A Smooth Version of the Zariski Topos

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The theory of manifolds goes back to Riemann's lecture "On the hypotheses which lie at the foundations of geometry," delivered in 1854 at the University of Göttingen. In fact, it was precisely the efforts to clarify and deepen Riemann's ideas (as understood by his successors) that led to manifolds and Riemannian spaces as we understand them today. Nevertheless, Riemann himself gave examples of manifolds in his sense (e.g., "the possibilities for a function in a given region") which are not (finite dimensional) manifolds in the modern sense. Here we touch upon one of the limitations of the category $\mathcal{M}$ of C^∞-manifolds and C^∞-maps: $\mathcal{M}$ is *not cartesian closed*; in particular the space of C^∞-maps between two C^∞-manifolds is not a C^∞-manifold.

A limitation of a different nature is the *absence of a convenient language* to describe things in the "infinitely small." In particular infinitesimals, which had played such an important role in analysis and geometry until the beginning of this century, have been exorcized by the modern theory of manifolds, although they are still mentioned as a heuristic help in understanding. In this way, they have been forced to play, literally, the role of "ghosts of departed quantities" that Bishop Berkeley had assigned them.

If we look at the works of geometers like Darboux, Lie, and Cartan, as well as those of contemporary engineers and physicists, we find (at least) two kinds of infinitesimals; the *nilpotent infinitesimals* (e.g., "first-order infinitesimals") which are used to deal with notions like forms and parallel transport, and the *invertible infinitesimals*, employed for instance in the theory of improper functions of which the δ function of Dirac is the best known example. Furthermore, these invertible infinitesimals come together with *infinitely large natural numbers*, used already by Leibniz and Euler to deal with series, infinite products, and the like.

Several attempts have been made to remove (some of) the limitations of

the theory of manifolds. The first, chronologically, is Non-Standard Analysis (NSA), initiated by A. Robinson in the 1960s (cf. Robinson [17]). Its aim is to create a convenient language for invertible infinitesimals and infinitely large natural numbers in order to deal efficiently with notios such as limits, onvergence, etc. Nowadays, NSA is a very active area of research which has achieved important results, both by simplifying old proofs and by proving new results. We should notice, however, that NSA cannot accomodate nilpotent infinitesimals and hence we cannot assert that NSA provides the desired language to describe things at the infinitesimal level. Furthermore, NSA is not concerned with the other limitations of the theory of manifolds already mentioned, namely the lack of a convenient category of smooth structures which is (at least) cartesian closed.

Two rather recent approaches which aim to construct such a cartesian closed category are the theory of *differentiable spaces* of Chen (cf. Chen [3, 4]), and the theory of *convenient vector spaces* as developed by Frölicher, Kriegl, and others (cf. Frölicher [7], Kriegl [9], Michor [10], and other references cited there). In particular, Chen has used his category of differentiable spaces quite efficiently to develop the calculus of variations. Once again, however, but quite independent from the interest of these approaches, they do not aim at creating an adequate language for "infinitesimal" structures, and we are forced to look elsewhere.

The aim of this paper is to sketch an alternative to the theory of manifolds, by constructing a cartesian closed category $\mathscr{X}$ which contains all the manifolds (more precisely, there is a full embedding of the category of (separable) manifolds in $\mathscr{X}$), just as in the two previous approaches. In addition to this, however—and this distinguishes our approach from all the others—$\mathscr{X}$ contains spaces of nilpotent infinitesimals, invertible infinitesimals and infinitely large natural numbers. Moreover, our category $\mathscr{X}$ is not just cartesian closed, but is in fact a Grothendieck topos. This implies that we can use set-theoretical language and arguments (provided they are constructive) to describe our structures directly, by literally adopting classical definitions and arguments, rather than guessing what the right analogue is for sheaves of a particular kind. In this way, one can carry out arguments using infinitesimals in a mathematically rigorous way. For example, "synthetic" arguments like the ones used by E. Cartan and others can be interpreted literally and word by word in a category like $\mathscr{X}$.

Our approach follows the lines already laid out by F. W. Lawvere at the end of the 1960s. The basic idea is to apply the functorial approach to algebraic geometry of Grothendieck and others (see, e.g., Demazure and Gabriel [5]) in the context of differential geometry, but using the set-theoretical language developed for Grothendieck toposes in the early 1970s. (See Kock [8] for an exposition of some aspects of this approach to differential geometry.)

Coming back to the use of set-theoretic language and arguments in the context of Grothendieck topos in a more philosophical vein, one could say that topos theory has brought to light and given the means to exploit a complementarity (or duality) principle between *logic* and *structure*. A mathematical theory $\mathscr{T}$ (e.g., differential geometry) is usually specified by two components:

(1) The type of structures $\mathscr{S}$ to which the notions of the theory belong.

(2) The canonical interpretations $\mathscr{I}$ of $\mathscr{S}$ in Sets which gives the settheoretical interpretations of the structures in question.

The first component is given by axioms and definitions in the language of set theory; the second is obtained via the "tautological" Tarski semantics. We shall symbolically write $\mathscr{T} = \mathscr{S} + \mathscr{I}$. Now, topos theory and categorical logic offer the possibility of considering interpretations of $\mathscr{S}$ into toposes such as $\mathscr{Z}$ (and not only Sets). These interpretations are obtained via sheaf semantics, rather than Tarski semantics.

The complementarity principle asserts that, when the interpretation is generalized in this fashion, no component is uniquely determined by $\mathscr{T}$, but several choices of $\mathscr{S}, \mathscr{I}$ are possible to specify the theory. However, once that a component has been chosen, the other is determined by the equation

$$\mathscr{T} = \mathscr{S} + \mathscr{I}.$$

In particular, if we complicate the interpretation from settheoretical to sheaf semantics, we may expect a corresponding simplification of $\mathscr{S}$.

The interpretation $\mathscr{I}$ specifies not only to which structures the definitions and axioms of $\mathscr{S}$ refer, but also the logical axioms and rules of inference that are valid for this interpretation, i.e., the *underlying logic* of the interpretation. In the case of sheaf semantics $\mathscr{I}$ also specifies the arithmetical axioms and rules of inference that are valid for this interpretation, since the natural numbers are interpreted as the natural number object (NNO) of the topos in question. What one gets is "full higher order Heyting arithmetic."

A new feature of our interpretation is that we interpret the natural numbers, or rather the integers, as the object $Z = \{ x \in R \mid \sin(\pi x) = 0 \}$ of the topos $\mathscr{Z}$, called the object of *smooth integers*. This object Z is *different* from the object of integers $\mathbb{Z}$ constructed in the usual way from the NNO $\mathbb{N}$ of $\mathscr{Z}$. Thus, by working with the object $N = \{ x \in Z \mid x \geqslant 0 \}$ of smooth natural numbers, we have weakened the underlying arithmetic of the interpretation, and it is precisely this feature that accounts for the possibility of having a natural model for analysis containing both nilpotent and invertible infinitesimals.

In the model on which most of the attention in the literature on "synthetic differential geometry" (SDG) has been focused so far, the model of Dubuc [6], Z is just the object of standard integers $\mathbb{Z}$, and $N = \{x \in Z \mid x \geqslant 0\}$ coincides with the natural number object $\mathbb{N}$; there are no infinitely large numbers in N and the object of invertible infinitesimals is empty. Thus, this model provides no means for contrasting the set of standard numbers $\mathbb{N}$ and its "nonstandard" counterpart N. (For an extensive discussion of Dubuc's model, the topos $\mathscr{G}$, see Moerdijk and Reyes [12].)

In the model $\mathscr{Z}$ discussed in this paper, however, N and $\mathbb{N}$ are quite different, and this model shows what is may be the main point of this paper, namely that it is N and not $\mathbb{N}$ that one needs in analysis, despite the drawback of having one's hands tied down to a weaker arithmetical theory. At a foundational level, there are obvious reasons for this claim: Z and N are inherent to the C^∞-structure of R, while $\mathbb{Z}$ and $\mathbb{N}$ are not, and on a strictly axiomatic approach to SDG, it is only Z (and N) that can be defined by elementary means (firstorder logic, finite inverse limits). On a mathematical level, this claim is supported by the fact that in the model $\mathscr{Z}$, the degree of a map is a smooth integer, and not necessarily a standard one (this was proved synthetically for endomaps of the circle in Bélair [1]). More generally, we will show that for homology theory we obtain the expected results in $\mathscr{Z}$ only if we use smooth integers everywhere, and change the basic algebraic notions, like that of a free ring, accordingly.

The plan of this paper, then, is as follows. In the first section, we describe our model, the so-called *smooth Zariski topos*, and give some of its basic properties. The definition of this topos is similar to the definition of the usual Zariski topos from algebraic geometry, but based on the algebraic theory of C^∞-rings. Section two deals with the mathematical consequences of working with Z instead of $\mathbb{Z}$ all the time. Here we discuss some of the basic constructions in "smooth algebra," i.e., algebra based on Z (in a sense making Z the free "smooth group" on one generator) that we will need in the third section. There, we will prove the degree theorem by using a simplicial homology theory with coefficients in the smooth integers Z.

1. Description of the Smooth Zariski Topos $\mathscr{Z}$

In this section, we will describe some of the basic properties of the Zariski smooth topos that we will need later on. This topos, denoted by $\mathscr{Z}$, is the analog of the usual Zariski topos of algebraic geometry which classifies local k-algebras (k, the ground field), but with the theory of k-algebras replaced by that of C^∞-rings. It was already described in Reyes [16], Moerdijk and Reyes [12], but we quickly repeat its definition here.

(For some details about C^∞-rings, see for example Dubuc [6], Moerdijk and Reyes [12].)

The category $\mathbb{L}$ of *loci* or *formal C^∞-varieties* is the opposite of the category of finitely generated C^∞-rings and C^∞-homomorphisms. In other words, *objects* of $\mathbb{L}$ are duals $\bar{A}$ of C^∞-rings A which are (isomorphic to a ring) of the form

$$A = C^\infty(\mathbb{R}^n)/I,$$

where $C^\infty(\mathbb{R}^n)$ is the ring of smooth functions $\mathbb{R}^n \to \mathbb{R}$, and I is an arbitrary ideal. Morphisms of $\mathbb{L}$ from one such dual $\overline{C^\infty(\mathbb{R}^n)/I}$ to another $\overline{C^\infty(\mathbb{R}^m)/J}$ are equivalence classes of smooth functions $\mathbb{R}^n \xrightarrow{\varphi} \mathbb{R}^m$ with the property that

$$\mathbb{R}^m \xrightarrow{f} \mathbb{R} \in J \Rightarrow f \circ \varphi \in I,$$

two such functions φ and φ' being equivalent if all their components are equivalent modulo J, i.e., for each projection π_i $(i = 1,...,m)$, $\pi_i \circ \varphi - \pi_i \circ \varphi' \in J$.

$\mathbb{L}$ can be made into a site by equipping it with the Grothendieck topology which forces the generic C^∞-ring R in the presheaf topos $\text{Sets}^{\mathbb{L}^{op}}$ (R is "the line," defined by $R(\bar{A}) = A$) to be a local ring, just as in the case of the usual Zariski topos. Thus, this Grothendieck topology is generated by covers of the form

$$\left\{ \overline{A\left[\frac{1}{a_i}\right]} \to \bar{A} \right\}_{i=1}^m,$$

where $1 \in (a_i)$, the ideal generated by $a_1,..., a_m$. Equivalently, this topology may be described as generated by covering families of the form

$$\{ \overline{C^\infty(U_i)/(I \mid U_i)} \subsetneq \overline{C^\infty(\mathbb{R}^n)/I} \}_{i=1}^m,$$

where $\{U_1,..., U_m\}$ is an open cover of $\mathbb{R}^n$.

LEMMA 1. *This topology on $\mathbb{L}$ is subcanonical.*

Proof. Although in general a C^∞-homomorphism $C^\infty(U)/I \to C^\infty(V)/J$ need not come from a smooth map $V \to U$, this is true if $U = \mathbb{R}^n$. From this we easily derive that representables are sheaves: Let $\{U_1,..., U_k\}$ be an open cover of $\mathbb{R}^n$ inducing a cover of $C^\infty(\mathbb{R}^n)/I$, and let

$$f_i: \overline{C^\infty(U_i)/(I \mid U_i)} \to \bar{B}$$

be a compatible family of maps into the dual $\bar{B}$ of a ring $B = C^\infty(\mathbb{R}^m)/J$. Each f_i comes from a smooth map $f_i = (f_i^1,..., f_i^m): U_i \to \mathbb{R}^m$. The unique

$f = (f^1, ..., f^m)$: $\overline{C^\infty(\mathbb{R}^n)}/I \to \overline{C^\infty(\mathbb{R}^m)}/J$ is now obtained as $f = \sum_{i=1}^k f_i \cdot \rho_i$, where $\{\rho_i\}$ is a partition of unity subordinate to the cover $\{U_i\}$ of $\mathbb{R}^n$. Further details are straightforward. ∎

As usual, the category of (separable) manifolds is fully and faithfully embedded in $\mathbb{L}$, and hence in $\mathscr{Z}$ by the Yoneda-embedding (using Lemma 1), and we write s for this embedding. Explicitly, if M is a manifold represented as a closed subspace of $\mathbb{R}^n$,

$$s(M) = \overline{C^\infty(M)} \cong \overline{C^\infty(\mathbb{R}^n)}/m_M^0,$$

where m_M^0 is the ideal of functions vanishing on M. Thus we obtain a diagram of categories and functors

$$\mathscr{M} \overset{s}{\hookrightarrow} \mathscr{Z} \underset{\overset{\longleftarrow}{B}}{\overset{\overset{\overset{\Delta}{\longleftarrow}}{\underset{\Gamma}{\longrightarrow}}}{}} \text{Sets},$$

where Γ is the global sections functor, Δ the constant sheaf functor left adjoint to Γ, and B the right adjoint to Γ, which exists since Γ preserves arbitrary colimits. Note also that s preserves transversal pullbacks and finite open covers.

The generic ring R, i.e., the representable object $\overline{C^\infty(\mathbb{R})}$, is an ordered local ring object in the topos $\mathscr{Z}$, with the strict order given by

$$x < y \qquad \text{iff} \quad \exists \text{ invertible } z \; z^2 = y - x.$$

One can show easily that in the model, this comes down to the following: for an element $\alpha: \bar{A} \to R$ of R at stage $\bar{A} \in \mathbb{L}$, $\bar{A} \Vdash \alpha > 0$ iff α factors through $R_{>0} = \overline{C^\infty(\mathbb{R}^2)}/(y\chi(x) - 1) \cong \overline{C^\infty(\mathbb{R}_{>0})}$, where $\chi: \mathbb{R} \to \mathbb{R}$ is the "take-off" function defined by $\chi(x) = e^{-1/x^2}$ if $x > 0$, $\chi(x) = 0$ if $x \leq 0$. This order relation satisfies the usual requirements of being compatible with the ring structure on R (i.e., $0 < 1$, $0 < x$ and $0 < y$ imply both $0 < x \cdot y$ and $0 < x + y$) and being total in the sense that x is invertible iff $x < 0 \vee x > 0$.

There are several (pre-)orders $\leq$ on R that are compatible with both $<$ and the ring structure of R. We will take the one represented by the sub-object $\overline{C^\infty(\mathbb{R}_{\geq 0})} = \overline{C^\infty(\mathbb{R})}/m_{[0,\infty)}^0 \subseteq R$. (Compatibility of this $\leq$ with the ring structure was shown in Quê and Reyes [15].) Thus, the closed unit interval $[0, 1]$ is interpreted in $\mathscr{Z}$ by the object $\overline{C^\infty(\mathbb{R})}/m_{[0,1]}^0$.

As for any topos, the natural number object $\mathbb{N} = \mathbb{N}_\mathscr{Z}$ of the topos $\mathscr{Z}$ is the constant sheaf $\Delta(\mathbb{N}_{\text{Sets}})$, i.e., $\mathbb{N}$ is the sheaf of *bounded* functions into $\overline{C^\infty(\mathbb{R})}/m_\mathbb{N}^0$. Thus, R is *not* Archimedean, that is

$$\mathscr{Z} \models \neg \, \forall x \in R \; \exists n \in \mathbb{N} \, x < n.$$

so the order topology on R does not coincide with the rational interval topology. When we speak about topological properties of R, we will always do this with respect to the order topology. In fact, the order topology can also be defined in purely logical terms: recall that a subobject $U \subset X$ is Penon-open if $\forall x \in U, \forall y \in X \ (x \neq y \lor y \in U)$.

PROPOSITION 2. $\mathscr{Z} \models \forall U \subset R(U$ is Penon-open iff U is open in the order topology).

Proof. We check this for Penon-, (resp. order-) neighbourhoods of $0 \in R$.

$\Leftarrow$ is easy: reason in $\mathscr{Z}$, and suppose $0 \in (-\delta, \delta) \subset U$ for some $\delta > 0$. Then U is a Penon-neighbourhood of 0, since $\forall y \in R((0 < y \lor y < 0)$ or $(-\delta < y \land y < \delta))$ is valid in $\mathscr{Z}$. Conversely, suppose U is a subobject of R at stage $\bar{A} \in \mathbb{L}$, with $A = C^\infty(\mathbb{R}^n)/I$, such that

$$\bar{A} \Vdash 0 \in U \quad \text{is Penon-open in } R.$$

Then also

$$\bar{A} \times R \Vdash \pi_2 \neq 0 \lor \pi_2 \in U,$$

so there is an open cover $\{W_1, W_2\}$ of $\mathbb{R}^n \times \mathbb{R}$ such that

$$W_1 \cap \bar{A} \times R \Vdash \pi_2 \neq 0, \qquad W_2 \cap \bar{A} \times R \Vdash \pi_2 \in U.$$

Thus $W_1 \cap \bar{A} \times \{0\} = \varnothing$, so $\bar{A} \times \{0\} \subset W_2$, i.e., there is a finitely generated $I_0 \subset I$ such that $Z(I_0) \subseteq V = W_2 \cap \mathbb{R}^n \times \{0\}$. Choose a smooth $\delta: V \to \mathbb{R}_{>0}$ such that

$$x \in V \quad \text{and} \quad -\delta(x) < y < \delta(x) \Rightarrow (x, y) \in W_2.$$

Then δ defines a positive element of R at stage $\bar{A}$, and $\bar{A} \times R \Vdash (-\delta < \pi_2 < \delta \Rightarrow \pi_2 \in U)$, so by genericity of π_2, $\bar{A} \Vdash (-\delta, \delta) \subset U$. ∎

$\mathscr{Z}$ is an adequate model for SDG. For example, it satisfies the Kock–Lawvere axiom

$$R \times R \cong R^D,$$

where D is the representable object $\overline{C^\infty(\mathbb{R})/(x^2)}$ of first-order infinitesimals, the integration axiom holds (cf. Quê and Reyes [15]), and $(-)^D$ has a right adjoint $(-)_D$ (Lawvere's amazing right adjoint).

In $\mathscr{Z}$, we have the usual infinitesimal subspaces

$$D \subset D_2 \subset D_3 \subset \cdots \subset D_\infty$$

of R, $D_n = \{x \in R \mid x^{n+1} = 0\}$, all contained in

$$\Delta = \{x \in R \mid x \text{ is not invertible}\} = \neg\neg\{0\}.$$

The special feature that distinguishes $\mathscr{Z}$ from the topos $\mathscr{G}$ mentioned in the introduction is that Δ does *not* coincide with the *object of infinitesimals*

$$\triangle = \bigcap_{n \in \mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \overline{C_g^\infty(\mathbb{R})} \subset R,$$

where $C_g^\infty(\mathbb{R})$ is the ring of germs at 0 of smooth functions $\mathbb{R} \to \mathbb{R}$. Indeed, not every element in $\triangle$ is noninvertible, or equivalently, the object of *invertible infinitesimals*

$$\Pi = \triangle - \{0\},$$

represented by the dual of the ring $C^\infty(\mathbb{R}^2)/(xy - 1, m_{\{0\}}^g(x))$ ($m_{\{0\}}^g(x)$ is the ideal of functions $\mathbb{R} \to \mathbb{R}$ with vanishing germ at 0) is nontrivial. In fact,

$$C^\infty(\mathbb{R}^2)/(xy - 1, m_{\{0\}}^g(x)) \cong C^\infty(\mathbb{R})/K,$$

where K is the ideal of functions of compact support, and the isomorphism is induced by

$$\mathbb{R}\backslash\{0\} \times R \to \mathbb{R}, \qquad (x, y) \mapsto \frac{1}{x}.$$

Since the product of two nonzero objects of $\mathbb{L}$ is nonzero, in particular $\bar{A} \times \Pi \neq 0$ if $\bar{A} \neq 0$, we have

PROPOSITION 3. *In $\mathscr{Z}$ it holds that* $\neg\neg\exists x\,(x \in \Pi)$.

Of course, Π does not have globals sections. We can add a generic global element to Π, by passing to the slice topos $\mathscr{Z}/\Pi$. From the usual embedding $\mathscr{M} \subset^s \mathscr{Z}$ we obtain a functor $\mathscr{M} \to^{s_0} \mathscr{Z}/\Pi$ by composing with the canonical logical functor $\mathscr{Z} \to \mathscr{Z}/\Pi$. This functor $\mathscr{Z} \to \mathscr{Z}/\Pi$ is not faithful, since $\Pi \to 1$ is not epic, but when restricted to $\mathscr{M}$ it is:

PROPOSITION 4. $\mathscr{M} \to_{s_0} \mathscr{Z}/\Pi$ *is faithful, and preserves transversal pullbacks and finite open covers.*

Proof. The last two properties follow immediately from the corresponding ones for $\mathscr{M} \subset^s \mathscr{Z}$. Faithfulness follows from the following explicit description of $\mathscr{Z}/\Pi(s_0(M), s_0(N))$ for $M, N \in \mathscr{M}$: There is a natural 1–1 correspondence between natural transformations $s_0(M) \to s_0(N)$ and

equivalence classes of smooth maps $\varphi: M \times \mathbb{R}^* \to N$ under the equivalence relation

$$\varphi \sim \psi \quad \text{iff} \quad \exists \varepsilon > 0, \; \varphi|_{M \times (\mathbb{R}^* \cap (-\varepsilon, \varepsilon))} = \psi|_{M \times (\mathbb{R}^* \cap (-\varepsilon, \varepsilon))}. \quad \blacksquare$$

2. SMOOTH INTEGERS IN $\mathscr{Z}$

Despite the fact that $\mathscr{Z}$ is a model for the basic axioms of SDG, it seems at first sight rather hard to do analysis in $\mathscr{Z}$. For example, the basic ingredients for doing some (synthetic) homology theory in the topos $\mathscr{G}$ are the compactness of $[0, 1]$ the fact that R is Archimedean (cf. Moerdijk and Reyes [13]), but neither of these is valid in $\mathscr{Z}$. Nonvalidity of Archimedeanness has already been pointed out above, and compactness of $[0, 1]$ cannot be valid in $\mathscr{Z}$ since it is not valid in $\mathscr{Z}/\Pi$: for the generic $\delta \in \Pi$, $\{(x - \delta, x + \delta) \mid x \in [0, 1]\}$ clearly cannot have a finite subcover.

Quite surprisingly, despite the fact that $[0, 1]$ is not compact, we claimed in Moerdijk and Reyes [12] that

THEOREM 1. *In* Sets$^{\mathbb{L}^{op}}$, *hence also in* $\mathscr{Z}$, *it is valid that all functions* $[0, 1] \to R$ *are uniformly continuous.*

Proof. We show

$$\mathscr{Z} \models \forall f \in R^{[0,1]} \quad \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x, \, y \in [0, 1](|x - y| < \delta \to |fx - fy| < \varepsilon).$$

(Although the absolute value $|-|$ does not exist in $\mathscr{Z}$, we can use it as an obvious shorthand.) So choose such f and ε at stage $\bar{A} \in \mathbb{L}$, i.e., $\bar{A} \times [0, 1] \xrightarrow{f} R$ and $\bar{A} \xrightarrow{\varepsilon} R_{>0}$, represented by $F: \mathbb{R}^n \times R \to \mathbb{R}$ modulo $(I(x), m^0_{[0,1]})$ and $E: \mathbb{R}^n \to R$ mod I (where $A = C^\infty(\mathbb{R}^n)/I$). Since $R_{>0}$ is the dual of the finitely presented C^∞-ring $C^\infty(\mathbb{R}^2)/(y\chi(x) - 1)$, we can find a finitely generated ideal $I_0 \subset I$ such that at $\bar{A}_0 \supseteq \bar{A}$, $A_0 = C^\infty(\mathbb{R}^n)/I_0$, E still represents a positive element of R, i.e., ε extends to a map

$$\bar{A}_0 \xrightarrow{\varepsilon} R_{>0}.$$

We now work with f as a map $\bar{A}_0 \times [0, 1] \to R$.

Choose any $\mu > 0$, and consider f as a map $\bar{A}_0 \times [-\mu, 1 + \mu] \to R$. By continuity of F and compactness of $[-\mu, 1 + \mu]$ we find for each $x \in Z(I_0)$ a $\delta_x > 0$ and a neighbourhood U_x such that

$$\forall y \in \mathbb{R}^n \; \forall s, \, t \in [-\mu, 1 + \mu]:$$

$$y \in U_x \, \& \, |s - t| < \delta_x \to |F(y, s) - F(y, t)| < E(x).$$

And by a partition of unity argument, we find an open neighbourhood V of $Z(I_0)$ and a smooth $D: \mathbb{R}^n \to R$ such that $D > 0$ on V and

(i) $\forall x \in V$ $\forall s, t \in [-\mu, 1 + \mu](|s - t| < D(x) \to |F(x, s) - F(x, t)|$ $<$ $E(x))$.

Now D corresponds to an element δ of R at $\bar{A}_0$ with $\bar{A}_0 \Vdash \delta > 0$, and we claim that

$$\bar{A}_0 \Vdash \forall \alpha, \beta \in [0, 1](|\alpha - \beta| < \delta \to |F(\alpha) - F(\beta)| < \varepsilon).$$

To prove the claim, take

$$\bar{B} \xrightarrow{g} \bar{A}_0 \qquad \text{and} \qquad \alpha, \beta: \bar{B} \to [0, 1],$$

such that $\bar{B} \Vdash |\alpha - \beta| < \delta \restriction g$. Say $B = C^\infty(\mathbb{R}^m)/J$. As before we find a finitely generated $J_0 \subset J$ such that already at $\bar{B}_0$,

$$\bar{B}_0 \Vdash |\alpha - \beta| < \delta \circ g,$$

where $B_0 = C^\infty(\mathbb{R}^m)/J_0$, and since $J \supseteq g^*(I_0) = \{\varphi \circ g \mid \varphi \in I_0\}$, we may without loss assume that $J_0 \supseteq g^*(I_0)$ (since $g^*(I_0)$ is finitely generated). Also, since $\bar{B} \Vdash 0 \leqslant \alpha, \beta \leqslant 1$ and $(-\mu, 1 + \mu)$ is finitely generated as an object of the site (since it is open), we may enlarge J_0 if necessary such that

$$\bar{B}_0 \Vdash -\mu < \alpha, \beta < 1 + \mu.$$

We now need to show that

$$\bar{B}_0 \Vdash |(F \restriction g)(\alpha) - (F \restriction g)(\beta)| < \delta \circ g.$$

But since J_0 is finitely generated it suffices to check this at the points of $Z(J_0)$, i.e., we need to show

(ii) $\forall y \in Z(J_0) |F(g(y), \alpha(y)) - F(g(y), \beta(y))| < \delta(g(y))$.

But if $y \in Z(J_0)$ then $\alpha(y), \beta(y) \in (-\mu, 1 + \mu)$ and $g(y) \in Z(I_0)$ by the properties of J_0, so (ii) follows immediately from (i). ∎

The explanation for the fact that despite the lack of compactness we still get *uniform* continuity comes from the existence of Lebesgue numbers in both $\mathrm{Sets}^{\mathbb{L}^{op}}$ and $\mathscr{L}$. (Even if one would only be interested in $\mathscr{L}$, the proof naturally splits into the two cases $\mathrm{Sets}^{\mathbb{L}^{op}}$ and $\mathscr{L}$.)

THEOREM 2. $\mathrm{Sets}^{\mathbb{L}^{op}} \models$ *every open cover $\mathscr{U}$ of $[0, 1]$ has a Lebesgue number.*

Proof. Suppose $\mathcal{U}$ is a cover of $[0, 1]$ at stage $\bar{A}$, with $A = C^\infty(\mathbb{R}^n)/I$, and assume $\mathcal{U}$ consists of open intervals. Then

$$\bar{A} \times [0, 1] \Vdash \exists a, b \ a < \pi_2 < b \text{ and } (a, b) \in \mathcal{U},$$

so there are $a(x, y), b(x, y): \mathbb{R}^n \times R \to \mathbb{R}$ such that

$$\bar{A} \times [0, 1] \Vdash a < \pi_2 < b \qquad \text{and} \qquad (a, b) \in \mathcal{U}.$$

Hence there are finitely generated ideals $I_0 = I_0(x) \subset I$ and $J = J(y) \subset m^0_{[0,1]}$ such that

(i) $\quad \bar{A}_0 \times \bar{B} \Vdash a < \pi_2 < b$,

where $A_0 = C^\infty(\mathbb{R}^n)/I_0$ and $\bar{B} = C^\infty(\mathbb{R})/J$, while without loss $Z(J) = [0, 1]$ (add a function having $[0, 1]$ as zero set to J). But $\bar{A}_0 \times \bar{B}$ is the dual of a finitely presented C^∞-ring $A_0 \otimes_\infty B$, so (i) is equivalent to

$$a(x, y) < y < b(x, y) \qquad \text{for all } (x, y) \in Z(I_0) \times Z(J) = Z(I_0) \times [0, 1].$$

Hence for each $(x_0, y_0) \in Z(I_0) \times [0, 1]$ there is a $\delta > 0$ such that $\delta < \min(y - a(x, y), b(x, y) - y)$ on a neighbourhood of (x_0, y_0), so by compactness of $[0, 1]$ we find an open cover $\{W_\alpha\}_\alpha$ of $Z(I_0)$ and reals $\delta_\alpha > 0$ such that

(ii) $\quad \forall x \in W_\alpha \ \forall y \in [0, 1] \ 0 < \delta_\alpha < \min(y - a(x, y), b(x, y) - y),$

and we may without loss assume that $\{W_\alpha\}_\alpha$ is neighbourhood-finite. Now let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{W_\alpha\}_\alpha$, and let

$$\delta = \sum_\alpha \rho_\alpha \delta_\alpha: W \to \mathbb{R}_{>0}$$

on the neighbourhood $W = \bigcup_\alpha W_\alpha$ of $Z(I_0)$. Then

(iii) $\quad \forall (x, y) \in W \times [0, 1] \ 0 < \delta(x) < \min(y - a(x, y), b(x, y) - y),$

δ defines an element of R at stage $\bar{A}_0$ such that $\bar{A}_0 \Vdash \delta > 0$, and moreover by (iii),

$$\bar{A}_0 \times [0, 1] \Vdash a < \pi_2 - \delta < \pi_2 < \pi_2 + \delta < b,$$

so

$$\bar{A}_0 \times [0, 1] \Vdash \exists U \in \mathcal{U} \qquad (\pi_2 - \delta, \pi_2 + \delta) \subset U.$$

But δ depends on x only, i.e., exists at stage $\bar{A}_0$ already, and π_2 is generic, so at $\bar{A}_0$, and a fortiori at $\bar{A}$, we have

$$\bar{A} \Vdash \forall \alpha \in [0, 1] \qquad \exists U \in \mathcal{U} \ (\alpha - \delta, \alpha + \delta) \in U,$$

so

$$\bar{A} \Vdash \exists \delta > 0 \quad \forall \alpha \in [0, 1], \quad \exists U \in \mathcal{U} \quad (\alpha - \delta, \alpha + \delta) \in U. \quad \blacksquare$$

To obtain the corresponding result for $\mathcal{Z}$, we use

LEMMA 3. *A downwards closed cover $\mathcal{U}$ of $[0, 1]$ in $\mathcal{Z}$ is already a cover in* Sets$^{\mathbb{L}^{op}}$. *More precisely, if $\mathcal{U}$ is a subsheaf of the sheaf of open intervals of R in $\mathcal{Z}$ at stage $\bar{A}$ such that $\bar{A} \Vdash (\mathcal{U}$ covers $[0, 1])$ in $\mathcal{Z}$, then already $\bar{A} \Vdash (\mathcal{U}$ covers $[0, 1])$ in* Sets$^{\mathbb{L}^{op}}$.

Proof. Let $\mathcal{U}$ be an open cover in $\mathcal{Z}$ at stage $\bar{A}$, and assume $\mathcal{U}$ consists of open intervals of R and is downwards closed. Thus

$$\bar{A} \times [0, 1] \Vdash \exists U \in \mathcal{U} \ \pi_2 \in U$$

($\Vdash$ refers to $\mathcal{Z}$ all the time), so we find a finite cover $\{W_1,..., W_k\}$ of $\mathbb{R}^n \times \mathbb{R}$, and $a_i, b_i \in R$ at $\bar{A}_i = (\bar{A} \times [0, 1] \cap W_i$ such that

$$\bar{A}_i \Vdash a_i < \pi_2 < b_i \quad \text{and} \quad (a_i, b_i) \in \mathcal{U}.$$

Hence (as in the two proofs above) there is a finitely generated ideal J_i,

$$J_i \subseteq (I(x), m^0_{[0,1]}(y), z \cdot \psi(x, y) - 1)$$

(where ψ_i is a characteristic function for W_i) such that for

$$A'_i = C^\infty(\mathbb{R}^{n+2})/J_i, \quad \bar{A}'_i \Vdash a_i < \pi_2 < b_i.$$

Thus we find finitely generated $I_i(x) \subset I(x)$, $M_i(y) \subset m^0_{[0,1]}(y)$, such that $J_i \subseteq (I_i(x), M_i(y), z \cdot \psi_i(x, y) - 1)$ for $i = 1,..., k$, and by putting the I_i's and M_i's together we find finitely generated $I' \subset I$ and $M \subset m^0_{[0,1]}$ with

$$J_i \subseteq (I'(x), M(y), z\psi_i(x, y) - 1) \quad \text{each} \quad i = 1,..., k$$

(and without loss $Z(M(y)) = [0, 1]$). Now let

$$B_i = C^\infty(\mathbb{R}^{n+2})/(I'(x), M(y), z\psi_i(x, y) - 1)$$

which is a finitely presented C^∞-ring. Then

$$\gamma(\bar{B}_i) := Z(I'(x), M(y), z\psi_i(x, y) - 1) = \pi^{-1}(Z(I') \times [0, 1]) \cap \hat{W}_i$$

(where $\hat{W}_i = Z(z\psi_i(x, y) - 1 \subseteq \mathbb{R}^{n+2})$, and since $\bar{B}_i \subset \bar{A}'_i$

$$\bar{B}_i \Vdash a_i < \pi_2 < b_i,$$

which is equivalent (since B_i is finitely presented) to

$$a_i(x, y, z) < y < b_i(x, y, z) \qquad \text{on } \gamma \bar{B}_i.$$

Let us write $B = C^\infty(\mathbb{R}^n \times \mathbb{R})/(I'(x), M(y))$, so that the $\bar{B}_i$ form an open cover of $\bar{B}$.

We now use the following lemma, which is proved using a straightforward application of partitions of unity.

LEMMA. *Let $X \subset \mathbb{R}^n$ be closed, $\{W_1, ..., W_k\}$ an open cover of X, and $a_i : W_i \to \mathbb{R}$ be smooth functions with $a_i > 0$ on $W_i \cap X$. Then there is a refinement $\{V_1, ..., V_k\}$ and a smooth $a : \mathbb{R}^m \to \mathbb{R}$ such that on $V_i \cap X$,*

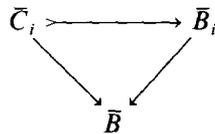$$0 < a|_{V_i \cap X} < a_i|_{V_i \cap X}, \qquad i = 1, ..., k.$$

Now apply this lemma to the cover $\{W_1, ..., W_k\}$ of $Z(I') \times [0, 1] = \gamma(\bar{B})$ and the functions $y - \tilde{a}_i$ (resp. $\tilde{b}_i - y$), where $\tilde{a}_i, \tilde{b}_i : W_i \to \mathbb{R}$ are the functions

$$\tilde{a}_i(x, y) = a_i(x, y, \psi_i(x, y)^{-1}), \qquad \tilde{b}_i(x, y) = b^i(x, y, \psi_i(x, y)^{-1}),$$

so as to find a cover $\{V_1, ..., V_k\}$ of $Z(I') \times [0, 1]$ and functions $a, b : \mathbb{R}^{n+1} \to \mathbb{R}$ such that for each i,

$$\tilde{a}_i(x, y) < a(x, y) < y < b(x, y) < \tilde{b}_i(x, y)$$

$$\text{for} \quad (x, y) \in V_i \cap Z(I') \times [0, 1]. \tag{*}$$

The V_i's induce an open cover $\{\bar{C}_i\}$ of $\bar{B}$ in the site

$$
\begin{array}{ccc}
\bar{C}_i & \rightarrowtail & \bar{B}_i \\
 & \searrow \quad \swarrow & \\
 & \bar{B} &
\end{array}
$$

and we have that

$$\bar{C}_i \Vdash a_i \upharpoonright C_i < a < \pi_2 < b < b_i \upharpoonright C_i$$

(as follows easily from (*) and spelling out the map $\bar{C}_i \rightarrowtail \bar{B}_i$ in the diagram). Hence since $\mathcal{U}$ is downwards closed,

$$\bar{C}_i \Vdash (a, b) \in \mathcal{U} \qquad \text{and} \qquad \pi_2 \in (a, b)$$

and therefore since the $\bar{C}_i$ cover $\bar{B}$ and $\bar{B} \supseteq \bar{A} \times [0, 1]$,

$$\bar{A} \times [0, 1] \Vdash (a, b) \in \mathcal{U} \wedge \pi_2 \in (a, b).$$

But this says that already in the presheaf topos $\text{Sets}^{\mathbb{L}^{op}}$ we force

$$\bar{A} \times [0, 1] \Vdash \exists U \in \mathcal{U} \pi_2 \in \mathcal{U}$$

(since a, b no longer exist just on a cover). π_2 is generic, so $\mathcal{U}$ covers $[0, 1]$ in $\text{Sets}^{\mathbb{L}^{op}}$. ∎

COROLLARY 4. *In the Zariski smooth topos $\mathcal{Z}$, every open cover $\mathcal{U}$ of $[0, 1]$ has a Lesbesgue number $\delta > 0$.*

Proof. Given a cover $\mathcal{U}$ in $\mathcal{Z}$, we find by Theorem 2 and Lemma 3 a Lebesgue number δ in $\text{Sets}^{\mathbb{L}^{op}}$. Clearly, the same δ is a Lebesgue number in $\mathcal{Z}$. ∎

Remark 5. More generally, one can prove the existence of Lebesgue numbers for open covers of $s(M)$ in $\mathcal{Z}$ for any compact manifold M, see Moerdijk [11].

Morally, the existence of Lebegue numbers should give compactness in some sense. We can indeed obtain compactness of $[0, 1]$ in $\mathcal{Z}$ by changing the notion of *finiteness*. Instead of $\mathbb{N}$, we use the object N of *smooth natural numbers* defined as the subobject $\{x \in Z \mid x \geqslant 0\}$ of the object Z of *smooth integers*,

$$Z = \{x \in R \mid \sin(\pi x) = 0\}.$$

In the topos $\mathcal{Z}$, Z is the representable object

$$Z = \overline{C^{\infty}(\mathbb{R})/\sin(\pi x)} \cong \overline{C^{\infty}(\mathbb{R})/m_Z^0}$$

(these two being isomorphic because 0 is a regular value of $\sin(\pi x)$), so

$$N \cong \overline{C^{\infty}(\mathbb{R})/m_N^0} \cong \overline{C^{\infty}(\mathbb{N})}.$$

This object N is different from the object $\mathbb{N}$ of standard natural numbers in $\mathcal{Z}$. N contains nonstandard elements, such as the canonical inclusion

$$\overline{C^{\infty}(\mathbb{N})/T} \overset{i}{\hookrightarrow} N,$$

where T is the ideal of functions vanishing on a tail, which is the generic infinitely large (smooth, nonstandard) natural number of $\mathcal{Z}$.

Now if we interpret *finite* as "a quotient of an initial segment $\{0,..., n-1\}$ for some $n \in N$" (call this smooth-finite, *s-finite*), we obtain *s*-compactness of $[0, 1]$ in $\mathcal{Z}$, i.e., the validity in $\mathcal{Z}$ of the assertion that every open cover $\mathcal{U}$ of $[0, 1]$ in $\mathcal{Z}$ has an *s*-finite refinement. This follows

immediately from the existence of a Lebesgue number together with the fact that R is Archimedean for smooth integers, i.e.,

$$\mathscr{Z} \models \forall x \in R \quad \exists n \in N \, x < n.$$

More generally (cf. Remark 5), it is proved in Moerdijk [11].

THEOREM 6. *For every compact manifold M, $\mathscr{Z} \models s(M)$ is s-compact.*

The idea now is to replace $\mathbb{N}$ by N consistently when doing analysis inside the topos $\mathscr{Z}$. As said in the introduction, we will show in the next section that for homology theory we obtain the correct results in $\mathscr{Z}$ only if we use smooth integers everywhere, and change the basic algebraic notions, like that of free ring, accordingly.

As a preparation to this next section, we will now discuss some of the basic properties of smooth integers in the topos $\mathscr{Z}$.

We have already seen that the topological notion of compactness should really be replaced by its smooth analog. In fact, the notion of topological space can be adjusted accordingly:

PROPOSITION 7. *R is an s-topological space, i.e., the intersection of s-finitely many opens is again open (for the order topology). Similarly, so is every space $s(M)$.*

Proof. See Moerdijk [11]. ∎

N satisfies all primitive recursive arithmetic in $\mathscr{Z}$. In fact, many objects perceive N as being the natural number object as far as definition by recursion is concerned, and this is precisely what we need to do analysis and algebra based on Z. For example,

THEOREM 8. *For any $\bar{A}, \bar{B} \in \mathbb{L}$, with B finitely presented,*

$$\mathscr{Z} \models \forall f \in \bar{B}^{\bar{A}} \, \forall g \in \bar{B}^{B \times \bar{A}} \, \exists! \, h \in \bar{B}^{N \times \bar{A}} (\forall x \in \bar{A}$$

$$h(0, x) = f(x) \wedge \forall n \in N \, \forall x \in \bar{A} \, h(n+1, x) = g(h(n, x), x)).$$

For the proof, we need

LEMMA 9. *Let $\varphi \colon \mathbb{R} \to \mathbb{R}$ be a smooth function with $(\varphi) = m_{\mathbb{N}}^{0}$, i.e., $C^{\infty}(\mathbb{R})/(\varphi) = N$. If $I \subseteq C^{\infty}(\mathbb{R}^{n})$ is a finitely generated ideal and $f \in C^{\infty}(\mathbb{R}^{n} \times R)$ satisfies $f(x, n) \in I$ for all $n \in \mathbb{N}$, then $f(x, y) \in (I(x), \varphi(y))$.*

Proof. Since (I, φ) is germ-determined (cf. Kock [8, p. 231]), it is enough to check that the germ $f|_{(x_0, n)}$ is in $(I|_{x_0}, \varphi|_n)$ for each $x_0 \in Z(I)$ and $n \in \mathbb{N}$. But around (x_0, n), $f(x, y) = f(x, n) + (y - n) g(x, y)$ for some smooth g (by Hadamard), and $(y - n)|_n \in (\varphi)|_n$. ∎

Proof of Theorem 8.   Choose $A = C^\infty(\mathbb{R}^n)/I$ and $B = C^\infty(\mathbb{R}^m)/J$, with J a finitely generated ideal, and suppose we have maps $f \in \bar{B}^{\bar{A}}$ and $g \in \bar{B}^{\bar{B} \times \bar{A}}$ at stage $\bar{C} \in \mathbb{L}$. By replacing $\bar{A}$ by $\bar{A} \times \bar{C}$, we may assume $\bar{C} = 1$, i.e., $f \colon \bar{A} \to \bar{B}$ and $g \colon \bar{B} \times \bar{A} \to \bar{B}$ are global sections, represented by smooth maps $F(x) \colon \mathbb{R}^n \to \mathbb{R}^m$ and $G(y, x) \colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$. Define $H(z, x) \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ to be any smooth function such that $H(0, x) = F(x)$, $H(n + 1, x) = G(H(n, x), x)$. We claim that H restricts to a map $N \times \bar{A} \to \bar{B}$. Indeed, since J is finitely generated, F and G restrict to maps $\bar{A}_0 \to \bar{B}$ and $\bar{B} \times \bar{A}_0 \to \bar{B}$ for some $A_0 = C^\infty(\mathbb{R}^n)/I_0$ with $I_0 \subset I$ a finitely generated ideal. Now H maps $N \times \bar{A}_0 = \overline{C^\infty(\mathbb{R} \times \mathbb{R}^n)/(\varphi(z), I_0(x))}$ into $\bar{B}$, by Lemma 9. It is clear that the restriction $h \colon N \times \bar{A} \to \bar{B}$ of H satisfies the recursion equations.

For the uniqueness of h, suppose we have two maps h and k satisfying the requirements, represented by $H \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ and $K \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$. Since the condition $h(0, x) \equiv f(x)$ and $h(n + 1, x) \equiv g(h(n, x), x)$ is finitary (the first is clear, the second says that

$$
\begin{array}{ccc}
N \times \bar{A} & \xrightarrow{\;(-1) \times \mathrm{id}\;} & N \times \bar{A} \\[4pt]
{\scriptstyle (h, \pi_2)} \big\downarrow & & \big\downarrow {\scriptstyle h} \\[4pt]
\bar{B} \times \bar{A} & \xrightarrow{\qquad g \qquad} & \bar{B}
\end{array}
$$

commutes, i.e., that the m differences of the components are in (φ, I); all this involves only a finite part of I), and since J is finitely generated, there is a finitely generated $I_0 \subset I$ such that the restrictions h_0 and $k_0 \colon N \times \bar{A}_0 \to \bar{B}$ of H and K, respectively, also satisfy the requirements. But then $h_0 - k_0 \in (\varphi(z), I_0(x))$ by another application of Lemma 9. ∎

EXAMPLE 10.   Theorem 8 says, for example, that all manifolds perceive, in some sense, N as the natural numbers. This is not to say that for a manifold $M \in \mathcal{M}$, $s(M)^N \cong s(M)^{\mathbb{N}}$. The canonical restriction map $s(M)^N \to s(M)^{\mathbb{N}}$ is always epic, but it fails to be mono already for $M = \mathbb{R}$, i.e., $R^N \to R^{\mathbb{N}}$ is *not* injective.

To see this, look at the stage $\bar{A} = \overline{C^\infty(\mathbb{R})/m_{\{0\}}^g}$. Take a smooth $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$
f(x, n) = \begin{cases} 1 & \text{if } |x| \geqslant 2^{-(n-1)} \\ 0 & \text{if } |x| < 2^{-n}. \end{cases}
$$

f induces a map $\bar{A} \times N \xrightarrow{f} R$ in $\mathcal{Z}$, which has the property that $\bar{A} \Vdash \forall n \in \mathbb{N}, f(n) = 0$, but $\bar{A} \nVdash \forall n \in N, f(n) = 0$, i.e., as an element of $A \otimes_\infty C^\infty(\mathbb{R})/m_{\{0\}}^g, f \neq 0$. (In other words, Lemma 9 may fail if I is not finitely generated.)

As a consequence of Theorem 8, we can define for each sequence $p \in R^N$ (at stage $\bar{A}$ say, i.e., $p: \bar{A} \times N \to R$) a map $N \times \bar{A} \to^h R$ with $h(0, x) = 0$, $h(n + 1, x) = h(n, x) + p(x, n)$. Equivalently, R becomes equipped with an operation

$$N \times R^N \to R$$

which we write of course as $(n, p) \mapsto \sum_{i < n} p_i$. In other words (since N has decidable equality), we can take the sum of an *s-finite* number of elements of R. Together with the usual inverse, this gives R the structure of what we will call, for the time being (cf. remark 15 below), an *s-group* (smooth group).

A similar argument applies to any Lie-group in $\mathscr{M}$, since $s: \mathscr{M} \hookrightarrow \mathscr{Z}$ maps manifolds without boundaries to duals of finitely presented C^∞-rings, so we obtain

COROLLARY 11. *The embedding $s: \mathscr{M} \hookrightarrow \mathscr{Z}$ maps Lie-groups to s-groups.*

An analogous application of Theorem 8 shows that R is closed under s-finite products, i.e., for a smooth natural number $n \in N$ and an n-tuple $p_0, ..., p_{n-1} \in R$ we can define $\prod_{i < n} p_i \in R$, satisfying the obvious s-analogues of the ring-axioms, thus making R into an *s-ring*.

The following explicit description of the s-ring structure is sometimes helpful. If $\bar{A} \to^k N$ is a smooth natural number at stage $\bar{A}$ and $\bar{A} \times N \to^p R$ is a sequence of elements of R at this stage, $\sum_{i < k} p_i: \bar{A} \to R$ can be described as follows: write $A = C^\infty(\mathbb{R}^n)/I$, and suppose k and p are represented by $K: \mathbb{R}^n \to \mathbb{R}$ and $P: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. Since N is finitely presented, k can be extended to a map $\bar{A}_0 \to N$, where $A_0 = C^\infty(\mathbb{R}^n)/I_0$ for some finitely generated ideal $I_0 \subset I$. Being finitely generated, I_0 has "enough points," and this we exploit: cover $Z(I_0)$ by the disjoint opens $U_m = K^{-1}(m - \varepsilon, m + \varepsilon)$ for some fixed ε, $0 < \varepsilon \leqslant \frac{1}{2}$. Then $\bar{A}_0 \subset U = \bigcup_m U_m$, and the smooth map

$$Q: \quad U \to \mathbb{R}, \qquad Q(x) = \sum_{i < m} P(x, i) \quad \text{if } x \in U_m$$

restricts to a map $\bar{A} \to^q R$ of $\mathbb{L}$ which is precisely $\sum_{i < k} p_i$.

A similar description can be given of $\prod_{i < k} p_i$.

As we will see in the next section, to develop some of the homology theory in $\mathscr{Z}$ it will be necessary to replace all algebraic notions by their s-analogs. Thus, for example, a *free s-group* on a set X is defined in the obvious way using words $\sum_{i < n} x_i$ for n-tuples $x_0, ..., x_{n-1}$ in X, with n a *smooth* natural number. In this way, the object Z of smooth integers is the free s-group on one generator. Note, by the way, that Z is a sub-s-ring of

R. Similarly, an $R-s$-module is an s-abelian s-group equipped with the obvious s-analog of an R-module structure.

Usually, to establish the smooth analog of a certain ring theoretic property is much harder than showing its standard counterpart. For example, while R is a local ring object in $\mathscr{Z}$ by definition of the Grothendieck topology on $\mathbb{L}$, the fact that R is an s-local ring in $\mathscr{Z}$ is not quite as immediate:

THEOREM 12. R is an s-local ring in $\mathscr{Z}$, i.e.,

$$\mathscr{Z} \models \forall k \in N \ \forall (p_0,...,p_{k-1}) \in R^{\{0,...,k-1\}}$$

$$(p_0 + \cdots + p_{k-1} \text{ is invertible} \rightarrow \exists i < k \ p_i \text{ is invertible}).$$

For the proof, we need the following lemma from dimension theory, which is actually a special case of a theorem of Ostrand [14].

LEMMA 13.   Let $W \subseteq \mathbb{R}^n$ be an open subspace, and $\{W_1,..., W_k\}$ be an arbitrary finite open cover of W. Then there exists an open cover $\{V_1,..., V_{n+1}\}$ of W such that each V_i can be written as a disjoint union $\bigcup_{j=1}^{k} V_{ij}$ with each V_{ij} contained in the corresponding W_j.

Proof. Recall that since the topological dimension dim W of W is n, every finite open cover has a shrinking of order $\leqslant n$ (i.e., for every open cover $\mathscr{A} = \{A_1,..., A_s\}$ of W there is an open cover $\mathscr{B} = \{B_1,..., B_s\}$ of W with each $B_i \subseteq A_i$, such that any $n+2$ distinct sets from $\mathscr{B}$ have empty intersection). Using this, we prove the lemma by induction on k. The cases $k \leqslant n+1$ are of course trivial. Suppose the lemma has been proved for k, and that we are given an open cover $\{W_1,..., W_{k+1}\}$ of W, (and assume $W \neq \varnothing$). By induction hypothesis, they are open sets $V_1...,V_{n+1}$ covering $W_0 := W_1 \cup \cdots \cup W_k$, such that each V_i is a disjoint union $\bigcup_{j=1}^{k} V_{ij}$ with $V_{ij} \subseteq W_j$. Now $W = V_1 \cup \cdots \cup V_{n+1} \cup W_{k+1}$, so we can take a shrinking $V_i' \subset V_i$, $W_{k+1}' \subset W_{k+1}$ covering W of order $\leqslant n$, i.e., $V_1' \cap \cdots \cap V_{n+1}' \cap W_{k+1}' = \varnothing$. By normality of W, we can take a further shrinking $\{O_1,..., O_{n+1}, U\}$ covering W with $\bar{O}_i \subset V_i'$, $\bar{U} \subset W_{k+1}'$ (closure in W, not in $\mathbb{R}^n$!), and $O_1 \cap \cdots \cap O_{n+1} \cap U = \varnothing$. Now let $\tilde{V}_i = O_i \cup U \backslash \bar{O}_i$ for $i = 1,..., n+1$. Then $\tilde{V}_i$ is a disjoint union $\bigcup_{j=1}^{k+1} O_{ij}$, where $O_{ij} = O_i \cap V_{ij} \subset W_j$ if $j \leqslant k$, and $O_{ik+1} = U \backslash \bar{O} \subset W_{k+1}$, and $\{\tilde{V}_1,..., \tilde{V}_{n+1}\}$ covers W, for if $x \in W = O_1 \cup \cdots \cup O_{n+1} \cup U$ then either $x \in$ some $O_i \subset \tilde{V}_i$, or $x \in U$. But if $x \in U$ then $x \in W_{k+1}'$, so $x \notin V_i'$ for some $i \leqslant n+1$, i.e., $x \notin \bar{O}_i$ for this $i \leqslant n+1$.   ∎

Proof of Theorem 12.   Take a C^∞-ring $A = C^\infty(\mathbb{R}^n)/I$ and a smooth integer k at stage $\bar{A}$, i.e., $\bar{A} \rightarrow^k N$, represented by $K: \mathbb{R}^n \rightarrow \mathbb{R}$. We may without loss extend the given k-tuple $(p_i \mid i < k)$ at stage $\bar{A}$ to a map

$\bar{A} \times N \to^P R$ (i.e., an element of R^N at stage $\bar{A}$, since N has decidable order), which is then represented by a smooth map $\mathbb{R}^n \times \mathbb{R} \to^P \mathbb{R}$.

Since N is finitely presented, $\bar{A} \to^k N$ can be extended to a map $\bar{A}_0 \to^k N$, where $A_0 = C^\infty(\mathbb{R}^n)/I_0$ for some finitely generated ideal $I_0 \subset I$. Also, since $\bar{A} \Vdash (p_0 + \cdots + p_{k-1}$ is invertible$)$, there is a finitely presented extension of $\bar{A}$ forcing this, which we may assume to be $\bar{A}_0$ by choosing I_0 sufficiently big.

Let $U_m = K^{-1}(m - \frac{1}{4}, m + \frac{1}{4})$. Then $\bigcup_{m \in \mathbb{N}} U_m$ contains $Z(I_0)$, and by definition of the s-ring structure of R, $p_0 + \cdots + p_{k+1} : \bar{A}_0 \to R$ is represented by the smooth map $Q : U \to \mathbb{R}$ defined by

$$Q(x) = \sum_{i < m} P(x, i) \qquad \text{if} \quad x \in U_m.$$

Since $\bar{A}_0 \Vdash (p_0 + \cdots + p_{k-1}$ is invertible$)$, we may without loss (by choosing the U_m smaller if necessary) assume that $Q(x)$ is invertible for every $x \in U$. Now consider each U_m separately, and write

$$W_{m,j} = \{x \in U_m \mid P(x, j) \neq 0\}, \qquad j = 0, \dots, m - 1.$$

Then $U_m = \bigcup_{j=0}^{m-1} W_{m,j}$, so by the lemma above there is a cover $\{V_m^1, \dots, V_m^{n+1}\}$ of U_m such that each V_m^i can be written as a disjoint union $V_m^{i,0} \cup \cdots \cup V_m^{i,m-1}$ with $V_m^{i,j} \subset W_{m,j}$. (The point of applying Lemma 13 is, of course, that the number of the V_m^i's needed is $n + 1$, and does not depend on m.) Now define a finite open cover $\{V_1, \dots, V_{n+1}\}$ of U by

$$V_i = \bigcup_{m \in \mathbb{N}} V_m^i = \bigcup_{m \in \mathbb{N}} \bigcup_{j=0}^{m-1} V_m^{i,j}.$$

This union is disjoint, so the locally constant function

$$l_i : V_i \to \mathbb{N}, \qquad l_i(x) = j \quad \text{if} \quad x \in V_m^{i,j}$$

defines a smooth integer at V_i, and $l_i < m$ on U_m, so $V_i \cap \bar{A}_0 \Vdash l_i < k$. Moreover, since $P(x, j)$ is invertible on $V_m^{i,j}$, it is clear that

$$V_i \cap \bar{A}_0 \Vdash p_{l_i} \quad \text{is invertible.}$$

Hence, since $\{V_1, \dots, V_{n+1}\}$ covers $U \supset Z(I_0)$,

$$\bar{A} \Vdash \exists l < k \; p_l \text{ is invertible.} \quad \blacksquare$$

In the standard sense, R is not only a local ring object in $\mathscr{X}$, but even a field (in Kock's sense), i.e.,

$$\mathscr{X} \models \forall x_1, \dots, x_n (\neg (x_1 = 0 \wedge \cdots \wedge x_n = 0)$$

$$\to (x_1 \text{ is invertible } \vee \cdots \vee x_n \text{ is invertible}))$$

(this is again immediate from the definition of the Grothendieck topology).
But we have not been able to show that its smooth analog is valid:

QUESTION 14. Is R an s-field object in $\mathscr{Z}$, in the sense that

$$\mathscr{Z} \models \forall k \in N \ \forall (p_0,..., p_{k-1}) \in R^{\{0,...,k-1\}}$$

$$(\neg \forall i < k p_i = 0 \rightarrow \exists i < k \ p_i \text{ is invertible})?$$

(For k-tuples $(p_0,..., p_{k-1})$ defined at a germ-determined stage there is no
problem, since Lemma 13 can be applied. In other words, R is an s-field in
the topos $\mathscr{G}_{\text{fin}}$ of Moerdijk and Reyes [12]).

Remark 15.   Finally, a remark on the terminology of s-groups, s-rings,
etc. This terminology is really quite misleading, since it suggests that there
is a different sort of algebra, a subject as smooth algebra, s-algebra. The
only reason that we have to use names like s-group, s-free, s-ring, etc., is
that there is already a canonical interpretation of groups, free rings, etc., in
any Grothendieck topos, using the (standard) natural number object. If we
would work in a weaker theory than the full logic of toposes with a natural
number object, say in the theory of the site $\mathbb{L}$ only, or the Cartesian closed
completion of $\mathbb{L}$, we would not have a natural number object $\mathbb{N}$ available,
and the canonical thing to use instead would be the object $N \in \mathbb{L}$. What is
really going on here is that we do not only weaken the logic underlying the
mathematics to constructive, intuitionistic logic, as one has to do when
working in a Grothendieck topos, but we also weaken the theory of
arithmetic to that of N. In this metatheory, s-groups are just groups.

In connection to this remark, it is worthwhile to mention a question
which Lawvere asked us, but which we have not solved.

QUESTION 16. Is there a subcategory $\mathscr{E}$ of $\mathscr{Z}$ (of objects which think
that N is the natural number object) containing all the manifolds $s(M)$
such that $\mathscr{E}$ is an elementary topos with $N = s(\mathbb{N})$ as its natural number
object? And if so, to which extent is $\mathscr{E}$ still a model of SDG?

3. COHOMOLOGY AND DEGREES IN $\mathscr{Z}$

In Moerdijk and Reyes [13], we considered cohomology theories of
manifolds in the topos $\mathscr{G}$. We proved several "internal" versions of
De Rham-type theorems, as well as some comparison theorems relating the
internal cohomologies to the external ones. The proofs were based on the
fact that in $\mathscr{G}$, R is Archimedean and each standard simplex $\Delta_q = \{x_0,..., x_q) \in R^{q+1} \mid \sum x_i = 1\}$ is compact. Although these properties both

fail in $\mathscr{Z}$, their smooth analogs hold, and this allows us to perform arguments parallel to those of Moerdijk and Reyes [13], which we refer to as (MR) in the remainder of this section. Thus, we obtain

THEOREM 1 (De Rham's theorem in $\mathscr{Z}$).   *Let M be a manifold of finite type. Then the canonical s-linear map*

$$H^q(sM) \to H_q(sM; R)^*,$$

induced by integration, is an isomorphism.

We give some comments on how to prove Theorem 1; further details and definitions can be found in (MR). Recall (cf. Bott and Tu [2]) that a manifold is said to be of finite type if it has a finite good cover (all compact manifolds are of finite type). $H^q(sM)$ is the internal De Rham cohomology of the object sM, which is defined synthetically as in (MR). $H^q(sM)$ is not only an R-module, but an s-module over the s-ring structure of R. $H_q(sM; R)^*$ is the dual of the singular homology module $H_q(sM; R)$. We now proceed as in (MR), using an induction on M via the Poincaré lemma and the Mayer–Vietoris sequence, but with the following modifications. First, $s: \mathscr{M} \hookrightarrow \mathscr{Z}$ preserves *finite* covers only, so we have to assume that M is of finite type to be able to perform the induction. Second, we have to show that $S_q^{\{sU,sV\}}(sM)$ and $S_q(sM)$ have the same homology (where $\{U, V\}$ is an open cover of M), by a barycentric subdivision argument using the Lebesgue number. But we cannot do this in $\mathscr{Z}$ unless we use smooth integers everywhere! Since the Lebesgue number is (bigger than something) of the form $1/n$ for some $n \in N$, we have to iterate the barycentric subdivision map $sd: S_q(sM) \to S_q(sM)$ a nonstandard number of times (depending polynomially on n). To define these iterates sd^n for arbitrary $n \in N$, it suffices by naturality to define the nth subdivision of the identity map $\Delta_q \xrightarrow{\text{id}} \Delta_q \in S_q(\Delta_q)$ as a map

$$N \to S_q(\Delta_q), \qquad n \mapsto sd^n(\text{id}_{\Delta_q}).$$

This can be done by coding $S_q(\Delta_q)$ as a subobject of $(\Delta_q^N)^Z$, and then defining the transposed map $N \times Z \times N \to \Delta_q \subset R^{q+1}$ by recursion, using Theorem 8 of the previous section. (Details are tedious, but straightforward.) The number of simplices occurring in $sd^n(\text{id}_{\Delta_q})$ depends arithmetically on n, so we have to define $S_q(\Delta_q)$, *not* as the free R-module generated by $\Delta_q^{\Delta_q}$, but as the corresponding s-free $s - R$-module, so as to include sums of nonstandard length. And similarly for the definition of $S_q(sM)$ occurring in the theorem. After having replaced $\mathbb{N}$ by N systematically as just sketched, the remaining details are completely parallel to those in (MR).

The theory of differential forms does not depend on integers (whether smooth or not), so exactly as in (MR) we can show

THEOREM 2 (Comparison theorem for De Rham cohomology in $\mathscr{Z}$). *Let M be a manifold of finite type. Then*

$$H^q(sM) \simeq \Pi_S R \qquad in \ \mathscr{Z} \ iff \ H^q(M) \simeq \Pi_S \mathbb{R} \qquad in \ \text{Sets},$$

where S is a (finite) set (in Sets).

We have similar comparison theorems for singular homology with coefficients in R, proved as in (MR), but necessarily using smooth integers everywhere, as in the sketch of the proof of Theorem 1. For our present purposes, we need a comparison theorem for singular homology with coefficients in $\mathscr{Z}$.

THEOREM 3 (Comparison theorem for singular homology in $\mathscr{Z}$ with coefficients in $\mathscr{Z}$). *Let M be a manifold of finite type. Then $H_q(sM, Z)$ is of the form $s(G)$, for a finitely generated abelian group G in Sets, and we have*

$$H_q(sM, Z) \simeq s(G) \quad in \ \mathscr{Z} \ iff \ H_q(M, \mathbb{Z}) \simeq G \qquad in \ \text{Sets}.$$

Proof (Sketch). First, use Mayer–Vietoris induction on a good cover of M, exactly as in (MR) for the case of R, to show that $H_q(sM, Z) \simeq s(G)$ for a finitely generated abelian group G. (Recall the well-known facts that any such group is of the form $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}$ and that if two out of the three groups in a short exact sequence are finitely generated, then so is the third). For the elements of the good cover this is true by the Poincaré lemma, which remains valid in this context (cf. the constructive proof in (MR)). For the induction step, suppose we have an exact sequence of abelian groups in $\mathscr{Z}$,

$$s(A) \to s(B) \to G \to s(C) \to s(D),$$

where A, B, C, D are finitely generated abelian groups in Sets (regarded as discrete manifolds in $\mathscr{M}$). Applying Γ to this exact sequence, we obtain a short exact sequence

$$0 \to B' \to \Gamma(G) \to C' \to 0$$

in Sets, where B', C' are defined by

$$B \twoheadrightarrow B' \rightarrowtail \Gamma(G) \twoheadrightarrow C' \rightarrowtail C.$$

Hence $\Gamma(G)$ is finitely generated. Now, apply the five-lemma in $\mathscr{Z}$ to the diagram

$$
\begin{array}{ccccccccc}
s(A) & \longrightarrow & s(B) & \longrightarrow & G & \longrightarrow & s(C) & \longrightarrow & s(D) \\
\| & & \| & & \uparrow & & \| & & \| \\
s(A) & \longrightarrow & s(B) & \longrightarrow & s\Gamma(G) & \longrightarrow & s(C) & \longrightarrow & s(D),
\end{array}
$$

where $s\Gamma(G) \to G$ is the obvious map, and the bottow row is exact since s preserves exact sequences of discrete manifolds. Hence G is of the required form. The comparison result now follows immediately by applying Γ to the long exact Mayer–Vietoris sequence, as in (MR).

Using these comparison theorems, we can now straightforwardly prove that the degree of a map is a smooth integer. Let M be a compact, connected, oriented, q-dimensional manifold. From the corresponding classical results and the comparison theorems, we obtain

(1) $H^q(sM) \cong R$,

(2) $H_q(sM; Z) \cong Z$.

Now reason in the topos $\mathscr{Z}$, and let $f \in s(M)^{s(M)}$ (at an arbitrary stage $\bar{A}$). Recall the definition of degree: let σ be a generator of $H^q(sM)$ (the area-form, say). Then from (1),

(3) $f^*(\sigma) = \lambda \cdot \sigma + d\tau$

for a unique $\lambda \in R$. This λ is by definition the degree of f, $\deg(f)$. Now let γ be the generator of $H_q(sM; Z)$. Then from (3) we obtain by Stokes' theorem

(4) $\deg(f) \int_\gamma \sigma = \int_{f_*(\gamma)} \sigma$.

But $f_*(\gamma) = n\gamma + \partial\beta$ for a unique $n \in Z$, so $\deg(f) \int_\gamma \sigma = n \int_\gamma \sigma$, hence $\deg(f) = n \in Z$. For the record,

THEOREM 4.   *Let M be as above. Then*

$$\mathscr{Z} \models \forall f \in s(M)^{s(M)} \, \deg(f) \in Z.$$

Remark 5. We should point out here that there is an alternative proof of Theorem 4, using results of (MR) on the cohomology theory in $\mathscr{G}$, *without* proving similar results about $\mathscr{Z}$ first, as we did above, but instead transferring the map $f \in s(M)^{s(M)}$ to the topos $\mathscr{G}$. This argument seems less natural than the one given above, but it may be slightly quicker. Here is a sketch: suppose we are given a map $f \in s(M)^{s(M)}$ in $\mathscr{Z}$, at stage $\bar{A}$ say, with

$A = C^\infty(\mathbb{R}^n)/I$, i.e., $f: \bar{A} \times s(M) \to s(M)$. Since $s(M)$ is representable by the dual of a finitely presented C^∞-ring, f can be extended to a map

$$f_0: \bar{A}_0 \times s(M) \to s(M),$$

where $A_0 = C^\infty(\mathbb{R}^n)/I_0$ for a finitely generated ideal $I_0 \subset I$. $\bar{A}_0$ is an object of the site $\mathbb{L}$ for $\mathscr{Z}$ as well as of the site $\mathbb{G}$ for $\mathscr{G}$, and the coreflection $\lambda: \mathbb{L} \to \mathbb{G}$ (cf. Moerdijk and Reyes [12]) induces a geometric morphism

$$\mathscr{G}/\bar{A}_0 \xrightarrow{\varphi} \mathscr{Z}/\bar{A}_0.$$

Now compute the degree of $\varphi^*(f)$ in $\mathscr{G}/\bar{A}_0$. This is a standard integer $n \in \mathbb{Z}$ at stage $\bar{A}_0$, and the assertion that $n = \deg(\varphi^*(f))$ can be written as an equation involving $\int \varphi^*(f)$ as above. Hence it is reflected by φ^*, so the equation holds in $\mathscr{Z}/\bar{A}_0$. But n is a standard integer in $\mathscr{G}$ at $\bar{A}_0$, i.e., a map $\bar{A}_0 \to \mathbb{Z}$ in $\mathbb{G}$, hence it corresponds to a *smooth* integer in $\mathscr{Z}$ at stage $\bar{A}_0$. Thus the degree of f in $\mathscr{Z}$ is a smooth integer.

Finally, we would like to mention the following problem. Classically, a way of interpreting the degree of f is by counting the inverse image $f^{-1}(p)$ of a regular value p. Regular values are dense, by Sard's theorem, so such p's can always be found. For the topos $\mathscr{G}$, a similar explanation can be given for the degree of internal mappings $f \in s(M)^{s(M)}$, i.e., $\bar{A} \times s(M) \to^f s(M)$ for some $\bar{A} \in \mathbb{G}$, since in $\mathscr{G}$ it is valid that regular values exist (in fact, are dense), essentially by the stability of transversality (provided M is compact). In $\mathscr{Z}$, however, this argument does not work, not because we have more stages $\bar{A}$ (not just germdetermined ones), but because we have finite covers in the site only. Thus

QUESTION 6. Do regular values exist in $\mathscr{Z}$, i.e., for M a compact manifold as above, does

$$\mathscr{Z} \models \forall f \in s(M)^{s(M)}, \quad \exists x \in s(M) \quad (x \text{ is a regular value of } f)$$

hold?

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