SINGULARITIES WITH CRITICAL LOCUS A 1-DIMENSIONAL COMPLETE INTERSECTION AND TRANSVERSAL TYPE A_1

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In this paper we study germs of holomorphic functions f: $(\mathbb{C}^{n+1}, 0) \to \mathbb{C}$ with the following two properties:
   (i) the critical set Σ of f is a 1-dimensional isolated complete intersection singularity (icis);
   (ii) the transversal singularity of f in points of $\Sigma - \{0\}$ is of type A_1.
We first compute the homology of the Milnor fibre F of f in terms of numbers of special points in certain deformations. Next we show that the homotopy type of the Milnor fibre F of f is a bouquet of spheres.
There are two cases:
   (a) general case $S^n \vee \cdots \vee S^n$,
   (b) special case $S^{n-1} \vee s^n \vee \cdots \vee S^n$.

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non-isolated singularity   Milnor fibre

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1. Introduction

1.1. In this paper we study germs of holomorphic functions f: $(\mathbb{C}^{n+1}, 0) \to \mathbb{C}$ with the following two properties:
   (i) the critical set Σ of f is a 1-dimensional icis (isolated complete intersection singularity),
   (ii) the transversal singularity of f at points of $\Sigma - \{0\}$ is of type A_1.
In the paper Isolated Line Singularities [25] we considered the case where Σ was smooth and one-dimensional and showed that the Milnor fibre of an isolated line singularity was homotopy equivalent to a bouquet of spheres in the middle dimension, just as in the case of isolated singularities.

1.2. This paper is the next step in the program of studying special types of non-isolated singularities. The main idea is to use special deformations of f to get a generic approximation

$$f_s: \mathbb{C}^{n+1} \to \mathbb{C}$$

with nice properties:

(a) the induced fibrations above a small circle around the origin of f_s and f must be equivalent, and have therefore the same fibre.

(b) the singular locus of f_s and the local singularity type of f_s at points of the critical locus should be as simple as possible (or at least good enough to use in our computations and constructions). For isolated singularities this idea was used by Lê (cf. [3]) to determine the homotopy type of the Milnor fibre. The generic approximation was in that case a Morsification, having only isolated singularities of Morse type (A_1).

In this paper the elementary local singularities, which we allow for f_s, are

(a) for isolated singular points of f_s,

      type A_1, local formula: $f_s = w_0^2 + w_1^2 + \cdots + w_n^2$.

(b) for non-isolated singular points of f_s,

(b1) type A_∞, local formula $f_s = w_1^2 + \cdots + w_n^2$ (transversal Morse), or

(b2) type D_∞, local formula $f_s = w_0 w_1^2 + w_2^2 + \cdots + w_n^2$ (Whitney umbrella).

The homotopy types of the Milnor fibres are:

(a) S^n for A_1,

(b1) S^{n-1} for A_∞,

(b2) S^n for D_∞, cf. [25].

1.3. The thesis of Pellikaan [23] contains the deformation theory of the occurring types of non-isolated singularities. He also gives a formula, which relates the number of special points in the approximation with the dimension of the quotient of two ideals: I, the reduced ideal defining Σ, and $J(f) = (\partial f/\partial z_0, \ldots, \partial f/\partial z_n)$, the Jacobian ideal:

$$\dim_{\mathbb{C}}(I/J(f)) = {}^{\#}A_1 + {}^{\#}D_\infty.$$

${}^{\#}A_1$ denotes the number of A_1-points in the approximation, similarly ${}^{\#}D_\infty$ the number of D_∞-points.

This formula is analogous to

$$\dim_{\mathbb{C}}(\mathcal{O}_{n+1}/J(f)) = {}^{\#}A_1$$

for isolated singularities.

1.4. Let the singular locus of f_s consist of the connected components

$$\Sigma_0, \Sigma_1, \ldots, \Sigma_\sigma.$$

In order to compute the homology of the Milnor fibre F we show in Section 2 in a rather general context the formula

$$H_*(E, F) = \bigoplus_{i=0}^{\sigma} H_*(E^i, F^i)$$

where

$$E = \text{Milnor ball of } f, \quad E^i = \text{small Milnor tube along } \Sigma_i;$$

$$F = \text{Milnor fibre of } f, \quad F^i = \text{restriction of nearby fibre to } E^i.$$

(All homology is taken with coefficients in $\mathbb{Z}$)

1.5. In the case of Σ a 1-dimensional icis and transversal type A_1, the generic approximation f_s has the critical components:

Σ_0 is a Riemann surface with holes,

$\Sigma_1, \ldots, \Sigma_\sigma$ are points with local Milnor fibre S^n (type A_1),

and so

$$H_{k+1}(E^i, F^i) = \begin{cases} \mathbb{Z}, & k = n, \\ 0, & k \neq n \end{cases} \quad (i = 1, \ldots, \sigma).$$

Next we study the situation near Σ_0. In Section 4 a natural projection $w_0 : (E^0, F^0) \to \Sigma_0$ is defined from the tube onto Σ_0. This projection is locally trivial outside the D_∞-points on Σ_0. The regular fibre is the Milnor pair of the transversal A_1-singularity, which is homotopy equivalent to (D^n, S^{n-1}).

With a Mayer–Vietoris sequence one can combine the information about this fibre bundle and the local Milnor fibre at the D_∞-points (which is of homotopy type S^n) and get Theorem 5.7.

Theorem. *Let Σ be a 1-dimensional icis and $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ a holomorphic function with singular locus $\Sigma(f) = \Sigma$ and transversal singularity type A_1 on $\Sigma - \{0\}$, then the homology groups of the Milnor fibre F of f are as follows*:

Case A: $\ ^{\#}D_\infty > 0$:
$$\begin{cases} \tilde{H}_n(F, \mathbb{Z}) = \mathbb{Z}^{\mu(\Sigma) + \,^{\#}A_1 + 2\,^{\#}D_\infty - 1}, \\ \tilde{H}_k(F, \mathbb{Z}) = 0, \quad k \neq n; \end{cases}$$

Case B: $\ ^{\#}D_\infty = 0$:
$$\begin{cases} \tilde{H}_n(F, \mathbb{Z}) = \mathbb{Z}^{\mu(\Sigma) + \,^{\#}A_1}, \\ \tilde{H}_{n-1}(F, \mathbb{Z}) = \mathbb{Z}, \\ \tilde{H}_k(F, \mathbb{Z}) = 0, \quad k \neq n - 1, n. \end{cases}$$

Where $\mu(\Sigma)$ is the Milnor number of $(\Sigma, 0)$.

From the homology and additional information about the fundamental group of F one can conclude (Theorem 6.2):

Main Theorem. *Let f be as in Theorem 5.7 then the homotopy type of the Milnor fibre F of f is a bouquet of spheres*:

Case A: $\ ^{\#}D_\infty > 0$: $\quad F \overset{h}{=} S^n \vee \cdots \vee S^n$;

Case B: $\#D_\infty = 0 \quad F \overset{h}{\simeq} S^{n-1} \vee S^n \vee \cdots \vee S^n.$

1.6. For the study of non-isolated singularities in general we refer to the work of Lê [14, 16] and Randell [24]. The special case of singularities with a 1-dimensional critical locus is especially studied by Iomdin [8–11], who gave formulas for the Euler characteristic of the Milnor fibre; see also [16]. Kato and Matsumota [13] proved that in this case the Milnor fibre is $(n-2)$-connected. Moreover it follows from [18] that the Milnor fibre is simply connected when $n = 2$ and f is irreducible.

1.7. As a general reference for singularities of functions $\mathbb{C}^{n+1} \to \mathbb{C}$ we mention the book of Arnol'd–Gusein Zade–Varchenko [2]. The study of this paper was influenced by a remark of Arnol'd [1, pp. 88–89] about series of singularities. See also the remark by Wall [30].

1.8. Part of this work was done, while the author was a guest at the Institut des Hautes Etudes Scientifiques (I.H.E.S.) at Bûres-sur-Yvette (France). We thank this institute and its staff members for the support and the good mathematical atmosphere.

2. The relative homology of a holomorphic map

2.1. Let B_ε be the closed ε-ball in $\mathbb{C}^{n+1}$ and let $f\colon B_\varepsilon \to \mathbb{C}$ be a homomorphic function with critical values $b_0, \ldots, b_\sigma$ contained in a closed disc $\Delta \subset \mathbb{C}$.

We make the following assumption:

$$f^{-1}(t) \pitchfork \partial B_\varepsilon \quad \text{for all } t \in \Delta.$$

[If $t \in \{b_0, \ldots, b_\sigma\}$ this means $f^{-1} \pitchfork \partial B_\varepsilon$ as a stratified set.]

We choose:
 (i) a system of disjoint closed discs D_i around every b_i,
 (ii) points $t_i \in \partial D_i$ and $t \in \partial \Delta$,
 (iii) a system of paths $\gamma_0, \ldots, \gamma_\sigma$ from t to t_i (in the usual way, see Fig. 1).
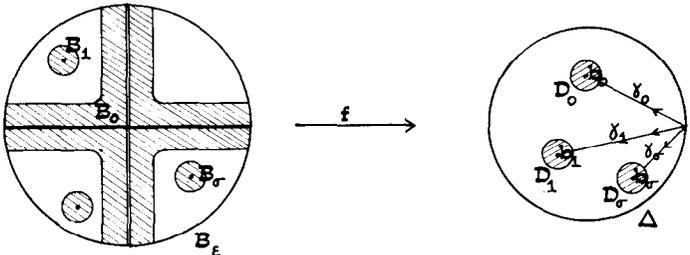

Fig. 1.

We use the following notations:

$$D = \bigcup D_i, \qquad \Gamma = \bigcup \gamma_i,$$

$$X_A = f^{-1}(A) \cap B_\varepsilon \quad \text{for } A \subset \Delta, \qquad X_s = f^{-1}(s) \cap B_\varepsilon \quad \text{for } s \in \Delta.$$

2.2. Lemma. $H_*(X_\Delta, X_t) \cong \overset{\sigma}{\underset{i=0}{\bigoplus}} H_*(X_{D_i}, X_{t_i}).$

Proof. Since f is locally trivial over the complement of D, (X_Δ, X_t) is homotopy equivalent to $(X_{D \cup \Gamma}, X_{\Gamma})$ by the homotopy lifting property.

So $H_*(X_\Delta, X_t) = H_*(X_{D \cup \Gamma}, X_\Gamma) = H_*(X_D, X_{D \cap \Gamma})$ by excision. Since the D_i are disjoint the homology is a direct sum. $\square$

This lemma tells us that the relative homology group $H_*(X_D, X_t)$ is equal to the sum of local contributions around the critical fibres.

2.3. Next we consider the local situation

$$f: (X_{D_i}, X_{t_i}) \to (D_i, t_i).$$

Let Σ_i be the critical locus of f inside X_{D_i}, $(i = 0, \ldots, \sigma)$. Suppose we are given a real analytic function r_i defined on a neighborhood of Σ_i with the properties

$$r_i(z) \geq 0, \qquad r_i(z) = 0 \Leftrightarrow z \in \Sigma_i.$$

Let $B_i(\varepsilon) = \{z \in B \mid r_i(z) \leq \varepsilon\}$.

2.4. Lemma. (a) *There exist ε_i such that for all $0 < \varepsilon \leq \varepsilon_i$: $X_{b_i} \pitchfork \partial B_i(\varepsilon)$.*

(b) *For every $0 < \varepsilon \leq \varepsilon_i$ there exist a $\tau_i = \tau_i(\varepsilon)$ such that $X_t \pitchfork \partial B_i(\varepsilon)$ for all $0 < |t - b_i| \leq \tau_i$.*

Proof. (a) is an application of the following lemma (cf. Milnor [20] or Lê [15]):

Let X be analytic and let $f: X \to \mathbb{R}$ be a polynomial in z and $\bar{z}$ then has $f | X_{\text{reg}}$ only a finite number of critical values.

(b) follows from the openness of the transversality condition. $\square$

2.5. We fix now $\varepsilon > 0$ and $\tau > 0$, which satisfy Lemma 2.4 and suppose that all of the $B_i(\varepsilon)$ and $D_i(\tau)$ which occur are disjoint and inside B_ε or Δ.

We write next:

$$B_i = B_i(\varepsilon), \qquad D_i = D_i(\tau),$$

$$E^i = B_i \cap X_{D_i}, \qquad E = X_\Delta,$$

$$F^i = B_i \cap X_{t_i}, \qquad F = X_t.$$

2.6. Lemma. (X_{D_i}, X_{t_i}) and $(E^i \cup X_{t_i}, X_{t_i})$ are homotopy equivalent (relative X_{t_i}).

Proof. Restrict f to $X_{D_i} \setminus \text{Int}(E^i)$. f satisfies transversality conditions on ∂B and ∂B_i and f has no critical points in the interior. We conclude by the Ehresmann fibriation theorem that $X_{D_i} \setminus \text{Int}(E^i)$ is a product of D_i and the fibre $X_{t_i} \setminus \text{Int}(E^i)$. □

2.7. Corollary. $H_*(X_{D_i}, X_{t_i}) \simeq H_*(E^i \cup X_{t_i}, X_{t_i}) \cong H_*(E^i, F^i)$.

Proof. From Lemma 2.6 and excision. □

Combining this with Lemma 2.2 we find the following proposition.

2.8. Proposition. $H_*(E, F) = \bigoplus_{i=0}^{\sigma} H_*(E^i, F^i)$.

This states that the relative group $H_*(E, F)$ is the sum of the relative groups around every singular set Σ_i.

2.9. Example (isolated singularity). Let $f: \mathbb{C}^{n+1} \to \mathbb{C}$ be a germ of a holomorphic function, with an isolated singularity at $0 \in \mathbb{C}^{n+1}$. The Milnor construction gives

$$f: f^{-1}(\Delta) \cap B_\varepsilon = X_\Delta \to \Delta$$

satisfying the usual transversality conditions. Consider a Morsification g of f such that the induced fibrations over $\partial \Delta$ are equivalent,

$$g: g^{-1}(\Delta) \cap B_\varepsilon = E \to \Delta.$$

The critical set of g consists of a finite number of points $c_1, \ldots, c_\mu$, where the local equation of f is equivalent to

$$f(w) = f(0) + w_0^2 + \cdots + w_n^2 \quad \text{(Morse point)}.$$

Take as distance functions $r_i(z) = |z - b_i|^2$. The local situation of the quadratic singularity is well known (cf [19] or [20]):

$$H_{k+1}(E^i, F^i) = \tilde{H}_k(F^i) = \begin{cases} \mathbb{Z}, & k = n, \\ 0, & k \neq n. \end{cases}$$

Since X_Δ and E can be chosen contractable we have

$$\tilde{H}_k(F) = H_k(E, F) = \bigoplus_{i=1}^{\mu} H_k(E^i, F^i) = \begin{cases} \mathbb{Z}^\mu, & k = n, \\ 0, & k \neq n. \end{cases}$$

2.10. Example. Let $f: \mathbb{C}^{n+1} \to \mathbb{C}$ be a function, satisfying the assumption of 2.1 and the extra property that $\Sigma_1, \ldots, \Sigma_\sigma$ are points, so locally we have there isolated singularities. Denote the Milnor numbers by $\mu_1, \ldots, \mu_\sigma$.

Then we have

$$\begin{cases} H_{n+1}(E, F) = H_{n+1}(E^0, F^0) \oplus \mathbb{Z}^{\mu_1} \oplus \cdots \oplus \mathbb{Z}^{\mu_\sigma}, \\ H_k(E, F) = H_k(E^0, F^0) \quad \text{if } k \neq n+1. \end{cases}$$

3. The singular locus of f is a 1-dimensional icis and the transversal type is A_1

3.1. Let Σ be a one-dimensional complete intersection with isolated singularity at $0 \in \mathbb{C}^{n+1}$. We consider $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ a holomorphic function germ with singular locus $\Sigma(f) = \Sigma$. This situation is treated (in more generality) in the thesis of Pellikaan [23].

On every branch of $\Sigma(f)$ there is for $z \neq 0$ a well-defined transversal singularity type. Let $g_1, \ldots, g_n$ define the complete intersection Σ as a reduced algebraic variety. Let $I = (g_1, \ldots, g_n)$.

Then we have:

$$f \text{ is singular on } \Sigma \Leftrightarrow f \in I^2 \quad [23, \text{I}(1.6)].$$

In this case we can write $f = \Sigma h_{ij} g_i g_j$ with $h_{ij} \in \mathcal{O}_{n+1}$ and $h_{ij} = h_{ji}$.

On I and I^2 acts the subgroup $\mathcal{D}_\Sigma$ of $\mathcal{D}$ of local diffeomorphisms defined by:

$$\mathcal{D}_\Sigma = \{\varphi \in \mathcal{D} \mid \varphi^*(g_1 \cdots g_n) \subset (g_1 \cdots g_n)\}$$

Let $\tau_\Sigma(f)$ be the tangentspace to the $\mathcal{D}_\Sigma$-orbit and

$$J_f = \left(\frac{\partial f}{\partial z_0}, \ldots, \frac{\partial f}{\partial z_n} \right) \quad \text{the Jacobian ideal of } f.$$

Define $j(f) = \dim_\mathbb{C}(I/J_f)$ and $c(f) = \dim_\mathbb{C}(I^2/\tau_\Sigma(f))$, the Jacobi number and the codimension.

3.2. Proposition [23]. *Equivalent are*:
  (a) $c(f) < \infty$,
  (b) $j(f) < \infty$,
  (c) *The transversal type of f along $\Sigma - \{0\}$ is A_1 (Morse point).*
Moreover if $c(f) < \infty$ then f is finitely determined inside I^2.

3.3. Let $G: (\mathbb{C}^{n+1} \times \mathbb{C}^p, 0) \to (\mathbb{C}^n \times \mathbb{C}^p, 0)$ be a versal deformation of $(\Sigma, 0)$ with

$$G(z, v) = (G_1(z, v), \ldots, G_n(z, v), v) \quad \text{and} \quad G_i(z, 0) = g_i(z).$$

(See e.g. [19 (6.5)])
For generic v and u the curve $\Sigma_{(u,v)}$ defined by

$$G_1(z, v) = u_1, \quad \ldots, \quad G_n(z, v) = u_n$$

is the Milnor fibre of $(\Sigma, 0)$.

From the theory of isolated complete intersection singularities we know that
  (i) $\Sigma_{(u,v)}$ is a connected one-dimensional complex manifold with boundary.
 (ii) $\Sigma_{(u,v)}$ has the homotopy type of a 1-dimensional CW-complex (so homotopy equivalent with a wedge of 1-spheres).
(iii) $H_1(\Sigma_{(u,v)}) = \mathbb{Z}^{\mu(g)}$ where $\mu(g)$ is the Milnor number of g.
Consider the following deformation of f:

$$f_s(z) = f_{(a,b,u,v)}(z)$$

$$= \Sigma(h_{ij}(z) + a_{ij} + \sum_{t=0}^{n} b_{ij}z_t\delta_{ij})(G_i(z, v) - u_i)(G_j(z, v) - u_j).$$

Let $S = \{(a, b, u, v) \mid a = (a_{ij})$ and $b = (b_{ij})$ matrices, $u \in \mathbb{C}^p$; $v \in \mathbb{C}^n\}$. The singular locus of f_s contains at least $\Sigma_{(u,v)}$ which we further denote by Σ_s.

3.4. Proposition [23]. *There exists a dense subset U of S and an open neighborhood $\mathcal{V}$ of 0 in $\mathbb{C}^{n+1}$ such that for all $s \in U$ sufficiently small,*
  (i) *f_s has only A_1-singularities in $\mathcal{V} \setminus \Sigma_s$*
 (ii) *Σ_s is a curve without singularities and f_s has only A_∞ and D_∞ singularities on $\mathcal{V} \cap \Sigma_s$.*
(iii) *$j_f = {}^\#\{A_1\text{-points of } f_s \text{ on } \mathcal{V} \setminus \Sigma_s\} + {}^\#\{D_\infty\text{-points of } f_s \text{ on } \mathcal{V} \cap \Sigma_s\}$.*

3.5. Let f: $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and ε_0 be an admissible radius for the Milnor fibration, that is $\varepsilon_0 > 0$ such that for all ε with $0 < \varepsilon \leq \varepsilon_0$ holds $f^{-1}(0) \pitchfork \partial B_\varepsilon$ (as a stratified set). For each admissible $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$f^{-1}(t) \pitchfork \partial B_\varepsilon \quad \text{for all } 0 < t \leq \delta(\varepsilon).$$

We fix now an $\varepsilon \leq \varepsilon_0$ and consider $0 < \delta \leq \delta(\varepsilon)$ and take the representative

$$f: X_\Delta = f^{-1}(\Delta) \cap B_\varepsilon \to \Delta$$

where Δ is the disc of radius δ.

3.6. Lemma. *Let f_s be as above. Consider the restriction*

$$f_s: X_{\Delta,s} = f_s^{-1}(\Delta) \cap B_\varepsilon \to \Delta.$$

For $s \in S$ and $\delta > 0$ sufficiently small, we have:
 (1) *$f_s^{-1}(t) \pitchfork \partial B_\varepsilon$ for all $t \in \Delta$.*
 (2) *Above the boundary circles $\partial\Delta$, the fibrations induced by f and f_s are equivalent.*
 (3) *X_Δ and $X_{\Delta,s}$ are homeomorphic.*

Proof. Let S_η be a small ball of radius η in the parameter space S. Take η so small that all A_1-points and D_∞-points of f_s are contained in $B_{\varepsilon/2}$.
  (1) Let $\Sigma^* = \bigcup_{s \in S} \Sigma_s$. In points of $\Sigma^* \cap \partial B_\varepsilon$ the functions f_s have type A_∞. Near such a point we can change the coordinates (z, s) of $\mathbb{C}^{n+1} \times S$ into (y, s), where $y = y(z, s)$ depends smoothly on s and moreover $f_s(y) = y_1^2 + \cdots + y_n^2$ (parametrized

version of the Morse lemma). In these coordinates one sees immediately that for $t \neq 0$ the tangent space to $f_s^{-1}(t)$ contains the line $L = \{y \mid y_0 = 0\}$. We apply the opposite coordinate change Φ and find locally that the tangentspace to $f_s^{-1}(t)$ contains $L_{z,s}$, the image of L under $d\Phi$, where $f_s(z) = t \neq 0$.

Note that $L_{z,s}$ is the tangentspace to Σ_s at z if $f_s(z) = t = 0$. Since we can suppose $\Sigma_s \pitchfork \partial B_\varepsilon$ for small s (Σ_s is for $s \neq 0$ the Milnor fibre of the complete intersection Σ) we find locally $L_{z,s} \pitchfork \partial B_\varepsilon$ for all $t \in \Delta$ (δ sufficiently small) and $z \in f_s^{-1}(t) \cap \partial B_\varepsilon$. At the other points we get $f_s^{-1}(t) \pitchfork \partial B_\varepsilon$ from the fact that $f_s^{-1}(0) \pitchfork \partial B_\varepsilon$ on $f_s^{-1}(0) \backslash \Sigma_s$.

(2) Consider $F(x, s) = (f_s(x, s), s)$; so $F: \mathbb{C}^{n+1} \times S \to \mathbb{C} \times S$. Define

$$Y_{\Delta, \eta} = F^{-1}(\Delta \times S_\eta) \cap B_\varepsilon \times S_\eta$$

and the restriction

$$F_{\Delta, \eta}: Y_{\Delta, \eta} \to \Delta \times S_\eta,$$

which defines for every $s \in S_\eta$

$$f_s: X_{\Delta, s} \to \Delta.$$

The map $F_{\Delta, \eta}$ is a submersion on the interior points

$$F^{-1}(\partial \Delta \times S_\eta) \cap (\text{int } B_\varepsilon \times S_\eta)$$

since df_s has maximal rank above $\partial \Delta$.

Also the restriction of $F_{\Delta, \eta}$ to the boundary $F^{-1}(\partial \Delta \times S_\eta) \cap (\partial B_\varepsilon \times S_\eta)$ is submersive since $f_s^{-1}(t) \pitchfork \partial B_\varepsilon$ for all $t \in \Delta$ and $s \in S_\eta$. Now we can apply a relative version of Ehresmann's fibration theorem (which is a special case of Thom's second isotopy lemma) and get to fact that all maps f_s define equivalent fibrations over $\partial \Delta$ ($s \in S_\eta$).

(3) From Thom's first isotopy lemma it follows that all $X_{\Delta, s}$ are homeomorphic ($s \in S_\eta$). ☐

4. The situation near the smooth 1-dimensional component Σ_s

4.1. From now on we choose s such that $f_s: X_{\Delta, s} \to \Delta$ satisfies the conditions of Proposition 3.4 and Lemma 3.6. We omit the suffix s and write again

$$f: X_\Delta \to \Delta.$$

We are now in the situation of Section 2.

The critical set of f consists of:

(a) A Riemann surface Σ_0 which is the Milnor fibre Σ_s of the 1-dimensional isolated complete intersection singularity Σ. The local singularities of f on Σ_0 are A_∞ or D_∞.

(b) $\Sigma_1 = \{c_1\}, \ldots, \Sigma_\sigma = \{c_\sigma\}$, where the local singularity of f is isolated of type A_1.

4.2. Since the situation at the A_1-point is well known we consider the situation along Σ_0.

We normalize coordinates as follows:

Write

$$\begin{cases} w_1(z) = G_1(z, v) - u_1, \\ w_n(z) = G_n(z, v) - u_n. \end{cases}$$

This defines a germ

$$w: \ (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^n, 0)$$

with $w^{-1}(0) = \Sigma_0$.

Let $r_0(z) = |w_1(z)|^2 + \cdots + |w_n(z)|^2$ then also $r_0^{-1}(0) = \Sigma_0$. We use the notation from Section 2

$$B_0 = \{z \in B \mid r_0(z) \leq \varepsilon\}.$$

4.3. Lemma. B_0 is (real) diffeomorphic to the product $\Sigma_0 \times Q^n$, where Q^n is a closed n-ball in $\mathbb{C}^n$.

Proof. Since $g(-, v)$ is a submersion in points of Σ_s, this implies that w is a submersion near Σ_0 and so Q is regular value of $w: \ B_0 \to \mathbb{C}^n$.

At the intersection points of B_0 with ∂B_ε we have the transversality property

$$\Sigma_0 = w^{-1}(0) \pitchfork \partial B_\varepsilon.$$

This follows from the fact that our original $f^{-1}(0)$ was transversal to ∂B_ε as a stratified set. So

$$\Sigma = g^{-1}(0) \pitchfork \partial B_\varepsilon$$

and during the deformation of Σ this property is kept for small values of the parameters. So we have that also $w: \ \partial B_0 \to \mathbb{C}^n$ is submersive. The lemma follows from the Ehresmann fibration theorem. We remark that it is possible to have locally a holomorphic product in a neighborhood of the D_∞-points. $\square$

4.4. We next use in B_0 coordinates $(w_0, w_1, \ldots, w_n)$ with $w_0 \in \Sigma_0$ and $(w_1, \ldots, w_n) \in Q^n$. $w_1, \ldots, w_n$ are holomorphic functions on B_0; w_0 is (real) differentiable. Σ_0 is defined by $w_1 = \cdots = w_n = 0$.

We use the distance function $r(w_0, w_1, \ldots, w_n) = |w_1|^2 + \cdots + |w_n|^2$ and define (E^0, F^0) as in 2.5. For the study of the local situation near Σ_0 it is necessary to choose the ε and τ involved in definition 2.5 a little bit more carefully.

We consider the projection $w_0: \ (E^0, F^0) \to \Sigma_0$.

Let $\Phi: \ B_0 \to \mathbb{C} \times \Sigma_0$ be defined by $\Phi(w) = (f(w), w_0)$.

The critical points of Φ are given by the (2×2) minors of the matrix:

$$\begin{pmatrix} \dfrac{\partial f}{\partial w_1} & \dfrac{\partial f}{\partial \bar{w}_1} & \cdots & \dfrac{\partial f}{\partial w_n} & \dfrac{\partial f}{\partial \bar{w}_n} \\ \dfrac{\overline{\partial f}}{\partial \bar{w}_1} & \dfrac{\overline{\partial f}}{\partial w_1} & \cdots & \dfrac{\overline{\partial f}}{\partial \bar{w}_n} & \dfrac{\overline{\partial f}}{\partial w_n} \end{pmatrix}.$$

The variety, defined by the (2×2) minors is the union of Σ_0 and the so-called polar variety Γ of f with respect to w_0. The next lemma is similar to [25, Lemma 3.7].

4.5. Lemma. *The polar variety Γ can cut Σ_0 only in the D_∞-points of f.*

Proof. We can write $f = \Sigma h_{ij} w_i w_j$ $(i, j \geq 1)$. So

$$\frac{\partial f}{\partial w_k} = \Sigma \frac{\partial h_{ij}}{\partial w_k} w_i w_j + \Sigma h_{jk} w_j, \qquad \frac{\partial f}{\partial \bar{w}_k} = \Sigma \frac{\partial h_{ij}}{\partial \bar{w}_k} w_i w_j.$$

The (2×2)-minors of the above matrix are as follows:

$$M_{p,q} = \frac{\partial f}{\partial w_p} \cdot \frac{\overline{\partial f}}{\partial w_q} - \frac{\partial f}{\partial \bar{w}_p} \cdot \frac{\overline{\partial f}}{\partial \bar{w}_q} \quad (p, q \geq 1),$$

$$N_{p,q} = \frac{\partial f}{\partial w_p} \cdot \frac{\overline{\partial f}}{\partial \bar{w}_q} - \frac{\partial f}{\partial w_p} \cdot \frac{\overline{\partial f}}{\partial \bar{w}_q} \quad (p, q \geq 1)$$

and their complex conjugates. We work in the ring $\mathscr{E}_{n+1}$ of differentiable function germs from $\mathbb{C}^{n+1}$ to $\mathbb{C}$. We define the following ideals:

$\quad m$ = maximal ideal,

$\quad A$ = the ideal generated by the (2×2) minors of the matrix in 4.4,

$\quad B$ = the ideal generated by $\{w_a \cdot \bar{w}_b \,|\, 1 \leq a, b \leq n\}$.

It follows that

$$M_{p,q} \in \Sigma h_{ip} \bar{h}_{jq} w_i w_j + m B.$$

In a small neighborhood of a A_∞-point we have that the matrix (h_{ij}) is invertible. It follows that for all $1 \leq a, b \leq n$

$$w_a \bar{w}_b \in A + m B.$$

So

$$B \subset A + m B.$$

According to Nakayama's lemma we have

$$B \subset A.$$

So the critical set of Φ is locally given by:

$$w_a \cdot \bar{w}_b = 0, \quad 1 \leq a, b \leq n.$$

It follows that $w_a = \bar{w}_a = 0$, $1 \leq a \leq n$.

So: in a neighborhood of a A_∞-point the singular locus of Φ coincides with Σ_0. $\quad\square$

4.6. Consider small disjoint 1-balls $W_1, \ldots, W_\tau$ in Σ_0 around the D_∞-points, let

$$W = W_1 \cup \cdots \cup W_\tau \quad \text{and} \quad M = \overline{\Sigma_0 \setminus W}.$$

So $\Sigma_0 = M \cup W$. Write

$$B_0 = B_M \cup B_W \quad \text{where } B_M = w_0^{-1}(M) \text{ and } B_W = w_0^{-1}(W),$$

$$E^0 = E_M \cup E_W \quad \text{where } E_M = E^0 \cap B_M \text{ and } E_W = E^0 \cap B_W,$$

$$F^0 = F_M \cup F_W \quad \text{where } F_M = F^0 \cap B_M \text{ and } F_W = F^0 \cap B_W.$$

Choose first E^0 so small that $\Gamma \subset E_W$.

4.7. Proposition. *For E^0 sufficiently small is w_0: $(E_M, F_M) \to M$ a locally trivial fibre bundle pair with fibres equivalent to the fibre pair $(\bar{E}, \bar{F})$ of the isolated, quadratic singularity*

$$w_1^2 + \cdots + w_n^2.$$

Proof. We remark first that:

(a) $E^0 = B_0 \cap X_{D_0}$ has boundaries $\partial_1 E^0 = \partial B_0 \cap X_{D_0}$ and $\partial_2 E^0 = B_0 \cap X_{\partial D_0}$ and corner $\partial_{12} E^0 = \partial B_0 \cap X_{\partial D_0}$;

(b) $F^0 = B_0 \cap X_{t_0}$ has boundary $\partial F^0 = \partial B_0 \cap X_{t_0}$.

We next consider M as a parameter space for a family of isolated singularities $f_{(c)}$: $B_M(c) \to \mathbb{C}$ $(c \in M)$ where $B_M(c) = B_M \cap w_0^{-1}(c)$. Since the type of $f_{(c)}$ is A_1 for $c \in M$, we can use, that for such families of quasi-homogeneous singularities there exists a stable radius for the Milnor construction (cf. [21] or [22]).

We now choose E^0 so small that the transversality condition

$$f_{(c)}^{-1}(t) \pitchfork \partial B_M(c)$$

holds in $w_0 = c$ for all $c \in M$ and for all $t \in D_0$.

It follows that

$$f^{-1}(t) \pitchfork \partial B_M(c) \quad \text{for all } c \in M, t \in D_0,$$

and

$$\partial_2 E^0 \pitchfork \partial B_M(c) \quad \text{for all } c \in M.$$

This implies that the projection

$$w_0: \partial_2 E^0 \to \Sigma_0 \quad \text{is a submersion above } M.$$

Since the kernal of w_0: $B_0 \to \Sigma_0$ is $\{w_0 = c\}$ and $\partial B_0 \pitchfork \{w_0 = c\}$ it follows that

$$w_0: \partial_1 E^0 \to \Sigma_0 \quad \text{and} \quad w_0: \partial_{12} E^0 \to \Sigma_0 \quad \text{and} \quad w_0: \partial F^0 \to \Sigma_0$$

are submersive above M.

The proposition follows now from Ehresmann's fibration theorem. □

4.8. Remark. We also choose W, B_0, D_0 and E^0 so small that on each $w_0^{-1}(W_i)$ the non-isolated singularity (of type D_∞) f: $w_0^{-1}(W_i) \to \mathbb{C}$ satisfies the conditions for the Milnor construction with respect to the polyball $W_i \times Q^n$ and such that the coordinate system $w_0, w_1, \ldots, w_n$ is holomorphic.
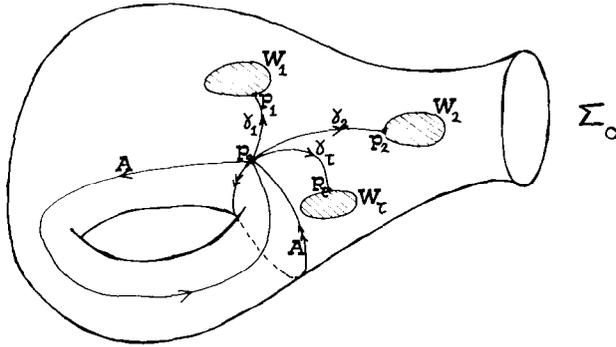
Fig. 2.

5. The homology of (E^0, F^0)

5.1. We choose a point $p_0 \in \Sigma_M$. Consider a system $\gamma_1, \ldots, \gamma_\tau$ of non-intersecting paths from p_0 to points p_i on the boundary of each W_i (see Fig. 2). Let $C = \bigcup \gamma_i$ and $W = \bigcup W_i$. Σ_0 has a deformation retract A consisting of a wedge of 1-spheres. This wedge can be choosen in such a way that the wedge point is p_0; $A \cap W = \emptyset$ and $A \cap C = \{p_0\}$. Let $B = A \cup C$.

In 4.4 we defined the map $w_0 : E^0 \to \Sigma_0$. Let $Y \subset \Sigma_0$ we denote $E_Y = w_0^{-1}(Y) \subset E^0$. Let $y \in \Sigma_0$ we denote $E_y = w_0^{-1}(y) \subset E^0$.

5.2. From [5, p. 51] we use the relative Mayer–Vietoris sequence. Let $(A; A_1, A_2) \subset (X; X_1, X_2)$ an excisive triad, then

$$H_*(X_1 \cup X_2, A_1 \cup A_2) \xrightarrow{d_*[-1]} H_*(X_1 \cap X_2, A_1 \cap A_2)$$

$$\nwarrow i_* \qquad \qquad j_* \nearrow$$

$$H_*(X_1, A_1) \oplus H_*(X_2, A_2)$$

is exact.

We apply this for the triples

$$(F_{W \cup B}; F_W, F_B) \subset (E_{W \cup B}; E_W, E_B),$$

and get

$$H_*(E_{W \cup B}, F_{W \cup B}) \xrightarrow{d_*[-1]} H_*(E_{W \cap B}, F_{W \cap B})$$

$$\nwarrow \qquad \qquad \nearrow \qquad \qquad \text{is exact}$$

$$H_*(E_W, F_W) \oplus H_*(E_B, F_B)$$

We remark that:

(i) $H_k(E_B, F_B) \cong H_k(E_A, F_A)$ since B and A are homotopy equivalent and so are (E_B, F_B) and (E_A, F_A) by the homotopy lifting property of $w_0 : (E_M, F_M) \to M$.

(ii) $H_k(E_W, F_W) = \bigoplus_{i=1}^{\tau} H_k(E_{W_i}, F_{W_i}) = \begin{cases} \mathbb{Z}^\tau & \text{if } k = n-1, \\ 0 & \text{if } k \neq n-1, \end{cases}$

since on each W_i we have a singularity of type D_∞, which has Milnor fibre S^n (cf. [25]).

(iii) $H_k(E_{W \cap B}, F_{W \cap B}) = \bigoplus_{i=1}^{\tau} H_k(E_{p_i}, F_{p_i}) = \bigoplus_{i=1}^{\tau} \tilde{H}_k(F_{p_i}) = \begin{cases} \mathbb{Z}^\tau & \text{if } k = n, \\ 0 & \text{if } k \neq n, \end{cases}$

since at each p_i we have a transversal A_1 singularity.

(iv) $H_k(E_{W \cup B}, F_{W \cup B}) = H_k(E^0, F^0)$ since $W \cup B$ is homotopy equivalent to Σ_0 (rel. W) and so are $(E_{W \cup B}, F_{W \cup B})$ and (E^0, F^0) by the homotopy lifting property of $w_0 \colon (E_M, F_M) \to M$.

5.3. We conclude now that

$$H_*(E^0, F^0) \xrightarrow{\;d_*[-1]\;} \bigoplus_{i=1}^{\tau} H_*(E_{p_i}, F_{p_i})$$

$$\bigoplus_{i=1}^{\tau} H_*(E_{W_i}, F_{W_i}) \oplus H_*(E_A, F_A)$$

is exact

5.4. We now concentrate on $H_*(E_A, F_A)$.

Case 1: The bundle $w_0 \colon (E_M, F_M) \to M$ is trivial. In this case $(E_A, F_A) = A \times (\bar{E}, \bar{F})$ where $(\bar{E}, \bar{F})$ is the Milnor pair for the transversal A_1-singularity. We find

$$H_{n+1}(E_A, F_A) = \mathbb{Z}^{\mu(\Sigma)}, \qquad H_n(E_A, F_A) = \mathbb{Z},$$

$$H_k(E_A, F_A) = 0, \quad k \neq n, n+1,$$

where $\mu(\Sigma)$ is the Milnor number of Σ.

Case 2: The bundle $w_0 \colon (E_M, F_M) \to M$ is non-trivial. Consider the restrictions of this bundle to the 1-spheres of the wedge A. At least one restriction must be non trivial. We find in this case

$$H_{n+1}(E_A, F_A) = \mathbb{Z}^{\mu(\Sigma)-1}, \qquad H_n(E_A, F_A) = \mathbb{Z}_2,$$

$$H_k(E_A, F_A) = 0, \quad k \neq n, n+1.$$

Both statements can be verified by a computation, which uses a cell decomposition of the relative CW-complex (E_A, F_A). The same computation shows that

$$\mathbb{Z} \cong H_n(E_{p_0}, F_{p_0}) \to H_n(E_A, F_A)$$

(which is induced by inclusion) is surjective.

5.5. We substitute this result in the Mayer–Vietoris sequence and we find the following lemma.

Lemma. $H_k(E^0, F^0) = 0$ *for* $k < n$ *and* $k > n+1$.

5.6. For the remaining homology groups we consider the following parts of the Mayer–Vietoris sequence (5.3).

Case 1: w_0: $(E_M, F_M) \to M$ trivial bundle.

$$0 \to \mathbb{Z}^\tau \oplus \mathbb{Z}^{\mu(\Sigma)} \to H_{n+1}(E^0, F^0) \to \mathbb{Z}^\tau \to \mathbb{Z} \to H_n(E^0, F^0) \to 0.$$

Case 2: w_0: $(E_M, F_M) \to M$ non-trivial bundle.

$$0 \to \mathbb{Z}^\tau \oplus \mathbb{Z}^{\mu(\Sigma)-1} \to H_{n+1}(E^0, F^0) \to \mathbb{Z}^\tau \to \mathbb{Z}_2 \to H_n(E^0, F^0) \to 0.$$

Consider moreover:

Case A: $\tau > 0$. There are D_∞-points. The maps $\mathbb{Z}^\tau \to \mathbb{Z}$ or $\mathbb{Z}^\tau \to \mathbb{Z}_2$ are surjective since

$$H_n(E_p, F_p) \xrightarrow{\cong} H_n(E_{p_0}, F_{p_0}) \twoheadrightarrow H_n(E_A, F_A).$$

It follows that

$$\begin{cases} H_n(F^0, E^0) = 0, \\ H_{n+1}(F^0, E^0) = \mathbb{Z}^{2\tau-1} \oplus \mathbb{Z}^{\mu(\Sigma)}. \end{cases}$$

Case B: $\tau = 0$. There are no D_∞-points.

Case 2: w_0: $(E_M, F_M) \to M$ non-trivial bundle.

$$\text{Case 1:} \quad \begin{cases} H_n(F^0, E^0) = \mathbb{Z}, \\ H_{n+1}(F^0, E^0) = \mathbb{Z}^{\mu(\Sigma)}; \end{cases} \qquad \text{Case 2:} \quad \begin{cases} H_n(F^0, E^0) = \mathbb{Z}_2, \\ H_{n+1}(F^0, E^0) = \mathbb{Z}^{\mu(\Sigma)-1}. \end{cases}$$

5.7. Theorem. *Let* Σ *be a 1-dimensional icis and* f: $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *a holomorphic function with singular locus* $\Sigma(f) = \Sigma$ *and transversal singularity type* A_1 *on* $\Sigma - \{0\}$ *then the homology groups of the Milnor fibre* F *of* f *are as follows:*

Case A: $^\# D_\infty > 0$:

$$\tilde{H}_n(F, \mathbb{Z}) = \mathbb{Z}^{2 ^\# D_\infty - 1} \oplus \mathbb{Z}^{\mu(\Sigma)} \oplus \mathbb{Z}^{\# A_1},$$

$$\tilde{H}_k(F, \mathbb{Z}) = 0 \quad \text{for } k \neq n;$$

Case B: $^\# D_\infty = 0$:

$$\tilde{H}_n(F, \mathbb{Z}) = \mathbb{Z}^{\mu(\Sigma)} \oplus \mathbb{Z}^{\# A_1},$$

$$\tilde{H}_{n-1}(F, \mathbb{Z}) = \mathbb{Z}, \qquad \tilde{H}_{n-1}(F, \mathbb{Z}) = 0 \quad \text{for } k \neq n-1, n,$$

where $\mu(\Sigma) = $ *the Milnor number of* $(\Sigma, 0)$, $^\# D_\infty = $ *number of* D_∞-*points of the approximation* f_s, *and* $^\# A_1 = $ *number of* A_1-*points of the approximation* f_s.

Proof. This follows from the preceding computations, Proposition 2.8 and Example 2.10. The Case B2 does not occur since the next Proposition 5.8 shows:

$$f_s = \bar{G}_1^2 + \cdots + \bar{G}_n^2.$$

Consider coordinates near Σ_0 from section 4.2. Then

$$B_0 = \Sigma_0 \times Q^n \quad \text{with coordinates } (w_0, w_1, \ldots, w_n)$$

and

$$f_s(w) = w_1^2 + \cdots + w_n^2 \quad \text{for all } w \in B_0.$$

This shows that we have a product situation for f_s and hence the bundle.

$$w_0: \ (E^0, F^0) = (E_M, F_M) \to \Sigma_0 \text{ is trivial.} \quad \square$$

5.8. Proposition. *Let f has no D_∞-points in the generic approximation, then we can find $\bar{G}_1, \ldots, \bar{G}_n$ such that*

$$f_s = \bar{G}_1^2 + \cdots + \bar{G}_n^2$$

in a neighborhood of O.

Proof. Let $f = \Sigma h_{ij} g_i g_j$. According to [23, I(7.17) and (7.20)] it follows that $\det h_{ij} \neq 0$. Write now

$$f_s = \Sigma H_{ij}^{(s)} G_i G_j,$$

then there is a neighborhood from O in the (z, s) space such that $\det H_{ij}^{(s)} \neq 0$.

By elementary constructions about quadratic forms, we can find

$$\bar{G}_1, \ldots, \bar{G}_n \quad \text{with } (\bar{G}_1, \ldots, \bar{G}_n) = (G_1, \ldots, G_n)$$

and

$$f_s = \bar{G}_1^2 + \cdots + \bar{G}_n^2.$$

6. The topology of the Milnor fibre

In this section we show that our Milnor fibre is homotopy equivalent to a wedge of spheres

$$S^n \vee \cdots \vee S^n \quad \text{if } {}^\# D_\infty > 0$$

or

$$S^{n-1} \vee S^n \vee \cdots \vee S^n \quad \text{if } {}^\# D_\infty = 0.$$

6.1. Proposition. *Let X be a $(n-2)$-connected CW-complex of dimension $n \geq 3$ with given homology*

$$H_n(X, \mathbb{Z}) = \mathbb{Z}^\mu, \qquad H_{n-1}(X, \mathbb{Z}) = \mathbb{Z},$$

$$\tilde{H}_k(X, \mathbb{Z}) = 0 \quad \textit{if } k \neq n-1, n.$$

Then we have a homotopy equivalence

$$X \overset{h}{\simeq} S^{n-1} \vee S^n \vee S^n \vee \cdots \vee S^n.$$

Proof. For $n \geq 3$ we have that X is simply connected. According to Hurewicz theorem

$$\pi_{n-1}(X) \simeq H_{n-1}(X) = \mathbb{Z}.$$

We attach a n-cell e_n corresponding to a generator φ of $\pi_{n-1}(X)$. Let $X^1 = X \cup_\varphi e_n$.
We have now

$$\pi_{n-1}(X^1) = 0 \quad \text{and} \quad \pi_k(X^1) = \pi_k(X) = 0, \quad k \leq n-2,$$

especially

$$H_{n-1}(X^1) = 0.$$

Consider the following Hurewicz diagram

$$
\begin{array}{ccc}
0 & & \\
\downarrow & & \downarrow \\
\mathbb{Z}^\mu = H_n(X) \longleftarrow & & \pi_n(X) \\
\Big\downarrow \alpha_1 & & \Big\downarrow \alpha_2 \\
H_n(X^1) \overset{\cong}{\longleftarrow} & & \pi_n(X^1) \\
\Big\downarrow \beta_1 & & \Big\downarrow \beta_2 \\
\mathbb{Z} = H_n(X^1, X) \overset{\cong}{\longleftarrow} & & \pi_n(X^1, X) \\
\Big\downarrow \gamma_1 & & \Big\downarrow \gamma_2 \\
\mathbb{Z} = H_{n-1}(X) \overset{\cong}{\longleftarrow} & & \pi_{n-1}(X) \\
\downarrow & & \downarrow \\
0 = H_{n-1}(X^1) \longleftarrow & & \pi_{n-1}(X^1) = 0
\end{array}
$$

γ_1 is an isomorphism (since γ_1 is surjective), so $\beta_1 = 0$ and α_1 must be an isomorphism. It follows that $H_n(X^1) = \mathbb{Z}^\mu$. This implies γ_2 is an isomorphism, $\beta_2 = 0$ and α_2 is surjective.

Let now $Y = S^{n-1} \vee S^n \vee \cdots \vee S^n$, and $Y^1 = D^n \vee S^n \vee \cdots \vee S^n$ where $\partial D^n = S^{n-1}$. Define $h: Y \to X$ and $h^1: Y^1 \to X^1$ as follows

$$h \mid S^{n-1} = \text{generator of } \pi_{n-1}(X),$$

$$h \mid S^n = \text{lifted generator of } \pi_n(X^1),$$

$$h \mid D = e_n.$$

The map h induces an isomorphism

$$H_*(Y) \cong H_*(X).$$

For $k > n$ and $k \leq n - 2$ this is trivial, for $k = n - 1$ this is by construction. For $k = n$ consider
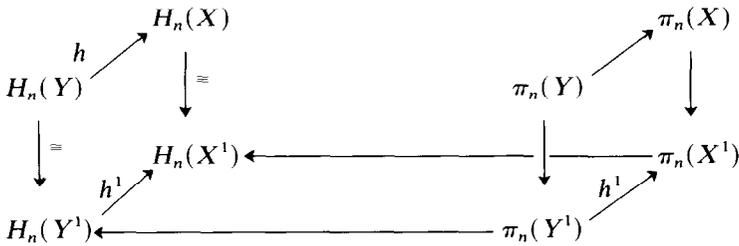
The following maps are isomorphisms:

$$h: \pi_n(Y^1) \to \pi_n(X^1) \quad \text{by construction,}$$

$$\pi_n(X^1) \to H_n(X^1) \quad \text{by Hurewicz-theorem,}$$

$$\pi_n(Y^1) \to H_n(Y^1) \quad \text{by Hurewicz-theorem,}$$

$$H_n(Y) \to H_n(Y^1) \quad \text{by exactness,}$$

$$H_n(X) \to H_n(X^1) \quad \text{by exactness.}$$

Since X and Y are simply connected and $h_*: H_*(Y) \cong H_*(X)$ it follows that h is a homotopy equivalence by a consequence of Whiteheads theorem. (Cf. [27, p. 406, Theorem 7.6.25].)

6.2. Main Theorem. *Let Σ be a 1-dimensional icis and $f: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ a holomorphic function with singular locus $\Sigma(f) = \Sigma$ and transversal singularity type A_1 on $\Sigma - \{0\}$ then the homotopy type of the Milnor fibre F of f is as follows:*
   Case A: $^\# D_\infty > 0$. F is homotopy equivalent to a bouquet of
      $$\mu_n(f) = 2^\# D_\infty - 1 + \mu(\Sigma) + {}^\# A_1 \text{ spheres } S^n.$$
   Case B: $^\# D_\infty = 0$. F is homotopy equivalent to a bouquet of one sphere S^{n-1} and
      $$\mu_n(f) = \mu(\Sigma) + {}^\# A_1 \text{ spheres } S^n.$$
   Here $\mu(\Sigma) = $ Milnor's number of $(\Sigma, 0)$, $^\# D_\infty = $ number of D_∞-points of the approximation f_s, $^\# A_1 = $ number of A_1-points of the approximation f_s.

Proof. The homology of F is already computed in Theorem 5.7. We need more information about $\pi_1(F)$. Let $n \geq 2$.

Case A: For $n \geq 3$, F is simply connected by Kato and Matsumoto [13].

For $n = 2$ we can use the remark of Lê and Saito [19] that the fundamental group $\pi_1(F)$ is free abelian of rank one less than the number of analytic components of $f^{-1}(0)$ in case the transversal type on $\Sigma - \{0\}$ is A_1. This implies $\pi_1(F) = H_1(F) = \{0\}$. So in both cases F is simply connected and we can follow the proof of Milnor's Theorem 6.5 [20] and we find that F is a bouquet of n-spheres $S^n \vee \cdots \vee S^n$ as an application of Whitehead's theorem.

Case B: For $n \geq 3$, F is simply connected [13] and we can apply Proposition 6.1 and find

$$F \stackrel{h}{\simeq} S^{n-1} \vee S^n \vee \cdots \vee S^n.$$

For $n = 2$, F has the homotopy type of a 2-dimensional CW-complex with the properties

$$\pi_1(F) = H_1(F) = \mathbb{Z} \quad \text{and} \quad H_2(F) = \mathbb{Z}^{\mu_2(f)}$$

Proposition 3.3 of Wall [29] (cf. also [6]) can be specialized to read that for a free group π of finite rank r every 2-dimensional CW-complex with fundamental group π has the homotopy type of a bouquet of r copies of S^1 and finitely many copies of S^2. In our case we find

$$F \stackrel{h}{\simeq} S^1 \vee S^2 \vee \cdots \vee S^2.$$

The case $n = 1$ we treat as follows: F has the homotopy type of a 1-dimensional CW-complex. In Case A ($^{\#}D_\infty > 0$) $H_0(F) = \mathbb{Z}$ so $F \stackrel{h}{\simeq} S^1 \vee \cdots \vee S^1$. In Case B ($^{\#}D_\infty = 0$) $H_0(F) = \mathbb{Z} \oplus \mathbb{Z}$ so $F \stackrel{h}{\simeq} (S^1 \vee \cdots \vee S^1) \cup (S^1 \vee \cdots \vee S^1)$ (disjoint union). This can be made more precise as follows:

$f: \mathbb{C}^2 \to \mathbb{C}$ with $^{\#}D_\infty = 0$ implies $f = g^2$ where $g = 0$ is a defining equation for Σ. So $f(x, y) = 1 \Leftrightarrow g(x, y) = \pm 1$ and F is the disjoint union of two Milnor fibres of g.

7. Remarks and related questions

7.1. Vanishing homology.

For a complete description of a non isolated singularity one should not only consider the Milnor fibre at the origin. Let $x \in \mathbb{C}^{n+1}$ and $F_x = f^{-1}(t) \cap B_x(\varepsilon)$ the local Milnor fibre, where $B_x(\varepsilon)$ a small ε-sphere around x and t sufficiently near to $f(x)$. From Theorem 5.7 it follows that

$$\text{For } x \in \Sigma - \{0\} \quad \begin{cases} \tilde{H}^{n-1}(F_x, \mathbb{Z}) = \mathbb{Z}, \\ \tilde{H}^k(F_x, \mathbb{Z}) = 0, \quad k \neq n-1; \end{cases}$$

$$\text{For } x = 0 \quad \begin{cases} \tilde{H}^n(F_x, \mathbb{Z}) = \mathbb{Z}^{\mu_n(f)}, \\ \tilde{H}^{n-1}(F_x, \mathbb{Z}) = 0 \quad \text{or } \mathbb{Z}, \\ \tilde{H}^k(F_x, \mathbb{Z}) = 0, \quad k \neq n-1, n; \end{cases}$$

If $x \notin \Sigma \quad \tilde{H}^k(F_x, \mathbb{Z}) = 0 \quad$ for all k.

In fact $\tilde{H}^k(F_x, \mathbb{Z})$ is the stalk of the sheaf of vanishing cycles $R^{\cdot}\psi$, defined by Deligne [4]. This sheaf is concentrated on Σ and is constructable; the strata are $\Sigma - \{0\}$ and $\{0\}$. On every branch of $\Sigma - \{0\}$ there is defined a local system with fibre $\tilde{H}^{n-1}(F_x, \mathbb{Z}) \simeq \mathbb{Z}$, corresponding to the transversal singularity of type A_1. This local system can have non trivial monodromy.

Examples.

$\quad A_\infty$: $f = z_1^2 + \cdots + z_{n_2}^2 \quad$ has trivial monodromy.

$\quad D_\infty$: $f = z_1 z_1^2 + \cdots + z_n^2 \quad$ has non-trivial monodromy.

We show this last assertion: Consider the loop $z_0(t) = z_0 e^{2\pi i t}$ in the singular locus. The induced monodromy of the transversal Milnor fibre

$$z_0 z_1^2 + \cdots + z_n^2 = 1$$

can be geometrically realized by

$$z(t) = (z_0 e^{2\pi i t}, z_1 e^{-\pi i t}, z_2, \ldots, z_n).$$

So

$$z(1) = (z_0, -z_1, z_2, \ldots, z_n).$$

Now the induced map $(R^n\psi)_x \to (R^n\psi)_x$ is the multiplication with -1 from $\mathbb{Z}$ to $\mathbb{Z}$.

A deformation argument shows that for isolated line singularities the monodromy of the local system is:

$\quad$ the identity if $^\#D_\infty$ is even, $\quad -(\text{identity})$ if $^\#D_\infty$ is odd.

In Sections 4 and 5 we have already studied the system of vanishing cycles transversal to the critical locus Σ_s of the generic approximation f_s: $\mathbb{C}^{n+1} \to \mathbb{C}$. A natural object to study in general is the sheaf of vanishing cycles for deformations F: $\mathbb{C}^{n+1} \times S \to \mathbb{C} \times S$, which have the property that the fibrations f_s are all equivalent.

7.2. Relative De Rham cohomology. Let $\Omega^{\cdot}$ be the sheaf of complexes of holomorphic differential forms on $\mathbb{C}^{n+1}$. For f: $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ a holomorphic germ one can define the relative De Rham complex sheaf $\Omega_f^{\cdot}$ as follows:

$$\Omega_f^{\cdot} = \Omega^{\cdot} / df \wedge \Omega^{\cdot - 1}$$

and its cohomology groups $H^k(\Omega_f^{\cdot})$

For isolated singularities these groups were considered by Brieskorn [3], who showed that:

$H^n(\Omega_f^{\cdot})$ is a free $\mathbb{C}\{f\}$-module of rank μ.

$H^k(\Omega_f^{\cdot}) = 0 \quad$ for $k \neq 0, n$.

For the non-isolated singularities, which we consider in this paper, we know the following.

Let f be of type A_∞. One can compute:

$$H^{\cdot}(\Omega_f^{\cdot}) = \mathbb{C}\{f\}, \qquad H^{n-1}(\Omega_f^{\cdot}) = \mathbb{C}\{f\},$$

$$H^k(\Omega_f^{\cdot}) = 0, \quad k \neq 0, n-1.$$

This shows that for dim $\Sigma = 1$ and transversal type A_1 we have:
 (a) $H^n(\Omega_f^{\cdot})$ is concentrated in $\{0\}$,
 (b) $H^{n-1}(\Omega_f^{\cdot})$ is a constructable sheaf concentrated on Σ or $\Sigma - \{0\}$.
 Van Straten [28] proved the following result:
 Let Σ be a 1-dim. icis, and $f \colon (\mathbb{C}^{n+1}, 0) \to \mathbb{C}$ has transversal type A_1 on $\Sigma - 0$, then
 (a) $H^n(\Omega_f^{\cdot})$ is a free $\mathbb{C}\{f\}$-module of dimension $b_n(F) = \mu_n(f)$,
 (b) $H^{n-1}(\Omega_f^{\cdot})$ is a free $\mathbb{C}\{f\}$-module of dimension $b_{n-1}(F)$

$$b_{n-1}(F) = \begin{cases} 0 & \text{if } {}^{\#}D_\infty > 0, \\ 1 & \text{if } {}^{\#}D_\infty = 0. \end{cases}$$

7.3. A similar approach as in this paper can lead directly to the homotopy type of the Milnor fibre. One considers deformations, which fix Σ and such that f_s has at the origin a 'central singularity' of a sufficient nice type. We report about this in [26]. Sometimes that method also gives results in the case, where Σ is not a complete intersection.

Example. Σ the three coordinate axis in $\mathbb{C}^3$. Central singularity $T_{\infty,\infty,\infty} \colon xyz$; Milnor fibre $S^1 \times S^1$.

Let

$$f = x^2 y^2 + y^2 z^2 + z^2 x^2.$$

f has singular locus Σ and the transversal type is A_1.

Consider the deformation

$$f_s = x^2 y^2 + y^2 z^2 + z^2 x^2 + sxyz.$$

The singular locus of f_s consist of:
 (a) 4 isolated A_1-points,
 (b) the 3 coordinate axis with each 2 D_∞-points and the central singularity at the origin.
The homotopy type of the Milnor fibre of f is a bouquet of 15 spheres S^2 (cf. [26]). The homology of this example can also be computed by methods of this paper.

We remark also that Pellikaan considers the same example in [23, I (7.7)]: A miniversal deformation of $(\Sigma, 0)$ is defined by the ideal of the (2×2)-minors of the matrix

$$\begin{pmatrix} x & y & z \\ x+s_1 & 2y+s_2 & 3z+s_3 \end{pmatrix}.$$

So

$$yz + s_3 y - s_2 z = 0, \qquad 2xz + s_3 x - s_1 z = 0, \qquad xy + s_2 x - s_1 y = 0.$$

The Milnor number of $(\Sigma, 0)$ is 2, so $\Sigma_s \overset{h}{\simeq} S^1 \vee S^1$.

Let

$$f_s(x, y, z) = (yz + s_3 y - s_2 z)^2 + (2xz + s_3 z - s_1 z)^2 + (xy - s_2 x - s_1 y)^2.$$

This deformation f_s has, for generic values of s, six A_1-points outside Σ_s and four D_∞-singularities on Σ_s and Σ_s non-singular. The constructions of Sections 4 and 5 also apply to this example. By Theorem, 5.7

$$H_n(F; \mathbb{Z}) = \mathbb{Z}^{15} \quad \text{and} \quad H_k(F; \mathbb{Z}) = 0, \quad k \neq n,$$

since $\mu(\Sigma) + {}^\# A_1 + 2^\# D_\infty - 1 = 15$.

Theorem 6.2 applies also; so the Milnor fibre is homotopy equivalent with a bouquet of 15 spheres S^2.

7.4. Bifurcation variety.

Goryonov [7] considered a versal deformation of isolated line singularities and defined a bifurcation variety in the base space of that deformation. He showed that for simple isolated line singularities [25] the complement of the bifurcation variety is a $K(\pi, 1)$.

A question is what happens for the more general case we consider in this paper, where Σ is a 1-dimensional icis and the transversal type is A_1. A first step in that direction should be a classification of simple singularities. Pellikaan [23, I§4] proved some determinacy theorems, which seem to be useful for that classification.

7.5. Other transversal types.

The methods of this paper can perhaps be applied to other transversal types. A first study in this direction was made by De Jong [12]. He considered as singular locus a smooth line and transversal types A_2, A_3, D_4, E_6, E_7 or E_8. Although the situation already become more complicated he found a similar behaviour and gets results of the expected flavor.

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