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# The Complexity of the Free Space for a Robot Moving Amidst Fat Obstacles \*

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## Abstract

The complexity of motion planning algorithms highly depends on the complexity of the robot's free space. Theoretically, the complexity of the free space can be very high, resulting in bad worst-case time bounds for motion planning algorithms. In practice, the complexity of the free space tends to be much smaller than the worst-case complexity. Motion planning algorithms with a running time that is determined by the complexity of the free space therefore become feasible in practical situations. We show that, under some realistic assumptions, the complexity of the free space of a robot with any fixed number of degrees of freedom moving around in a  $d$ -dimensional Euclidean workspace with fat obstacles is linear in the number of obstacles.

## 1 Introduction

Autonomous robots are one of the ultimate goals in the field of robotics. An autonomous robot must accept high-level descriptions of tasks and execute these tasks without further intervention from its environment. A user of such a robot is only bothered with specifying the tasks; he is not bothered with how these tasks are executed.

An obvious task for an autonomous robot would be to move from one place to another. While moving, the robot must avoid collision with the obstacles that are present. The problem of finding such a collision-free motion is referred to as the *motion planning problem*. (An overview of the state-of-the art in robot motion planning is given in [6].) In this paper, we consider the following version of the general motion planning problem.

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Given a robot  $\mathcal{B}$  moving amidst a collection of obstacles  $\mathcal{E}$ , and an initial placement  $Z_0$  and a desired final placement  $Z_1$  for  $\mathcal{B}$ , find a continuous motion for  $\mathcal{B}$  from  $Z_0$  to  $Z_1$  during which the robot avoids collision with the obstacles, or report that no such motion exists.

A robot  $\mathcal{B}$  moves around in a *workspace*  $W$ . This workspace, or physical space, usually corresponds to the Euclidean space of dimension two or three, since these are the most interesting cases from a practical point of view. The collection of obstacles  $\mathcal{E}$  is a closed subset of  $W$ .

The *configuration space*, i.e., the space of parametric representations of placements of the robot  $\mathcal{B}$ , is usually not the same as the workspace  $W$ . The dimension of the configuration space is determined by the number of degrees of freedom of  $\mathcal{B}$ . As an example, take a line segment (ladder) moving around in the plane. A placement of the ladder can be identified by the position of some reference point on the ladder in the plane and the orientation of the ladder. Hence, the configuration space is three-dimensional in this case.

The set  $FP$  of free placements, or *free space*, is a subset of the configuration space. It consists of the placements of the robot  $\mathcal{B}$  in which  $\mathcal{B}$  does not intersect  $\mathcal{E}$ . The motion planning problem now reduces to the problem of finding a continuous path between an initial configuration and a goal configuration in  $FP$  (i.e., between two points in  $FP$ ). Different techniques exist to find such a path.

The *cell decomposition* approach (see e.g. [2, 4, 5, 7, 12]) partitions the space  $FP$  into a finite number of simple connected cells, such that planning a motion between any two placements within a single cell is straightforward and such that uniform crossing rules can be defined for  $\mathcal{B}$  crossing from one cell into another. Each cell defines a vertex in the *connectivity graph*  $CG$ . Two vertices in  $CG$  are connected by an edge if their corresponding cells share a common boundary allowing direct crossing of the robot  $\mathcal{B}$ . Given the connectivity graph  $CG$ , the problem of motion planning is reduced to a graph problem: find the cells  $C_0$  and  $C_1$  in which the placements  $Z_0$  and  $Z_1$  lie and determine a path in  $CG$  between the vertices corresponding to the cells  $C_0$  and  $C_1$ , or report that no such path exists.

Applications of the cell decomposition technique include an  $\mathcal{O}(n^5)$  algorithm by Schwartz and Sharir [12] for planning the motion of a non-convex polygonal robot  $\mathcal{B}$  moving amidst polygonal obstacles  $\mathcal{E}$  in the plane (with a total number of  $n$  edges). Leven and Sharir [7] present a more efficient version of the algorithm by Schwartz and Sharir for the special case of a ladder moving amidst polygonal obstacles. Their algorithm runs in time  $\mathcal{O}(n^2 \log n)$ . Other results are given by Kedem and Sharir [5] and Avnaim, Boissonnat, and Faverjon [2]. Both algorithms have high worst-case time bounds: they run in roughly quadratic and cubic time respectively. In practical situations, some of the algorithms show better performance though.

A different approach to motion planning is the *retraction method* (see e.g. [9, 10, 13]). In this approach we define a lower dimensional subspace  $FP'$  of  $FP$  and show that there exists a retraction function  $Im$  from  $FP$  to  $FP'$ . The retraction function  $Im$  maps each placement in  $FP$  onto a placement in  $FP'$ . Hence, motion planning in  $FP$  is reduced to motion planning in  $FP'$ . By repeated retractions we obtain a one-dimensional network

$N \subseteq FP$ . Motion planning is again reduced to graph searching if we represent the one-dimensional network  $N$  as a graph.

Ó'Dúnlaing and Yap [9] and Ó'Dúnlaing, Sharir, and Yap [10] present algorithms for planning the motion of a disc and a ladder, based on retractions onto Voronoi diagrams. The algorithms run in time  $\mathcal{O}(n \log n)$  and  $\mathcal{O}(n^2 \log n \log^* n)$  respectively. Sifrony and Sharir [13] apply a variant of the retraction technique. They use a retraction that maps placements in  $FP$  onto particular vertices on the boundary of  $FP$ . The resulting algorithm runs in  $\mathcal{O}(K \log n)$ , where  $K$  is the number of feature pairs that is less than the length of the ladder apart. The complexity of this algorithm is determined by the complexity of the free space. Hence, not too much time is spent on other things than producing a description of the free space. Apparently, algorithms that have this property are close to optimal, because algorithms that run in less time than the time needed to produce an efficient representation of the free space can only be obtained in special cases where it is sufficient to compute only part (e.g. a single cell) of the free space. So, we must strive for algorithms with a complexity that is determined by the complexity of the free space.

As stated above, the complexity of a cell decomposition or a retraction network highly depends on the complexity of the free space  $FP$ . The complexity of the free space, on its turn, is determined by the number of multiple contacts of the robot  $\mathcal{B}$  and the obstacles. A multiple contact of the robot  $\mathcal{B}$  is a placement in which it touches more than one obstacle feature (e.g. edge, corner). Unfortunately the number of multiple contacts, and hence, the complexity of the free space, can be very high. If  $n$  is the number of obstacle features and  $f$  is the number of degrees of freedom of the robot (i.e., the dimension of the configuration space) and the number of robot features is bounded by some constant, then this complexity can be  $\Omega(n^f)$ . So, theoretically, these cell decomposition and retraction techniques are very expensive. Fortunately, in many practical situations the complexity of  $FP$  is much smaller and, hence, these methods might become feasible (assuming that the algorithms are sensitive to the size of  $FP$ , like e.g. the algorithm of Sifrony and Sharir [13]). A study of properties that limit the number of multiple contacts for the robot (and hence the complexity of  $FP$ ) is therefore of obvious importance.

In many practical cases the relative positions and the shapes of the obstacles are such that the number of multiple contacts for the robot  $\mathcal{B}$  is very low. Obstacles that are far apart clearly result in less double contacts for  $\mathcal{B}$  than obstacles that are cluttered. Similarly, obstacles that have no long and skinny parts will induce less double contacts than obstacles that do have such parts.

An obstacle that has no long and skinny parts is called *fat*. This informal description of fatness can be formalised in many different ways (see e.g. [1] and [8]). Our (slightly different) definition results in a free space with linear complexity for a robot moving amidst fat obstacles. The proof of the result is based on the following idea: we consider the smallest obstacle and prove a constant upper bound on the number of larger obstacles that lie close enough to this smallest obstacle so that both can be involved in a single multiple contact, and repeat this argument for every next smallest obstacle. Using this idea, we obtain a linear number of multiple contacts.

The importance of the linear complexity result becomes clear if we reconsider the

$\mathcal{O}(K \log n)$ -algorithm of Sifrony and Sharir [13]. Our result shows that the number of feature pairs that is less than the length of the ladder apart is linear in case of fat obstacles. As a consequence, the algorithm then runs in  $\mathcal{O}(n \log n)$  time, whereas it might take  $\mathcal{O}(n^2 \log n)$  time for non-fat obstacles.

The remainder of this paper is organised as follows. In Section 2 we explain our definition of fatness and compare it to a few alternative definitions. In Section 3 we show that under some realistic assumptions, the free space of a robot moving around in a workspace with fat obstacles has linear complexity. Section 4 concludes this paper.

## 2 Fat obstacles

The number of multiple contacts for a robot  $\mathcal{B}$ , and, hence, the complexity of the free space, is in most practical situations much lower than the worst case number of multiple contacts induced by the dimension of the configuration space. If, for example, the obstacles are far apart then the robot might not be able to touch two obstacles simultaneously, so there is no double contact at all. The complexity of the free space is only  $\mathcal{O}(n)$  (the number of single contacts) in this case. Long and skinny obstacles can obviously be involved in a larger number of multiple contacts than more “compact” obstacles without protuberances. These compact, or *fat*, obstacles will therefore result in a lower complexity of the free space.

Fatness turns out to be a very interesting property in computational geometry. Alt *et al.* [1] and Matoušek *et al.* [8] show that the upper bounds on the combinatorial complexity of the union of certain geometric figures are lower if these figures are fat. The union of geometric figures plays a role in many computational geometry applications. Fatness can lead to efficient algorithms. Overmars [11] shows that point location queries in fat subdivisions (no cell has long and skinny parts) in  $d$ -dimensional space can be performed in a simple way in  $\mathcal{O}(\log^{d-1} n)$  time with a data structure that uses  $\mathcal{O}(n \log^{d-1} n)$  storage.

Our definition of fatness in a  $d$ -dimensional Euclidean workspace involves  $d$ -dimensional closed spherical regions centered at some arbitrary point in an obstacle  $E$ . The closed spherical region with radius  $r$  centered at  $m$  will be denoted by  $S_{m,r}$ , so

$$S_{m,r} = \{x \in R^d \mid d(x, m) \leq r\};$$

the boundary of this region will be denoted by  $\partial S_{m,r}$ , so

$$\partial S_{m,r} = \{x \in R^d \mid d(x, m) = r\}.$$

Spherical regions centered at a point  $m$  with a boundary that has non-empty intersection with an obstacle  $E$  play a central role in our notion of fatness. Therefore, the following definition is useful.

**Definition 2.1** [ $U_{m,E}, U_E$ ]

Let  $m \in R^d$  and let  $E \subseteq R^d$  be an obstacle. The set  $U_{m,E}$  is defined as:

$$U_{m,E} = \{S_{m,r} \subseteq R^d \mid \partial S_{m,r} \cap E \neq \emptyset\}$$

The set  $U_E$  is defined as:

$$U_E = \bigcup_{m \in E} U_{m,E}$$

So,  $U_E$  is the set of all spherical regions with center inside  $E$  that do not fully contain  $E$ . Figure 1 shows a two-dimensional example of a circular region  $S_0$  that does not belong to  $U_E$  and a circular region  $S_1$  that does belong to  $U_E$ . Region  $S_0$  does not belong to  $U_E$  because the obstacle  $E$  lies entirely inside it. Region  $S_1$  is an element of  $U_E$  since its boundary has non-empty intersection with the obstacle  $E$ .

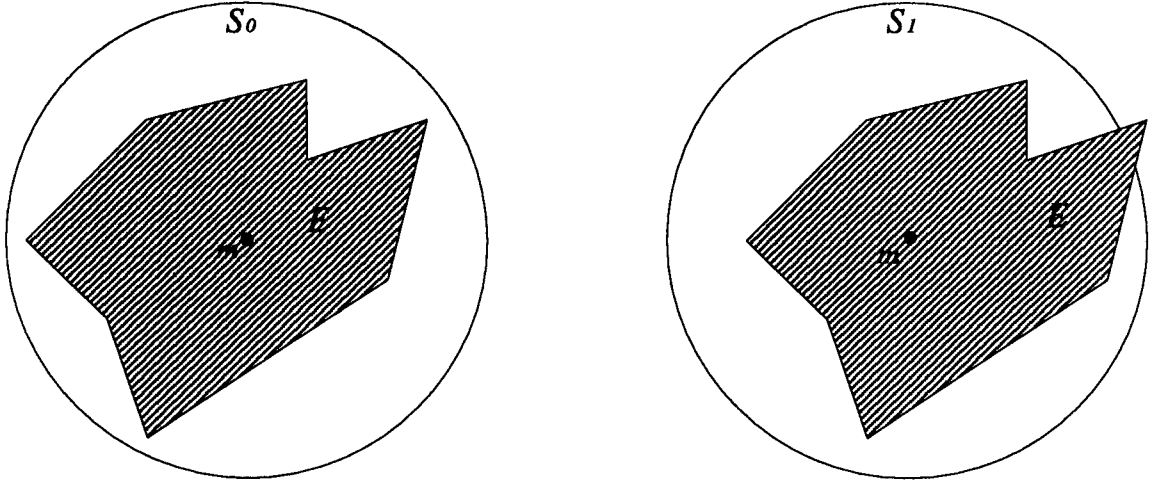


Figure 1: Illustration of the definition of  $U_E$ :  $S_0 \notin U_E$  and  $S_1 \in U_E$ .

We define fatness in a way such that obstacles are not only “compact” but also do not have extremely thin protuberances. The definition of fatness involves some positive number  $k$ . This number is a measure for the actual fatness of the obstacle. If the value of  $k$  is increased then the obstacle is allowed to be less fat. For obstacles with a boundary with infinitesimally thin protuberances (e.g. line segments) it is impossible to find such a  $k$ , so these obstacles can never be fat.

**Definition 2.2 [ $k$ -fatness]**

Let  $E \subseteq \mathbb{R}^d$  be an obstacle and let  $k$  be a positive constant. The obstacle  $E$  is  $k$ -fat if:

$$\forall S \in U_E \quad k \cdot \text{volume}(E \cap S) \geq \text{volume}(S).$$

Informally, an obstacle  $E$  is  $k$ -fat if the part of any spherical region  $S$  (with a boundary that intersects  $E$  and its center inside  $E$ ) covered by the obstacle  $E$  is at least a  $\frac{1}{k}$ <sup>th</sup> of  $S$ . The choice for spherical regions in the definition of fatness is rather arbitrary. In fact we could have used any compact region, e.g. hypercubic regions, regions bounded by simplices etc. Any  $k$ -fat obstacle according to one definition is easily seen to be  $k'$ -fat according to



another definition for some  $k'$  that is only a constant multiple of  $k$ . Note that there is a straightforward property that an obstacle that is  $k$ -fat is also  $k'$ -fat for  $k' \geq k$ .

No lower bound on the value of  $k$  is given in the definition of fatness. In fact the lower bound on  $k$  differs from dimension to dimension. There are for example no 1-fat obstacles at all; there can be 5-fat obstacles in a two-dimensional workspace but 5-fat obstacles in a three-dimensional workspace do not exist. This is inherent to the definition of  $k$ -fatness. Suppose we have a  $k$ -fat obstacle  $E$  with diameter  $\delta$ . The volume of this obstacle is bounded from above by the volume of a hypersphere with diameter  $\delta$  (or radius  $\delta/2$ ). The diameter of  $E$  is  $\delta$ , so there is a pair of points on the boundary of  $E$  that is a distance  $\delta$  apart; let  $m, m' \in E$  be these two points. The spherical region  $S_{m,\delta}$  is an element of  $U_E$  since  $m' \in \partial S_{m,\delta}$  and  $m' \in E$ . (Similarly, spherical region  $S_{m',\delta}$  is an element of  $U_E$ .) Hence, the set  $U_E$  contains an element  $S$  with radius  $\delta$ . We know that  $\text{volume}(E \cap S) \leq \text{volume}(E) \leq C_d \cdot (\delta/2)^d$  and  $\text{volume}(S) = C_d \cdot \delta^d$ , where  $C_d$  is the dimension-dependent multiplier in the volume formulae for hyperspheres<sup>1</sup>. Combination with Definition 2.2 ( $E$  is  $k$ -fat and  $S \in U_E$ ) yields  $k \geq 2^d$ . The boundary value  $2^d$ -fatness is only obtained for spherical obstacles; spherical obstacles have maximal fatness.

The definition of  $k$ -fatness given in Definition 2.2 has a very “local” character: a certain portion of the proximity of every point in the obstacle must be covered by the obstacle too. As stated before, this locality prohibits obstacles with infinitesimally thin protuberances, even if these protuberances are extremely short. A huge spherical obstacle with a very short line segment sticking out of its boundary will not be  $k$ -fat for any value of  $k$ . A more natural way to define fatness might be the more “global” type of fatness given in Definition 2.3. For convenience, we will refer to it as *thickness*. Here, we only compare the volume of the entire obstacle to the volume of its minimal (volume) enclosing hypersphere: the volume of the obstacle should be at least a certain portion of the minimal enclosing hypersphere of the obstacle. This more liberal definition allows obstacles with small protuberances. If  $E$  is an obstacle then we denote the minimal enclosing hypersphere of  $E$  by  $MES_E$ .

### Definition 2.3 [ $k$ -thickness]

Let  $E \subseteq R^d$  be an obstacle and let  $k \geq 1$  be a constant. The obstacle  $E$  is  $k$ -thick if:

$$k \cdot \text{volume}(E) \geq \text{volume}(MES_E).$$

The definition of  $k$ -thickness involves just one hypersphere instead of infinitely many. Note that not necessarily  $MES_E \in U_E$ : the minimal enclosing hypersphere of an obstacle can have its center outside the obstacle. Again we have the straightforward property that an obstacle that is  $k$ -thick is also  $k'$ -thick for  $k' \geq k$ . Spherical obstacles are 1-thick, because the minimal enclosing hyperspheres of such obstacles are the obstacles themselves.

Unfortunately, the notion of  $k$ -thickness does not result in low complexities of the free space. Hence, the definition is too general for our purposes. We could restrict ourselves to convex obstacles but, as we will see below, in that case thickness is equivalent to fatness.

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<sup>1</sup>For even dimension  $C_d = C_{2n} = \frac{\pi^n}{n!}$ . For odd dimension  $C_d = C_{2n+1} = \frac{2(2\pi)^n}{(2n+1)!!}$ . See, e.g., [3, Section 394].

Therefore, we have chosen to use the definition of fatness stated in Definition 2.2 because it also allows for non-convex obstacles. The property of the set  $U_{m,E}$  for a convex region  $E$  given in the next lemma is a useful tool in the proof of the equivalence of thickness and fatness for convex obstacles.

**Lemma 2.1** *Let  $E \subseteq \mathbb{R}^d$  be a convex obstacle and  $m \in E$ . Let  $S_{m,r} \in U_{m,E}$  and  $S_{m,R} \in U_{m,E}$  with  $r \leq R$ . Now the following inequality holds:*

$$\frac{\text{volume}(E \cap S_{m,r})}{\text{volume}(S_{m,r})} \geq \frac{\text{volume}(E \cap S_{m,R})}{\text{volume}(S_{m,R})}.$$

**Proof:** We use a polar coordinate frame with origin  $m$  and angles  $\phi, \theta_1, \dots, \theta_{d-2}$ , with  $0 \leq \phi < 2\pi$  and  $0 \leq \theta_1, \dots, \theta_{d-2} \leq \pi$ . Each combination of angles  $(\phi, \theta_1, \dots, \theta_{d-2})$  specifies a viewing direction from  $m$ . Since the obstacle  $E$  is convex, each point on the boundary of  $E$  can be seen from  $m$ . Therefore, the relation between the viewing direction and the distance to the boundary of  $E$  is a function. The same obviously holds for both spheres. So, there are three functions  $\rho_E, \rho_{S_{m,r}}, \rho_{S_{m,R}} : [0, 2\pi) \times [0, \pi]^{d-2} \rightarrow \mathbb{R}^+ \cup \{0\}$ , that give the distance from  $m$  to the boundary of  $E, S_{m,r}$ , and  $S_{m,R}$  respectively. The latter two functions are constant:  $\rho_{S_{m,r}}(\phi, \theta_1, \dots, \theta_{d-2}) = r$  and  $\rho_{S_{m,R}}(\phi, \theta_1, \dots, \theta_{d-2}) = R$ . Let  $f, F : [0, 2\pi) \times [0, \pi]^{d-2} \rightarrow [0, 1]$  be defined as follows:

$$f(\phi, \theta_1, \dots, \theta_{d-2}) = \min\left(\frac{\rho_E(\phi, \theta_1, \dots, \theta_{d-2})}{r}, 1\right),$$

$$F(\phi, \theta_1, \dots, \theta_{d-2}) = \min\left(\frac{\rho_E(\phi, \theta_1, \dots, \theta_{d-2})}{R}, 1\right).$$

Integrating the product of function  $f$  to some power  $i$  and some determinant function  $\Phi$  over all angular domains yields the ratio of  $\text{volume}(E \cap S_{m,r})$  and  $\text{volume}(S_{m,r})$  as given in the left-hand side of the inequality that is to be proven. The right-hand side is obtained by integrating the product of function  $F$  to the same power  $i$  and the same determinant function  $\Phi$ . This  $\Phi$  is a product of  $(\sin \theta_i)^j$ -terms. Since  $0 \leq \theta_1, \dots, \theta_{d-2} \leq \pi$ , function  $\Phi$ 's range is restricted to  $[0, 1]$ . Function  $f$  and  $F$  have the same range. If we can prove that  $f(\phi, \theta_1, \dots, \theta_{d-2}) \geq F(\phi, \theta_1, \dots, \theta_{d-2})$ , for all  $0 \leq \phi < 2\pi$  and  $0 \leq \theta_1, \dots, \theta_{d-2} \leq \pi$ , then, because  $\Phi, f$ , and  $F$  only have non-negative function values, the integral containing  $f$  will yield a larger value than the one containing  $F$ , and hence the inequality involving the volumes will be proved.

Relevant changes in the values of  $f$  and  $F$  obviously appear at  $\rho_E(\phi, \theta_1, \dots, \theta_{d-2}) = r$  and  $\rho_E(\phi, \theta_1, \dots, \theta_{d-2}) = R$ . Therefore, we consider three different ranges for the value of  $\rho_E(\phi, \theta_1, \dots, \theta_{d-2})$ .

1. If  $\rho_E(\phi, \theta_1, \dots, \theta_{d-2}) \leq r$  then:  
 $f(\phi, \theta_1, \dots, \theta_{d-2}) = \rho_E(\phi, \theta_1, \dots, \theta_{d-2})/r \geq \rho_E(\phi, \theta_1, \dots, \theta_{d-2})/R = F(\phi, \theta_1, \dots, \theta_{d-2}).$
2. If  $r \leq \rho_E(\phi, \theta_1, \dots, \theta_{d-2}) \leq R$  then:  
 $f(\phi, \theta_1, \dots, \theta_{d-2}) = 1 \geq \rho_E(\phi, \theta_1, \dots, \theta_{d-2})/R = F(\phi, \theta_1, \dots, \theta_{d-2}).$

3. If  $R \leq \rho_E(\phi, \theta_1, \dots, \theta_{d-2})$  then:  
 $f(\phi, \theta_1, \dots, \theta_{d-2}) = 1 = F(\phi, \theta_1, \dots, \theta_{d-2})$ .

Figure 2 shows a two-dimensional example of each of the three cases given above.

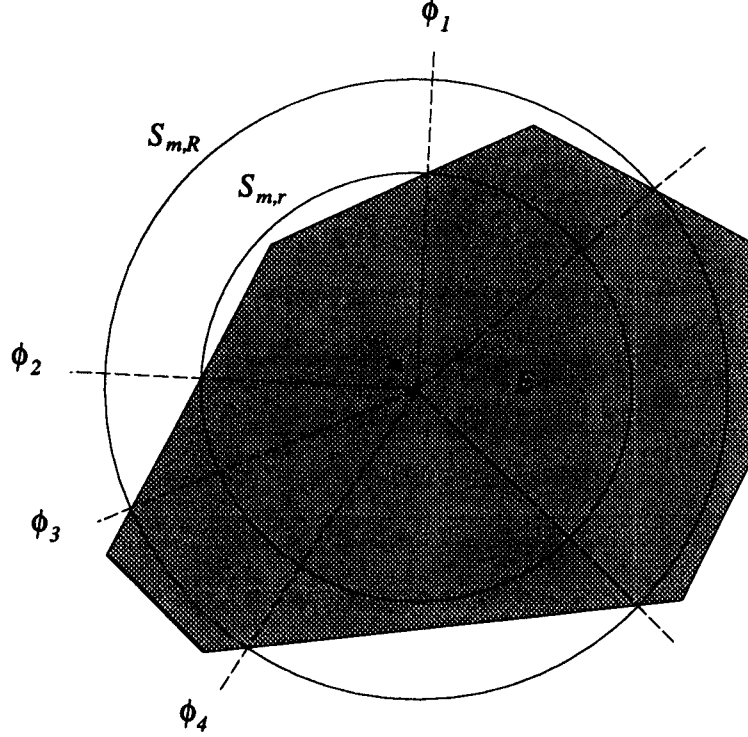


Figure 2: The angular interval  $[\phi_1, \phi_2]$  is an example of case (1), interval  $[\phi_2, \phi_3]$  is an example of case (2), and the angular interval  $[\phi_3, \phi_4]$  is an example of case(3).

Combining the three different ranges, we obtain  $f(\phi, \theta_1, \dots, \theta_{d-2}) \geq F(\phi, \theta_1, \dots, \theta_{d-2})$ , for all  $0 \leq \phi < 2\pi$  and  $0 \leq \theta_1, \dots, \theta_{d-2} \leq \pi$ .  $\square$

Lemma 2.1 shows that in each set  $U_{m,E}$  the portion of a spherical region that is covered by the obstacle  $E$  does not increase as the radius of the spherical region increases. The ratio is therefore minimal for region  $\sup U_{m,E}$  (the region in  $U_{m,E}$  with maximal volume). In the sequel we will use the abbreviation

$$ES_{m,E} = \sup U_{m,E}.$$

$ES$  stands for enclosing sphere because  $\sup U_{m,E}$  is actually the enclosing spherical region of the obstacle  $E$  centered at point  $m$ . A consequence of Lemma 2.1 is that if  $k \cdot \text{volume}(E \cap ES_{m,E}) \geq \text{volume}(ES_{m,E})$  holds then we can conclude that  $k \cdot \text{volume}(E \cap S) \geq \text{volume}(S)$  for all  $S \in U_{m,E}$ . Define the set  $ES_E$  of all enclosing spherical regions centered at some

point in the obstacle:

$$ES_E = \bigcup_{m \in E} \{ES_{m,E}\}.$$

It is clear that  $ES_E \subseteq U_E$ . Due to Lemma 2.1 we can reformulate the requirement given in Definition 2.2 for convex obstacles. Note that for all  $S \in ES_E$ , the obvious equality  $E \cap S = E$  holds. A convex obstacle  $E$  is  $k$ -fat if:

$$\forall S \in ES_E \quad k \cdot \text{volume}(E) \geq \text{volume}(S).$$

We are now ready to prove the equivalence of thickness and fatness of convex obstacles.

**Theorem 2.1** *Let  $E \subseteq \mathbb{R}^d$  be a convex obstacle. Then*

$$E \text{ is } k\text{-fat} \equiv E \text{ is } k'\text{-thick},$$

where  $k$  is only a constant multiple of  $k'$ .

**Proof:**

$\Rightarrow$  Choose some spherical region  $S \in ES_E$ . The obstacle  $E$  is  $k$ -fat and  $ES_E \subseteq U_E$ , so  $k \cdot \text{volume}(E) = k \cdot \text{volume}(E \cap S) \geq \text{volume}(S)$ . Region  $S$  is some enclosing spherical region of  $E$  and  $MES_E$  is defined as the minimal volume enclosing spherical region of  $E$ , so obviously  $\text{volume}(MES_E) \leq \text{volume}(S)$  holds. Combining both inequalities results in  $k \cdot \text{volume}(E) \geq \text{volume}(MES_E)$ , proving  $k'$ -thickness of  $E$ , with  $k' = k$ .

$\Leftarrow$  The convex obstacle  $E$  is  $k'$ -thick, so the inequality  $k' \cdot \text{volume}(E) \geq \text{volume}(MES_E)$  holds. By Lemma 2.1 and the convexity of  $E$  we know that it suffices to prove that  $\forall S \in ES_E: k \cdot \text{volume}(E) \geq \text{volume}(S)$ , for some constant  $k$ . Let  $\delta$  be the diameter of  $MES_E$  and let  $\epsilon$  be the diameter of the obstacle  $E$ . The obstacle  $E$  fits inside  $MES_E$  so trivially  $\epsilon \leq \delta$ . The diameter of the obstacle  $E$  is determined by two points  $m$  and  $m'$  on its boundary. The radius of a spherical region in  $ES_E$  is at most  $\epsilon$ . This is the radius of the largest regions  $ES_{m,E}$  and  $ES_{m',E}$ .

We have  $\text{volume}(MES_E) = C_d \cdot (\delta/2)^d$  and for all  $S \in ES_E: \text{volume}(S) \leq C_d \cdot \epsilon^d$ , where  $C_d$  is the dimension-dependent multiplication factor mentioned earlier in this section. Combination of all equalities and inequalities yields for all  $S \in ES_E$ :

$$\begin{aligned} & 2^d \cdot k' \cdot \text{volume}(E) \\ \geq & 2^d \cdot \text{volume}(MES_E) \\ = & C_d \cdot \delta^d \\ \geq & C_d \cdot \epsilon^d \\ \geq & \text{volume}(S), \end{aligned}$$

proving  $k$ -fatness of the convex obstacle  $E$ , with  $k = 2^d \cdot k'$ . □

A consequence of Theorem 2.1 is that the complexity results that we prove for convex obstacles that are  $k$ -fat also hold for convex obstacles that are  $k$ -thick. In the rest of this paper we will only consider fatness, not thickness.

Let us consider some examples of fat shapes. Spherical obstacles are obviously fat, because we have shown earlier that a  $d$ -sphere has maximal achievable  $2^d$ -fatness. Other shapes that are fat (in two-dimensional space) include squares, rectangles with bounded aspect ratio, and triangles with angle restriction.

In [8] an approximate motion planning algorithm is given for a rectangular robot. The complexity of the algorithm decreases as the ratio of the rectangle's sides gets closer to 1. It is obvious that a (non-degenerate) rectangle is fat according to our definition. The actual fatness obviously depends, like the complexity of the algorithm, on the ratio of the rectangle's sides. The value of  $k$  will increase as the rectangle becomes "narrow". The largest spherical region in  $U_E$  (centered at a rectangle corner and enclosing the rectangle) has radius  $\sqrt{a^2 + b^2}$ , so a rectangle with sides  $a$  and  $b$  is  $\frac{\pi(a^2+b^2)}{ab}$ -fat. Clearly, we get maximal fatness for squares and no fatness if either  $a = 0$  or  $b = 0$ .

In [1] a different notion of fatness is introduced for triangles by imposing a restriction on the angles in the triangle. A triangle is called  $\delta$ -fat if each of its three internal angles is at least  $\delta$ . A  $\delta$ -fat triangle is also fat according to our definition. Assume that we are given a  $\delta$ -fat triangle with a longest edge  $e$ . The triangle has minimal area if the other two angles have magnitudes  $\delta$  and  $\pi - 2\delta$ . This minimal area is  $\frac{1}{4}|e|^2 \tan \delta$ . The largest spherical region in  $U_E$  (centered at one of the end-points of  $e$  and enclosing the triangle) has radius  $|e|$ . Therefore, each  $\delta$ -fat triangle is  $\frac{4\pi}{\tan \delta}$ -fat according to our fatness definition. Maximal fatness is obviously obtained for equilateral triangles and there is no fatness if  $\delta = 0$ . The latter triangle will also be non-fat in [1].

Many other classes of shapes are fat. In three-dimensional space we can think of cubes, boxes with bounded aspect ratio's in each of their bounding rectangular faces, equilateral tetrahedra etc.

### 3 Multiple contacts for a robot

Let us now return to determining the complexity of the free space for some motion planning problem involving fat obstacles. The actual complexity of the free space depends on the number of intersections of hypersurfaces that bound the free space. Such a hypersurface is a set of placements of the robot  $\mathcal{B}$  in which a certain feature of  $\mathcal{B}$  is in contact with a certain feature of the boundary of  $\mathcal{E}$ . Each hypersurface can intersect any other hypersurface. Intersections of hypersurfaces correspond to multiple contacts of the robot  $\mathcal{B}$  with the boundary of  $\mathcal{E}$ . The intersections define faces on the hypersurfaces. The set of faces forms a description of the free space FP. If the number of possible multiple contacts of the robot  $\mathcal{B}$  is low then the complexity of the free space is also low.

We consider the situation where a robot  $\mathcal{B}$  moves amidst  $k$ -fat obstacles  $E \subseteq \mathcal{E}$ . We

assume that the obstacles are not infinitesimally small. Let the diameter of the smallest minimal enclosing hypersphere of any obstacle be  $d_{min}$ . The robot  $\mathcal{B}$  is assumed to be relatively small compared to the obstacles  $\mathcal{E}$ : the diameter of  $\mathcal{B}$  is at most  $b \cdot d_{min}$ , where  $b$  is some positive constant. Both assumptions are quite realistic. The assumption on the size of the obstacles basically rules out point obstacles, whereas the assumption on the size of the robot forbids unboundedly large robots. If we do not make the second assumption then one may choose a very large robot that would make the obstacles into “point” obstacles relative to the robot. In practical situations, both assumptions will be satisfied.

We assume that the number of features of the robot  $\mathcal{B}$  is bounded by a constant and the number of features of the obstacle set  $\mathcal{E}$  is  $n$ . As a consequence, the total number of hypersurfaces is  $\mathcal{O}(n)$ . The hypersurfaces are assumed to be of bounded degree, so that the number of intersections of two hypersurfaces is also bounded by a constant. This requirement for the degree of the hypersurfaces mainly means that the boundary of the robot and the obstacles must not be too irregularly shaped. As the hypersurfaces are of bounded degree this implies that the total complexity of the free space FP is bounded by  $\mathcal{O}(n^f)$ , where  $f$  is the dimension of the configuration space. The dimension of the configuration space equals the number of degrees of freedom of the robot. Each obstacle  $E \subseteq \mathcal{E}$  is assumed to have only a constant number of features, so the number of obstacles is  $\Omega(n)$ .

The assumptions in the previous two paragraphs are sufficient to prove that the number of multiple contacts for a robot  $\mathcal{B}$  is at most linear in the number of obstacles. We summarise these assumptions below.

- The workspace of the robot  $\mathcal{B}$  is the  $d$ -dimensional Euclidean space ( $R^d$ ).
- The workspace of the robot  $\mathcal{B}$  contains  $n$   $k$ -fat obstacles  $E \subseteq \mathcal{E} \subseteq R^d$ .
- The diameter of the smallest minimal enclosing hypersphere of any obstacle  $E \subseteq \mathcal{E}$  is  $d_{min}$ .
- The diameter  $d_{\mathcal{B}}$  of the robot  $\mathcal{B}$  is bounded:  $d_{\mathcal{B}} \leq b \cdot d_{min}$  (constant  $b > 0$ ).
- The robot  $\mathcal{B}$  has constant complexity.
- Each obstacle  $E \subseteq \mathcal{E}$  has constant complexity.
- The set of robot placements (hypersurface in the configuration space) in which a certain robot feature is in contact with a certain obstacle feature is of bounded degree.

### 3.1 The proximity of an obstacle

In this subsection we consider the proximity of a  $k$ -fat obstacle as a first step in finding an upper bound on the number of multiple contacts for a robot  $\mathcal{B}$ . It is obvious that two obstacles that are far (more than the diameter of the robot) apart can not be involved in

any multiple contact for  $\mathcal{B}$ . Hence, obstacles that do cause such a contact must lie in each other's proximity.

A strategy for proving a linear upper bound on the number of multiple contacts for the robot  $\mathcal{B}$  could be to prove that the number of multiple contacts involving a certain obstacle is only constant. Straightforward application of this strategy, however, would yield no result. If we have a situation with  $n - 1$  equally sized  $k$ -fat obstacles and one much larger  $k$ -fat obstacle, then considering the proximity of this large obstacle does not result in a constant upper bound on the number of obstacles that can participate in a multiple contact involving the large obstacle: all  $n - 1$  smaller obstacles might lie in the proximity of the large obstacle. Note, however that this strategy contains some redundancy: a multiple contact is counted more than once.

To avoid counting a single multiple contact more than once, we only count the number of larger obstacles that can participate in a multiple contact involving  $E$ . By starting with the smallest obstacle and repeatedly considering the next smallest obstacle we will count each multiple contact  $m$  for the robot exactly once. It turns out that the number of such multiple contacts involving any obstacle  $E$  is constant.

Before we focus on the problem of finding an upper bound on the number of multiple contacts, we first consider the notion of multiple contact itself. What kind of subspaces of the configuration space are defined by multiple contacts and how many obstacles can participate in a multiple contact?

The set of placements of the robot  $\mathcal{B}$  in which a certain feature of  $\mathcal{B}$  is in contact with a certain feature of the boundary of  $\mathcal{E}$  forms an  $(f - 1)$ -dimensional subspace (or hypersurface) in the  $f$ -dimensional configuration space. An intersection of two of these hypersurfaces corresponds to a simultaneous contact of the robot with two features of the boundary of the obstacle set  $\mathcal{E}$ . Such an intersection is an  $(f - 2)$ -dimensional subspace of the configuration space. Moreover, a  $j$ -fold contact of the robot defines an  $(f - j)$ -dimensional space. Consequentially, an  $f$ -fold contact appears at an isolated point in the configuration space, and, hence, fixes the position of the robot. Contacts that involve more than  $f$  obstacle features only appear accidentally and can be discarded without affecting the complexity of the free space. We see that a robot  $\mathcal{B}$  with  $f$  degrees of freedom can have up to  $f$  simultaneous contacts with the boundary of  $\mathcal{E}$ .

Any  $k$ -fat obstacle  $E'$  that participates in some multiple contact with a given  $k$ -fat obstacle  $E$  must lie close to this obstacle  $E$ . Lemma 3.1 states that the number of obstacles that lie in the proximity of the obstacle  $E_{\min}$  with the smallest minimal enclosing hypersphere is bounded by a constant.

**Lemma 3.1** *Let  $\mathcal{E} \subseteq R^d$  be a set of  $k$ -fat obstacles. Let  $E_{\min} \subseteq \mathcal{E}$  be the obstacle with the smallest minimal enclosing hypersphere; the diameter of this hypersphere is assumed to be  $d_{\min}$ . If the diameter  $d_{\mathcal{B}}$  of the robot is at most  $b \cdot d_{\min}$  then the number of obstacles  $E \subseteq R^d$  that lie close enough to  $E_{\min}$  so that  $\mathcal{B}$  can touch  $E_{\min}$  and  $E$  simultaneously is at most  $2^d \cdot k \cdot (b + 1)^d$ .*

**Proof:** Consider the obstacle  $E_{\min}$ . If the robot  $\mathcal{B}$  is in contact with  $E_{\min}$ , then it has non-empty intersection with the closed region  $MES_{E_{\min}}$ . In such a placement, the robot

$\mathcal{B}$  lies entirely inside a closed spherical region  $S$  with a boundary that is concentric with the boundary of the minimal enclosing spherical region  $MES_{E_{min}}$  and having a diameter  $d_{min} + 2d_B$  (see Figure 3 for an example in the two-dimensional case). An obstacle  $E$

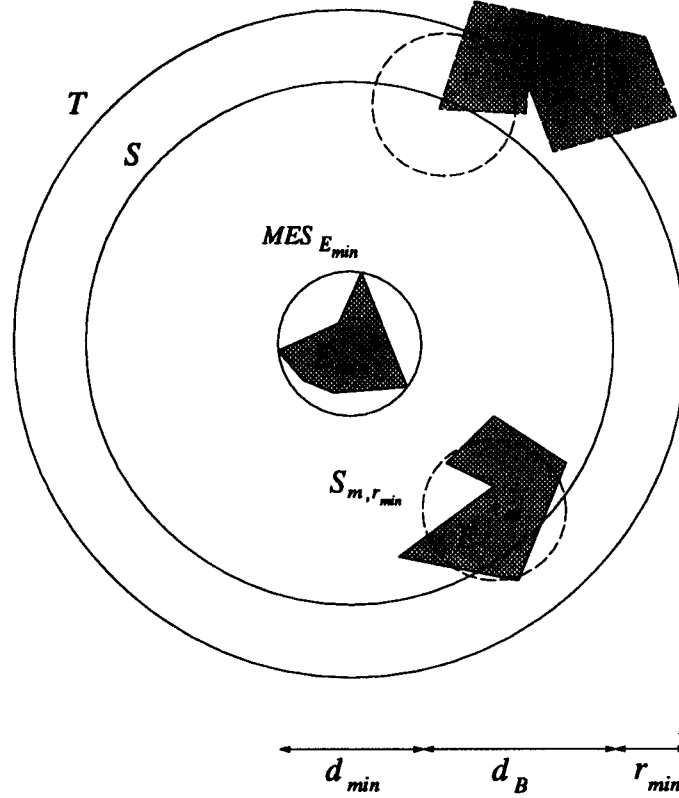


Figure 3: Illustration of the proof of Lemma 3.1.

participating in a multiple contact for  $\mathcal{B}$  will have non-empty intersection with region  $S$ . Hence, there is a point  $m \in E \cap S$ .

As a next step we consider  $S_{m, r_{min}}$ , with  $m \in E \cap S$  and  $r_{min} = \frac{1}{2}d_{min}$ . In order to be able to give a lower bound on the volume of the obstacle  $E$  that is inside  $S_{m, r_{min}}$ , we prove that  $S_{m, r_{min}} \in U_E$ . For a contradiction, we assume that  $\partial S_{m, r_{min}} \cap E = \emptyset$ . The center  $m$  of the region lies inside the obstacle  $E$ , so this would imply that the obstacle  $E$  lies entirely inside  $S_{m, r_{min}}$ . But then the minimal enclosing hypersphere of  $E$  would be smaller than  $MES_{E_{min}}$  which has diameter  $d_{min}$ . This contradicts the assumption that  $E_{min}$  is the obstacle with the smallest minimal enclosing hypersphere, so  $\partial S_{m, r_{min}} \cap E \neq \emptyset$ . Since  $m \in E$ , we conclude that  $S_{m, r_{min}} \in U_E$ .

We construct a spherical region  $S_{m, r_{min}}$  for each obstacle  $E$  that has non-empty intersection with  $S$ ; for each region we choose  $m \in E \cap S$ . Each such region  $S_{m, r_{min}}$  obviously lies entirely inside a closed spherical region  $T$  with a boundary that is concentric with the



boundary of  $S$  and  $MES_{E_{\min}}$  and having diameter  $2d_{\min} + 2d_{\mathcal{B}}$  and thus radius  $d_{\min} + d_{\mathcal{B}}$ . Now each obstacle  $E$  that participates in a multiple contact for  $\mathcal{B}$  has, because of its  $k$ -fatness and because  $S_{m,r_{\min}} \in U_E$  for  $m \in E \cap S$ , at least a part of volume  $\frac{1}{k} \cdot \text{volume}(S_{m,r_{\min}})$  inside region  $T$ , which is equal to  $k^{-1} \cdot C_d \cdot r_{\min}^d = 2^{-d} k^{-1} \cdot C_d \cdot d_{\min}^d$ , where  $C_d$  again is the dimension-dependent multiplier mentioned in Section 2. The volume of region  $T$  is  $C_d \cdot (d_{\min} + d_{\mathcal{B}})^d$ . Since diameter  $d_{\mathcal{B}}$  is bounded by  $b \cdot d_{\min}$ , the volume of region  $T$  is bounded from above by  $(b+1)^d \cdot C_d \cdot d_{\min}^d$ . Combining both bounds results in an upper bound of  $2^d \cdot k \cdot (b+1)^d$  on the number of obstacles  $E$  intersecting  $S$  and, hence, on the number of obstacles that lie close enough to the obstacle  $E_{\min}$  to participate in a multiple contact that involves both  $E$  and  $E_{\min}$ .  $\square$

In the sequel we shall use the abbreviation  $p_{k,b}$  for the upper bound given in Lemma 3.1. So  $p_{k,b} = 2^d \cdot k \cdot (b+1)^d$  is the maximal number of  $k$ -fat obstacles that can participate in a multiple contact involving the  $k$ -fat obstacle with the smallest minimal enclosing hypersphere.

The value of  $p_{k,b}$  is constant for fixed  $k$  and  $b$  but increases when the value of  $k$  or  $b$  increases. This is not surprising: the number of obstacles that can participate in a multiple contact of the robot  $\mathcal{B}$  increases when the obstacles become less fat or when the diameter of the robot increases.

### 3.2 A linear number of multiple contacts for $\mathcal{B}$

The proximity result given in Lemma 3.1 is the key to successful application of the proof strategy presented in the previous subsection. Using that strategy we start with the obstacle with the smallest enclosing hypersphere and repeatedly consider the obstacle with the next smallest minimal enclosing hypersphere. For each obstacle  $E$  we count the number of multiple contacts for the robot  $\mathcal{B}$  involving  $E$  and obstacles with larger minimal enclosing hyperspheres. Lemma 3.1 guarantees that we find a constant upper bound on this number for each obstacle. The resulting overall number of multiple contacts will be linear, which is stated in Theorem 3.1.

**Theorem 3.1** *Let  $\mathcal{E} \subseteq \mathbb{R}^d$  be a set of  $n$   $k$ -fat obstacles of constant complexity. The diameter of the minimal enclosing hypersphere of each obstacle  $E \subseteq \mathcal{E}$  is at least  $d_{\min}$ . Let  $\mathcal{B}$  be a robot of constant complexity with  $f$  degrees of freedom and with diameter  $d_{\mathcal{B}} \leq b \cdot d_{\min}$ . For each  $j$  ( $2 \leq j \leq f$ ), the number of  $j$ -fold contacts of the robot  $\mathcal{B}$  is linear in the number of obstacles:  $\mathcal{O}(n)$ .*

**Proof:** Consider the obstacle  $E_0 \subseteq \mathcal{E}$  with the smallest minimal enclosing hypersphere. The diameter of this hypersphere is  $d_{\min}$  in the worst case. We count the number of  $j$ -fold contacts of  $\mathcal{B}$  that involve  $E_0$  and obstacles with a larger minimal enclosing hypersphere. By Lemma 3.1 we know that  $p_{k,b}$  different obstacles can participate in such a  $j$ -fold contact.

A single  $j$ -fold contact is determined by  $j$  different pairs, each pair consisting of a robot feature and an obstacle feature. It is not determined by the robot and  $j$  different obstacles, because e.g. more than one feature of a single obstacle can be involved in a single  $j$ -fold

contact. Let us assume that the robot has  $x_B$  different features and that the number of features of each obstacle  $E$  is bounded by  $x_E$ .

The first contact is a contact between a robot feature and a feature of the obstacle  $E_0$ . Since the robot  $B$  and the obstacle  $E_0$  have  $x_B$  and  $x_E$  features respectively, we have  $x_B \cdot x_E$  choices for this first contact. For each of the  $j - 1$  next contacts we can choose the obstacle feature on each of the  $p_{k,b}$  obstacles in the proximity of  $E_0$ , which gives a total number of  $x_E \cdot p_{k,b}$  possibly involved obstacle features. For each contact we can again choose from all  $x_B$  robot features. An upper bound on the number of choices for the remaining  $j - 1$  contacts is therefore  $(x_B \cdot x_E \cdot p_{k,b})^{j-1}$ . Hence, the total number of  $j$ -fold contacts involving  $E_0$  is bounded by  $(x_B \cdot x_E \cdot p_{k,b})^{j-1} + x_B \cdot x_E$ , which is a constant.

Subsequently, we choose the obstacle  $E_i$  with the next smallest minimal enclosing hypersphere. Again we count the number of  $j$ -fold contacts of  $B$  that involve  $E_i$  and obstacles with a larger minimal enclosing hypersphere. The counting for the obstacle  $E_i$  obviously also results in at most  $(x_B \cdot x_E \cdot p_{k,b})^{j-1} + x_B \cdot x_E$   $j$ -fold contacts.

Adding all the  $n$  constant upper bounds results in an overall upper bound on the number of  $j$ -fold contacts of  $n \cdot ((x_B \cdot x_E \cdot p_{k,b})^{j-1} + x_B \cdot x_E)$ , which is  $\mathcal{O}(n)$ , since  $x_B$ ,  $x_E$ ,  $p_{k,b}$ , and  $j$  are constants.  $\square$

Note that the value of  $j$  in Theorem 3.1 ranges from 2 to  $f$ . The number of single contacts is also linear; a different hypersurface corresponds to each contact between a robot feature and an obstacle feature, giving a total of  $\mathcal{O}(n)$  hypersurfaces since there are  $n$  obstacles and because of the constant complexity of the robot and each obstacle.

The  $(f - j)$ -dimensional subspace defined by a single  $j$ -fold contact is not necessarily connected. Figure 4 shows an example for  $f = 3$  and  $j = 2$ , where it is impossible for the robot to move from  $Z_0$  to  $Z_1$  without losing contact with either the upper or the lower obstacle feature. The 1-dimensional subspace induced by the contact with both features is therefore non-connected. Our assumption that all contact hypersurfaces are of bounded degree, however, implies that the number of different connected subspaces induced by a single multiple contact is bounded by some small constant. The complexity of the free space is now solely determined by the number of multiple contacts. Variable  $j$  in Theorem 3.1 can only have  $f - 1$  different values, so the total number of multiple contacts is linear, and, hence, the free space has linear complexity.

**Corollary 3.1** *Let  $\mathcal{E} \subseteq \mathbb{R}^d$  be a set of  $n$   $k$ -fat obstacles of constant complexity. The diameter of the minimal enclosing hypersphere of each obstacle  $E \subseteq \mathcal{E}$  is at least  $d_{\min}$ . Let  $B$  be a robot of constant complexity with  $f$  degrees of freedom and with diameter  $d_B \leq b \cdot d_{\min}$ . The free space for the robot  $B$  moving amidst the  $k$ -fat obstacles of set  $\mathcal{E}$  has linear complexity.*

The constant that we obtained in Theorem 3.1 can be quite high: the first contact for the robot is a feature of  $E$ , but each of the other  $j - 1$  contacts are chosen from all features in the proximity of  $E$ . In practice, this approach clearly yields a bound that is far from tight for a number of reasons. A robot touching a feature  $w$  of  $E$  might not be able to touch all features in the proximity of  $E$  because some of them are just too far away.

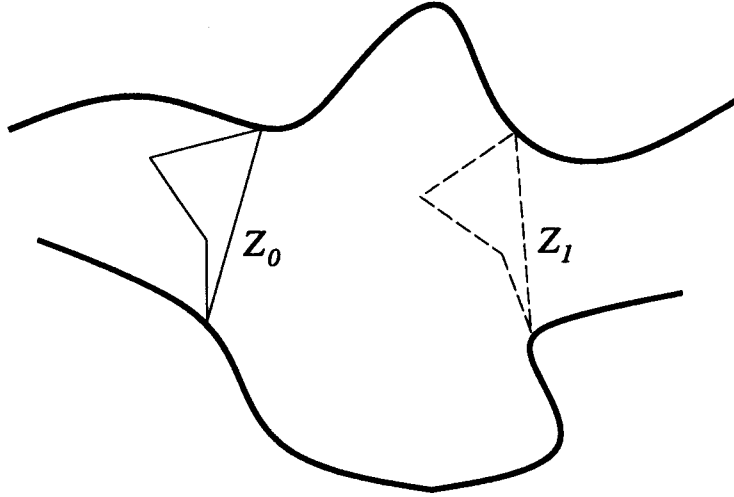


Figure 4: There is no continuous motion of the robot from  $Z_0$  to  $Z_1$  during which it remains in contact with both features.

Moreover, the robot  $\mathcal{B}$  will not be able to touch a large number of features that lie close enough to  $w$  but cannot be touched by  $\mathcal{B}$ , simply because  $\mathcal{B}$ 's shape does not allow it. By fixing a certain contact for  $\mathcal{B}$ , some features will stop being candidates for being involved in the  $j$ -fold contact for  $\mathcal{B}$ , because of either one of both reasons. Hence, fixing a contact reduces the set of candidates for the remaining contacts. Clearly, the number of actual  $j$ -fold contacts for  $\mathcal{B}$  will remain far below the upper bound of Theorem 3.1.

## 4 Conclusion

In this paper we have shown that, under some realistic assumptions, the complexity of the free space of a robot  $\mathcal{B}$  moving amidst  $k$ -fat obstacles is linear. The assumptions include constant complexity requirements for the robot and the obstacles, a lower bound on the size of the smallest obstacle, and an upper bound on the diameter of the robot. In most practical situations, all these constraints will be satisfied. The assumption on the size of the smallest obstacle in fact only forbids point obstacles. The upper bound on the size of the robot depends on the size of the smallest obstacle. It basically states that the robot should not be too big compared to the obstacles. The constant complexity requirement and an additional bound on the degree of the hypersurfaces defined by robot-obstacle contacts can be translated into a requirement that the boundary of the obstacles and the robot do not have a too complex shape. Both assumptions are common assumptions in motion planning.

If an obstacle does not meet the complexity assumption, the linear complexity result

presented in this paper can still be applicable. First of all, it may very well be possible to partition the obstacle into a constant number of sub-obstacles, such that each sub-obstacle satisfies the complexity assumption for obstacles. After the partitioning, each sub-obstacle is treated as a separate obstacle. If the complexity is too high to apply this procedure, an adequate constant complexity outer approximation of the obstacle might be found with a volume that is not too much larger than the volume of the obstacle itself. If we can find such an outer approximation then we can do motion planning for this approximation at the cost of a relatively small reduction of the number of solutions. A constant complexity outer approximation is certainly superior to a simple approximation like a minimal enclosing hypersphere with respect to minimising the reduction of the solution space. Similar procedures can be applied if the complexity of the robot is too high.

The linear complexity result leads to better time bounds for a specific class of motion planning algorithms for a robot moving amidst fat obstacles. This class consists of algorithms with a running time that depends on the complexity of the free space. The motion planning algorithm of Sifrony and Sharir [13] for a ladder moving in two-dimensional space is an example of such an algorithm, because its running time is determined by the number of feature pairs that is less than the length of the ladder apart, which, on its turn, is among the factors that determine the complexity of the free space. If the complexity of the free space is linear, then the number of such feature pairs must also be linear. Hence, we obtain an algorithm with improved running time in case of fat obstacles. Algorithms that consider *all* features obviously do not benefit from our result. In [14] we will present an efficient motion planning algorithm with a running time that depends on the complexity of the free space for a robot moving amidst fat obstacles.

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