Isomorphisms between Predicate and State Transformers

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Abstract

We study the relation between state transformers based on directed complete partial orders and predicate transformers. Concepts like 'predicate', 'liveness', 'safety' and 'predicate transformers' are formulated in a topological setting. We treat state transformers based on the Hoare, Smyth and Plotkin powerdomains and consider continuous, monotonic and unrestricted functions. We relate the transformers by isomorphisms thereby extending and completing earlier results and giving a complete picture of all the relationships.

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Contents

1 Introduction

In this paper we give a full picture of the relationship between state transformers and predicate transformers. For the state transformers we consider the Hoare, Smyth and Plotkin powerdomains. We give a full picture in the sense that

- we consider algebraic directed complete partial orders (with an bottom element) (and not only flat domains),
- we consider not only continuous state transformers, but also monotonic transformers and the full function space,
- we do not restrict to bounded nondeterminism,
- we treat all the three powerdomains with or without empty set.

The first item is important when we want to use domains used in concurrency semantics. The second and third item give more freedom in the sense that we can use these transformations also for specification purposes without constraints on computability. Having the empty set in a powerdomain can be important to treat deadlock. Our treatment includes the Plotkin powerdomain. The motivation for also including full function spaces (not restricted to monotone or continuous functions) stems a transfer lemma we have used in [BK92]. This lemma allows to find by iteration least fixed points also for some non-monotonic functions. We will give a version of this lemma in the section with mathematical preliminaries and hope that this lemma gives motivation for studying the unrestricted function space.

For state transformers we use an extension of the standard powerdomains. For predicate transformers we start from the (informal) classification of predicates in liveness and safety predicates of Lamport [Lam77]. Later Smyth [Smy83] followed by [AS85, Kwi91] used topology to formalize this classification. We use also topology for defining predicates and predicate transformers, and obtain three kinds of them: safety and liveness predicate transformers and moreover, we consider also predicate transformers with predicates that are the intersection of safety and liveness predicates.

We prove that the Hoare state transformers are isomorphic to safety predicate transformers, the Smyth state transformers are isomorphic to the liveness predicate transformers, and that the Plotkin state transformers are isomorphic to the "intersection" predicate transformers. So for the first time we are able to give a full picture of all the relationships filling several gaps that were present in the literature.

Next we discuss how this paper is related to previous work. Powerdomains for ω-algebraic cpo's were introduced in [Plo76], [Smy78] and [Plo81]. Our power domains are slightly more general in the sense that we consider algebraic dcpo's, no restriction to (Scott-) compact sets, and we can add the empty set in all the three variants, where for the Smyth case is added as a top element [Smy83, Plo81], in the Hoare case is added as a bottom element and in the Plotkin case is added as an element apart, comparable only with itself and with the bottom [HP79, MM79, Plo81, Abr91]. Moreover we propose a variant of the Smyth power domain by treating the empty set as an in the Plotkin case [BK92].

Predicate transformers were introduced in [Dij76]. We use a definition of predicate transformer that is more general because we only require monotonicity. Back and von Wright [Bac80, vW90] use the same restriction on predicate transformers, but they consider only the flat case. They use predicate transformers for refinement and provide a nice lattice theoretical framework. Nelson [Nel89] has (for the flat case) used pairs of predicate transformers for giving semantics to a language with backtracking. Smyth [Smy83] introduced predicate transformers (with the Dijkstra healthiness conditions) for non-flat domains.

Isomorphisms between state and predicate transformers have been given for the flat case of the Smyth power domain in [Plo79] (and for countable nondeterminism in [AP86]), for the flat case of the Hoare power domain [Plo81]. Also de Bakker and de Roever [Bak80, Roe76] study (from a semantical point of view) for the flat case the relation between state transformer and predicate transformer semantics. Moreover, for the flat case of the Plotkin powerdomain we have proposed an isomorphism in [BK92].

For the general case of the compact Smyth powerdomain in the paper [Smy83] an isomorphism is given for continuous state transformers. He uses a topological technique which Plotkin later used in [Plo81] for the continuous Hoare state transformers and which we also use in this paper. As far as we know no isomorphism was known for the non-flat Plotkin power domain (as for example is remarked in the extended Pisa lecture notes [Plo81] and in [Smy83]). Recent work includes an operational point of view in van Breugel [Bre93].

2 Mathematical Preliminaries

We introduce some basic notions on domain theory and topology. For a more detailed discussions on domain theory consult for example [Plo81], and for topology we refer to [Eng77].

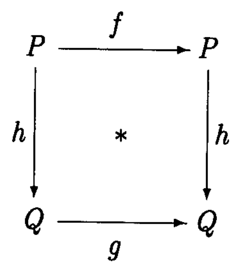
Let P a partial order, $x \in P$ and A a subset of P. Define $x \uparrow = \{y | y \in P \;\wedge\; x \sqsubseteq y\}$ and $A \uparrow = \bigcup\{x \uparrow | x \in A\}$. A set A is called upper-closed if $A = A \uparrow$. A subset A of P is called an *ω-chain* if there is a enumeration $x_0, x_1, ...$ of the elements of A such that $x_i \sqsubseteq x_{i+1}$ for every i. A generalization of ω-chain is the concept of directed set; $A \subseteq P$ is said to be *directed* if it is non empty and every finite subset of A has an upper bound in A. P is a *directed complete partial order* (dcpo) if there exists a least element $\perp$ and every directed subset A of P has least upper bound (lub) $\bigsqcup A$. A directed set A is *eventually constant* if $\bigsqcup A \in A$.

Clearly every directed complete partial order is a complete partial order. Their difference is a question of cardinality since a partial order is a complete partial order if and only if it has all least upper bound of countable directed sets [SP82].

An element b of a dcpo P is *finite* if for every directed set $A \subseteq P$, $b \sqsubseteq \bigsqcup A$ implies $b \sqsubseteq x$ for some $x \in A$. The set of all finite elements of P is denoted by B_P and is called *base*. A complete partial order P is *algebraic* if for every element $x \in P$ the set $\{b | b \in B_P \ \wedge \ b \sqsubseteq x\}$ is directed and has least upper bound x; it is *ω-algebraic* if it is algebraic and its base is denumerable.

Let P, Q be two partial orders. A function $f : P \to Q$ is *monotone* (denoted by $f : P \to_m Q$) if for all $x, y \in P$ with $x \sqsubseteq_P y$ we have $f(x) \sqsubseteq_Q f(y)$. If P, Q are two dcpo we say f is *continuous* (denoted by $f : P \to_c Q$) if $f(\bigsqcup A) = \bigsqcup f(A)$ for each directed set $A \subseteq P$; moreover f is *stabilizing* (denoted by $f : P \to_{c_s} Q$) if it is continuous and for every directed set $A \subseteq P$ $f(A)$ is an eventually constant directed set in Q. If $f : P \to_c Q$ is continuous then f is monotone. We say f is *strict* (denoted by $f : P \to_s Q$) if and only if $f(\bot_P) = \bot_Q$; dually f is *top preserving* (denoted by $f : P \to_t Q$) if and only if $f(\top_P) = \top_Q$. If f is onto and monotone then it is also strict.

Let $f : P \to P$, we denote by $\mu.f$ the *least fixed point* of f, that is, $f(\mu.f) = \mu.f$ and for every other $x \in P$ such that $f(x) = x$ then $\mu.f \sqsubseteq x$. For a monotone function $f : P \to_m P$, where P is a dcpo, the least fixed point of f always exists and can be calculated by iteration, that is, there exists an ordinal λ such that $\mu.f = f^\lambda$, where the α-iteration of f is defined by $f^\alpha = f(\bigsqcup_{k<\alpha} f^k)$ for every ordinal α [HP72]. If f is also continuous then $\lambda \leq \omega$. It can also be of interest to consider also non-monotonic functions, at least when they are representation as quotient of some monotonic functions between dcpo, as shown in [BK92]: let P be a dcpo and Q be a partial order, $f : P \to_m P$ be a monotone function, $h : P \to_c Q$ be an onto and continuous function and $g : Q \to Q$ be a (possibly non monotone) function such that it is a *representation as quotient* of f with respect to h, that is, the following diagram commutes:



Then for every ordinal α the α-iteration from the bottom element g^α exists. Moreover, if for each $y \in Q$ the partial order determined by $h^{-1}(y) \subseteq P$ is finite or has either the bottom or the top element then the smallest fixed point $\mu.g$ exists and $\mu.g = h(\mu.f)$.

We now introduce some basic topological notions. A *topology* $\mathbf{O}(X)$ on a set X is a collection of subsets of X that is closed under finite intersections and arbitrary unions. The pair $(X, \mathbf{O}(X))$ is called *topological space* and the elements of $\mathbf{O}(X)$ are the *open sets* of the space X. A *base* of a topology $\mathbf{O}(X)$ on X is a subset $\mathbf{B} \subseteq \mathbf{O}(X)$ such that every open set is the union of elements of $\mathbf{B}$. The topologies on a set X form a complete lattice if ordered by inclusion, with bottom element the *trivial* topology $\mathbf{O}_t(X) = \{\emptyset, X\}$ and top the *discrete* topology $\mathbf{O}_d(X) = \mathcal{P}(X)$. A set $S \subseteq X$ is *dense* if and only if $X \setminus S$ contains no non-empty open sets. A G_δ*-set* is a countable (finite or infinite) intersection of open sets.

For example, given a partial order X, its *Alexandroff topology* $\mathbf{O}_{Al}(X)$ consists of all the upper-closed subsets of X. If X is a dcpo, a finer topology of X is the Scott topology $\mathbf{O}_{Sc}(X)$, where $o \in \mathbf{O}_{Sc}(X)$ if and only if o is upper-closed and for any directed set $S \subseteq X$ if $\bigsqcup S \in o$ then $S \cap o \neq \emptyset$.

We can describe a topology by its closed sets instead of its open sets. A subset of a set X is *closed* if and only if it is the complement of an open set of a given topology on X. The collection of closed sets of a topological space is denoted by $\mathbf{C}(X)$ and, dually to the case of open sets, is closed under finite unions and arbitrary intersections. For every $A \subseteq X$ there exists a closed set c and a dense set d such that $A = c \cap d$.

For example, given a partial order P the closed sets of the Alexandroff topology are all the lower closed sets, while a set $c \subseteq P$ is closed with respect to the Scott topology if c is lower closed and for every directed set $S \subseteq P$ if $S \subseteq c$ then $\bigsqcup S \in c$.

Let X be an algebraic dcpo and let $\mathbf{O}(X)$ be a topology on X. A subset $A \subseteq X$ is *compact* in $\mathbf{O}(X)$ if and only if for every collection of open sets $o_i \in \mathbf{O}(X)$ with $i \in I$ such that $A \subseteq \bigcup_I o_i$ there exists a finite subcollection o_j such that $A \subseteq \bigcup_J o_j$. For example $A \subseteq X$ is compact in $\mathbf{O}_d(X)$ if and only if it is a finite set. The intersection of a closed set with a compact one is always compact.

A subset $\mathcal{F} \subseteq \mathbf{O}(X)$ is a *filter* if and only if

1. $\forall o_1 \in \mathcal{F}, o_2 \in \mathbf{O}(X) : o_1 \subseteq o_2 \Rightarrow o_2 \in \mathcal{F}$,
2. $\forall o_1, o_2 \in \mathcal{F} : o_1 \cap o_2 \in \mathcal{F}$.

A filter $\mathcal{F} \subseteq \mathbf{O}(X)$ is *proper* if $\emptyset \notin \mathcal{F}$ (or equivalently $X \neq \mathcal{F}$), while it is an *open filter* of $\mathbf{O}(X)$ if $\mathcal{F}$ is open in the Scott topology of $\mathbf{O}(X)$.

3 Predicates and Predicate Transformers

A *predicate* P is a function from a set X to the boolean set $\{tt, ff\}$ or, equivalently, is a subset of X. Topology provides an elegant way of expressing predicates of programs (see [Smy83], [Kwi91]) in which the open sets of a topological space X are seen as the *computable predicates.*

Theorem 3.1 *Let* $\mathbf{Bool} = \{tt, ff\}$ *with* $ff \sqsubseteq tt$. *Then* $\mathbf{O}_d(X)$ *is isomorphic to the set of all the predicates from* X *to* $\mathbf{Bool}$, $\mathbf{O}_{Al}(X)$ *is isomorphic to the set of all the monotone predicates from* X *to* $\mathbf{Bool}$ *and* $\mathbf{O}_{Sc}(X)$ *is isomorphic to the set of all the continuous (and clearly stabilizing) predicates from* X *to* $\mathbf{Bool}$.

In this sense taking different topologies corresponds to different restrictions on the function space. The theorem can be generalized as follows: take on the left hand side an arbitrary topology X and on the right hand side the continuous (in topological sense) predicates from X to **Bool**, where **Bool** is equipped with the Scott topology.

[Lam77] introduces two classes of predicates: *safety* and *liveness predicates.* In the topological view of Smyth [Smy83, Smy] closed sets represent *safety predicates* while *liveness predicates* are intersections of open sets. Due to computability, Smyth uses closedness under countable intersection (such specifications are known in topology as G_δ sets: countable intersections of open sets). We take arbitrary intersections of open sets as liveness predicates. This differs from [AS85] where liveness predicates are dense sets (the complement does not contain non-empty open sets). In [Kwi91] liveness predicates are also G_δ-sets.

In this paper we consider algebraic dcpo's together with the Scott topology. The Scott-closed sets are the safety predicates. The Scott-open sets of a dcpo Y represent the computable predicates and are *finitary* in the sense that $y \in o$ if and only if there exists a $b \in B_Y$ such that $b \in o$ and

$b \sqsubseteq y$. In other words, a predicate P is finitary if we can test P holds for y by testing only the finite elements smaller than y. Liveness predicates are the arbitrary intersection of Scott-open sets (that is Alexandroff open sets).

An example might clarify this: consider the sequence domain with the prefix ordering. Safety predicate: always a is described by the set $\{x | x = a^* \vee x = a^\omega\}$ which is Scott closed. Liveness predicate: eventually a is described by the set $\{xa | x \in \Sigma^*\} \uparrow$ which is Alexandroff open.

Let $P(Y)$ and $P(X)$ be two collections of predicates on the space Y and X, respectively. We define *predicate transformers* as the monotone functions from $P(Y)$ to $P(X)$. Another natural restriction (besides monotonicity) in the case that $Y \in P(Y)$ is to require that a predicate transformer must be top-preserving.

We now define a restricted version of the cartesian product on predicate transformers by requiring monotonicity on the intersection.

Definition 3.2 *Let $P_1(X)$, $P_2(X)$ be two collections of predicates on X and $Q_1(Y)$, $Q_2(Y)$ be two collections of predicates on Y. Define $(P_1(X) \rightarrow_m Q_1(Y)) \otimes (P_2(X) \rightarrow_m Q_2(Y))$ as the following subset of $(P_1(X) \rightarrow_m Q_1(Y)) \times (P_2(X) \rightarrow_m Q_2(Y))$:*

$$\begin{aligned}\{(\pi, \rho) | \quad & \forall\, p, p' \in P_1(X), q, q' \in P_2(X) : \\ & p \cap q \subseteq p' \cap q' \; \Rightarrow \pi(p) \cap \rho(q) \subseteq \pi(p') \cap \rho(q')\},\end{aligned}$$

ordered componentwise.

If $P(X)$ is a collection of predicates closed under arbitrary intersection, then a predicate transformer $\pi : P(X) \rightarrow P(Y)$ is *multiplicative* (denoted by $\pi : P(X) \rightarrow_M P(Y)$ if and only if for every index set $I \neq \emptyset$ and family of predicates P_i, with $i \in I$, we have

$$\pi(\bigcap_I p_i) = \bigcap_I \pi(p_i).$$

(This is "intersection"-multiplicativity and not "meet"-multiplicativity.) Given a predicate transformer $\pi : P(X) \rightarrow P(Y)$ its *dual* $\pi^\circ : (X \setminus P(X)) \rightarrow (Y \setminus P(Y))$ is given by $\pi^\circ(S) = Y \setminus \pi(X \setminus S)$, for every $S \in (X \setminus P(X))$. If $P(X)$ is a collection of predicates closed under arbitrary union, a predicate transformer is *additive* (denoted by $\pi : P(X) \rightarrow_A P(Y)$) if and only if its dual is multiplicative.

Intuitively, multiplicative predicate transformers preserve the logical '$\forall$' on predicates, while the additive ones preserve the logical '$\exists$'.

Now we come to the definition of safety and liveness predicate transformers used in this paper.

Definition 3.3 *Let X and Y be algebraic dcpo's. The liveness predicate transformers are*

$$\mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_d(X)$$

ordered pointwise by subset inclusion. The safety predicate transformers are

$$\mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_d(X)$$

ordered pointwise by superset inclusion

Note that we could have defined the safety predicate transformers equivalently by the dual $\mathbf{O}_{Sc}(Y) \rightarrow_{sA} \mathbf{O}_d(X)$, ordered pointwise by subset inclusion.

An interesting restriction on the liveness predicate transformers is to consider only the continuous ones with respect to directed sets S contained in the Scott topology of Y because they correspond to finite nondeterminism [Smy83]. We denote this collection by $\mathbf{O}_{Al}(Y) \rightarrow_{\hat{c}tM} \mathbf{O}_d(X)$

4 State Transformers

In this section we give generalizations of the three 'classical' power domains on ω-algebraic dcpo's, the so-called Hoare, Smyth and Plotkin powerdomains ([Plo76], [Smy78] and [Plo81]).

Definition 4.1 *Let X be an algebraic dcpo and $A \subseteq X$. Define*

- $\overline{A} = \{x | \forall b \in B_X : b \sqsubseteq x \Rightarrow \exists x_b \in A : b \sqsubseteq x_b\}$,
- $A^* = \{x | (\exists x' \in A : x' \sqsubseteq x) \wedge (\forall b \in B_X : b \sqsubseteq x \Rightarrow \exists x_b \in A : b \sqsubseteq x_b)\}$.

For every $A \subseteq X$ we have $\overline{A} = A$ if and only if $A \in \mathbf{C}_{Sc}(X)$. Further $A \subseteq A^*$, $(A^*)^* = A^*$, and $A = A^* \Leftrightarrow A = A\uparrow \cap \overline{A}$.

Next we define the powerdomains:

Definition 4.2 *Let X be an algebraic dcpo. Define*

- *the Hoare power domain* $\mathcal{H}(X) = \langle \{A | A \subseteq X \wedge A = \overline{A}\}, \sqsubseteq_H \rangle$, *where* $A \sqsubseteq_H B$ *if* $A \subseteq B$,
- *the Smyth power domain* $\mathcal{S}(X) = \langle \{A | A \subseteq X \wedge A = A\uparrow\}, \sqsubseteq_S \rangle$, *where* $A \sqsubseteq_S B$ *if* $A \supseteq B$,
- *the Plotkin power domain* $\mathcal{P}(X) = \langle \{A | A \subseteq X \wedge A = A^*\}, \sqsubseteq_P \rangle$, *where* $A \sqsubseteq_P B$ *if* $A\uparrow \sqsubseteq_S B\uparrow$ *and* $\overline{A} \sqsubseteq_H \overline{B}$.

An interesting variant of the Smyth power domain is the so called deadlock power domain $\mathcal{D}(X)$ introduced in [BK92] and defined by $\langle \{A | A \subseteq X \wedge A = A\uparrow\}, \sqsubseteq_D \rangle$, where $A \sqsubseteq_D B$ if $A = X \vee (A = B = \emptyset) \vee (A \supseteq B)$.

If we want to consider only bounded nondeterminism then we can restrict the powerdomains above considering only those sets which are compact in the Scott topology (denoted by the subscript *comp*). Since every Scott closed set of a dcpo is compact in the Scott topology we have $\mathcal{H}_{comp}(X) = \mathcal{H}(X)$. The standard definitions of the Hoare, Smyth and Plotkin powerdomains are $\mathcal{H}^+(X), \mathcal{S}^+_{comp}(X)$ and $\mathcal{P}^+_{comp}(X)$, where the superscript $^+$ denotes that the powerdomains should be taken without the empty set.

State transformers are functions (ordered pointwise) from an algebraic dcpo X to one of the powerdomains over an algebraic dcpo Y.

5 Relations

In this section we give the isomorphisms between the state transformer and predicate transformer domains. We start with the relation between safety predicate transformers and the Hoare state transformers:

Theorem 5.1 *Let X and Y be two algebraic dcpo's. We have the following isomorphisms between the partial orders:*

1. $X \to \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_d(X)$,
2. $X \to \mathcal{H}^+(Y) \cong \mathbf{C}_{Sc}(Y) \to_{stM} \mathbf{C}_d(X)$,
3. $X \to_m \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_{Al}(X)$,
4. $X \to_c \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_{Sc}(X)$

In all cases the isomorphism is given by the function γ:

$$\gamma(m)(c) = \{x | m(x) \subseteq c\}$$

The function γ is the generalization of the weakest liberal precondition and its inverse is given by $\gamma^{-1}(\rho)(x) = \bigcap\{c | x \in \rho(c)\}$. Because the isomorphism is always the same we can combine cases of the theorem (for example combining 2. and 4. we get the result of [Plo81]:

$$X \to_c \mathcal{H}^+(Y) \cong \mathbf{C}_{Sc}(Y) \to_{stM} \mathbf{C}_{Sc}(X)).$$

Now we relate liveness predicate transformers and Smyth state transformers:

Theorem 5.2 *Let X and Y be two algebraic dcpo's. We have the following isomorphisms between the partial orders:*

1. $X \to \mathcal{S}(Y) \cong \mathbf{O}_{Al}(Y) \to_{tM} \mathbf{O}_d(X)$,
2. $X \to \mathcal{S}^+(Y) \cong \mathbf{O}_{Al}(Y) \to_{stM} \mathbf{O}_d(X)$,
3. $X \to \mathcal{S}_{comp}(Y) \cong \mathbf{O}_{Al}(Y) \to_{\hat{c}tM} \mathbf{O}_d(X)$,
4. $X \to_m \mathcal{S}(Y) \cong \mathbf{O}_{Al}(Y) \to_{tM} \mathbf{O}_{Al}(X)$,
5. $X \to_{c_s} \mathcal{S}(Y) \cong \mathbf{O}_{Al}(Y) \to_{tM} \mathbf{O}_{Sc}(X)$.

In all cases the isomorphism is given by the function ω:

$$\omega(m)(o) = \{x | m(x) \subseteq o\}$$

The function ω is a generalization of the weakest precondition and its inverse is given by $\omega^{-1}(\pi)(x) = \bigcap\{o | x \in \pi(o)\}$. Also in this case we can combine 2., 3. and 5. to obtain the result of [Smy83]. We use stabilizing functions as counterpart of the finiteness condition of the Scott topology. We can remove the stability condition if we consider only compact sets.

To prove these theorems we need the following *stability lemma* due to Plotkin [Plo79, AP86]:

Lemma 5.3 *Let $\pi : P(X) \rightarrow_M P(Y)$ be a multiplicative predicate transformer with $P(X)$ a collection of predicates closed under arbitrary intersection. Then*

$$y \in \pi(\hat{p}) \Leftrightarrow \bigcap\{p | y \in \pi(p)\} \subseteq \hat{p}$$

for every $\hat{p} \in P(X)$ and $y \in Y$.

Finally we relate the Plotkin state transformers with pairs of safety and liveness predicate transformers:

Theorem 5.4 *Let X and Y be two algebraic dcpo's. We have the following isomorphisms between partial orders:*

1. $X \rightarrow \mathcal{P}(Y) \cong (\mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_d(X)) \otimes (\mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_d(X))$
2. $X \rightarrow \mathcal{P}^+(Y) \cong (\mathbf{O}_{Al}(Y) \rightarrow_{stM} \mathbf{O}_d(X)) \otimes (\mathbf{C}_{Sc}(Y) \rightarrow_{stM} \mathbf{C}_d(X))$,
3. $X \rightarrow \mathcal{P}_{comp}(Y) \cong (\mathbf{O}_{Al}(Y) \rightarrow_{\hat{c}tM} \mathbf{O}_d(X)) \otimes (\mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_d(X))$,
4. $X \rightarrow_m \mathcal{P}(Y) \cong (\mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_{Al}(X)) \otimes (\mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_{Al}(X))$,
5. $X \rightarrow_{c_s} \mathcal{P}(Y) \cong (\mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_{Sc}(X)) \otimes (\mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_{Sc}(X))$.

In all cases the isomorphism is given by the function η:

$$\eta(m)(o, c) = (\{x | m(x) \uparrow \subseteq o\}, \{x | \overline{m(x)} \subseteq c\})$$

The inverse of η is given by

$$\eta^{-1}((\pi, \rho))(x) = \bigcap\{o | x \in \pi(o)\} \cap \bigcap\{c | x \in \rho(c)\}$$

To prove this theorem we need a different *stability lemma*:

Lemma 5.5 *Let $(\pi, \rho) : (P_1(X) \rightarrow_M Q_1(Y)) \otimes (P_2(X) \rightarrow_M Q_2(Y))$ for $P_1(X), P_2(X)$ two arbitrary collection of predicates over X closed under arbitrary intersection and $Q_1(Y), Q_2(Y)$ two collections of predicates over Y. Then for every $y \in Y$, $\hat{p} \in P_1(Y)$, and $\hat{q} \in P_2(Y)$ we have:*

1. $y \in \pi(\hat{p}) \Leftrightarrow \bigcap\{p | y \in \pi(p)\} \cap \bigcap\{q | y \in \rho(q)\} \subseteq \hat{p}$,
2. $y \in \rho(\hat{q}) \Leftrightarrow \bigcap\{p | y \in \pi(p)\} \cap \bigcap\{q | y \in \rho(q)\} \subseteq \hat{q}$.

6 Conclusion and Future Work

We have given a formal definition of safety and liveness predicates and of predicate transformers, given generalizations of the standard definitions of powerdomains and of state transformers, and have given a complete series of isomorphisms between predicate and state transformers (including the Plotkin state transformers). This gives us many insights in these concepts. One can say that weakest (liberal) preconditions and state transformers are very tightly connected and not at all arbitrary concepts. The asymmetry between the definition in the Hoare and the Smyth powerdomains (Scott closed versus Alexandroff open) is due to the fact that the intersection of Scott closed sets is again a Scott closed set, while the intersection of Scott open sets is not necessarily Scott open. Another example of an insight is that the Plotkin powerdomain is equivalent to predicate transformers that act on combinations of safety and liveness predicates. Further, it has been shown that monotonicity of predicate transformers is in some way necessary and does not implies monotonicity of state transformers.

Future work includes:

- Definition of an OR between predicate transformers with a corresponding notion on state transformers.
- Characterization of predicates that are not safe nor live nor belong to the intersection of safe and live (and are they interesting at all?).
- Generalization of the results to arbitrary topological spaces.
- Applications of predicate transformers to non-flat domains for concurrency and communication.

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References

[Abr91] S. Abramsky. A domain equation for bisimulation. *Information and Computation*, 92:161–218, 1991.

[AP86] K. R. Apt and G. Plotkin. Countable nondeterminism and random assignment. *Journal of the ACM*, 33(4):724–767, October 1986.

[AS85] B. Alpern and F.B. Schneider. Defining liveness. *Information Processing Letters*, 21:181–185, 1985.

[Bac80] R.-J.R. Back. *Correctness Preserving Program Refinements: Proof Theory and Applications*, volume 131 of *Mathematical Centre Tracts*. Mathematical Centre, Amsterdam, 1980.

[Bak80] J. W. de Bakker. *Mathematical Theory of Program Corretness.* Prentice-Hall, 1980.

[BK92] M. Bonsangue and J. N. Kok. Semantics, orderings and recursion in the weakest precondition calculus. Technical Report CS-R9267, Centre for Mathematics and Computer Science, Amsterdam, 1992. Extended abstract to appear in the proceedings of the Rex Workshop '92 'Semantics: Foundations and Applications' LNCS, 1993.

[Bre93] F. van Breugel. Relating state transformation semantics and predicate transformer semantics for parallel programs. To appear, 1993.

[Dij76] E.W. Dijkstra. *A Discipline of Programming.* Prentice-Hall, 1976.

[Eng77] R. Engelking. *General Topology.* Polish Scientific Publishers, 1977.

[HM81] K. Hoffman and M. Mislove. Local compactness and continuous lattices. In Banaschewski and Hoffman, editors, *Continuous lattices - Proceedings Bremen 1979*, number 871 in Lecture Notes in Mathematics, pages 209–248, 1981.

[HP72] P. Hitchcock and D. Park. Induction rules and termination proofs. In *International Conference on Automata, Languages and Programming*, 1972.

[HP79] M. Hennessy and G. D. Plotkin. Full abstraction for a simple parallel programming language. In J. Becvar, editor, *Proc. 8th Int'l Symp. on Mathematical Foundations on Computer Science*, volume 74 of *Lecture Notes in Computer Science*, pages 108–120. Springer-Verlag, Berlin, 1979.

[Kwi91] M.Z. Kwiatowska. On topological characterization of behavioural properties. In G.M. Reed, A.W. Roscoe, and R.F. Wachter, editors, *Topology and Category Theory in Computer Sciences - Proc. Oxford Topology Symposioum, June 1989*, pages 153–177. Oxford Science Pubblications, 1991.

[Lam77] L. Lamport. Proving the correctness of a multiprocess programs. *IEEE Transaction on Software Eng.*, SE-3:125–143, 1977.

[MM79] G. Milne and R. Milner. Concurrent processes and their syntax. *J. ACM*, 26, 2:302–321, 1979.

[Nel89] G. Nelson. A generalization of Dijkstra's calculus. *ACM Transaction on Programming Languages and Systems*, 11 - 4:517–561, 1989.

[Plo76] G. D. Plotkin. A powerdomain construction. *SIAM J. Comput.*, 5:452–487, 1976.

[Plo79] G. D. Plotkin. Dijkstra's predicate transformer and Smyth's powerdomain. In *Proceedings of the Winter School on Abstract Software Specification*, volume 86 of *Lecture Notes in Computer Science*, pages 527–553. Springer-Verlag, Berlin, 1979.

[Plo81] G.D. Plotkin. Post-graduate lecture notes in advanced domain theory (incorporating the "Pisa Notes"). Department of Computer Science, Univ. of Edinburgh, 1981.

[Roe76] W.P. de Roever. Dijkstra's predicate transformer, non-determinism, recursion, and terminations. In *Mathematical foundations of computer science*, volume 45 of *Lecture Notes in Computer Science*, pages 472–481. Springer-Verlag, Berlin, 1976.

[Smy] M.B. Smyth. Topology. To appear in Handbook of Logic in Computer Science, Oxford University Press.

[Smy78] M.B. Smyth. Power domains. *J. Comput. Syst. Sci.*, 16,1:23–36, 1978.

[Smy83] M.B. Smyth. Power domains and predicate transformers: A topological view. In *Proceeding of ICALP '83 (Barcelona)*, volume 154 of *Lecture Notes in Computer Science*, pages 662–675. Springer-Verlag, Berlin, 1983.

[SP82] M.B. Smyth and G.D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM J. Comput.*, 11:761–783, 1982.

[vW90] J. von Wright. *A Lattice-theoretical Basis for Program Refinement.* PhD thesis, Abo Akademi, 1990.

A Appendix: Some proofs

In this appendix we give the proof of the some of a selection of Theorems and Lemmas introduced in section 5. We start by giving a proof of the two stability Lemmas.

Proof of Lemma 5.3

$\Rightarrow$) If $y \in \pi(\hat{p})$ then $\hat{p} \in \{p | y \in \pi(p)\}$ and hence

$$\bigcap\{p|y \in \pi(p)\} \subseteq \hat{p}.$$

$\Leftarrow$) Since π is multiplicative we have

$$y \in \bigcap_{y\in\pi(p)} \pi(p) = \pi(\bigcap\{p|y \in \pi(p)\})$$

Suppose $\bigcap\{p|y \in \pi(p)\} \subseteq \hat{p}$. Since $\pi \in P(X) \to_M Q(Y)$ it is monotone, thus we have

$$y \in \pi(\bigcap\{p|y \in \pi(p)\}) \subseteq \pi(\hat{p}),$$

that is $y \in \pi(\hat{p})$.

□

Proof of Lemma 5.5

1. $\Rightarrow$) If $y \in \pi(\hat{p})$ then $\hat{p} \in \{p|y \in \pi(p)\}$ and hence

$$\bigcap\{p|y \in \pi(p)\} \cap \bigcap\{q|y \in \rho(q)\} \subseteq \bigcap\{p|y \in \pi(p)\} \subseteq \hat{p}.$$

$\Leftarrow$) Since π, ρ are multiplicative we have

$$y \in \bigcap_{y\in\pi(p)} \pi(p) = \pi(\bigcap\{p|y \in \pi(p)\})$$

and also

$$y \in \bigcap_{y\in\rho(q)} \rho(q) = \rho(\bigcap\{q|y \in \rho(q)\}).$$

Suppose $\bigcap\{p|y \in \pi(p)\} \cap \bigcap\{q|y \in \rho(q)\} \subseteq \hat{p}$, then

$$\bigcap\{p|y \in \pi(p)\} \cap \bigcap\{q|y \in \rho(q)\} \subseteq \hat{p} \cap \bigcap\{q|y \in \rho(q)\}.$$

But $(\pi, \rho) \in (P_1(X) \to_M Q_1(Y)) \otimes (P_2(X) \to_M Q_2(Y))$, thus we have

$$y \in \pi(\bigcap\{p|y \in \pi(p)\}) \cap \rho(\bigcap\{q|y \in \rho(q)\}) \subseteq \pi(\hat{p}) \cap \rho(\bigcap\{q|y \in \rho(q)\}),$$

that is $y \in \pi(\hat{p})$.

2. Similar as above.

□

Next we prove the Theorem 5.1. Other lemmas, necessary for this proof, are introduced and proved.

Proof of Theorem 5.1

5.1.1: $X \to \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_d(X)$.

We need the following lemmas:

Lemma A.1 *For every $m \in X \to \mathcal{H}(Y)$ the function $\gamma(m)$ is top preserving and multiplicative.*

Proof: The function $\gamma(m)$ is top preserving since

$$\gamma(m)(Y) = \{x|m(x) \subseteq Y\} = X.$$

Let now $I \neq \emptyset$ be an arbitrary index set and for every $i \in I$ let $c_i \in \mathbf{C}_{Sc}(Y)$. If $x \in \bigcap_I \gamma(m)(c_i)$ then $x \in \gamma(m)(c_i)$, and hence $m(x) \subseteq c_i$, for every $i \in I$. But then $m(x) \subseteq \bigcap_I c_i$ and hence $x \in \gamma(m)(\bigcap_I c_i)$.

If, instead, $x \in \gamma(m)(\bigcap_I c_i)$, then $m(x) \subseteq \bigcap_I c_i \subseteq c_i$ for every $i \in I$. Thus $x \in \gamma(m)(c_i)$ for every $i \in I$, and hence $x \in \bigcap_I \gamma(m)(c_i)$. □

Lemma A.2 *The function γ is monotone.*

Proof: Let $m_1 \sqsubseteq_H m_2$, that is $m_1(x) \subseteq m_2(x)$ for every $x \in X$. Thus, for every $c \in \mathbf{C}_{Sc}(Y)$ if $x \in \gamma(m_2)(c)$ then $m_1(x) \subseteq m_2(x) \subseteq c$. Therefore for every $c \in \mathbf{C}_{Sc}(Y)$

$$\gamma(m_2)(c) = \{x|m_2(x) \subseteq c\} \subseteq \{x|m_1(x) \subseteq c\} = \gamma(m_1)(c),$$

□

Lemma A.3 *For every $\rho \in \mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_d(X)$ and for every $x \in X$ the function $\gamma^{-1}(\rho)(x) \in \mathcal{H}(Y)$.*

Proof: Clear, because the intersection of Scott-closed sets is again a Scott-closed set. □

Lemma A.4 *The function γ^{-1} is monotone.*

Proof: Let $\rho_1, \rho_2 \in \mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_d(X)$ such that $\rho_1(c) \supseteq \rho_2(c)$ for every $c \in \mathbf{C}_{Sc}(Y)$. Then $\{c|x \in \rho_1(c)\} \supseteq \{c|x \in \rho_2(c)\}$ for every $x \in X$. Therefore

$$\gamma^{-1}(\rho_1)(x) = \bigcap\{c|x \in \rho_1(c)\} \subseteq \bigcap\{c|x \in \rho_2(c)\} = \gamma^{-1}(\rho_2)(x),$$

that is $\gamma^{-1}(\rho_1) \sqsubseteq_H \gamma^{-1}(\rho_2)$. □

Finally we have the required isomorphism:

Theorem A.5 *The function γ is an isomorphism of partial order between $X \rightarrow \mathcal{H}(Y)$ and $\mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_d(X)$ with inverse the function γ^{-1}.*

Proof: We have shown that γ and γ^{-1} are monotone, thus we have to prove they are inverses of each other.

- $\gamma^{-1} \circ \gamma = id_{(X \rightarrow \mathcal{H}(Y))}$)

  Let $m \in X \rightarrow \mathcal{H}(Y)$. Then

$$
\begin{aligned}
& \gamma^{-1}(\gamma(m))(x) \\
= & \{ \text{ definition } \gamma^{-1} \} \\
& \bigcap\{c | x \in (\gamma(m))(c)\} \\
= & \{ \text{ definition } \gamma \} \\
& \bigcap\{c | m(x) \subseteq c\} \\
= & \{ \ m(x) \in \mathcal{H}(Y) \equiv \mathbf{C}_{Sc}(Y) \ \} \\
& m(x)
\end{aligned}
$$

- $\gamma \circ \gamma^{-1} = id_{(\mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_d(X))}$)

  Let $\rho \in \mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_d(X)$. Then

$$
\begin{aligned}
& \gamma(\gamma^{-1}(\rho))(\hat{c}) \\
= & \{ \text{ definition } \gamma \} \\
& \{x | \gamma^{-1}(\rho)(x) \subseteq \hat{c}\} \\
= & \{ \text{ definition } \gamma^{-1} \} \\
& \{x | \bigcap\{c | x \in \rho(c)\} \subseteq \hat{c}\} \\
= & \{ \text{ stability lemma 5.3 } \} \\
& \{x | x \in \rho(\hat{c})\} \\
= & \\
& \rho(\hat{c}).
\end{aligned}
$$

□

5.1.2: $X \rightarrow \mathcal{H}^+(Y) \cong \mathbf{C}_{Sc}(Y) \rightarrow_{stM} \mathbf{C}_d(X)$.

It is enough to prove the following two lemmas:

Lemma A.6 *Let $m \in X \rightarrow \mathcal{H}^+(Y)$. Then $\gamma(m)(\emptyset) = \emptyset$.*

Proof: By an easy calculation we have

$$\gamma(m)(\emptyset) = \{x | m(x) \subseteq \emptyset\} = \emptyset.$$

because $m(x) \neq \emptyset$ for every $x \in X$. $\square$

Lemma A.7 *Let $\rho \in \mathbf{C}_{Sc}(Y) \rightarrow_{stM} \mathbf{C}_d(X)$. Then $\gamma^{-1}(\rho)(x) \neq \emptyset$ for every $x \in X$.*

Proof: Assume $\gamma^{-1}(\rho)(x) = \bigcap\{c|x \in \rho(c)\} = \emptyset$. Then we get following contradiction:

$$x \in \rho(\bigcap\{c|x \in \pi(c)\}) = \rho(\emptyset) = \emptyset.$$

$\square$

5.1.3: $X \rightarrow_m \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_{Al}(X)$.

We need to prove the following two lemmas:

Lemma A.8 *Let $m \in X \rightarrow_m \mathcal{H}(Y)$. Then $\gamma(m)(c) \in \mathbf{C}_{Al}(X)$ for every $c \in \mathbf{C}_{Sc}(Y)$.*

Proof: Let $x_2 \in \gamma(m)(c)$ and $x_1 \sqsubseteq x_2$. Then $m(x_1) \sqsubseteq_H m(x_2)$, that is, $m(x_1) \subseteq m(x_2) \subseteq c$, thus $x_1 \in \gamma(m)(c)$. Therefore $\gamma(m)(c)$ is lower closed. $\square$

Lemma A.9 *Let $\rho \in \mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_{Al}(X)$. Then $\gamma^{-1}(\rho)$ is monotone.*

Proof: Let $x_1 \sqsubseteq x_2$ and assume $x_2 \in \rho(c)$. Then $x_1 \in x_2 \downarrow \subseteq \rho(c) \in \mathbf{C}_{Al}(X)$. Thus $\{c|x_2 \in \rho(c)\} \subseteq \{c|x_1 \in \rho(c)\}$ and hence

$$\gamma^{-1}(\rho)(x_2) = \bigcap\{c|x_2 \in \rho(c)\} \supseteq \bigcap\{c|x_1 \in \rho(c)\} = \gamma^{-1}(\rho)(x_1),$$

that is, $\gamma^{-1}(\rho)(x_1) \sqsubseteq_H \gamma^{-1}(\rho)(x_2)$. $\square$

5.1.4: $X \rightarrow_c \mathcal{H}(Y) \cong \mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_{Sc}(X)$.

It is enough to prove the following two lemmas:

Lemma A.10 *Let $m \in X \rightarrow_c \mathcal{H}(Y)$. Then $\gamma(m)(c) \in \mathbf{C}_{Sc}(X)$ for every $c \in \mathbf{C}_{Sc}(Y)$.*

Proof: If m is continuous then it is also monotone, thus $\gamma(m)(c) \in \mathbf{C}_{Al}(X)$ for every $c \in \mathbf{C}_{Sc}(Y)$, and hence is lower closed. Let now $S \subseteq X$ be a directed set and suppose $x_i \in \gamma(m)(c)$ for every $x_i \in S$. Then $m(x_i) \subseteq c$ for every $x_i \in S$ and hence $\bigcup_{x_i \in S} m(x_i) \subseteq c$. Therefore, applying the Scott closure operator we obtain:

$$\overline{\bigcup_{x_i \in S} m(x_i)} = \bigsqcup_{x_i \in S} m(x_i) = m(\bigsqcup S) \subseteq c$$

because m is continuous, $\overline{(.)}$ is monotone and $c \in \mathbf{C}_{Sc}(Y)$. Thus $\bigsqcup S \in \gamma(m)(c)$ and hence $\gamma(m)(c) \in \mathbf{C}_{Sc}(X)$. $\square$

Lemma A.11 *Let $\rho \in \mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_{Sc}(X)$. Then $\gamma^{-1}(\rho)$ is continuous.*

Proof: Since $\rho(c) \in \mathbf{C}_{Sc}(X) \subseteq \mathbf{C}_{Al}(X)$ we have $\gamma^{-1}(\rho)$ monotone. Thus

$$\bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i) \sqsubseteq_H \gamma^{-1}(\rho)(\bigsqcup S),$$

for every directed set $S \subseteq X$. Appling the stability lemma 5.3 we have $x_i \in \rho(\bigcap\{c | x_i \in \rho(c)\}) = \rho(\gamma^{-1}(\rho)(x_i))$ for every $x_i \in S$. But

$$\gamma^{-1}(\rho)(x_i) \subseteq \bigcup_{x_i \in S} \gamma^{-1}(\rho)(x_i) \subseteq \overline{\bigcup_{x_i \in S} \gamma^{-1}(\rho)(x_i)} = \bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i) \in \mathbf{C}_{Sc}(Y),$$

and thus, because ρ is multiplicative, and hence monotone, we have

$$x_i \in \rho(\gamma^{-1}(\rho)(x_i)) \subseteq \rho(\bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i)),$$

for every $x_i \in S$. Since $\rho(c) \in \mathbf{C}_{Sc}(X)$ for every $c \in \mathbf{C}_{Sc}(Y)$, we obtain

$$\forall x_i \in S : x_i \in \rho(\bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i)) \;\Rightarrow\; \bigsqcup S \in \rho(\bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i)).$$

Thus, applying again the stability lemma 5.3 we have

$$\gamma^{-1}(\rho)(\bigsqcup S) = \bigcap\{c | \bigsqcup S \in \rho(c)\} \subseteq \bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i),$$

that is $\gamma^{-1}(\rho)(\bigsqcup S) \sqsubseteq_H \bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i)$. Therefore $\bigsqcup_{x_i \in S} \gamma^{-1}(\rho)(x_i) = \gamma^{-1}(\rho)(\bigsqcup S)$. □

Next we prove Theorem 5.2 introducing other Lemmas when necessary.

Proof of the Theorem 5.2

5.2.1: $X \to \mathcal{S}(Y) \cong \mathbf{O}_{Al}(Y) \to_{tM} \mathbf{O}_d(X)$.

We need the following lemmas:

Lemma A.12 *For every $m \in X \to \mathcal{S}(Y)$ the function $\omega(m)$ is top preserving and multiplicative.*

Proof: The function $\omega(m)$ is top preserving since

$$\omega(m)(Y) = \{x | m(x) \subseteq Y\} = X.$$

Let now $I \neq \emptyset$ be an arbitrary index set and for every $i \in I$ let $o_i \in \mathbf{O}_{Al}(Y)$. If $x \in \bigcap_I \gamma(m)(o_i)$ then $x \in \omega(m)(o_i)$, and hence $m(x) \subseteq o_i$, for every $i \in I$. But then $m(x) \subseteq \bigcap_I o_i$ and hence $x \in \omega(m)(\bigcap_I o_i)$.

If, instead, $x \in \omega(m)(\bigcap_I o_i)$, then $m(x) \subseteq \bigcap_I o_i \subseteq o_i$ for every $i \in I$. Thus $x \in \omega(m)(o_i)$ for every $i \in I$, and hence $x \in \bigcap_I \omega(m)(o_i)$. □

Lemma A.13 *The function ω is monotone.*

Proof: Let $m_1 \sqsubseteq_S m_2$, that is $m_1(x) \supseteq m_2(x)$ for every $x \in X$. Thus, for every $o \in \mathbf{O}_{Al}(Y)$ if $x \in \omega(m_1)(o)$ then $m_2(x) \subseteq m_1(x) \subseteq o$. Therefore for every $o \in \mathbf{O}_{Al}(Y)$

$$\omega(m_1)(o) = \{x | m_1(x) \subseteq o\} \subseteq \{x | m_2(x) \subseteq c\} = \omega(m_2)(o),$$

□

Lemma A.14 *For every $\pi \in \mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_d(X)$ and for every $x \in X$ the function $\omega^{-1}(\pi)(x) \in \mathcal{S}(Y)$.*

Proof: It is enough to prove $\omega^{-1}(\pi)(x)$ is upper closed. Let $y_1 \sqsubseteq_Y y_2 \in Y$ with $y_1 \in \omega^{-1}(\pi)(x)$. Then $y_1 \in \bigcap\{o | x \in \pi(o)\}$ and hence for every $o \in \mathbf{O}_{Al}(Y)$ if $x \in \pi(o)$ then $y_1 \in o$. But $o \in \mathbf{O}_{Al}(Y)$, thus if $x \in \pi(o)$ then $y_2 \in y_1 \uparrow \subseteq o$. Therefore $y_2 \in \bigcap\{o | x \in \pi(o)\} = \omega^{-1}(\pi)(x)$. □

Lemma A.15 *The function ω^{-1} is monotone.*

Proof: Let $\pi_1, \pi_2 \in \mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_d(X)$ such that $\pi_1(o) \subseteq \pi_2(o)$ for every $o \in \mathbf{O}_{Al}(Y)$. Then $\{o | x \in \pi_1(o)\} \subseteq \{o | x \in \pi_2(o)\}$ for every $x \in X$. Therefore

$$\omega^{-1}(\pi_1)(x) = \bigcap\{o | x \in \pi_1(o)\} \supseteq \bigcap\{o | x \in \pi_2(o)\} = \omega^{-1}(\pi_2)(x),$$

that is $\omega^{-1}(\pi_1) \sqsubseteq_S \omega^{-1}(\pi_2)$. □

Theorem A.16 *The function ω is an isomorphism of partial orders between $X \rightarrow \mathcal{S}(Y)$ and $\mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_d(X)$ with inverse the function ω^{-1}.*

Proof: We have already proved ω and ω^{-1} monotone, thus we have to prove they are inverses of each other.

- $\omega^{-1} \circ \omega = id_{(X \rightarrow \mathcal{S}(Y))}$)

  Let $m \in X \rightarrow \mathcal{S}(Y)$. Then

  $\omega^{-1}(\omega(m))(x)$

  $=$ { definition of ω^{-1} }

  $\bigcap\{o | x \in (\omega(m))(o)\}$

  $=$ { definition of ω }

  $\bigcap\{o | m(x) \subseteq o\}$

  $=$ { $m(x) \in \mathcal{S}(Y)$ which is equivalent to $\mathbf{O}_{Al}(Y)$ with the reversed order }

  $m(x)$.

- $\omega \circ \omega^{-1} = id_{(\mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_d(X))}$)

  Let $\pi \in \mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_d(X)$. Then

  $\omega(\omega^{-1}(\pi))(\hat{o})$

  $=$ { definition of ω }

  $\{x | \omega^{-1}(\pi)(x) \subseteq \hat{o}\}$

  $=$ { definition of ω^{-1} }

$$\{x \mid \bigcap\{o \mid x \in \pi(o)\} \subseteq \hat{o}\}$$

= { stability lemma 5.3 }

$$\{x \mid x \in \pi(\hat{o})\}$$

=

$$\pi(\hat{o}).$$

□

5.2.2: $X \to \mathcal{S}^+(Y) \cong \mathbf{O}_{Al}(Y) \to_{stM} \mathbf{O}_d(X)$.

It is enough to prove the following two lemmas:

Lemma A.17 *Let $m \in X \to \mathcal{S}^+(Y)$. Then $\omega(m)(\emptyset) = \emptyset$.*

Proof: By an easy calculation we have

$$\omega(m)(\emptyset) = \{x \mid m(x) \subseteq \emptyset\} = \emptyset.$$

because $m(x) \neq \emptyset$ for every $x \in X$. □

Lemma A.18 *Let $\pi \in \mathbf{O}_{Al}(Y) \to_{stM} \mathbf{O}_d(X)$. Then $\omega^{-1}(\pi)(x) \neq \emptyset$ for every $x \in X$.*

Proof: Assume $\omega^{-1}(\pi)(x) = \bigcap\{o \mid x \in \pi(o)\} = \emptyset$. Then we get following contradiction:

$$x \in \pi(\bigcap\{o \mid x \in \pi(o)\}) = \pi(\emptyset) = \emptyset.$$

□

5.2.3: $X \to \mathcal{S}_{comp}(Y) \cong \mathbf{O}_{Al}(Y) \to_{\hat{c}tM} \mathbf{O}_d(X)$.

It is enough to prove the following two lemmas:

Lemma A.19 *Let $m \in X \to \mathcal{S}_{comp}(Y)$ and let $S \subseteq \mathbf{O}_{Sc}(Y)$ be a directed set. Then $\omega(m)(\bigcup S) = \bigcup_{o \in S} \omega(m)(o)$.*

Proof: Let $S \subseteq \mathbf{O}_{Sc}(Y)$ be a directed set. Since $\omega(m)$ is multiplicative, it is also monotone, and we have

$$\bigcup_{o \in S} \omega(m)(o) \subseteq \omega(m)(\bigcup S).$$

Consider $x \in \omega(m)(\bigcup S)$, that is $m(x) \subseteq \bigcup S$. As $m(x)$ is compact in $\mathbf{O}_{Sc}(Y)$ there exists a finite subset $S' \subseteq S$ such that $m(x) \subseteq \bigcup S'$. But S is directed, thus for every finite $S' \subseteq S$ we have $\bigcup S' \in S$, that means there exists $o \in S$ such that $\bigcup S' \subseteq o$. Hence $m(x) \subseteq o$ and $x \in \bigcup_{o \in S} \omega(m)(o)$. □

To prove the converse we need the following theorem which relates open filters with compactness (see [HM81] and also [Smy83, Smy]):

Theorem A.20 *Let P be an algebraic complete partial order. If $\mathcal{F} \subseteq \mathbf{O}(X)$ is an open filter of $\mathbf{O}(X)$ then $\bigcap \mathcal{F}$ is compact in $\mathbf{O}(X)$.*

Lemma A.21 *Let $\pi \in \mathbf{O}_{Al}(Y) \rightarrow_{\hat{c}tM} \mathbf{O}_d(X)$. Then $\omega^{-1}(\pi)(x)$ is compact in $\mathbf{O}_{Sc}(Y)$.*

Proof: We prove first $\{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$ is an open filter:

1. Let $o_1 \in \{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$ and $o_1 \subseteq o_2 \in \mathbf{O}_{Sc}(Y)$. Then $x \in \pi(o_1) \subseteq \pi(o_2)$ because π is multiplicative, and hence monotone. Thus $o_2 \in \{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$.
2. Let $o_1, o_2 \in \{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$. Then $x \in \pi(o_1)$ and $x \in \pi(o_2)$, that is $x \in \pi(o_1) \cap \pi(o_2) = \pi(o_1 \cap o_2)$, because π is multiplicative. Thus $o_1 \cap o_2 \in \{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$.
3. By 1., $\{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$ is upper closed. If S is a directed set in $\mathbf{O}_{Sc}(Y)$ such that $\bigcup S \in \{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$, then $x \in \pi(\bigcup S) = \bigcup_{o \in S} \pi(o)$ since π is continuous with respect to directed sets in $\mathbf{O}_{Sc}(Y)$. Therefore $x \in \pi(o)$ for some $o \in S$, that is, $o \in \{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$.

Thus $\{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$ is an open filter, and hence by theorem A.20 we have $\bigcap\{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$ is Scott compact. Now we prove also $\omega^{-1}(\pi)(x) = \bigcap\{o \in \mathbf{O}_{Al}(Y) | x \in \pi(o)\}$ is Scott compact. Assume $\omega^{-1}(\pi)(x) \subseteq \bigcup i \in I o_i$ for $o_i \in \mathbf{O}_{Sc}(Y)$, that is $\bigcup i \in I o_i$ is a Scott cover of $\omega^{-1}(\pi)(x)$. Then by stability lemma 5.3 we have $x \in \pi(\bigcup i \in I o_i)$, and hence $\bigcup i \in I o_i \in \{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$. But $\{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\} \subseteq \{o \in \mathbf{O}_{Al}(Y) | x \in \pi(o)\}$ and hence

$$\bigcap\{o \in \mathbf{O}_{Al}(Y) | x \in \pi(o)\} \subseteq \bigcap\{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\} \subseteq \bigcup i \in I o_i.$$

Further, $\bigcap\{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\}$ is Scott compact, thus there exists a finite $J \subseteq I$ such that

$$\bigcap\{o \in \mathbf{O}_{Al}(Y) | x \in \pi(o)\} \subseteq \bigcap\{o \in \mathbf{O}_{Sc}(Y) | x \in \pi(o)\} \subseteq \bigcup i \in J o_i.$$

This proves $\omega^{-1}(\pi)(x)$ is Scott compact. □

5.2.4: $X \rightarrow_m \mathcal{S}(Y) \cong \mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_{Al}(X)$.

It is enough to prove the following two lemmas:

Lemma A.22 *Let $m \in X \rightarrow_m \mathcal{S}(Y)$. Then $\omega(m)(o) \in \mathbf{O}_{Al}(X)$ for every $o \in \mathbf{O}_{Al}(Y)$.*

Proof: Let $x_1 \in \omega(m)(o)$ and $x_1 \sqsubseteq x_2$. Then $m(x_1) \sqsubseteq_S m(x_2)$, that is, $m(x_2) \subseteq m(x_1) \subseteq o$, thus $x_2 \in \omega(m)(o)$. Therefore $\omega(m)(o)$ is upper-closed. □

Lemma A.23 *Let $\pi \in \mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_{Al}(X)$. Then $\omega^{-1}(\pi)$ is monotone.*

Proof: Let $x_1 \sqsubseteq x_2$ and assume $x_1 \in \pi(o)$. Then $x_2 \in x_1 \uparrow \subseteq \pi(o) \in \mathbf{O}_{Al}(X)$. Thus $\{o | x_1 \in \pi(o)\} \subseteq \{o | x_2 \in \pi(o)\}$ and hence

$$\omega^{-1}(\pi)(x_1) = \bigcap\{o | x_1 \in \pi(o)\} \supseteq \bigcap\{o | x_2 \in \pi(o)\} = \omega^{-1}(\pi)(x_2),$$

that is, $\omega^{-1}(\pi)(x_1) \sqsubseteq_S \omega^{-1}(\pi)(x_2)$. □

5.2.5: $X \rightarrow_{c_s} \mathcal{S}(Y) \cong \mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_{Sc}(X)$.

It is enough to prove the following two lemmas:

Lemma A.24 *Let $m \in X \to_{c_s} \mathcal{S}(Y)$. Then $\omega(m)(o) \in \mathbf{O}_{Sc}(X)$ for every $o \in \mathbf{O}_{Al}(Y)$.*

Proof: If m is continuous and stabilizing, then it is also monotone, thus $\omega(m)(o) \in \mathbf{O}_{Al}(X)$ for every $o \in \mathbf{O}_{Al}(Y)$. Let now $S \subseteq X$ be a directed set and $\bigsqcup S \in \omega(m)(o)$. Then $m(\bigsqcup S) = \bigsqcup_{x_i \in S} m(x_i) \subseteq o$ because m is continuous. But it is also stabilizing, thus there exists $x_k \in S$ such that $m(\bigsqcup S) = \bigsqcup_{x_i \in S} m(x_i) = m(x_k) \subseteq o$, that is, $x_k \in \omega(m)(o)$. This proves $\omega(m)(o) \in \mathbf{O}_{Sc}(X)$. □

The following example shows the condition of m to be stabilizing is necessary in order to have $\omega(m)(o) \in \mathbf{O}_{Sc}(X)$ for $o \in \mathbf{O}_{Al}(X)$:

Example: Let $X = \{x_i | i \in N\} \cup \{x_\omega\}$ ordered by $x_i \sqsubseteq_X x_j$ if and only if $i \leq j$ or $j = \omega$, and let $Y = \{y_i | i \in N\} \cup \{y_\omega, \bot\}$ ordered by $y_i \sqsubseteq_X x_j$ if and only if $y_i = \bot$ or $y_i = y_j$. Define $m : X \to_c \mathcal{S}(Y)$ by

$$\begin{aligned} m(x_\omega) &= \{y_\omega\} \\ m(x_n) &= \{y_\omega\} \cup \{y_i | i \in N \setminus \{0, ..., n\}\}, \end{aligned}$$

for every $n \in N$. Then

$$m(x_0) \supseteq m(x_1) \supseteq ... \supseteq m(x_\omega) = m(\bigsqcup x_i) = \bigsqcup m(x_i).$$

Thus m is continuous, but not stabilizing. Consider $\{y_\omega\} \in \mathbf{O}_{Al}(Y)$ and we have:

$$\omega(m)(\{y_\omega\}) = \{x | m(x) \subseteq \{y_\omega\}\} = \{x_\omega\} \notin \mathbf{O}_{Sc}(X),$$

even if $\{x_\omega\} \in \mathbf{O}_{Al}(X)$.

Lemma A.25 *Let $\pi \in \mathbf{O}_{Al}(Y) \to_{tM} \mathbf{O}_{Sc}(X)$. Then $\omega^{-1}(\pi)$ is continuous and stabilizing.*

Proof: Since $\pi(o) \in \mathbf{O}_{Sc}(X) \subseteq \mathbf{O}_{Al}(X)$ we have $\omega^{-1}(\pi)$ monotone. Thus

$$\bigsqcup_{x_i \in S} \omega^{-1}(\pi)(x_i) \sqsubseteq_S \omega^{-1}(\pi)(\bigsqcup S),$$

for every directed set $S \subseteq X$. Consider $y \in \bigsqcup_{x_i \in S} \omega^{-1}(\pi)(x_i) = \bigcap \omega^{-1}(\pi)(x_i)$ then

$$\forall x_i \in S, \forall o \in \mathbf{O}_{Al}(Y) : x_i \in \pi(o) \Rightarrow y \in o.$$

But if $\bigsqcup S \in \pi(o)$ then there exists $x_k \in S$ such that $x_k \in \pi(o)$ because it is Scott open. Hence by above we have:

$$\forall o \in \mathbf{O}_{Al}(Y) : \bigsqcup S \in \pi(o) \Rightarrow \exists x_k \in \pi(o) \Rightarrow y \in o.$$

Hence $y \in \bigcap \{o | \bigsqcup S \in \pi(o)\} = \omega^{-1}(\pi)(\bigsqcup S)$, that means $\omega^{-1}(\pi)(\bigsqcup S) \supseteq \bigsqcup_{x_i \in S} \omega^{-1}(\pi)(x_i)$, or equivalently $\omega^{-1}(\pi)(\bigsqcup S) \sqsubseteq_S \bigsqcup_{x_i \in S} \omega^{-1}(\pi)(x_i)$. It remains to prove that $\omega^{-1}(\pi)$ stabilizes for every directed set $S \subseteq X$. By stability lemma 5.3 we have $\bigsqcup S \in \pi(\bigcap \{o | \bigsqcup S \in \pi(o)\})$. Hence there exists $x_k \in S$ such that $x_k \in \pi(\bigcap \{o | \bigsqcup S \in \pi(o)\})$ because it is Scott open. Hence again by stability lemma 5.3 we obtain

$$x_k \in \pi(\bigcap \{o | \bigsqcup S \in \pi(o)\}) \Rightarrow \bigcap \{o | x_k \in \pi(o)\} \subseteq \bigcap \{o | \bigsqcup S \in \pi(o)\},$$

that is, $\omega^{-1}(\pi)(\bigsqcup S) \sqsubseteq_S \omega^{-1}(\pi)(x_k)$ for some $x_k \in S$. But $\omega^{-1}(\pi)$ is monotone, and $x_k \sqsubseteq \bigsqcup S$, thus $\omega^{-1}(\pi)(x_k) \sqsubseteq_S \omega^{-1}(\pi)(\bigsqcup S)$ and hence $\omega^{-1}(\pi)(x_k) = \omega^{-1}(\pi)(\bigsqcup S)$. □

Finally we prove only the first item of the Theorem 5.4. The other items are a combination of the results of the Theorem 5.1 and Theorem 5.2 and are left to the reader.

Proof of Theorem 5.4

In this part of the appendix we only prove only the first item of the Theorem 5.4. The other items are left to the reader and are just a combination of the results of the Theorem 5.1 and Theorem 5.2. Other lemmas, necessary for this proof, are introduced.

5.4.1: $X \to \mathcal{P}(Y) \cong (\mathbf{O}_{Al}(Y) \to_{tM} \mathbf{O}_d(X)) \otimes (\mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_d(X))$

We need the following lemmas:

Lemma A.26 *Let $m : X \to \mathcal{P}(Y)$. Then*

1. $\eta_1(m) \in \mathbf{O}_{Al}(Y) \to_{tM} \mathbf{O}_d(X)$,
2. $\eta_2(m) : \mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_d(X)$,
3. *for every $o_1, o_2 \in \mathbf{O}_{Al}(Y)$ and for every $c_1, c_2 \in \mathbf{C}_{Sc}(Y)$ we have* $o_1 \cap c_1 \subseteq o_2 \cap c_2 \;\Rightarrow\; \eta_1(m)(o_1) \cap \eta_2(m)(c_1) \subseteq \eta_1(m)(o_2) \cap \eta_2(m)(c_2)$.

Proof:

1. Similar to Lemma A.12.
2. Similar to Lemma A.1.
3. Let $o_1, o_2 \in \mathbf{O}_{Al}(Y)$ and $c_1, c_2 \in \mathbf{C}_{Sc}(Y)$ such that $o_1 \cap c_1 \subseteq o_2 \cap c_2$. Then

$$\eta_1(m)(o_1) \cap \eta_2(m)(c_1)$$
$$= \{\text{definition of } \eta = (\eta_1, \eta_2)\ \}$$
$$\{x | m(x) \uparrow \subseteq o_1\} \cap \{x | \overline{m(x)} \subseteq c_1\}$$
$$=$$
$$\{x | m(x) \uparrow \subseteq o_1 \ \wedge \ \overline{m(x)} \subseteq c_1\}$$
$$=$$
$$\{x | m(x) \uparrow \cap \overline{m(x)} \subseteq o_1 \cap c_1\}$$
$$=$$
$$\{x | m(x) \subseteq o_1 \cap c_1\}.$$

But $m(x) \subseteq o_1 \cap c_1 \subseteq o_2 \cap c_2 \subseteq o_2$ implies $m(x) \uparrow \subseteq o_2$ and also $m(x) \subseteq o_1 \cap c_1 \subseteq o_2 \cap c_2 \subseteq c_2$ implies $\overline{m(x)} \subseteq c_2$, thus $\{x | m(x) \subseteq o_1 \cap c_1\} \subseteq \{x | m(x) \uparrow \subseteq o_2\}$ and also $\{x | m(x) \subseteq o_1 \cap c_1\} \subseteq \{x | \overline{m(x)} \subseteq c_2\}$, that is

$$\eta_1(m)(o_1) \cap \eta_2(m)(o_2)$$
$$=$$
$$\{x | m(x) \subseteq \leq_1 \cap c_1\}$$
$$\subseteq$$
$$\{x | m(x) \uparrow \subseteq o_2\} \cap \{x | \overline{m(x)} \subseteq c_2\}$$
$$=$$

$\eta_1(m)(o_2) \cap \eta_2(m)(c_2)$.

$\square$

Lemma A.27 *The function η is strictly monotone.*

Proof: Let us prove first η is monotone. Let $m_1 \sqsubseteq_P m_2$, that is $m_1(x)\uparrow \supseteq m_2(x)\uparrow$ and $\overline{m_1(x)} \subseteq \overline{m_2(x)}$. Thus, for every $o \in \mathbf{O}_{Al}(Y)$ we have

$$\eta_1(m_1)(o) = \{x | m_1(x)\uparrow \subseteq o\} \subseteq \{x | m_2(x)\uparrow \subseteq o\} = \eta_1(m_2)(o),$$

and for every $c \in \mathbf{C}_{Sc}(Y)$

$$\eta_2(m_2)(c) = \{x | \overline{m_2(x)} \subseteq c\} \subseteq \{x | \overline{m_1(x)} \subseteq c\} = \eta_2(m_1)(c),$$

that is $(\eta_1(m_1), \eta_2(m_1)) \sqsubseteq_N (\eta_1(m_2), \eta_2(m_2))$.
Suppose now $m_1 \not\sqsubseteq_P m_2$. Then there exist an $x \in X$ such that $m_1(x)\uparrow \not\supseteq m_2(x)\uparrow$ or $\overline{m_1(x)} \not\subseteq \overline{m_2(x)}$. But $m_1(x)\uparrow \in \mathbf{O}_{Al}(Y)$ and $\overline{m_2(x)} \in \mathbf{C}_{Sc}(Y)$, thus we have

$$\eta_1(m_1)(m_1(x)\uparrow) = \{x | m_1(x)\uparrow \subseteq m_1(x)\uparrow\} \not\subseteq \{x | m_2(x)\uparrow \subseteq m_1(x)\uparrow\} = \eta_1(m_2)(m_1(x)\uparrow),$$

or

$$\eta_2(m_2)(\overline{m_2(x)}) = \{x | \overline{m_2(x)} \subseteq \overline{m_2(x)}\} \not\subseteq \{x | \overline{m_1(x)} \subseteq \overline{m_2(x)}\} = \eta_2(m_1)(\overline{m_2(x)}).$$

$\square$

Lemma A.28 *For every $(\pi, \rho) \in \mathbf{O}_{Al}(Y) \rightarrow_{tM} \mathbf{O}_d(X) \otimes (\mathbf{C}_{Sc}(Y) \rightarrow_{tM} \mathbf{C}_d(X))$ and for every $x \in X$ the function $\eta^{-1}((\pi,\rho))(x) \in \mathcal{P}(Y)$.*

Proof: We have to prove that $\eta^{-1}((\pi,\rho))(x)$ is $*$-closed. By definition of * we have

$$\eta^{-1}((\pi,\rho))(x) \subseteq (\eta^{-1}((\pi,\rho))(x))^*.$$

Let $y \in (\eta^{-1}((\pi,\rho))(x))^*$. Then by definition

$$(\exists \hat{y} \in \eta^{-1}((\pi,\rho))(x) : \hat{y} \sqsubseteq y) \ \wedge \ (\forall b \in B_Y : b \sqsubseteq Y \Rightarrow \exists y_b \in \eta^{-1}((\pi,\rho))(x) : b \sqsubseteq y_b).$$

Using the definition of $\eta^{-1}((\pi,\rho))(x)$ we obtain

$$\forall o \in \mathbf{O}_{Al}(Y) : x \in \pi(o) \Rightarrow \hat{y} \in o,$$

and also

$$\forall c \in \mathbf{C}_{Sc}(Y) : x \in \rho(c) \Rightarrow y_b \in c.$$

But $y \in \hat{y}\uparrow \subseteq o$ as $o \in \mathbf{O}_{Al}(Y)$ and hence

$$\forall o \in \mathbf{O}_{Al}(Y) : x \in \pi(o) \Rightarrow y \in o,$$

that is $y \in \bigcap\{o|x \in \pi(o)\}$. Moreover $b' \sqsubseteq y_b$ implies $b' \in y_b \downarrow\subseteq c$ because $c \in \mathbf{C}_{Sc}(Y)$, hence

$$\forall c \in \mathbf{C}_{Sc}(Y) : x \in \rho(c) \ \wedge \ b \sqsubseteq y \Rightarrow b \in c.$$

But Y is an algebraic dcpo, hence $y = \bigsqcup\{b|b \in B_Y \wedge b \sqsubseteq_Y y\}$ and all these finite elements are elements of c. Since c is Scott closed we obtain $y = \bigsqcup\{b|b \in B_Y \ \wedge \ b \sqsubseteq_Y y\} \in c$. This means $y \in \bigcap\{c|x \in \rho(c)\}$. Therefore

$$y \in \bigcap\{o|x \in \pi(o)\} \cap \bigcap\{c|x \in \rho(c)\} = \eta^{-1}(x).$$

□

Lemma A.29 *For every $A \subseteq Y, o \in \mathbf{O}_{Al}(Y)$ and $c \in \mathbf{C}_{Sc}(Y)$ we have*

1. $A \uparrow\subseteq o \Leftrightarrow A \subseteq o$,
2. $\overline{A} \subseteq c \Leftrightarrow A \subseteq c$.

□

Proof: Trivial.

Finally we have the required isomorphism:

Theorem A.30 *The function η is an isomorphism of partial order between $X \to \mathcal{P}(Y)$ and $(\mathbf{O}_{Al}(Y) \to_{tM} \mathbf{O}_d(X)) \otimes (\mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_d(X))$ with inverse the function η^{-1}.*

Proof: We have already proved η strictly monotone thus we have to prove that η^{-1} is the inverse of $\eta = (\eta_1, \eta_2)$.

- $\eta^{-1} \circ \eta = id_{(X \to \mathcal{P}(Y))}$)

  Let $m \in X \to \mathcal{P}(Y)$. Then

  $\eta^{-1}(\eta_1(m), \eta_1(m))(x)$

  $= \{$ definition of η^{-1} $\}$

  $\bigcap\{o|x \in (\eta_1(m))(o)\} \cap \bigcap\{c|x \in (\eta_2(m))(c)\}$

  $= \{$ definition of η $\}$

  $\bigcap\{o|m(x) \uparrow\subseteq o\} \cap \bigcap\{c|\overline{m(x)} \subseteq c\}$

  $=$

  $m(x) \uparrow \cap \overline{m(x)}$

  $= \{$ because $m(x)$ is *-closed $\}$

  $m(x)$

- $\eta \circ \eta^{-1} = id_{(\mathbf{O}_{Al}(Y) \to_{tM} \mathbf{O}_d(X)) \otimes (\mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_d(X))}$)

  Let $(\pi, \rho) \in (\mathbf{O}_{Al}(Y) \to_{tM} \mathbf{O}_d(X)) \otimes (\mathbf{C}_{Sc}(Y) \to_{tM} \mathbf{C}_d(X))$. Then

  $\eta(\eta^{-1}(\pi, \rho))(\hat{o}, \hat{c})$

  $= \{$ definition of η $\}$

  $(\{x|\eta^{-1}(\pi, \rho)(x) \subseteq \hat{o}\}, \{x|\eta^{-1}(\pi, \rho)(x) \subseteq \hat{c}\})$

$= \{$ definition of η^{-1} $\}$

$$(\{x|(\bigcap\{o|x \in \pi(o)\} \cap \bigcap\{c|x \in \rho(c)\}) \uparrow\subseteq \hat{o}\},$$
$$\{x|\overline{(\bigcap\{o|x \in \pi(o)\} \cap \bigcap\{c|x \in \rho(c)\})} \subseteq \hat{c}\})$$

$= \{$ Lemma A.29 $\}$

$$(\{x|\bigcap\{o|x \in \pi(o)\} \cap \bigcap\{c|x \in \rho(c)\} \subseteq \hat{o}\},$$
$$\{x|\bigcap\{o|x \in \pi(o)\} \cap \bigcap\{c|x \in \rho(c)\} \subseteq \hat{c}\})$$

$= \{$ stability lemma 5.5 $\}$

$$(\{x|x \in \pi(\hat{o})\}, \{x|x \in \pi(\hat{c})\})$$

$=$

$$(\pi(\hat{o}), \pi(\hat{c})).$$

□