Perfect colorings

Erwin M. Bakker, Jan van Leeuwen, Richard B. Tan

RUU-CS-90-35
November 1990

Utrecht University

Department of Computer Science

Padualaan 14, P.O. Box 80.089,
3508 TB Utrecht, The Netherlands,
Tel. : ... + 31 - 30 - 531454

Perfect colorings

Erwin M. Bakker, Jan van Leeuwen, Richard B. Tan

Technical Report RUU-CS-90-35
November 1990

Department of Computer Science
Utrecht University
P.O.Box 80.089
3508 TB Utrecht
The Netherlands

ISSN:0924-3275

Perfect Colorings *

Erwin M. Bakker , Jan van Leeuwen
Department of Computer Science, University of Utrecht
P.O.Box 80.089, 3508 TB Utrecht, the Netherlands

Richard B. Tan
Department of Computer Science, University of Utrecht
and
University of Science & Arts of Oklahoma
Chickasha, OK 73018, USA

Abstract

We study a problem that arises when one considers the segmentation of a file in a network without redundancy. This problem translates to a graph coloring problem: perfect k-colorability. A perfect k-coloring of a graph is a coloring with k colors such that for every node of the graph every color is used exactly once for this node or one of its neighbors. Perfect k-colorings can only exist for $(k-1)$-regular graphs. Determining whether a perfect coloring of a $(k-1)$-regular graph exists is equivalent to finding a distance-2 coloring of the $(k-1)$-regular graph which uses exactly k colors, and to finding a covering of the $(k-1)$-regular graph on K_k.
In this paper we prove several basic results for perfectly k-colorable graphs. We show how to generate all perfectly k-colorable graphs, and we characterize some simple classes of perfectly k-colorable graphs. On the other hand, we prove that the problem of deciding whether a given $(k-1)$-regular graph is perfectly k-colorable is NP-complete for every fixed $k \geq 4$.

1 Introduction

In distributed databases and file-systems data is replicated to achieve greater availability and fault-tolerance. Here the problem of strategically distributing a file over

*This work was partially supported by the ESPRIT Basic Research Actions of the EC under contract no. 3075 (project ALCOM).

a given communication network is of great importance. Typically, communication-, memory-, query- and update-costs have to be minimized by this *strategic* distribution of the file. Constraints like upperbounds on the available memory per site, upperbounds on the number of messages to retrieve data, etc., limit the solution space. Several of the problems that arise have been reviewed in [DF82]. In general these problems are NP-complete.

In this paper we study a very simple variant of the file distribution problem. We assume that the files that must be distributed over the communication network are large and that it is not appropriate to assign a complete copy of the file to every node of the network, as this would mean too much of a burden on the available memory at the nodes. At the same time we do not want to spend too many messages on data retrieval, which means that a centralized solution is not desired either. This leads to the following segmentation problem for files in a network: *Given a connected network modelled by a graph $G = (V, E)$ and a file F, assign to each node $v \in V$ a proper segment $F_v \subsetneq F$ such that for all $v \in V$ the union of F_v and all the filesegments F_w assigned to nodes w within a fixed distance d of node v is equal to F.*

Although there are many interesting versions of this problem, we only study a very restricted form of it here. We assume that the file is to be divided into a fixed number of nonoverlapping segments F_i and we ask for an assignment of these segments to the nodes of the network such that (i) each node holds exactly one segment and (ii) each node together with all its neighbors contains all the segments without any redundancy. This problem can be reformulated as a "perfect" coloring problem for the underlying graph. *A perfect k-coloring of a graph is a node coloring of the graph with k colors such that for every node of the graph we have that every color is used exactly once for this node or one of its neighbors.* It is shown in [KL90a] that several classical regular processor interconnection-networks such as the d-dimensional hypercube, the d-dimensional torus, the cube connected cycles and the chordal ring are perfectly colorable. A main result of this paper states that the problem of deciding the perfect colorability for regular graphs in general is NP-complete.

The paper is organized as follows. In Section 2 we state several elementary results concerning perfectly colorable graphs, and characterize some simple classes of perfectly colorable graphs. In Section 3 two alternative characterizations of perfectly k-colorable graphs are given. Firstly, we show that a $(k-1)$-regular graph G is perfectly k-colorable iff G has a distance-2 coloring with k colors. Secondly, we prove that G is perfectly k-colorable iff there exists a covering of G on the complete graph K_k. In Section 4 we give an effective procedure for generating all perfectly colorable graphs. In Section 5 we show that the problem of deciding whether a given $(k-1)$-regular graph is perfectly k-colorable is NP-complete for every fixed $k \geq 4$. Finally in Section 6 some conclusions and open problems are given.

2 Basic Properties of Perfectly Colorable Graphs

In this section we introduce *perfectly (k-)colorable graphs* and discuss some of their elementary properties.

Notation: Let $G = (V, E)$ be a graph, with $V = V(G)$ the set of nodes and $E = E(G)$ the set of edges. We usually denote $|V|$ by n. Let $C = \{c_1, ..., c_k\}$ be a set of k colors, $k \in \mathbb{N}$, $k \geq 1$. Let the nodes of G be colored with the colors from C. We denote the color of a node v by $c(v)$. For each set of nodes $V' \subseteq V$ we let $\mathcal{C}(V') \subseteq C$ denote the set of colors of the nodes $v \in V'$, i.e., $\mathcal{C}(V') = \{c(v) \mid v \in V'\}$.

Notation: Let $G = (V, E)$ be a graph. For every node $v \in V$ the set of immediate neighbors of v is defined as the set $N_1(v) = \{w \mid w \in V \text{ and } (v, w) \in E\}$, and we let $N_1[v] = N_1(v) \cup \{v\}$.

Definition 2.1 *Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$, $k \geq 1$. G is perfectly k-colorable iff there exists a coloring of G using colors from $C = \{c_1, ..., c_k\}$ such that for all $v \in V : \mathcal{C}(N_1[v]) = C$ and there do not exist $w_1, w_2 \in N_1[v]$, $w_1 \neq w_2$: $c(w_1) = c(w_2)$.*

Note that a perfect k-coloring is a very special kind of k-coloring. We will assume throughout that $k \in \mathbb{N}$, $k \geq 1$ whenever (perfect) k-colorings are considered.
Consider a perfect k-coloring of a perfectly k-colorable graph $G = (V, E)$. Then for all nodes $v \in V : \mathcal{C}(N_1[v]) = C$ and every two nodes of the neighborhood $N_1[v]$ have different colors. It follows that $|N_1[v]| = k$. Hence G necessarily is a $(k-1)$-regular graph.

Definition 2.2 *Let $G = (V, E)$ be a graph with a given perfect k-coloring with colors $C = \{c_1, ..., c_k\}$. For all $i \in \{1, ..., k\}$ we define the* colorclass *$V_i \subseteq V$ as the set of nodes $v \in V$ that have color c_i.*

Notation: If a graph G has a perfect k-coloring with colorclasses $V_1, ..., V_k$, then we denote the perfect k-coloring by $[V_1, ..., V_k]$.

Recall that a dominating set of a graph G is any set of nodes such that every node of G is either an element of this set or a neighbor of at least one element of this set. The next theorem states that every colorclass of a perfectly colored graph is a minimum dominating set of this graph. From the proof it is obvious that this minimum dominating set is also a maximal independent set, i.e., a maximal set such that no two nodes of this set are adjacent. (In fact, it can be observed that no two nodes of this independent set can have distance less than 2 from each other.) Hence every perfect coloring consists of a collection of rather special, mutually disjoint maximal independent sets.

Theorem 2.3 *Let $G = (V, E)$ be a graph with a perfect k-coloring $[V_1, ..., V_k]$. Then V_i is a minimum dominating set and of size n/k, for every $i \in \{1, ..., k\}$.*

Proof. Let $i \in \{1, ..., k\}$. By definition we have $\mathcal{C}(N_1[w]) = C$ for all $w \in V - V_i$. Thus for every $w \in V$ there exists a $v \in V_i$ such that $w \in N_1[v]$. Hence V_i is a dominating set of G and $\bigcup_{v \in V_i} N_1[v] = V$. By definition, there exists no node that has two neighbors with the same color. It follows that for all $v_1, v_2 \in V_i$, $v_1 \neq v_2$: $N_1[v_1] \cap N_2[v_2] = \phi$. From this we conclude that $\sum_{v \in V_i} |N_1[v]| = |V|$. G necessarily is a $(k-1)$-regular graph, thus $|N_1[v]| = k$, for every $v \in V$. It follows that $|V_i| .k = |V|$ and thus $|V_i| = n/k$ for every i. This is the minimum size a dominating set of a $(k-1)$-regular graph can have. Therefore V_i must be a minimum dominating set of G. □

A perfect coloring $[V_1, ..., V_k]$ of a graph consists of a collection of mutually disjoint maximal independent sets V_i. It is well-known [AM70] that if a graph $G = (V, E)$ has M maximal independent sets, these sets can be enumerated in $O(\text{poly}(|V|).M)$ time. Now one can do a backtrack search on the M maximal independent sets to search for a perfect coloring of G. Hence for all classes of graphs with a polynomially bounded number of maximal independent sets this gives a polynomial time algorithm for deciding perfect colorability. For example, the class of chordal graphs is such a class.
However, this may not be the most efficient way to determine whether a given graph is perfectly colorable. In [KL90a] several classes of graphs have been studied for which there exist nice characterizations of the perfectly colorable members. There it is shown that the following regular interconnection networks are perfectly colorable:

- The hypercube C_n, if and only if $n = 2^i - 1, i > 0$.
- The d-dimensional torus of size $l_1 \times ... \times l_d$ if $l_i \bmod q = 0$, with q such that $\sqrt[r]{2d+1} \mid q$ for some integer $r > 0$.
- The cube connected cycles CCC_d, if and only if $d > 2$, $d \neq 5$.
- The chordal ring network with chord length $4p-1 (p > 0)$ and $4kp-t (0 \leq t < p)$ nodes, if and only if :
  1. k and t are even and (if $t > 0$) $\frac{t}{\gcd(t,p)}$ is even, or
  2. k, $\frac{t}{\gcd(t,p)}$ and $\frac{p}{\gcd(t,p)}$ are odd and $t + p$ is even.
- The hexagonal network of size $m \times n$, if and only if $m, n \bmod 7 = 0$.

Thus, perfect colorability is an interesting notion for further analysis.
We now derive a number of general properties of perfectly colorable graphs. The following simple, but useful fact is immediate from Theorem 2.3.

Corollary 2.4 *If $G = (V, E)$ is perfectly k-colorable, then $k|n$ (k divides n).*

A similar technique of node-partitioning as used in the proof of Theorem 2.3 can be applied to obtain the following interesting characteristic of all perfectly k-colorable graphs (for all $k \geq 3$).

Theorem 2.5 *Every connected perfectly k-colorable graph is biconnected (for $k \geq 3$).*

Proof. Let G be a connected perfectly k-colorable graph, and assume that some perfect k-coloring of G is given (some $k \geq 3$). Suppose that G is not biconnected. Consider any biconnected component H of G that is an *end-block*, i.e., a biconnected component that contains exactly one cutpoint x of G. (The existence of end-blocks follows e.g. by considering the block-cutpoint tree of G, cf. [B79].) Clearly x is the only node in H with an "incomplete" neighborhood, as it is the only node that has one or more neighbors outside of H. Yet all the remaining nodes of H must have $(k-1)$ neighbors inside H each, by the perfect coloring. This is not possible, by the following argument. Let x have ε neighbors inside H, for some $1 \leq \varepsilon \leq (k-2)$, including some node y. (Note that y must have at least one more neighbor inside H, thus H is a non-trivial block.) Let $c(x) =$"red" and $c(y) =$"blue".
Let H contain r red nodes (including x) and b blue nodes (including y). By the perfect coloring, the neighborhoods of the red nodes partition H and thus : $|H| = (r-1)k + 1 + \varepsilon$. But so do the neighborhoods of the blue nodes, which implies that $|H| = bk$. We conclude that $(r-1)k + 1 + \varepsilon = bk$ and hence that $k \mid (1 + \varepsilon)$. Contradiction. □

It would be of interest to have a complete characterisation of the perfectly k-colorable graphs in simple, graph-theoretic terms. It is easy to see that for $k = 1, 2$ a graph G is perfectly k-colorable iff G is $(k-1)$-regular.

Proposition 2.6 *A connected graph $G = (V, E)$ is perfectly 3-colorable iff G is 2-regular and $3|n$.*

Proof. $\Leftarrow$ The only connected graphs that are 2-regular are rings. If the number of nodes of a ring is a multiple of 3 then a perfect 3-coloring of the ring is right at hand: assign the colors a, b and c alternatingly clockwise around the ring.
$\Rightarrow$ If a graph is perfectly 3-colorable, then it must be 2-regular. By Corollary 2.4 we know that $3|n$. □

A similar characterization of the perfectly 4-colorable graphs is not known. In fact, we will show in Section 5 that the problem of deciding whether a 3-regular graph is perfectly 4-colorable is NP-complete, which makes it very unlikely that an easy characterization of perfect 4-colorability exists or, indeed, of perfect k-colorability for any $k \geq 4$. We do have the following necessary condition.

Lemma 2.7 *If* $G = (V, E)$ *is perfectly* 4*-colorable, then* G *cannot have simple cycles of length* 5.

Proof. Assume G is perfectly 4-colorable. Let (v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1) be a simple cycle in G of length 5. Then we can assume w.l.o.g. that v_1, v_2 and v_3 are colored with different colors c_1, c_2 and c_3, respectively. Node v_4 cannot have color c_2 or c_3, nor can it have color c_1. Therefore v_4 must have color c_4. But similarly v_5 cannot have colors c_1, c_2, c_3 and c_4. Hence this node must be colored with a fifth color, in contradiction to the fact that G is perfectly 4-colorable. □

By a similar argument one shows that a perfectly 4-colorable graph cannot have $K_{2,3}$ as a subgraph. ($K_{2,3} = (\{v_1, ..., v_5\}, \{v_1, v_2\} \times \{v_3, v_4, v_5\})$ is a complete bipartite graph.)
If one colorclass of a perfectly 4-colorable graph is deleted, a perfectly 3-colorable graph remains, i.e., a collection of rings of size divisible by 3. In general the following holds.

Theorem 2.8 *Let* $G = (V, E)$ *be a perfectly* k*-colorable graph, and let* $[V_1, ..., V_k]$ *be a perfect* k*-coloring of* G*. Then for every* $i \in \{1, ..., k\}$ *the subgraph* G' *of* G *induced by* $V - V_i$ *is perfectly* $(k-1)$*-colorable and* $[V_1, ..., V_{i-1}, V_{i+1}, .., V_k]$ *is a perfect* $(k-1)$*-coloring of* G'*.*

Proof. Let $i \in \{1, ..., k\}$. Let $G' = (V', E')$ be the subgraph of G induced by $V - V_i$. Consider the coloring $[V_1, ..., V_{i-1}, V_{i+1}, .., V_k]$ of G'. Then $\mathcal{C}(N_1[v]_{G'}) = \mathcal{C}(N_1[v]_G) - \{c_i\}$ for every $i \in \{1, ..., k\}$. The condition that colors of neighboring nodes are different is unaffected. Hence it is a perfect $(k-1)$-coloring of G'. □

Looking at its colored neighborhoods, every perfectly k-colorable graph seems to have some resemblances to the complete graph K_k. For example, every perfectly 3-colorable graph contains subgraphs H that are homeomorphic to K_3 (see e.g. [H69] for the definition of homeomorphic graphs) and for every node v a subgraph H can be found that is homeomorphic to K_3 and that contains v. (This follows immediately from the simple structure of perfectly 3-colorable graphs.) On the other hand, this is not true in general. For example, there are perfectly 5-colorable graphs that contain no homeomorphic copy of K_5 at all. The following result appears to be the strongest that one can get.

Theorem 2.9 *For* $k \geq 4$*, every perfectly* k*-colorable graph* G *contains a subgraph* H *that is homeomorphic to* K_4 *and for every node* v *a subgraph* H *can be found that is homeomorphic to* K_4 *and contains* v*.*

Proof. Let G be a perfectly k-colorable graph, v a node of G, and assume that v is colored "d". Choose three additional colors "a", "b" and "c" and omit all nodes that have a color that is not in the set $\{a, b, c, d\}$. By the previous theorem it is clear

that this leaves a set of perfectly 4-colored components. As we will no longer be referring to the original graph, we let G denote the component that contains v. We will be exploiting the following kind of argument. Let α, β and γ be three different colors and w a node with $c(w) = \alpha$. Starting at w and going to α-, β- and γ-colored neighbors alternatingly, traces a uniquely defined path through G (because of the perfect coloring condition) that must ultimately close itself at w again. It is thus appropriate to speak of the "$\alpha\beta\gamma$-cycle" that starts at w. (Note e.g. that the $\alpha\gamma\beta$-cycle that starts at w traces the same cycle, in opposite order.)
Omitting the d-colored nodes for a moment, the nodes colored a, b and c divide into perfectly 3-colored components which are exactly the abc-cycles of G. Thus G consists of "islands" (abc-cycles) and d-colored nodes, and edges between islands and d-colored nodes such that every d-node has exactly one neighbor of the colors a, b and c respectively.
We will now show that there is a subgraph H that is homeomorphic to K_4 and contains v. Let v_a, v_b and v_c be the neighbors of v in G. We assume w.l.o.g. that $c(v_a) = a$, $c(v_b) = b$ and $c(v_c) = c$. Let D be the dca-cycle that starts at v. Observe that it first visits v_c, traces through a number of islands (entering a c-colored node and leaving at the neighboring a-colored node), and ends by visiting v_a and making a final step from v_a to v. Now consider the dba-cycle that starts at v. (Call it E.) E first visits v_b and then traces through G until it reaches the b-colored neighbor of v_a, steps to v_a and finishes at v. Trace E from v onward, and let C' be the first island reached that is also visited by D. (Note that C' exists, because D and E ultimately meet on at least one island, namely the island that contains v_a. Also note that E cannot hit D for the first time in a d-colored node, because it would violate the requirement that this node has only one a-colored neighbor.) Let w_c and w_a be the two consecutive nodes on C' as they appear along D, and let w_b be the node where E hits on C'. We necessarilly have $c(w_c) = c$, $c(w_a) = a$ and $c(w_b) = b$. It is now easy to see that the nodes v, w_a, w_b and w_c span a homeomorphic copy of K_4. (Note that w_a, w_b and w_c are connected along C', and v is connected to w_c and w_a by the disjoint halves of D and to w_b by the part of E that leads to it.) □

Observe that if the graph G in Theorem 2.8 is bipartite then G' is also bipartite because no new edges have been inserted. For bipartite graphs we have the following necessary conditions to be perfectly colorable.

Lemma 2.10 *Let $G = (V, E)$ be a perfectly k-colorable bipartite graph, then $2k|n$.*

Proof. Let G be bipartite and let $[V_1, ..., V_k]$ be a perfect k-coloring of G. By Corollary 2.4 we know that for all $i \in \{1, ..., k\} : |V_i| = n/k$. Let $G' = (V', E')$ be the subgraph of G induced by $V - (V_4 \cup ... \cup V_k)$. From Theorem 2.8 and the previous observation it is clear that G' is a perfectly 3-colorable bipartite graph. Hence G' consists of one or more bipartite perfectly 3-colorable rings R_j. As R_j is bipartite, we have that $2|\,|R_j|$. And as R_j is perfectly 3-colorable we also have that $3|\,|R_j|$.

Hence for all $i \in \{1, ..., k\}$ we have: $6 \mid |V'| = 3|V_i|$, hence $2 \mid |V_i| = n/k$ and thus $2k \mid n$. □

Theorem 2.11 *If $G = (V, E)$ is a $(k-1)$-regular bipartite graph with $2k$ nodes, then G is perfectly k-colorable.*

Proof. By induction. It is easy to see that for $k = 1, 2$ the theorem holds. Let $k > 2$. As G is a $(k-1)$-regular bipartite graph there exists a partition of V into, say, V_1 and V_2 with $|V_1|, |V_2| \geq (k-1)$ such that the edges of G connect between nodes of V_1 and V_2 only. Because G is regular, it is easily seen that V_1 and V_2 must have the same size. Hence $|V_1| = |V_2| = k$, and there must be nodes $v \in V_1$ and $w \in V_2$ which are independent, as shown in Figure 1.

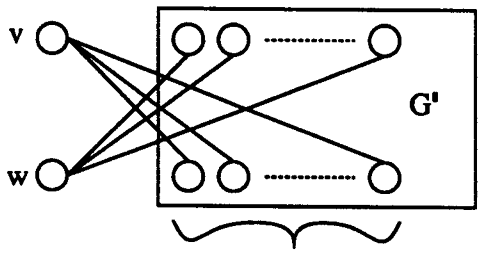

Figure 1

Let G' be the subgraph induced by $V - \{v, w\}$. Note that G' is a $(k-2)$-regular bipartite graph with $2(k-1)$ nodes, and that v is connected to all nodes in the bottom row and w is connected to all nodes in the upper row. By induction G' is perfectly $(k-1)$-colorable, say with perfect coloring $[V_1, ..., V_{(k-1)}]$. Then it is clear that $[V_1, ..., V_{(k-1)}, \{v, w\}]$ is a perfect k-coloring of G. □

We note that Theorem 2.11 cannot be generalized. There exist $(k-1)$-regular bipartite graphs $G = (V, E)$ with $2k \mid n$ that are not perfectly k-colorable. An example is given in Figure 2.

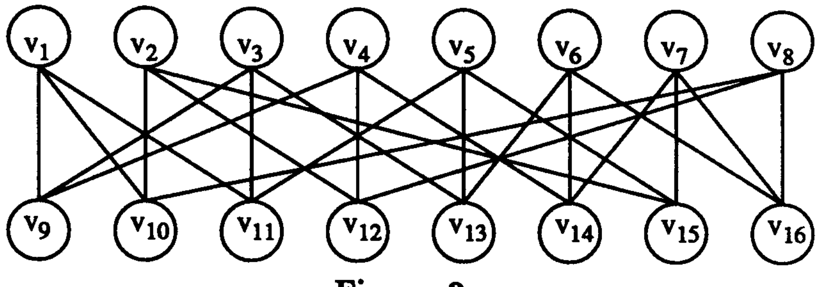

Figure 2

The graph in Figure 2 is a 3-regular bipartite graph with 16 nodes that is not perfectly 4-colorable, although $2.4 \mid 16$.

Recall that a Hamiltonian circuit in a graph G is a simple circuit that includes all nodes of G. If G has a Hamiltonian circuit, then G is called Hamiltonian. We have the following result.

Theorem 2.12 *Every perfectly k-colorable graph with $\leq 2k$ ($k \geq 3$) nodes is Hamiltonian.*

Proof. The result is obvious for perfectly k-colorable graphs with k nodes. For perfectly k-colorable graphs with $2k$ nodes we will prove the theorem by induction on k. It is easy to verify that for $k = 3$ the theorem holds. Let $k > 3$ and $G = (V, E)$ be a perfectly k-colored graph with $2k$ nodes. Let $v, w \in V$ be two nodes that have the same color. Let G' be the subgraph of G induced by $V - \{v, w\}$. Then G' is perfectly $(k-1)$-colorable and has $2(k-1)$ nodes. By induction there exists a Hamiltonian circuit H in G'. Consider H as a subgraph of G. Let V_v and V_w be the sets of nodes that are connected to v and w, respectively. By the perfect coloring $V_v \cup V_w = V'$ and necessarily $|V_v| = |V_w| = (k-1)$. If there are nodes $v_1, v_2 \in V_v$ such that $(v_1, v_2) \in H$, then there must also exist nodes $w_1, w_2 \in V_w$ such that $(w_1, w_2) \in H$. In this case it follows that G is Hamiltonian: just traverse H and, upon reaching v_1 and w_1, side-step to v and w and return at v_2 and w_2 respectively and continue along H. On the other hand, if there do not exist nodes $v_1, v_2 \in V_v$ such that $(v_1, v_2) \in H$, then there do not exist $w_1, w_2 \in V_w$ such that $(w_1, w_2) \in H$. In this case, let $u_1, ..., u_4 \in V'$ such that $(u_1, u_2), (u_2, u_3), (u_3, u_4) \in H$. W.l.o.g. we may assume that $u_1 \in V_v$, and hence that $u_3 \in V_v$ and $u_2, u_4 \in V_w$. It is now easily verified that $H' = (H - \{(u_1, u_2), (u_3, u_4)\}) \cup \{(u_1, v), (v, u_3), (u_2, w), (w, u_4)\}$ constitutes a Hamiltonian circuit in G. □

3 Distance-2 Colorings, Coverings and Perfect Colorability

In this section we consider two different ways of looking at perfect colorings. We first consider the relation to distance-d coloring, which is a well-known generalization of the traditional vertex coloring problem. Applications of this problem can be found in e.g. computing approximations to sparse Hessian matrices [Mc81]. It turns out that perfect colorability and distance-2 colorability are strongly related notions.

Definition 3.1 *Let $G = (V, E)$ be a graph and $d \in \mathbb{N}, d \geq 1$. A distance-$d$ coloring of G is a coloring of G such that if $v, w \in V$ are nodes such that $c(v) = c(w)$, then $d_G(v, w) > d$. (Here $d_G(.,.)$ is the usual distance metric on graphs.)*

For our purposes only distance-2 colorings are interesting. These colorings have also be termed "strong colorings" and have been studied in detail in [KL90b].

Lemma 3.2 *Let $G = (V, E)$ be a $(k-1)$-regular graph. Then G is perfectly k-colorable if and only if G has a distance-2 coloring with k colors.*

Proof. $\Rightarrow$ Let $[V_1, ..., V_k]$ be a perfect k-coloring of G. Let v be a node of G. If $v \in V_i$, then none of its neighbors is an element of V_i or has another node of V_i as neighbor. Hence if $v, w \in V_i$, then $d_G(v, w) > 2$.
$\Leftarrow$ Assume G has a distance-2 coloring with k colors. G is $(k-1)$-regular, thus for all $v \in V(G) : |\mathcal{C}(N_1[v])| = k$. From this it is clear that the coloring is also a perfect k-coloring of G. □

In [Mc81] it is shown that for arbitrary graphs the problem of deciding whether a given graph is distance-d colorable with k colors is NP-complete for every $d \geq 1$. In Section 5 we shall prove that the problem of deciding whether a given regular graph is perfectly k-colorable is NP-complete for every fixed $k \geq 4$. Hence the distance-2 colorability problem is NP-complete even when restricted to the class of $(k-1)$-regular graphs, for every fixed $k \geq 4$.
The following result shows that distance-2 colorings essentially are "incomplete" perfect colorings.

Theorem 3.3 *A graph is distance-2 colorable with k colors if and only if it is a subgraph of a perfectly k-colorable graph.*

Proof. $\Leftarrow$ Every subgraph of a perfectly k-colorable, hence distance-2 k-colorable graph clearly is distance-2 k-colorable again.
$\Rightarrow$ Let G be any graph that is distance-2 colorable with k colors. Consider any distance-2 coloring of G with k colors $c_1, ..., c_k$ and let $V_1, ..., V_k$ be the corresponding colorclasses. Assume that the coloring is not perfect, otherwise we are done. In this case the coloring can be viewed as an "incomplete" perfect coloring, in the sense that some neighborhoods lack some of the colors (and thus some nodes lack the necessary "neighbors" for some of the colors). We show that G can be extended to a $(k-1)$-regular graph H that is perfectly k-colorable.
Consider the colorclasses $V_1, ..., V_k$ and let the largest one contain s nodes. Add new nodes to the other colorclasses, when necessary, so they all have size s. This defines the node-set of H and the intended coloring. To define the edge-set of H, we begin by adding in the edges of G. We will now argue that we can add in further edges such that all neighborhoods get all the different colors and the resulting graph is perfectly k-colored. The procedure is simple : whenever there is a node v that "lacks" a color c_i in its neighborhood, look for a node $w \in V_i$ that does not yet have an edge to a node with color $c(v)$ and add the edge (v, w) to H. It is clear that such a node w can always be found because, if all s nodes of V_i had a neighbor colored $c(v)$ already, then all of these neighbor nodes would be distinct (by the fact that we maintain a distance-2 coloring) and all s nodes colored $c(v)$, including v itself, would already have a c_i-colored neighbor. Thus the procedure can continue as long

as there is an incomplete neighborhood and terminates with a graph H that extends G and is perfectly k-colored. □

Observe from the proof of Theorem 3.3 that if G is distance-2 colorable with k colors, then it is a subgraph of a perfectly k-colorable graph H with at most kn nodes.

Corollary 3.4 *A graph G is distance-2 colorable with k colors such that $|V_1| = \ldots = |V_k|$ if and only if G is a subgraph of a perfectly k-colorable graph H with exactly the same number of nodes.*

Theorem 3.3 can be extended, although it tends to require a considerably larger graph H. In [KL90b] the following result is shown.

Theorem 3.5 *A graph G is distance-2 colorable with k colors if and only if it is an induced subgraph of a perfectly k-colorable graph H.*

Another way of approaching perfect colorings is through coverings. A covering is a mapping that maps the nodes of a given graph onto the nodes of an other (smaller) graph such that the local structure of the graph is preserved. Coverings have applications in various fields. For example, in the design of parallel algorithms it is usually assumed that the processors are connected by suitable networks with a number of nodes depending on the size of the problem to be solved. In practice these algorithms will be executed on networks that are much smaller. Therefore in some way the larger, virtual network has to be simulated on the smaller actual network. This problem can be formalized with the help of coverings (cf. [BL86] , [B89]). In this section we will show that perfect k-colorability is related to coverings of K_k, the complete graph on k nodes.

Definition 3.6 *Let $G = (V, E)$ and $H = (V', E')$ be graphs. A mapping $f : V(G) \to V(H)$ covers G on H iff*

1. *f is surjective.*
2. *for all $v, w \in V(G)$: if $(v, w) \in E(G)$, then $(f(v), f(w)) \in E(H)$.*
3. *for all $v \in V(G)$: $|N_1(v)| = |N_1(f(v))|$ and for every $w' \in N_1(f(v))$ there exists a $w \in N_1(v)$ such that $f(w) = w'$.*

We say that G covers H iff there exists a mapping f that covers G on H (or: a covering of G on H).

Theorem 3.7 *A graph $G = (V, E)$ is perfectly k-colorable if and only if G covers K_k.*

Proof. $\Rightarrow$ Let $[V_1, ..., V_k]$ be a perfect k-coloring of the graph G. Let $\{v_1, ..., v_k\}$ be the set of nodes of the complete graph K_k. Define the mapping $f : V(G) \rightarrow V(K_k)$ by $f(v) = v_i$ iff $v \in V_i$. Clearly f is surjective. Furthermore if $(v, w) \in E(G)$, then v and w belong to different colorclasses. Thus $f(v) \neq f(w)$ and $(f(v), f(w)) \in E(K_k)$. G is $(k-1)$-regular and for every $v \in V(G)$ all nodes $w \in N_1[v]$ belong to a different colorclass. Thus it is clear that for all $v \in V(G) : |N_1(v)| = (k-1) = |N_1(f(v))|$ and for every $w' \in N_1(f(v))$ there exists a $w \in N_1(v)$ such that $f(w) = w'$. Hence f covers G on K_k.
$\Leftarrow$ Let $f : V(G) \rightarrow V(K_k)$ be a covering of G on the complete graph K_k. Then for all $v \in V : |N_1(v)| = |N_1(f(v))| = (k-1)$. Thus G is $(k-1)$-regular. Define for every $i \in \{1, ..., k\} : V_i = f^{-1}(v_i) = \{v \mid f(v) = v_i\}$. As f is a covering, the images of a node v and all its neighbors are all different, i.e., they all belong to different classes V_i. Hence it is clear that $[V_1, ..., V_k]$ constitutes a perfect k-coloring of G. □

Corollary 3.8 *Let $H = (V', E')$ be a perfectly k-colorable graph. If $G = (V, E)$ covers H, then G is perfectly k-colorable.*

Proof. It is given that G covers H. H is perfectly k-colorable thus, by Theorem 3.7, it covers K_k. In [B89] it is proven that if G covers H and H covers a further graph I, then G covers I. We conclude that G covers K_k. Hence by Theorem 3.7 G is perfectly k-colorable. □

Define for every $k \in \mathbb{N}, k \geq 1$ the graph $B_{2k} = (V', E')$ with $V' = \{v_1^1, ..., v_k^1, v_1^2, ..., v_k^2\}$, with $(v_1^1, v_i^2), (v_1^2, v_i^1) \in E'$ for all $i \in \{2, ..., k\}$ and $(v_i^1, v_j^2) \in E'$ for all $i, j \in \{2, ..., k\}$, $i \neq j$. The graph B_{2k} is the smallest perfectly k-colorable bipartite graph (see Theorem 2.11). In fact, up to isomorphism it is the unique graph with $2k$ nodes that is bipartite and perfectly k-colorable. This graph can be used to characterize the class of perfectly k-colorable bipartite graphs.

Theorem 3.9 *A graph $G = (V, E)$ is bipartite and perfectly k-colorable if and only if G covers B_{2k}.*

Proof. $\Leftarrow$ We know that B_{2k} is perfectly k-colorable (Theorem 2.11). By Corollary 3.8 it follows that G is perfectly k-colorable. Let $f : V \rightarrow V'$ be a covering of G on B_{2k}. B_{2k} is bipartite, hence we can divide its set of vertices V' into two disjoint sets V_1' and V_2' of equal size such that if v, w are neighbors, then they do not belong to the same set V_1', V_2'. Define $W_i := \{v \mid v \in V \text{ and } f(v) \in V_i'\}$. Then it is clear that W_1, W_2 define a partition on G and that the edges of G connect nodes of W_1 and W_2 only. Hence G is bipartite.
$\Rightarrow$ Let G be bipartite and $[V_1, ..., V_k]$ a perfect coloring of G. Let $V(G)$ be partitioned into W_1 and W_2 respectively. Let V' be divided into the two disjoint sets $V_1' = \{v_1^1, ..., v_k^1\}$ and $V_2' = \{v_1^2, ..., v_k^2\}$. Define the mapping $f : V \rightarrow V'$ by: $f(v) = v_i^1$

iff $v \in V_i$ and $v \in W_1$, and $f(v) = v_i^2$ iff $v \in V_i$ and $v \in W_2$. It is now straightforward to check that f is a covering of G on B_{2k}. □

The preceding results show that the study of perfect k-colorings is really the study of coverings on K_k. There also is a connection between perfect k-colorings and edge-colorings of a graph. Recall that an *edge-coloring* of a graph G is a coloring of the edges such that all edges adjacent to the same node have different colors. The minimum number of colors that are required in an edge-coloring of G is called the *chromatic index* of G.

Lemma 3.10 *Let $G = (V, E)$ and $H = (V', E')$ be graphs such that G covers H. If H is edge-colorable with k colors, then G is edge-colorable with k colors.*

Proof. Let $f : V \to V'$ be a covering of G on H. Assume we have an edge-coloring of H with k colors. By definition of f, if $(v, w) \in E$, then $(f(v), f(w)) \in E'$. Define a coloring of the edges of G by assigning the color of $(f(v), f(w))$ to (v, w), for every $(v, w) \in E$. To verify that this indeed defines a (valid) edge-coloring of G, let v an arbitrary node of G. All its neighbors w are mapped by f to different nodes $f(w)$ in H that are all neighbors of node $f(v)$. Because H was edge-colored, all edges adjacent to $f(v)$ have different colors. Hence all edges adjacent to v must have different colors. We conclude that G has an edge-coloring with k colors. □

A theorem of Vizing [V64] implies that every $(k-1)$-regular graph is edge-colorable with $(k-1)$ or k colors. In [LG83] it is proven that it is NP-hard to determine the chromatic index of regular graphs. We can however conclude the following result, first observed in [KL90b].

Theorem 3.11 *If G is perfectly k-colorable and k is even, then G is edge-colorable with $(k-1)$ colors.*

Proof. Let $k \in \mathbb{N}, k \geq 1$, be even. Then K_k is edge-colorable with $(k-1)$ colors [K36] . As G is perfectly k-colorable we know from Theorem 3.7 that G covers K_k. By Lemma 3.10 we conclude that G is edge-colorable with $(k-1)$ colors. □

Note that $(k-1)$-regular bipartite graphs are always edge-colorable with $(k-1)$ colors [K36]. However, for every k odd it is possible to construct perfectly k-colorable graphs that are not bipartite but edge-colorable with $(k-1)$ colors. The problem of finding an efficient algorithm that computes the chromatic index of perfectly colorable graphs is still open.

4 Generating All Perfectly Colorable Graphs

There are several approaches to the generation of perfectly colorable graphs. One approach uses the fact that, if we delete one colorclass of a perfectly k-colorable graph, we obtain a graph that is perfectly $(k-1)$-colorable (Theorem 2.8). In this approach we start with a perfectly 1-colorable graph, and by inserting new color-classes we can build every perfectly k-colorable graph. The other approach uses the fact that for a perfectly k-colorable graph G there exists a covering on K_k and that all perfectly k-colorable graphs are characterized in this way (Theorem 3.7). Here we start with the appropriate number of complete graphs K_k and by rearranging pairs of edges we can build every perfectly k-colorable graph. In this paper we only discuss the last method.

Lemma 4.1 *Let $G = (V, E)$ be a perfectly k-colorable graph with perfect coloring $[V_1, ..., V_k]$. Let $v_1, v_2 \in V_i$ and $w_1, w_2 \in V_j, (i \neq j)$ be such that $(v_1, w_1), (v_2, w_2) \in E$. Then the graph $G' = (V, E')$ with $E' = (E - \{(v_1, w_1), (v_2, w_2)\}) \cup \{(v_1, w_2), (v_2, w_1)\}$ has perfect coloring $[V_1, ..., V_k]$ also.*

Proof. By deleting the edges (v_1, w_1) and (v_2, w_2) the given coloring of G is not a perfect coloring anymore. This is because nodes v_1 and v_2 do not have any neighbor with color $c(w_1)$ and $c(w_2)$, respectively, anymore and similarly nodes w_1 and w_2 do not have any neighbor with color $c(v_1)$ and $c(v_2)$, respectively. This situation is resolved by inserting the new edges (v_1, w_2) and (v_2, w_1). □

Let $k \in \mathbb{N}, k \geq 1$ be given. By Theorem 2.3 it is clear that only perfectly k-colorable graphs $G = (V, E)$ can be constructed for which $|V| = tk$ for some $t \in \mathbb{N}$. This and the previous lemma are the main motivations for the following algorithm. On inputs $k, t \in \mathbb{N}$, this algorithm constructs a perfectly k-colorable graph with tk nodes.

Procedure Generate-Perfectly-Colorable-Graph(k, t)
{ This procedure generates a perfectly k-colorable graph G_i with tk nodes. }
$G_0 := \{H_i \mid i \in \{1, ..., t\}$, each H_i a "new" copy of $K_k\}$;
Let $[V_1, ..., V_k]$ be a perfect k-coloring of G_0 ;
$i := 1$;
repeat
Take two different edges $(v_1, w_1), (v_2, w_2) \in E(G_{(i-1)})$
such that $c(v_1) = c(v_2)$ and $c(w_1) = c(w_2)$;
$G_i := (G_{(i-1)} - \{(v_1, w_1), (v_2, w_2)\}) \cup \{(v_1, w_2), (v_2, w_1)\}$;
$i := i + 1$
until i = finished
{ "finished" depends on the desired number of iterations, which can be taken to be some random number. }

It is easily shown by induction that every graph obtained from the procedure must be perfectly k-colorable (and has tk nodes). Because no pair of edges needs to be selected more than once, the number of iterations in the **repeat-until** loop need not be larger than $O(t^2k^4)$. In the next theorem we prove that the procedure is capable of generating every possible perfectly k-colorable graph with tk nodes.

Theorem 4.2 *Every perfectly k-colorable graph* $G = (V, E)$ *can be constructed by the procedure* Generate-Perfectly-Colorable-Graph.

Proof. Let $G = (V, E)$ be an arbitrary perfectly k-colorable graph with perfect coloring $[V_1, ..., V_k]$. Assume w.l.o.g. that $t = |V_1|$. By Theorem 2.3 we know that $P = \{N_1[v] \mid v \in V_1\}$ is a partition of G into t disjoint sets. Let $G' = G_0 = \{H_i \mid i \in \{1, ..., t\}, H_i$ a copy of $K_k\}$. It is clear that G' is perfectly k-colorable. Let $[V_1', ..., V_k']$ be a perfect coloring of G'. By mapping the elements of P to the elements of G' in a coloring preserving way, it follows that there is a function $f : V_G \to V_{G'}$ such that:

1. f is a bijection.

2. for all $i \in \{1, ..., k\} : (v \in V_i$ iff $f(v) \in V_i')$.

3. for all $v \in V_1$ and $w \in V$: if $(v, w) \in E_G$, then $(f(v), f(w)) \in E_{G'}$.

Color all edges $(v, w) \in E_G$ with $(f(v), f(w)) \in E_{G'}$ "white" and color all the remaining edges of G "black". If there are no black edges in G, then G and G' are isomorphic and we are done. Thus assume there are black edges in G. We will show that there exists an execution of the algorithm such that in each iteration of the **repeat-until** loop the number of black edges in G decreases by at least one, and which stops the moment all edges of G are colored white.
Assume that $(v, w) \in E_G$ is a black edge, i.e., $(f(v), f(w)) \notin E_{G'}$. We want the procedure to pick the right edges such that it transforms G' into a new graph that continues to be perfectly colored but contains the edge $(f(v), f(w))$. As G' is perfectly colored there exist neighbors w' and v' of $f(v)$ and $f(w)$, respectively, with $c(w') = c(f(w))$ and $c(v') = c(f(v))$. Observe that $(v', w') \notin E_{G'}$ and also that $(f^{-1}(w'), v) \notin E_G$ and $(f^{-1}(v'), w) \notin E_G$ because v and w already block the corresponding colors in their neighborhoods. Now let the procedure choose the edges $(f(v), w')$ and $(f(w), v')$ and transform G' into the graph $(G' - \{(f(v), w'), (f(w), v')\}) \cup \{(f(v), f(w)), (v', w')\}$. Then the edge (v, w) is colored white in the new graph G'. Furthermore, if $(f^{-1}(v'), f^{-1}(w')) \in E_G$, then it is clear this was a black edge that is now colored white also, as (v', w') is now an edge of G'. If $(f^{-1}(v'), f^{-1}(w')) \notin E_G$, then there are no extra changes of colors of edges of G. This shows that there exists an execution of the procedure such that at each iteration of the **repeat-until** loop the number of black edges decreases by at least one. If we let the procedure stop the moment all edges of G are colored white, the function f is an isomorphism from G to G'. Hence G can be constructed by the procedure. □

Observe that the procedure could take any perfectly k-colored graph on tk nodes as a start instead of the particular graph G_0, and the result of Theorem 4.2 would continue to hold. It would be interesting to analyze the probability that a certain perfectly k-colorable graph with tk nodes is generated by the procedure above and to determine whether it is sufficiently "random". Another interesting open problem is to determine the exact number of different perfectly k-colorable graphs with tk nodes.

5 The Complexity of Perfect Colorability

For $k = 1, 2, 3$ we were able to completely characterize the classes of perfectly k-colorable graphs. This characterization gave us an easy test to determine whether a given graph from one of these classes is perfectly k-colorable. For $k \geq 4$, only some necessary conditions for a graph to be perfectly k-colorable are known. We consider the following problem (for fixed $k \geq 4$).

Problem: *PERFECT k-COLORABILITY*
Instance: A $(k-1)$-regular graph $G = (V, E)$.
Question: Is the graph perfectly k-colorable ?

Our main result is the following. (We assume that the reader is familiar with the theory of NP-completeness, cf. [GJ79].)

Theorem 5.1 *PERFECT 4-COLORABILITY is NP-Complete.*

Proof. The perfect 4-colorability problem is clearly in NP, since a non-deterministic algorithm can always guess a coloring of the given graph and check whether this coloring is a perfect 4-coloring in polynomial time.
To prove the problem NP-complete we use a transformation from 3-SAT:

Problem: 3-*SAT*
Instance: A set $X = \{x_1, ..., x_n\}$ of variables. A collection $C = \{c_1, ..., c_m\}$ of clauses over X such that each clause $c \in C$ has $|c| = 3$.
Question: Is there a satisfying truth assignment for C ?

Assume an instance of 3-SAT is given. We show that the instance of 3-SAT can be transformed to a graph $G = (V, E)$ such that: G is perfectly 4-colorable iff there is a satisfying truth assignment for the set of clauses of the given instance of 3-SAT. The transformation incorporates several types of "building blocks", i.e., subgraphs, that are used to construct the graph G. There are 4 different types of subgraphs.

The subgraphs of Type 1, Type 2, Type 3 and Type 4 are shown in Figure 3a, 4a, 5a and 6a, respectively. The graph G and its important components are pictured in Figure 7 and beyond. To simplify the drawings, the diagrams shown in Figures 3b, 4b, 5b and 6b are used to denote the subgraphs of Type 1, Type 2, Type 3 and Type 4, respectively.

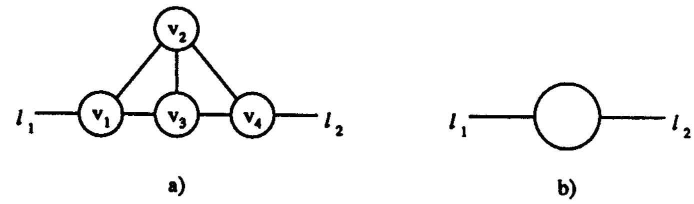

Figure 3

In Figure 3a the subgraph of Type 1 is shown. A pair of these subgraphs will correspond to a variable x_i in the given instance of 3-SAT. In G the nodes v_1 and v_4 are connected via links l_1 and l_2 respectively, to other subgraphs of different types. Clearly, if G is perfectly 4-colorable, then every node of this subgraph will have a different color. Thus the colors of the nodes v_2, v_3 form a pair that is one of 6 possible colorpairs.

Let $C = \{c_1, ..., c_4\}$ be the set of colors to be used in a perfect 4-coloring of G, then for all $i, j \in \{1, ..., 4\}, i \neq j$, we define $p_{ij} = p_{ji} = \{c_i, c_j\}$, and $\overline{p_{ij}} = C - p_{ij}$. If G has a perfect 4-coloring such that nodes v_2 and v_3 of a subgraph of Type 1 have colors c_i and c_j, with $i, j \in \{1, ..., 4\}, i \neq j$, then we say that this subgraph has a coloring $p_{ij}(= p_{ji})$. The colorings of the subgraph will (intuitively) correspond to the boolean values of the variables used in the instance of 3-SAT.

The nodes v_1 and v_4, which we shall call *boundary nodes*, are colored with the colors $\overline{p_{ij}}$. Whereas the colors of v_2 and v_3 can be interchanged without further implications for the rest of the coloring, this is not the case for the colors of the boundary nodes. This is one of the reasons to introduce the following "building block".

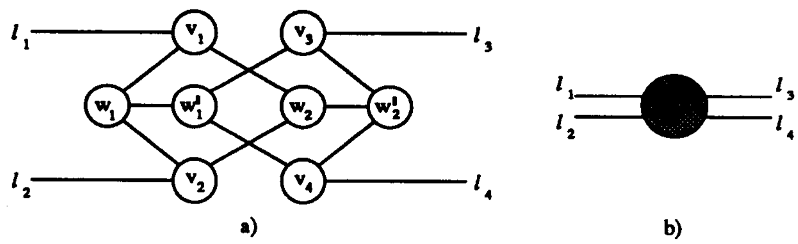

Figure 4

In Figure 4a the subgraph of Type 2 is shown. In G the nodes $v_1, ..., v_4$ are connected via links $l_1, ..., l_4$ to 4 different subgraphs $G_1, ..., G_4$ of Type 1. In a perfect 4-coloring of G the subgraph of Type 1 that is connected via link l_1 has a coloring, say p_{ij}. It is clear that in a perfect coloring of G, the nodes w_1 and w_2 must have colors such

that $\{c(w_1), c(w_2)\} = p_{ij}$. Furthermore, as v_1 and v_2 are connected to w_1 and w_2, node w_1' is the only neighbor of w_1 that can be colored with color $c(w_2)$. Similarly, we must have $c(w_2') = c(w_1)$. Now it is obvious that in this perfect coloring of G we have $\{c(v_1), c(v_2)\} = \{c(v_3), c(v_4)\} = \overline{p_{ij}}$. Hence the 4 neighboring subgraphs of Type 1 must all have the same coloring p_{ij}.
The boundary nodes of $G_1, ..., G_4$ are all colored with colors $\in \overline{p_{ij}}$. If we interchange the colors of the boundary nodes of G_1 this necessarily means that we have to interchange the colors of the boundary nodes of G_2. But the colors of the boundary nodes of G_3 and G_4 may remain the same.

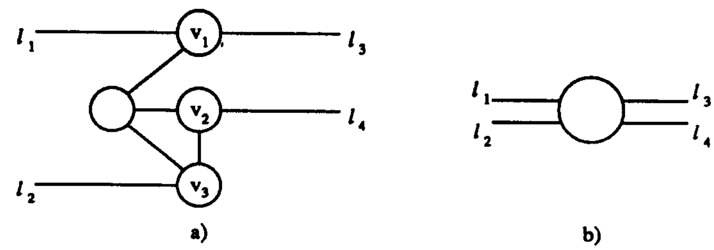

Figure 5

In Figure 5a the subgraph of Type 3 is shown. In G the nodes v_1, v_2, v_3 are connected via links $l_1, ..., l_4$ to 4 different subgraphs $G_1, ..., G_4$ respectively, that are of Type 1. In G the subgraphs G_1 and G_2 will also be connected to the same subgraph of Type 2, and the subgraphs G_3 and G_4 to another subgraph of Type 2. Hence if we have a perfect 4-coloring of G, then G_1 and G_2 must have the same coloring p_{ij}, and G_3 and G_4 must have the same coloring, say p_{kl}. By a similar reasoning as above we can show that the connection to the subgraph of Type 3 ensures that $p_{ij} \neq p_{kl}$ and $p_{ij} \neq \overline{p_{kl}}$. On the other hand, if the colorings p_{ij} and p_{kl} are such that $p_{ij} \neq p_{kl}$ and $p_{ij} \neq \overline{p_{kl}}$, then it is always possible to give the boundary nodes the proper colors. (We have this "freedom of choice" in G because $G_1, ..., G_4$ are connected to subgraphs of Type 2.)

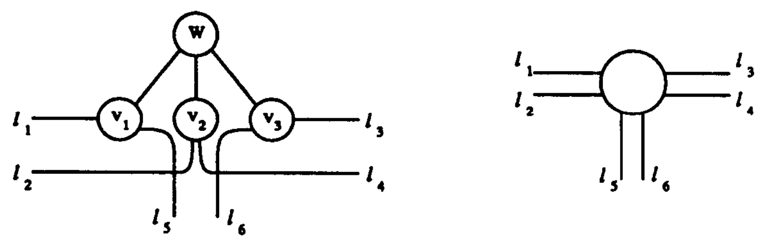

Figure 6

In Figure 6a the subgraph of Type 4 is shown. In G the nodes v_1, v_2, v_3 are connected via links $l_1, ..., l_6$ to 6 different subgraphs $G_1, ..., G_6$ respectively, that are of Type 1. Again the subgraphs form pairs $(G_1, G_2), (G_3, G_4)$ and (G_5, G_6), which are connected to different subgraphs of Type 2. Thus the two elements of the pairs must

have the same coloring. Let the pairs $(G_1, G_2), (G_3, G_4)$ and (G_5, G_6), have colorings p_{ij}, p_{kl} and p_{mn} respectively. It is easy to verify that the following must hold: $c(w) = p_{ij} \cap p_{kl} \cap p_{mn}$, and the three colorings are distinct $(p_{ij} \neq p_{kl} \neq p_{mn} \neq p_{ij})$. Therefore the connection of the pairs to the subgraph of Type 4 ensures that none of the colorings and the complement of the colorings are equal.
Again if, on the other hand, the colorings p_{ij}, p_{kl}, p_{mn} are distinct such that $c(w) = p_{ij} \cap p_{kl} \cap p_{mn}$, then it is always possible to give the boundary nodes the proper colors.

Let $G = (V, E)$ be defined as the graph pictured in Figure 7.

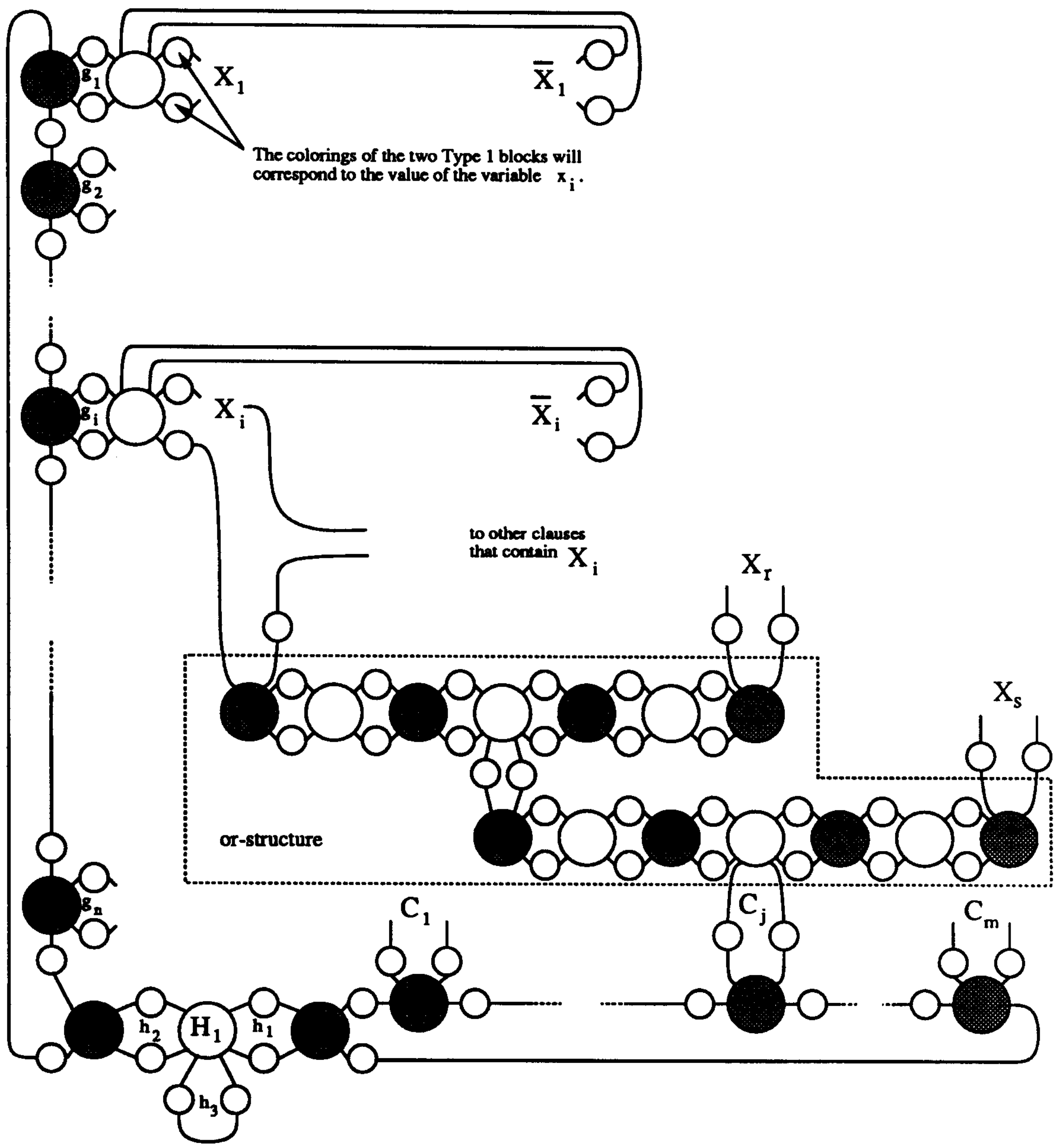

Figure 7

In this graph there are components that correspond to the variables x_i and the clauses c_j of the given instance of 3-SAT. These components are denoted by X_i and C_j respectively. Each variable x_i will be identified with two subgraphs of Type 1 (see Figure 9). The coloring of these subgraphs will correspond to the value of the variable. Also each clause c_j will be identified with two subgraphs of Type 1 (See Figure 8.). Again the coloring of these subgraphs will correspond to the value of the clause. The two subgraphs of Type 1 that correspond to a clause c_j are connected to a structure that has 3 inputs (See Figure 10.). This structure will play the role of an or-gate, and is therefore called an *or-structure.* If clause c_j equals (x_i, x_r, x_s), then the 3 inputs of this structure are connected via subgraphs of Type 1 and 2 with X_i, X_r, X_s respectively, i.e., a component that corresponds to a variable x_i is connected via subgraphs of Type 1 and 2 to an input of every or-structure that is connected to a component corresponding to a clause containing x_i.

Claim: G is perfectly 4-colorable if and only if there is a satisfying truth assignment for C the set of clauses of the given instance of 3-SAT.

Proof. $\Leftarrow$ Assume there is a satisfying truth assignment for C. Assign appropriate truth-values to the variables $x_i \in X$ such that C is satisfied. In the graph G a coloring p_{12} of a subgraph of Type 1 will correspond to **true** and a coloring p_{13} to **false**. A coloring p_{14} is used as an intermediate.

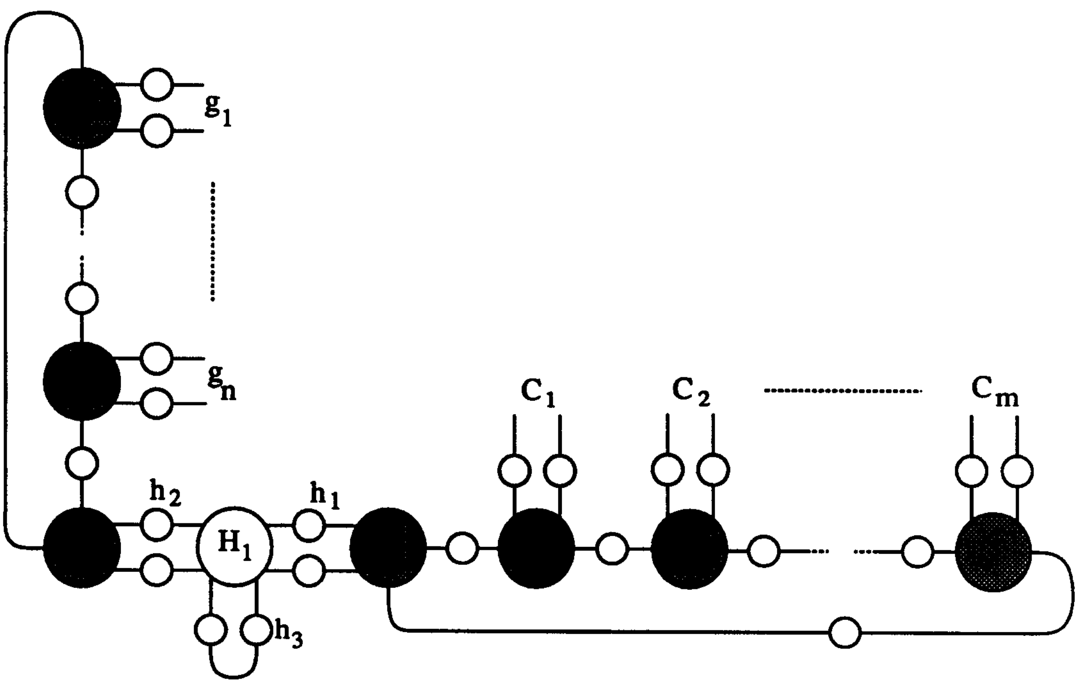

Figure 8

As all the clauses $c \in C$ are true, we color all the subgraphs C_i with coloring p_{12}. Consider Figure 8. Subgraph H_1 is of Type 4 thus we can color the subgraphs h_1, h_2 and h_3 with colorings p_{12}, p_{14} and p_{13}, respectively. Now it is clear that subgraphs $g_1, ..., g_n$ can all be colored with coloring p_{14}.
This leaves us the choice of coloring subgraphs X_i and $\overline{X_i}$ with colorings p_{12} and p_{13}, or p_{13} and p_{12}, respectively. (See Figure 9.) Obviously we color X_i ($\overline{X_i}$) with coloring p_{12} (p_{13}) if the value **true** is assigned to the variable x_i, and we color X_i ($\overline{X_i}$) with coloring p_{13} (p_{12}) if the value **false** is assigned to the variable x_i.

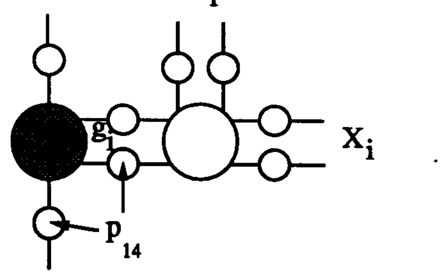

Figure 9

Consider the following subgraph of G:

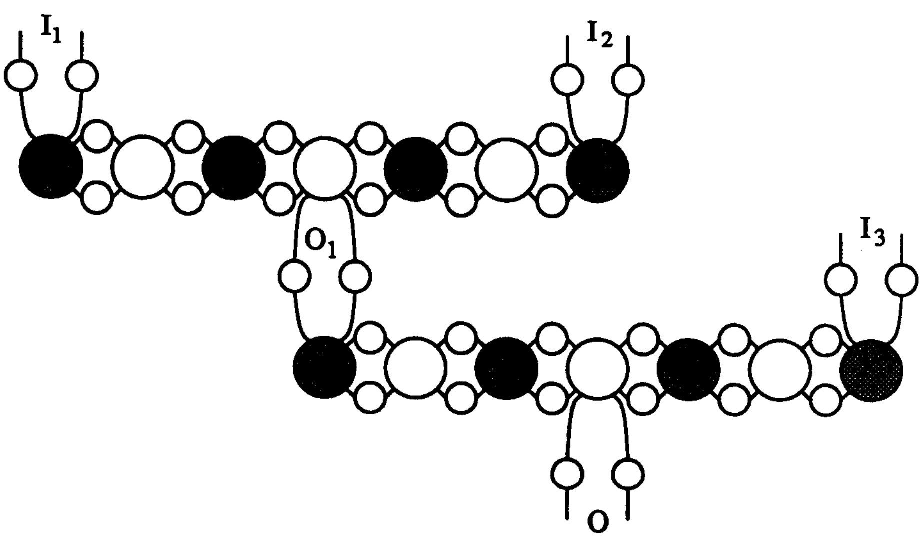

Figure 10

The subgraphs I_1, I_2 and I_3 will have colorings $\in \{p_{12}, p_{13}\}$. It is easy to verify that if both I_1 and I_2 have coloring p_{13}, then subgraph O_1 must have coloring p_{13} (or $\overline{p_{13}}$, but we are free to choose p_{13} in this case). But if at least one of the subgraphs I_1, I_2 has coloring p_{12}, then O_1 can be colored with p_{12}. The truth assignment is such that for each $j \in \{1, ..., m\}$ the clause c_j is true. Therefore, at least one of the subgraphs I_1, I_2, I_3 must have coloring p_{12}. If O_1 cannot be colored with p_{12}, then I_3 must have coloring p_{12}. Thus O can always be colored with coloring p_{12}. Hence

G is perfectly 4-colorable.
$\Rightarrow$ Assume G is perfectly 4-colorable. Let a perfect 4-coloring of G with colors $\{c_1, ..., c_4\}$ be given. Consider a partition of G as depicted in Figure 7. Subgraph H_1 is of Type 4. The 3 neighboring pairs of subgraphs of Type 1, h_1, h_2 and h_3 must all have different colorings, say p_{ij}, p_{kl}, and p_{mn}, respectively. It follows that $p_{ij} \cap p_{kl} \cap p_{mn} \neq \phi$, hence w.l.o.g. $p_{ij} = p_{12}$, $p_{kl} = p_{14}$, and $p_{mn} = p_{13}$. We may also assume that the subgraphs $C_1, ..., C_m$ have coloring p_{12} and that the subgraphs $g_1, ..., g_n$ have colorings p_{14}.
Since g_i is connected to a subgraph of Type 4, the subgraphs X_i and $\overline{X_i}$ can only have colorings $p_{12}, p_{13}, \overline{p_{12}}$ or $\overline{p_{13}}$. Hence these are the possible colorings of the subgraphs I_1, I_2 and I_3 as depicted in Figure 10. It is straightforward to verify that if both I_1 and I_2 have coloring p_{13} or $\overline{p_{13}}$, then subgraph O_1 can only have coloring p_{13} or $\overline{p_{13}}$. Thus if both I_1, I_2, and I_3 have coloring p_{13} or $\overline{p_{13}}$, O would have coloring p_{13} or $\overline{p_{13}}$. But O must have the same coloring as subgraph C_i which is p_{12}. Hence this can never yield a valid perfect 4-coloring. From this we conclude that at least one of the subgraphs I_1, I_2, I_3 must have coloring p_{12} or $\overline{p_{12}}$.
Assign **true** to the variable x_i if X_i has coloring p_{12} or $\overline{p_{12}}$, and assign **false** to the variable x_i if X_i has coloring p_{13} or $\overline{p_{13}}$. It can easily be verified that this defines a satisfying truth assignment for the given instance of 3-SAT. □

Theorem 5.2 *For every fixed $k \in \mathbb{N}, k \geq 4$, PERFECT k-COLORABILITY is NP-complete.*

Proof. Let $k \in \mathbb{N}, k \geq 5$. Clearly PERFECT k-COLORABILITY is in NP. To prove the problem NP-complete we use a transformation from 3-SAT. Again we transform an instance of 3-SAT to a suitable graph G. To construct G we use the same type of "building blocks" as in the previous theorem. The subgraph of Type 1 shown in Figure 11 consists of $K_k - \{w_1, w_2\}$. This subgraph is connected to other types of subgraphs via a link l_1 adjacent to w_1 and a link l_2 adjacent to w_2.
Let $C = \{c_1, ..., c_k\}$ be the set of colors to be used in a perfect coloring of G. If $U = \{c \mid \text{there is an } i \in \{1, ..., (k-2)\} \text{ such that } c(v_i) = c \in C\}$ is the set of colors of $v_1, ..., v_{(k-2)}$ in the subgraph of Type 1 (see Figure 11a), then we say that this subgraph has coloring p_U.

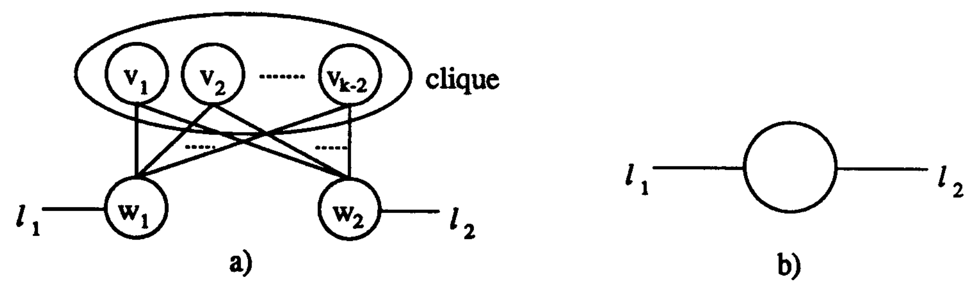

Figure 11

The subgraph of Type 2 is shown in Figure 12a. This subgraph has the same function as the subgraph of Type 2 in the previous theorem, i.e., subgraphs of Type 1 connected to this subgraph must all have the same coloring p_U for some $U \subset C$ with $|U| = (k-2)$.

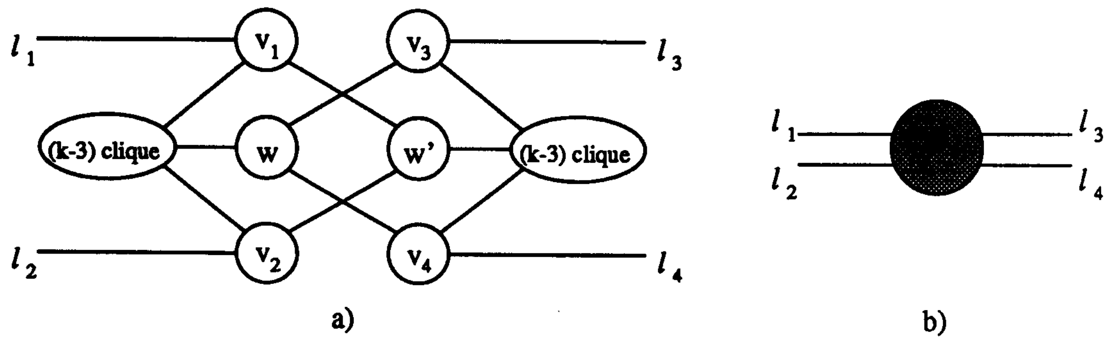

Figure 12

The subgraph of Type 3 is shown in Figure 13a.

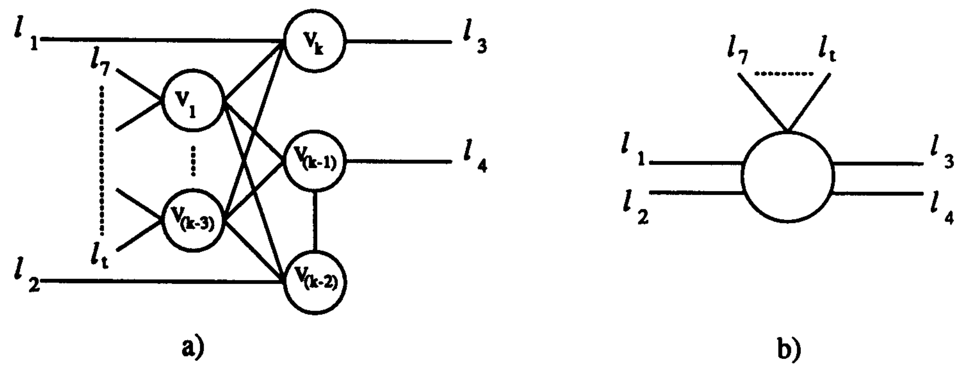

Figure 13

The subgraph of Type 4 is shown in Figure 14a.

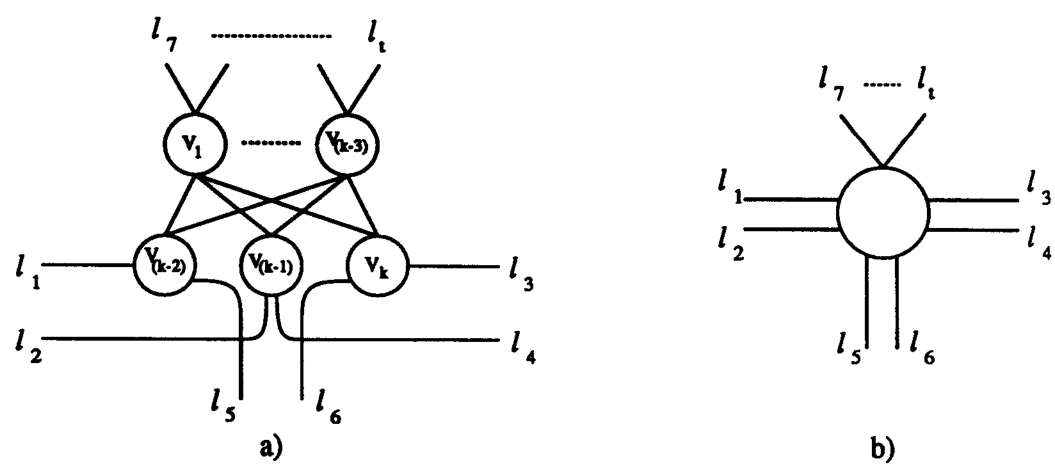

Figure 14

We transform the given instance of 3-SAT into the graph G as in the previous theorem. But instead of using the subgraphs of Type 1, ..., 4 of the previous theorem we now use the subgraphs of Type 1, ..., 4 respectively, as described above.
In this way the nodes $v_1, ..., v_{(k-3)}$ of the subgraphs of Type 3 and 4 are left with "free" links $l_7, ..., l_t$ respectively. These "free" links are used to connect all the subgraphs G_1,...,G_s of Type 3 and 4 in structures as shown in Figure 15. Every node v_i ($i \in \{3, ..., (k-3)\}$) of subgraph G' is connected via a subgraph of type 1 to node $v_{(i \bmod (k-3)+1)}$ of a first subgraph G_1 of type 3 or 4. Every next subgraph of type 3 or 4 is connected to the previous subgraph of type 3 or 4 in a similar way, i.e., v_i of the previous subgraph is connected to $v_{(i \bmod (k-3)+1)}$ of this next subgraph. The last subgraph G_s is then connected to G' following the same procedure. With this extra construction it is enforced that in a perfect k-coloring of G there is a set of $(k-3)$ colors that can only be used for the nodes $v_1, ..., v_{(k-3)}$ in the subgraphs of Type 3 and 4. This has as side effect that the functionality of these subgraphs is the same as of the subgraphs of Type 3 and 4 in the previous theorem.

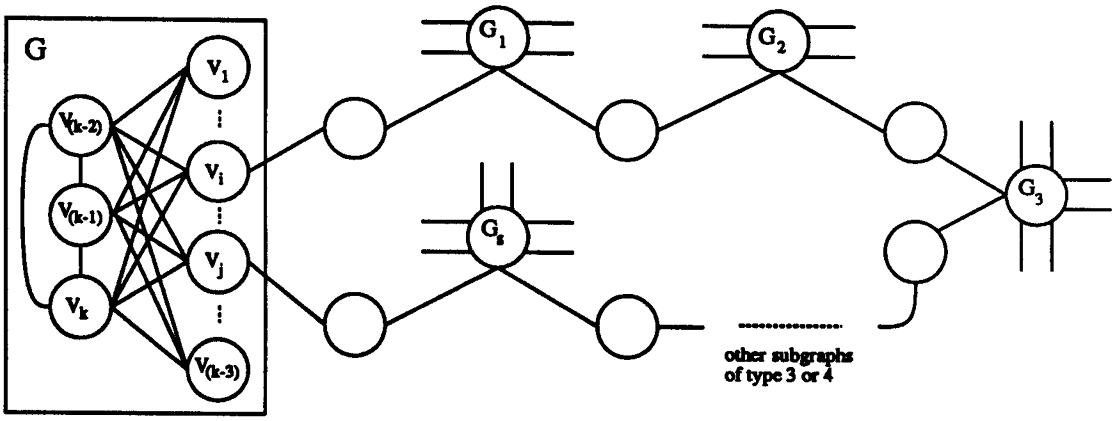

Figure 15

From here on we can adopt the previous proof. It follows that PERFECT k-COLORABILITY is NP-complete for every fixed $k \in \mathbb{N}, k \geq 4$. □

From Theorem 5.2 we can conclude from Lemma 3.2 that also the following problem is NP-complete.

Problem: *DISTANCE-2 COLORING WITH k COLORS, restricted to $(k-1)$-regular graphs. (Fixed $k \in \mathbb{N}, k \geq 4$.)*
Instance: A $(k-1)$-regular graph $G = (V, E)$.
Question: Is the graph distance-2 colorable with k colors ?

Furthermore, with Theorem 3.7 we can also conclude that the following problem is NP-complete.

[B89] H.L.Bodlaender. *The Classification of Coverings of Processor Networks.* Journal of Parallel and Distributed Computing, Vol.6, 1989, pp.166-182.

[DF82] L.W.Dowdy, D.V.Foster. *Comparative Models of the File Assignment Problem.* ACM Computing Surveys, Vol.14, 1982, pp. 287-313.

[GJ79] M.R.Garey, D.S.Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* W.H.Freeman, San Francisco, California, 1979.

[GWW88] W.Gutjahr, E.Welzl, G.Woeginger. *Polynomial Graph-Colorings.* In: B.Monien, R.Cori (Eds.), STACS 89, 6th Annual Symposium, Lecture Notes in Computer Science, Vol.349, Springer-Verlag, 1989, pp. 108-119.

[H69] F.Harary. *Graph Theory.* Addison-Wesley Publ. Comp., Reading, Mass., 1969.

[HN90] P.Hell, J.Nešetril. *On the Complexity of H-Coloring.* Journal of Combinatorial Theory (B), Vol.48, No.1, February 1990, pp. 92-110.

[KL90a] G.Kant, J.van Leeuwen. *The File Distribution Problem for Processor Networks.* In: J.R.Gilbert and R.Karlson (Eds.), Proc. 2nd Scandinavian Workshop on Algorithm Theory, Lecture Notes in Computer Science, Vol.447, Springer-Verlag, 1990, pp. 48-59.

[KL90b] G.Kant, J.van Leeuwen. *Strong Colorings of Graphs.* Technical Report RUU-CS-90-15, Dept. of Computer Science, University of Utrecht, Utrecht, 1990.

[K36] D.König. *Theorie der Endlichen und Unendlichen Graphen.* Leipzig, 1936, Reprinted Chelsea, New York, 1950.

[LG83] D.Leven, Z.Galil. *NP-Completeness of Finding the Chromatic Index of Regular Graphs.* Journal of Algorithms, Vol.4, 1983, pp. 35-44.

[Mc81] S.T.McCormick. *Optimal Approximation of Sparse Hessians and Its Equivalence to a Graph Coloring Problem.* Technical Report Sol 81-22, Dept. of Operations Research, Stanford University, Stanford, 1981.

[V64] V.G.Vizing. *On the Estimate of the Chromatic Class of a p-Graph.* Discret. Anal. ,Vol.3, 1964, pp. 25-30.