

Stochastic Models for Uncertain Flexible Systems*†

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Abstract—If a spectral operator is perturbed by an infinite-dimensional white noise process, it generates a stochastic evolution operator which has well defined second order properties. This type of stochastic bilinear spectral evolution equation may be used to model uncertainty of the higher modes in flexible systems. Three typical examples of flexible beams with bilinear white noise perturbations are analysed in detail and conditions are given for the existence of a steady state covariance operator.

1. Introduction

IN SYSTEMS THEORY an important class of problems arises from the control of large flexible space structures whose behaviour is very complex and difficult to model. Although for control purposes it is not essential to have a perfect physical model to achieve one's control objective, the model does need to contain enough essential features of the dynamics. The process of arriving at a "good enough" model is ill defined and usually depends critically on the particular application. For large flexible space structures second order partial differential equation models have received a lot of attention in the literature, but as pointed out by Bernstein and Hyland (1985), among others, in practice one cannot obtain accurate partial differential equation models. Only the lower modes can be obtained with any reliability, while the higher modes may have a 30–50% error in frequency. In Bernstein and Hyland (1985) it was suggested that one way of allowing for this uncertainty in the model is to formulate a stochastic partial differential equation in which the stochastic perturbations reflect this uncertainty in the higher modes. For more details of their control design philosophy for uncertain flexible systems we refer to their account in Bernstein and Hyland (1985). There the analysis was restricted to finite-dimensional second order systems and this motivates the problem of the appropriate mathematical formulation of an infinite-dimensional bilinear stochastic partial differential equation.

In Curtain and Kotelenez (1986) we developed a theory for a class of stochastic bilinear spectral evolution equations, which one can formally think of as having a stochastic generator of the form $A + B(\cdot, n)$, where A is a discrete spectral operator on Hilbert space H , n is "white noise" on

the Hilbert space K and B is an H -valued bilinear form on $H \times K$ which leaves the spectral subspaces of A invariant. Spectral systems include a wide class of distributed parameter systems on a compact domain and so form an important class for applications, as pointed out by Curtain (1984).

In this paper we use the theory developed in Curtain and Kotelenez (1986) to investigate the well posedness of uncertain models for flexible systems which could be useful in robust controller design as done in Bernstein and Hyland (1985). In Section 2 we define the class of deterministic flexible systems considered and we discuss three examples, which are typical partial differential descriptions of damped, flexible beams. In Section 3 it is shown how various modelling uncertainties can be formulated as Stratonovich bilinear stochastic partial differential equations. For the three examples we give conditions for well posedness of the stochastic bilinear systems with two types of uncertainty which are motivated by engineering experience with models of flexible structures. Comparisons with a different approach by Balakrishnan (1983) for the same examples are given in Section 4. Finally some tentative conclusions are drawn in Section 5 and a technical lemma is proved in an Appendix.

2. Second order models for flexible systems

Large flexible systems are often modelled by a second order partial differential equation of the form

$$\ddot{y} + J\dot{y} + A_0 y = 0 \quad (2.1)$$

where A_0 is a self adjoint discrete spectral operator on a Hilbert space H_0 and J is a linear, possibly unbounded, operator on H_0 such that the following operator

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -J \end{pmatrix} \quad (2.2)$$

is a discrete spectral operator on the product space, $H = D(A_0^{1/2}) \times H_0$. We recall from Curtain (1984) that A is a discrete spectral operator on a complex separable Hilbert space H if it has a compact resolvent and its generalized projections $\{\pi_i\}$ form a basis for H .

This means that for all $x \in H$, we have the representation

$$x = \sum_{i=1}^{\infty} \pi_i x \quad (2.3)$$

and $\{\pi_i\}$ forms a *resolution of the identity* on H .

If A has the simple real eigenvalue λ_i with corresponding eigenvector ϕ_i , then the corresponding spectral projection π_i is given by

$$\pi_i x = \langle x, \psi_i \rangle \phi_i \quad (2.4)$$

where $\langle \cdot, \cdot \rangle$ is the inner product on H and ψ_i is the eigenvector of A^* corresponding to λ_i such that

$$\langle \phi_i, \psi_j \rangle = \delta_{ij} \quad (2.5)$$

$\{\phi_i, \psi_j; j = 1, \dots, \infty\}$ is called a *biorthogonal system* for H .

If A has the simple conjugate pair $(\lambda_i, \bar{\lambda}_i)$ as eigenvalues with corresponding eigenvectors $(\phi_i, \bar{\phi}_i)$ and if the

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corresponding eigenvectors of A^* are (ψ_{-i}, ψ_i) then we find it convenient to consider the joint spectral projection for $(\lambda_i, \bar{\lambda}_i)$ defined by

$$\pi_i x = \langle x, \psi_i \rangle \phi_i + \langle x, \psi_{-i} \rangle \phi_{-i}. \quad (2.6)$$

This implies that for $x \in D(A)$

$$Ax = \sum_{i=1}^{\infty} \Delta_i \pi_i x \quad (2.7)$$

where $\Delta_i = \lambda_i$ for real eigenvalues, and for the complex eigenvalue pair $(\lambda_i, \bar{\lambda}_i)$ we define

$$\Delta_i \pi_i x = \lambda_i \langle x, \psi_i \rangle \phi_i + \bar{\lambda}_i \langle x, \psi_{-i} \rangle \phi_{-i}. \quad (2.8)$$

If $\sup \operatorname{Re} |\lambda_i| < \infty$, then A generates a strongly continuous semigroup e^{At} given by Curtain (1984):

$$e^{At} x = \sum_{i=1}^{\infty} e^{\Delta_i t} \pi_i x \quad (2.9)$$

where $e^{\Delta_i t} = e^{\lambda_i t}$ for real eigenvalues and for the complex eigenvalue pair $(\lambda_i, \bar{\lambda}_i)$ we define it to be the following

$$e^{\Delta_i t} \pi_i x = e^{\lambda_i t} \langle x, \psi_i \rangle \phi_i + e^{\bar{\lambda}_i t} \langle x, \psi_{-i} \rangle \phi_{-i}. \quad (2.10)$$

Equation (2.1) has the state space form on H of

$$\dot{x} = Ax; \quad x(0) = x_0 \quad (2.11)$$

and its unique solution is $x(t) = e^{At} x_0$.

The spectral character of A means that it has a particularly convenient infinite matrix modal representation with respect to $\{\phi_i\}_{i=1}^{\infty}$ as a basis for H :

$$A = \operatorname{diag} (\Delta_1, \Delta_2, \dots); \quad (2.12)$$

$$e^{At} = \operatorname{diag} (e^{\Delta_1 t}, e^{\Delta_2 t}, \dots)$$

where for complex eigenvalues $(\lambda_j, \bar{\lambda}_j)$, we have $\Delta_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \bar{\lambda}_j \end{pmatrix}$ and $\Delta_j = \lambda_j$ for real eigenvalues.

The eigenvalues distribution depends strongly on the damping as we can see in the following three examples.

Example 1

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} + 2\alpha \frac{\partial y}{\partial t} = 0 \quad (2.13)$$

$$y(0, t) = 0 = y(1, t) = y_{xx}(0, t) = y_{xx}(1, t).$$

This is a special case of the example in Balakrishnan (1983). Then for A_0 on $L_2(0, 1)$ we choose

$$A_0 h = \frac{d^4 h}{dx^4}; \quad D(A_0) = \{h \in H^2(0, 1):$$

$$h(0) = 0 = h(1), \quad h_{xx}(0) = 0 = h_{xx}(1)\} \quad (2.14)$$

and $J = 2\alpha I$ is bounded on $L_2(0, 1)$. Then on the state space $H = D(A_0^{1/2}) \times L_2(0, 1)$, the following operator is a discrete spectral and an infinitesimal generator on H :

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -2\alpha I \end{pmatrix}. \quad (2.15)$$

A has the eigenvalues $\lambda_n = -\alpha \pm i\sqrt{(n\pi)^4 - \alpha^2}$ and the eigenvectors

$$\phi_n = \begin{pmatrix} \sin n\pi x \\ \lambda_n \sin n\pi x \end{pmatrix}, \quad \phi_{-n} = \begin{pmatrix} \sin n\pi x \\ \bar{\lambda}_n \sin n\pi x \end{pmatrix}.$$

$$\psi_n = \begin{pmatrix} \sin n\pi x \\ -\bar{\lambda}_n \sin n\pi x \end{pmatrix}, \quad \psi_{-n} = \begin{pmatrix} \sin n\pi x \\ -\lambda_n \sin n\pi x \end{pmatrix}$$

are the eigenvectors of A^* and (ϕ_n, ψ_n) ; $n = \pm 1, \pm 2, \dots$, forms a biorthogonal system when suitably normalized. So A satisfies our assumptions and has the diagonal form

$$A = \operatorname{diag} (\Delta_1, \Delta_2, \dots);$$

$$\Delta_j = \begin{pmatrix} -\alpha + i\sqrt{(j\pi)^4 - \alpha^2} & 0 \\ 0 & -\alpha - i\sqrt{(j\pi)^4 - \alpha^2} \end{pmatrix}. \quad (2.16)$$

Example 2

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} + 2\alpha \frac{\partial^3 y}{\partial t \partial x^2} = 0$$

$$y(0, t) = 0 = y(1, t) = y_{xx}(0, t) = 0 = y_{xx}(1, t). \quad (2.17)$$

This example was studied in Curtain (1984) and it can be formulated on the same state space as Example 1 with the same A_0 . In this case $J = 2\alpha d^2/dx^2$ is unbounded. A given by (3.2) has the eigenvalues

$$\lambda_n = -\alpha(n\pi)^2 \pm (n\pi)^2 i\sqrt{1 - \alpha^2}$$

and the eigenvectors

$$\phi_n = \begin{pmatrix} \sin n\pi x \\ \lambda_n \sin n\pi x \end{pmatrix}, \quad \phi_{-n} = \begin{pmatrix} \sin n\pi x \\ \bar{\lambda}_n \sin n\pi x \end{pmatrix}$$

and

$$\psi_{-n} = \begin{pmatrix} \sin n\pi x \\ -\bar{\lambda}_n \sin n\pi x \end{pmatrix} \quad \text{and} \quad \psi_n = \begin{pmatrix} \sin n\pi x \\ -\lambda_n \sin n\pi x \end{pmatrix}$$

are the eigenvectors of A^* and $\{\phi_n, \psi_n\}$, $n = \pm 1, \pm 2, \dots$, forms a biorthogonal system for H if suitably normalized.

Example 3. Although this example does not strictly fall within our definition of a discrete spectral operator it does have a well-defined spectral decomposition as does the semigroup, and therefore the theory applies here too. The following was studied in Curtain *et al.* (1985):

$$\left\{ \begin{aligned} \frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} + \alpha \frac{\partial^4}{\partial x^4} \left(\frac{\partial y}{\partial t} \right) &= 0 \\ y(0) = 0 = y_x(0); \quad y_{xx}(1) = 0 = y_{xxx}(1). \end{aligned} \right\} \quad (2.18)$$

With $H_0 = L_2(0, 1)$ we define the operator A_0 by

$$A_0 h = \frac{d^4 h}{dx^4};$$

$$D(A_0) = \left\{ h \in H^4(0, 1) \right. \\ \left. h(0) = 0 = h_x(0) = h_{xx}(1) = h_{xxx}(1) \right\} \quad (2.19)$$

and we take $J = \alpha A_0$. Then A given by (3.2) on $H = D(A_0^{1/2}) \times L_2(0, 1)$ has eigenvalues

$$\lambda_j^+ = -\frac{\alpha}{2} \mu_j^4 + \left(\frac{\alpha^2}{4} \mu_j^8 - \mu_j^4 \right)^{1/2}$$

$$\lambda_j^- = -\frac{\alpha}{2} \mu_j^4 - \left(\frac{\alpha^2}{4} \mu_j^8 - \mu_j^4 \right)^{1/2},$$

where μ_j are the roots of $1 + \cos \mu_j \cosh \mu_j = 0$. For $\mu_j^4 < 4/\alpha^2$, $(\lambda_j^+, \lambda_j^-)$ form a complex conjugate pair, while for large values of μ_j , both λ_j^+ and λ_j^- are real and as $\mu_j \rightarrow \infty$, we have

$$\lambda_j^+ \rightarrow -\frac{1}{\alpha}; \quad \lambda_j^- \rightarrow \frac{1}{\alpha} - \alpha \pi^4 (j + \frac{1}{2})^4. \quad (2.20)$$

So the eigenvalues have finitely many complex conjugate pairs and a *finite accumulation* point in $-(1/\alpha)$ and an accumulation point at $-\infty$. The eigenvectors of A corresponding to λ_j^+ and λ_j^- are

$$\psi_j = \begin{pmatrix} v_j \\ \lambda_j^+ v_j \end{pmatrix}, \quad \psi_{-j} = \begin{pmatrix} v_j \\ \lambda_j^- v_j \end{pmatrix}$$

where

$$v_j(x) = \sin \mu_j x + A_j \cos \mu_j x - \sinh \mu_j x - A_j \cosh \mu_j x \quad (2.21)$$

$$A_j = -\left(\frac{\sin \mu_j + \sinh \mu_j}{\cos \mu_j + \cosh \mu_j} \right) \quad (2.22)$$

and so $\{\phi_j, \phi_{-j}\}$ are complete in H , and together with the eigenvectors of A^* form a biorthogonal system for H if suitably normalized. Now although A does not have a compact resolvent by virtue of the accumulation of the spectrum, by a separation of variable type solution of (2.19) it is clear that A does have a spectral decomposition of the type (2.12).

3. Stochastic bilinear models for flexible systems

Unfortunately one has imperfect models for flexible systems and so the eigenvalues and eigenvectors are not known exactly. This presents a problem for the design of active controllers which must be designed for a given nominal model and then applied on an unknown real system. Engineers have suggested many ways of solving this problem, but all rely on assuming some knowledge about the model uncertainty. Bernstein and Hyland (1985) suppose that one can model the uncertainty as a stochastic perturbation of (2.1)–(2.11) as follows:

$$\begin{aligned} dz &= Az \, dt + \sum_{j=1}^{\infty} \gamma_j B_j \pi_j z \, d\beta_j(t) + F \, dw(t) \\ z(0) &= z_0 \end{aligned} \quad (3.1)$$

where A and π_j are as in (2.4), γ_j are real scalars, B_j are matrices $B_j: \pi_j H \rightarrow \pi_j H$ ($\|B_j\| = 1$), $\beta_j(t)$ are mutually independent scalar Wiener processes of unit variance, $w(t)$ is a Wiener process on the separable Hilbert space K , $F \in L(K, H)$ and z_0 is an H -valued second order random variable. z_0 , $w(t)$ and $\beta_j(t)$ are assumed to be mutually independent and $w(t)$ has the nuclear covariance operator W .

Notice that there are two types of stochastic perturbations involved: $F \, dw(t)$ is linear and state independent and this type of perturbation is well understood (Curtain and Pritchard, 1978); the other form is bilinear and affects each spectral component of A independently. The motivation for this bilinear type of stochastic perturbation lies in the engineers' experience that while the lower eigenvalues are usually known quite well, the accuracy of the high order eigenvalues deteriorates rapidly as $j \rightarrow \infty$. Furthermore, one is sometimes reasonably sure about the asymptotes of the eigenvalues, but not about their exact location. This structural information concerning the uncertainty can be incorporated in one's model by a suitable choice of the structure matrices B_j and the scalar weights γ_j ($\gamma_j \rightarrow \infty$ as $j \rightarrow \infty$).

Following Bernstein and Hyland (1985), we interpret (3.1) in the Stratonovich sense as is often done in engineering modelling. This is always a modelling choice, but we shall see that it is the correct one for the theory of robust controllers followed in Bernstein and Hyland (1985). There finite-dimensional bilinear stochastic models are used to design robust controllers, which stabilize a whole class of systems whose lower order eigenmodes are close, but whose higher order eigenmodes are extremely uncertain. The question posed there was whether this philosophy would carry over to the infinite-dimensional case. Mathematically this translates into the following questions.

- (1) For which perturbations γ_j , B_j does (3.1) have a well defined solution as a sample continuous stochastic process with values in H ?
- (2) For which perturbations γ_j , B_j does (1) hold and does the covariance operator of the solution converge strongly as $t \rightarrow \infty$?

Question (1) is of course a well posedness question, while the strong convergence is necessary for the solution of the controller problem. For the details of the robust control design we refer to Bernstein and Hyland (1985). Here we answer questions (1) and (2) for two types of matrices B_j and three different types of second order systems using the theory of Curtain and Kotelenez (1986). To use the results in Curtain and Kotelenez (1986) we need the Itô version of (3.1), which looks exactly the same except that the system operator becomes

$$\tilde{A} = A + \frac{1}{2} \sum_{j=1}^{\infty} \gamma_j^2 B_j^2 \pi_j \quad (3.2)$$

or in diagonal representation

$$\tilde{A} = \text{diag}(\tilde{\Delta}_1, \tilde{\Delta}_1, \dots), \quad \text{and} \quad \tilde{\Delta}_j = \Delta_j + \frac{1}{2} \gamma_j^2 B_j^2. \quad (3.3)$$

Type 1. Uncertain damping $B_j = I$

The perturbed operator $\Delta_j + \gamma_j I$ has eigenvalues $(\lambda_j + \gamma_j, \bar{\lambda}_j + \gamma_j)$ which represents a change in the damping of the

modes of the system. From Curtain and Kotelenez (1986), (3.1) will be well posed if

$$\sup_j 2 \operatorname{Re} \lambda_j + 2\gamma_j^2 = \bar{\gamma} < \infty. \quad (3.4)$$

From the Appendix the covariance operator has a strong limit as $t \rightarrow \infty$ if $\bar{\gamma} < 0$, and these conditions can mean a restriction on the special weights γ_j . In Example 1, we require $\gamma_j^2 < \alpha$, which means that we cannot model the phenomenon that the higher modes become increasingly uncertain. In Example 2, we require $\gamma_j^2 < \alpha(j\pi)^2$, which allows $\gamma_j \rightarrow \infty$ as $j \rightarrow \infty$. In Example 3 the λ_j^+ modes cannot become increasingly uncertain, whereas the λ_j^- modes may become increasingly uncertain as $j \rightarrow \infty$.

That the results depend strongly on B_j is seen from the second type of perturbation.

Type 2. Uncertain frequency $B_j = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

The perturbed operator $\Delta_j = \gamma_j \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ has the eigenvalues $(\gamma_j + \gamma_j i, \bar{\lambda}_j + \gamma_j i)$ which represents a shift in the frequency. $\Delta_j = \Delta_j - \frac{1}{2} \gamma_j^2 I$ and from Curtain and Kotelenez (1986), (3.1) will be well posed if

$$\sup_j 2 \operatorname{Re} \lambda_j = \gamma < \infty \quad (3.5)$$

and from the Appendix the covariance operator has a strong limit as $t \rightarrow \infty$ if $\gamma > 0$. Notice that (3.5) is independent of the weights γ_j , and so for all three examples in Section 2 we can allow $\gamma_j \rightarrow \infty$ as $j \rightarrow \infty$.

For different B_j we obtain other conditions, but if $B_j = B \, \forall j$, the theory of Curtain and Kotelenez (1986) gives (3.4) as a general sufficient condition, which may be conservative.

4. Comparisons with the Balakrishnan theory

Comparisons between theories considering different classes of systems under different assumptions are difficult to make, but in this case there is sufficient overlap to make it a worthwhile exercise. Balakrishnan (1983) considers the stochastic bilinear system

$$\dot{z} = Az + B(z, N) \quad (4.1)$$

where A is the infinitesimal generator of a strongly continuous semigroup on H , N is a "white noise" on a Hilbert space K and B is a Hilbert Schmidt bilinear form on $H \times K$, such that

$$B(B(z, N, N_1), N_2) = B(B(z, N_2), N_1) \quad (4.2a)$$

$$B(z, N): D(A) \rightarrow D(A) \quad (4.2b)$$

$$AB(z, N) + B(Az, N)$$

$$\text{extends to a continuous bilinear form on } H \times K. \quad (4.2c)$$

The class of system operators A is much more general than our spectral class, but we shall see that the assumptions (4.2) on the Hilbert Schmidt bilinear form can be restrictive in some examples.

If we model a white noise on K formally as

$$N(t) = \sum_{i=0}^{\infty} \dot{\beta}_i(t) e_i \quad (4.3)$$

for some orthonormal basis $\{e_i\}$ for K , where $\beta_i(t)$ are mutually independent standard Wiener processes, and consider the bilinear form

$$B(z, N) = \sum_{j=1}^{\infty} \gamma_j B_j \pi_j z n_j \quad (4.4)$$

for all

$$N = \sum_{j=1}^{\infty} n_j e_j \in K, \quad n_j = \dot{\beta}_j(t)$$

and B_j , π_j , γ_j defined as in Section 2, then we can make some interesting comparisons between the two approaches. First we remark that B defined by (4.4) is Hilbert Schmidt if

and only if

$$\sup_j \gamma_j < \infty \quad (4.5)$$

and this is much stronger than our assumptions (3.4) or (3.5). (4.2a) and (4.2b) are automatically satisfied for our bilinear form and (4.2c) leads to

$$\sup_j \gamma_j \|\Delta_j B_j - B_j \Delta_j\| < \infty \quad (4.6)$$

which may impose further restrictions on γ_j .

Let us now consider the well posedness of this formulation for our examples in Section 3.

Notice that for type 1 and type 2 perturbations (4.6) is automatically satisfied and so we only need consider (4.5), a uniform bound on the weights. This is stronger than our assumptions for the type 2 perturbation where no assumption on γ_j was needed. For type 1 perturbations we get the same restriction on γ_j in Example 1, but in Examples 2 and 3 we can allow $\gamma_j \rightarrow \infty$. So for type 1 and 2 stochastic perturbations our results give weaker assumptions on γ_j in general, and for the modelling of uncertain flexible systems this is what is desired.

On the other hand, our theory can only treat discrete spectral operators A and perturbations B with a very special structure. The theory of Balakrishnan (1985) is more general with respect to the structure of A and B . By exploiting the special spectral structure of A and B one could probably obtain results similar to ours using the white noise approach of Balakrishnan (1985); the spectral structure was a key assumption in our work.

5. Conclusions

The aim of this paper was to examine if it was possible to develop a suitable theory for stochastic bilinear evolution equations to model uncertainties in large flexible systems as proposed by Bernstein and Hyland (1985). We have seen that using the theory expanded in Curtain and Kotelenez (1986) it is certainly possible to model stochastic bilinear perturbations for spectral systems. The class of spectral systems is adequate to cover most large flexible structures and the allowed stochastic perturbation is of the right form to model all sorts of uncertainties in the spectrum. For the special case of uncertainty in frequency it is possible to model the experience of engineers that the higher modes become increasingly uncertain as $j \rightarrow \infty$; one always has well defined steady state covariance operators independently of the size of the uncertainty in the j th frequency, γ_j . For other types of uncertainty in the eigenvalues, for example in the damping, the weights γ_j must be less than the order of the real part, $\text{Re } \lambda_j$, of the corresponding eigenmode in order to assure the existence of steady state covariance operators. If $\text{Re } \lambda_j$ is of the order of a finite constant, then this means that with our theory we cannot model increasing uncertainty in the damping in the higher modes. For systems such as analytic systems with $\text{Re } \lambda_j \rightarrow -\infty$ as $j \rightarrow \infty$, we can always model increasing uncertainty in the higher modes. For this special class of spectral systems and spectral type perturbations we can allow for a larger noise covariance than by using the theory in Balakrishnan (1983). This theory, however, is applicable to a much wider class of systems than our approach: general C_0 -semigroup systems.

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Appendix: technical lemma

Using the results in Curtain and Kotelenez (1986) it is straightforward to show that the mild solution of (3.1) has a bounded covariance operator $Q(t)$ given by

$$Q(t) = E\{U(t, 0) \text{Cov}\{z_0\} U^*(t, 0)\} + E\left\{\int_0^t U(t, s) F W F^* U^*(t, s) ds\right\} \quad (A.1)$$

where $U(t, s)$ is the stochastic evolution operator generated by $A + \sum \gamma_j^2 B_j^2 \pi_j \beta_j$.

Lemma. If $\sup_j 2 \text{Re } \lambda_j + \gamma_j^2 = \alpha < 0$, then $Q(t)$ converges strongly as $t \rightarrow \infty$.

Proof. We recall from Curtain and Kotelenez (1986) the following properties of $U(t, s)$.

$$U(t, s)z = \sum_{j=1}^{\infty} U_j(t, s) \pi_j z \quad (A.2)$$

where $U_j(t, s)x$ is the unique solution of the one- or two-dimensional bilinear Itô differential equation.

$$dy = \Delta_j y dt + \gamma_j B_j y d\beta_j(t); \quad y(0) = x \quad (A.3)$$

whence

$$E\{|U_j(t, s)|^2\} \leq \exp(2 \text{Re } \lambda_j + \gamma_j^2)(t - s). \quad (A.4)$$

First consider $R(t) = E\{U(t, 0) K U^*(t, 0)\}$ for K a self adjoint trace class operator on H . Then $R(t)$ is positive, self adjoint and

$$\begin{aligned} \langle R(t)x, x \rangle &= E\{\|K^{1/2} U^*(t, 0)x\|^2\} \\ &= E\left\{\left\|\sum_{j=1}^{\infty} K^{1/2} U_j^*(t, 0) \pi_j x\right\|^2\right\} \quad \text{by (A.2)} \\ &= E\left\{\sum_{i=1}^{\infty} \left\langle \sum_{j=1}^{\infty} K^{1/2} U_j^*(t, 0) \pi_j x, e_i \right\rangle^2\right\} \end{aligned}$$

where $\{e_i\}$ is any orthonormal basis for H . This is further equal to

$$E\left\{\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle \pi_j x, U_j(t, 0) \pi_j K^{1/2} e_i \rangle^2\right\}$$

and since β_j are mutually independent, we have the further equality

$$\begin{aligned} &= E \sum_{j=1}^{\infty} \|K^{1/2} U_j^*(t, 0) \pi_j x\|^2 \\ &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} e^{(2 \text{Re } \lambda_j + \gamma_j^2)t} \|\pi_j x\|^2 \|K^{1/2} e_i\|^2 \quad \text{by (A.4)} \\ &= \text{trace } K \sum_{j=1}^{\infty} e^{(2 \text{Re } \lambda_j + \gamma_j^2)t} \|\pi_j x\|^2 \\ &\leq \text{trace } K e^{\alpha t} \|x\|^2. \end{aligned}$$

Hence $\|R(t)\| \rightarrow 0$ as $t \rightarrow \infty$ since $\alpha < 0$.

Now let $T(t) = E\{\int_0^t U(t, s)FWF^*U^*(t, s) ds\}$, and consider

$$\begin{aligned}\langle T(t)x, x \rangle &= E\left\{\int_0^t \|W^{1/2}F^*U^*(t, s)x\|^2 ds\right\} \\ &\leq \text{trace } FWF^* \int_0^t \sum_{j=1}^{\infty} e^{(2\operatorname{Re} \lambda_j + \gamma_j^2)s} \|\pi_j x\|^2 ds\end{aligned}$$

which uses the same argument as for $R(t)$. This shows that

$\|T(t)\|$ is uniformly bounded for $t \geq 0$ since $\alpha < 0$. Now $\langle T(t)x, x \rangle$ is monotonically increasing in t , since $E\{\|U(t, s)x\|^2\}$ depends only on $(t-s)$. So the self adjoint positive operator $T(t)$ has a strong limit as $t \rightarrow \infty$. Combining this fact with the strong convergence of $R(t)$ to zero we have proved that $Q(t)$ converges strongly as $t \rightarrow \infty$ to

$$E\left\{\int_0^{\infty} U(t, 0)FWF^*U^*(t, 0) dt\right\}.$$