n'm}(s_1; z - q'^2),$$
(5)

with

$$q_1^2 = \frac{2}{8} q^2 + \frac{1}{8} q'^2 + \frac{3}{4} qq'x; \quad q_2^2 = \frac{2}{8} q'^2 + \frac{1}{8} q^2 + \frac{3}{4} qq'x; \quad s_1^2 = \frac{3}{2} q^2 + \frac{1}{2} q'^2 + \sqrt{3} qq'x;$$

$$s_2^2 = \frac{3}{2} q'^2 + \frac{1}{2} q^2 + \sqrt{3} qq'x; \quad z_1 = z - \left(\frac{2}{8} q^2 + \frac{2}{8} q'^2 + \frac{3}{4} qq'x \right); \quad z_2 = z - \left(\frac{3}{2} q^2 + \frac{3}{2} q'^2 + \sqrt{3} qq'x \right).$$

In eqs. (4-5) z is the total energy of the system, while the four-particle amplitude can be expressed linearly in A and B . Furthermore, the kernel of eq. (4) contains the (3+1)- and (2+2)-eigenvalues η_n and ζ_n and their corresponding eigenfunctions w_{nm} and v_{nm} . They are given by

$$\eta_n(z) w_{nm}(p; z) = 4\pi \sum_{m''} \int_0^\infty p''^2 dp'' V_{mm''}(p, p''; z) d_m''(z - p''^2) w_{nm''}(p''; z),$$

$$\zeta_n(z) v_{nm}(p; z) = 4\pi \sum_{m''} \int_0^\infty p''^2 dp'' U_{mm''}(p, p''; z) d_m''(z - p''^2) v_{nm''}(p''; z),$$
(6)

where

$$V_{mm''} = \left(\frac{2}{\sqrt{3}} \right)^3 \int_{-1}^1 dx g_m(p_2; z - p^2) g_{m''}(p_1; z - p''^2) [z - (\frac{4}{3} p^2 + \frac{4}{3} p''^2 + \frac{4}{3} pp''x)]^{-1},$$

$$U_{mm''} = g_m(p; z - p^2) g_{m''}(p''; z - p''^2) [z - (p^2 + p''^2)]^{-1},$$
(7)

with

$$p_1^2 = \frac{4}{3} p^2 + \frac{1}{3} p''^2 + \frac{4}{3} pp''x; \quad p_2^2 = \frac{1}{3} p^2 + \frac{4}{3} p''^2 + \frac{4}{3} pp''x.$$

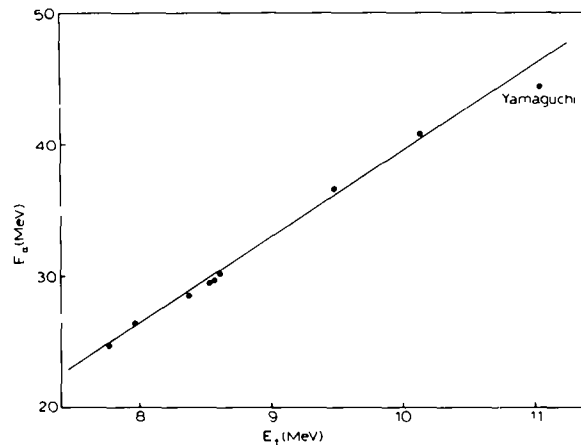


Fig. 1. Compilation of the results for the three- and four-particle binding energies for various approximations to the local interactions.

In the derivation of (4) the angular momentum $l \neq 0$ contributions of the subsystem in the $(3+1)$ -amplitude are neglected. The structure of the equations obtained in the presence of spin-dependent central forces is essentially the same as (4) since the spin-isospin algebra can be done independently from the orbital angular momentum part.

From (4) we see that the number of integral equations depends on the number N_η and N_ζ of eigenfunctions w and v which are kept in the four-body equations. In addition, the kernel of (4) depends explicitly on the number N_λ of the two-particle eigenfunctions g . The resulting equations for the 0^+ -channel have been studied in detail in the boundstate region for the local interactions (1). Some results are shown in table 1 in the approximation of keeping only the terms corresponding to the largest positive eigenvalue of the $(3+1)$ - and $(2+2)$ -amplitudes[†]. According to this table, the convergence of the expansion in $N_\lambda^{(\pm)}$ is reasonable. $N_\lambda^{(\pm)}$ are the number of terms with positive and negative eigenvalues taking into account in eq. (2).

An important difference between the Yamaguchi potential and the local interaction is the presence of negative two-particle eigenvalues in the latter case. As a consequence the eigenvalues η and ζ can in general become complex and can cross each other as a function of their argument. Obviously, in the simplest case of $N_\lambda^{(+)} = 1$, $N_\lambda^{(-)} = 0$, η and ζ are real. For this the convergence rate with respect to N_η and N_ζ is shown in table 2. As is seen the rate is similar to the separable case of ref. [2]. On increasing the number of separable terms in the two-body T -matrix the convergence in N_η and N_ζ improves considerably. For the case of $N_\lambda^{(+)} = 2$, $N_\lambda^{(-)} = 2$ the inclusion of additional terms than the most relevant one in the $(3+1)$ - and $(2+2)$ -amplitudes change the binding energy of the groundstate by less than 0.1 MeV. By increasing each N_λ , N_η and N_ζ up to six terms one finds as the final result 29.6 MeV for the binding energy of the groundstate of the 0^+ -state. This is in remarkable agreement with experimental value of 28.34 MeV for ${}^4\text{He}$, especially since the Coulomb energy has not been taken here into account. Furthermore, we see from table 1 that stronger overbinding of the triton results in also stronger overbinding of ${}^4\text{He}$.

In fig. 1 we show the combined results for E_t and E_α for various N_λ , keeping only the most relevant separable term in the $(3+1)$ - and $(2+2)$ -amplitudes. From this we see an almost linear relation between these quantities. For completeness also the Yamaguchi result is shown. This results suggests that if the triton binding energy is reproduced correctly by a given two-particle interaction, this will also be the case for ${}^4\text{He}$. Moreover, as a consequence the more realistic interactions like the Reid potential would give considerable underbinding of ${}^4\text{He}$. We now turn to the discussion of excited states of ${}^4\text{He}$. From table 1 we see that for this case there is an excited

[†] We always take here the number of terms related to the spin-singlet amplitude the same as in the spin-triplet amplitude.

boundstate at the position of -9.65 MeV. However, if we consider the cases which have smaller binding energy for the groundstate no excited states are found. It is expected for these cases that it will show up in the continuum as a resonance, which is in agreement with the experiment giving a resonance at 0.4 MeV above $n-t$ threshold. Finally, by varying the distribution and number of integration points the numerical accuracy in the binding energy was estimated to be of the order of 0.5% .

To summarize, we have shown that the Hilbert-Schmidt expansion is an useful and practical way to solve the four-body equations for the case of local Yukawa potentials and we may hope that this method can also be applied to the more realistic nucleon-nucleon interactions.

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