# HIERARCHICAL PARALLEL MEMORY-SYSTEMS, AND MULTI-PERIODIC SKEWING SCHEMES

Gerard Tel and Harry A.G. Wijshoff

RUU-CS-85-24
August 1985

Rijksuniversiteit Utrecht

**Vakgroep informatica**

Budapestlaan 6 3584 CD Utrecht
Corr. adres: Postbus 80.012 3508 TA Utrecht
Telefoon 030-53 1454
The Netherlands

# HIERARCHICAL PARALLEL MEMORY-SYSTEMS, AND MULTI-PERIODIC SKEWING SCHEMES

Gerard Tel and Harry A.G. Wijshoff

Technical Report RUU-CS-85-24
August 1985

Department of Computer Science
University of Utrecht
P.O. Box 80.012
3508 TA Utrecht, the Netherlands

[illegible]

[illegible] been studied from the view[illegible] [illegible] we are concerned with [illegible] [illegible] new [illegible] housing schemes [illegible]

# HIERARCHICAL PARALLEL MEMORY-SYSTEMS, AND MULTI-PERIODIC SKEWING SCHEMES

Gerard Tel and Harry A.G. Wijshoff

Department of Computer Science, University of Utrecht
P.O. Box 80.012, 3508 TA Utrecht, the Netherlands

*Abstract.* The theory of skewing schemes deals with the problem of distributing data in parallel memories, in such a way that parallel computations can proceed efficiently. Until now skewing schemes have been studied from the viewpoint of the BSP and ILLIAC IV architectures. In the present paper we are concerned with hierarchically organized parallel memory-systems, for which a new class of skewing schemes is introduced, namely the multi-periodic skewing schemes. We show that multi-periodic skewing schemes are an extension of both the periodic skewing schemes and the diamond schemes for traditional parallel memories. It is also shown that the schemes work out very well for many applications and, in particular, a bound on the minimum number of memory banks needed for certain applications is derived. Furthermore, multi-periodic skewing schemes can be represented at the cost of only a small amount of space, which makes them of practical interest.

## 1. Introduction.

The availability of data in multi-processor computations heavily influences the actual performance of such computations. For this reason considerable attention has been given to the problem of storing data in such a way that the data can be retrieved rapidly and without much overhead cost. Until now the problem has been studied mainly in the context of the BSP and ILLIAC IV architectures, which contain a number of memory banks that are directly connected (via an interconnection network) to a number of processors [1,3]. See figure 1. In this context classes of data mappings known as skewing schemes have been defined, and a study was initiated of the validity of certain skewing schemes for collections of data templates. Suppose there are M parallel memories and let 0, 1, ..., M−1 represent the memory banks.

**Definition** 1.1. A *skewing scheme* s is a map from $\mathbb{Z}^d$ to the (finite) set $\{0,1,...,M-1\}$.

Thus a skewing scheme s denotes how the elements of a d-dimensional array, which is the most common data structure in numerical computations, have to be distributed over the memory banks. (We ignore the assignment of individual addresses within the memory banks.)

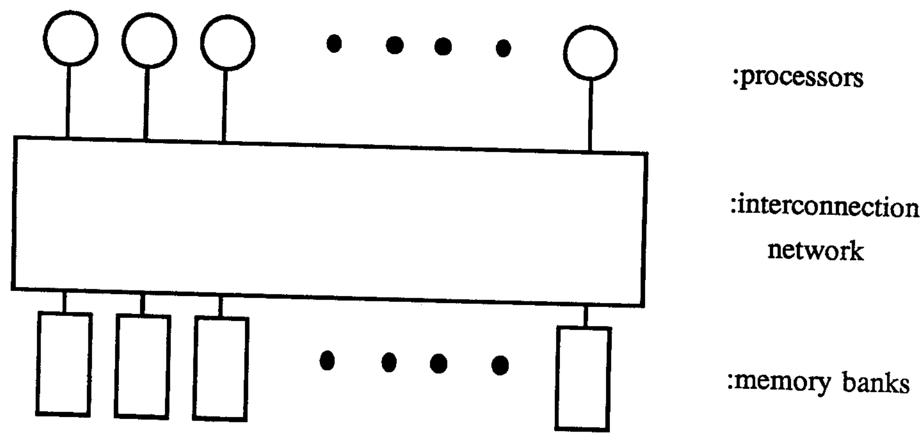


Figure 1 .

**Definition** 1.2.

(i) A *data-template* T is defined to be any finite set $T=\{\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_{p-1}\}$ with $\vec{a}_i \in \mathbb{Z}^d$ and $\vec{a}_0 = \vec{0}(=(0,0,\ldots,0))$.

(ii) An *instance* $T(\vec{x})$ of a data-template T ($\vec{x} \in \mathbb{Z}^d$) is defined by $T(\vec{x}) = \{\vec{a}_0+\vec{x}, \vec{a}_1+\vec{x}, \ldots, \vec{a}_{p-1}+\vec{x}\}$.

(iii) A *skewing scheme* s is valid for T iff $\forall \vec{x}\ s \upharpoonright T(\vec{x})$ is an injection.

A template denotes a "vector" of array elements which the processors want to access in parallel. If a skewing scheme is valid for a template T, then the processors can fetch (or store) each instance of T in one memory cycle.

In many of today's supercomputers the memory system is more complex and hierarchically organized, e.g. the CRAY-1 comprises a main memory, divided into 16 memory banks, feeding data to and from a set of scalar and vector registers. In figure 2 the overall structure of such a system is depicted. In a hierarchically structured memory it is obvious that one should not consider all the instances of a certain template T, but only those which do not overlap too much. For, whenever a particular instance of T has been processed, it is likely that this instance is still kept in the data-buffers. If the next instance of T which has to be processed overlaps the previous instance, then only the non-overlapped part of it needs to be fetched from the memory banks.

For this purpose, we shall consider only those instances $T(\vec{x})$ of T, with $\vec{x} \in L$ and L a lattice $\subseteq \mathbb{Z}^d$.

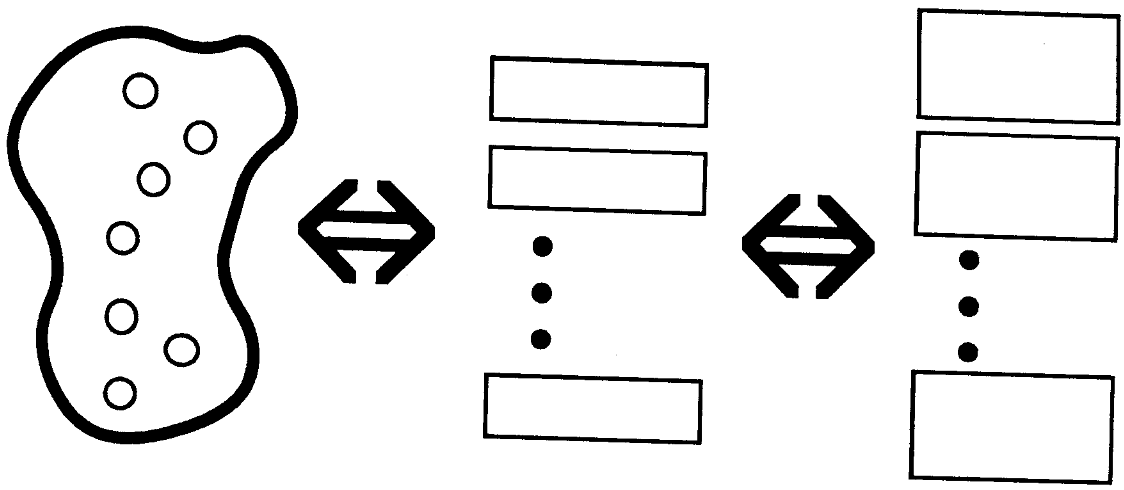


Figure 2 .

**Definition** 1.3. A *lattice* $L \subseteq \mathbb{Z}^d$ is defined to be any set $L=\{\lambda_1.\vec{x}_1+\lambda_2.\vec{x}_2+...+\lambda_d.\vec{x}_d \mid \lambda_i \in \mathbb{Z}\}$, with $\vec{x}_1, \ldots, \vec{x}_d \in \mathbb{Z}_d$, and $\det(\vec{x}_1, \ldots, \vec{x}_d) \neq 0$. $\{\vec{x}_1, \ldots, \vec{x}_d\}$ is called the basis of L, and the determinant $\Delta(L)$ of L is defined by $\Delta(L)= |\det(\vec{x}_1..\vec{x}_d)|$.

It can be shown that $\Delta(L)$ is independent of the particular basis chosen for L. Points $p=(p_1, p_2, \ldots, p_d)$ and $q=(q_1, q_2, \ldots, q_d)$ are said to be equivalent modulo L, notation $p \equiv_L q$, if $p-q \in L$.
It appears that for commonly used templates T there always exists a lattice $L \subseteq \mathbb{Z}^d$, such that (i) $\Delta(L)=O(|T|)$, and (ii) for all $\vec{y} \in \mathbb{Z}^d$ there exists a $\vec{x} \in L$ such that $\vec{y} \in T(\vec{x})$. See figure 3 for some examples (d=2).

**Definition** 1.4. Let L be a lattice $\subseteq \mathbb{Z}^d$. A skewing scheme s is L-*valid* for T iff $\forall \vec{x} \in L$ $s \restriction T(\vec{x})$ is an injection.

General skewing schemes are not much of practical interest if a high price must be paid for representing the scheme and computing the bank number. Periodic and linear skewing schemes were introduced to remedy this problem [1, 4, 5, 6]. These schemes are completely defined by finite tabular information, and can be represented by a single arithmetic formula [6].

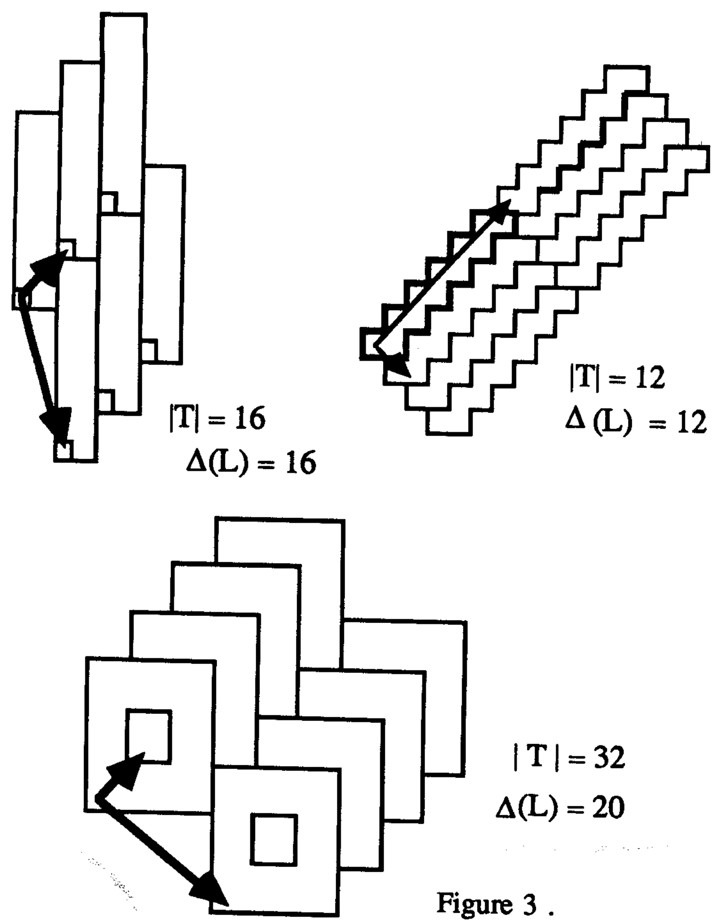


Figure 3 .

In section 2 of this paper we show that periodic skewing schemes are too "strong" for just obtaining L-validity. Therefore we introduce a weaker and more general version of these schemes: the multi-periodic skewing schemes. It turns out that the multi-periodic skewing schemes also are an extension of the diamond schemes of Jalby et. al. [2], and thus appear to be a suitable unified class of schemes for a great variety of purposes. We show in section 3 that multi-periodic schemes are quite suitable for skewing a collection of templates, such that they are L-valid for this collection of templates. In section 4 we show that multi-periodic skewing schemes can be compactly represented, like ordinary periodic schemes.

## 2. Multi-periodic skewing schemes.

We first recall the definition of periodic skewing schemes. A thorough study of these schemes can be found in [5, 6].

**Definition 2.1.** A skewing scheme s: $\mathbb{Z}^d \to \{0,1,...,M-1\}$ is called *periodic* if there exist $\vec{a}_0, \vec{a}_1, \ldots, \vec{a}_{M-1} \in \mathbb{Z}^d$ and a lattice $L \subseteq \mathbb{Z}^d$ such that for all $i \in \{0,1,...,M-1\}$ $s^{-1}(i)=\{\vec{a}_i+\vec{x} \mid \vec{x} \in L\}$.

The following proposition directly follows from [5, prop.4.1].

**Proposition 2.2.** Given a periodic skewing scheme s, a template T and a lattice $L \subseteq \mathbb{Z}^d$. Then s is L-valid for T iff s is valid for T.

From proposition 2.2 follows that, while the property of L-validity is weaker than unrestricted validity, the two notions are equivalent for periodic skewing schemes. To take full advantage of the weaker requirement of L-validity we need a class of skewing schemes, which are as attractive as the periodic schewing schemes, but which are not as strictly tied to the lattice structure. The skewing schemes which fit these conditions are the multi-periodic skewing schemes, introduced in the following definitions.

**Definition 2.3.** Let $L \subseteq \mathbb{Z}^d$ be a lattice, with basis $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d$. For every $\vec{a} \in \mathbb{Z}^d$ let $\vec{a}+L = \{\vec{a}+\vec{x} \mid \vec{x} \in L\} \cong \mathbb{Z}^d$. Let $\eta_L^{\vec{a}}$ be the isomorphism: $\eta_L^{\vec{a}}: \vec{a}+L \to \mathbb{Z}^d$, with $\eta_L^{\vec{a}}(\vec{a}+i_1.\vec{x}_1+i_2.\vec{x}_2+...+i_d.\vec{x}_d) = (i_1, i_2, \ldots, i_d)$. The *$\vec{a}$-reduced skewing scheme* $s_L^{\vec{a}}$ of a skewing scheme s: $\mathbb{Z}^d \to \{0,1,...,M-1\}$ is defined by $s_L^{\vec{a}}: \mathbb{Z}^d \to A$, $A \subseteq \{0,1,...,M-1\}$, and $s_L^{\vec{a}}(\vec{y})=s((\eta_L^{\vec{a}})^{-1}(\vec{y}))$.

**Definition 2.4.** A skewing scheme s: $\mathbb{Z}^d \to \{0,1,...,M-1\}$ is called *multi-periodic* if there exists a lattice $L \subseteq \mathbb{Z}^d$ such that for all $\vec{a} \in \mathbb{Z}^d$ $s_L^{\vec{a}}$ is periodic. (L is called the underlying lattice of s.)

Note that the periodic skewing schemes are multi-periodic.

**Proposition 2.5.** Given a skewing scheme s: $\mathbb{Z}^d \to \{0,1,...,M-1\}$. If s is periodic, then s is multi-periodic.

**Proof:**

Take $L=\mathbb{Z}^d$. □

We show that the multi-periodic skewing schemes are an extension of the diamond schemes as well. Diamond schemes were introduced by Jalby et. al.[2]. We give the definition in our lattice-framework. Let $S_M$ be the symmetric group, i.e., the group of permutations on M elements.

**Definition 2.6.** A skewing scheme s: $\mathbb{Z}^d \rightarrow \{0,1,...,M-1\}$ is called a *diamond scheme* iff there exists a lattice L $\subseteq \mathbb{Z}^d$ with basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d\}$, and d commuting permutations $\lambda_1, \lambda_2, \ldots, \lambda_d \in S_M$ such that for all $\vec{x} \in \mathbb{Z}^d$ and for all $1 \le i \le d$: $s(\vec{x}+\vec{x}_i) = \lambda_i(s(\vec{x}))$. (L is called the underlying lattice of s.)

Jalby et.al.[2] actually defined the diamond schemes only for the case that the underlying lattice L is defined by an orthogonal basis. In order to prove that the multi-periodic schemes are an extension of the diamond schemes, we need the following characterisation of periodic schemes.

**Lemma 2.7.** Let s: $\mathbb{Z}^d \rightarrow \{0,1,...,M-1\}$ be a skewing scheme using M memory banks. Then s is periodic iff $\forall\, \vec{x} \in \mathbb{Z}^d\ \exists\, \sigma_{\vec{x}} \in S_M\ \forall\, \vec{y} \in \mathbb{Z}_d\ s(\vec{y}+\vec{x}) = \sigma_{\vec{x}}(s(\vec{y}))$.

**Proof:**

The proof makes use of the following fact.

**Fact 2.7.1.** [5, prop. 2.3] A skewing scheme s is periodic iff for all $\vec{x}, \vec{y} \in \mathbb{Z}^d$: if $s(\vec{x}) = s(\vec{y})$, then for all $\vec{x}', \vec{y}'$, with $\vec{x}'-\vec{y}' = \vec{x}-\vec{y}$, $s(\vec{x}') = s(\vec{y}')$.

($\Rightarrow$) Let s be a periodic skewing scheme, and $\vec{x} \in \mathbb{Z}^d$. Define $\sigma_{\vec{x}}$ as follows: if $s(\vec{y}) = p$, $\vec{y} \in \mathbb{Z}^d$, then $\sigma_{\vec{x}}(p) = s(\vec{x}+\vec{y})$. This definition is sound because, if $s(\vec{y})=p$ and $s(\vec{z})=p$ for $\vec{y} \neq \vec{z}$ then from fact 2.7.1 follows that $s(\vec{x}+\vec{y})=s(\vec{x}+\vec{z})$. It is obvious that $\sigma_{\vec{x}}$ fits the conditions.

($\Leftarrow$)Let $\vec{y}, \vec{z}, \vec{y}', \vec{z}'$ be such that $s(\vec{y})=s(\vec{z})$ and $\vec{y}'-\vec{z}'=\vec{y}-\vec{z}$. Define $\vec{x}=\vec{y}'-\vec{y}(=\vec{z}'-\vec{z})$. Then $s(\vec{y}')=\sigma_{\vec{x}}(s(\vec{y}))$ and $s(\vec{z}')=\sigma_{\vec{x}}(s(\vec{z}))$. Hence $s(\vec{y}')=s(\vec{z}')$. From fact 2.7.1 it follows that s is periodic. □

Let Ran $(s_L^{\vec{c}})$ denote the set $\{s_L^{\vec{c}}(\vec{x}) \mid \vec{x} \in \mathbb{Z}^d\}$.

**Theorem 2.8.** Let s be a skewing scheme defined on $\mathbb{Z}^d$. s is a diamond scheme with a underlying lattice L iff

(1) s is a multi-periodic skewing scheme with underlying lattice L, and

(2) for all $\vec{a}, \vec{b} \in \mathbb{Z}^d$, with $\vec{a} \not\equiv_L \vec{b}$ is valid that Ran $(s_L^{\vec{a}}) \cap$ Ran $(s_L^{\vec{b}}) = \emptyset$ , or, there exists a $\vec{y} \in \mathbb{Z}^d$ such that for all $\vec{x} \in \mathbb{Z}^d$ $s_L^{\vec{a}}(\vec{x}) = s_L^{\vec{b}}(\vec{x}+\vec{y})$.

**Proof:**

($\Rightarrow$) Let s:$\mathbb{Z}^d \rightarrow \{0, 1, ..., M-1\}$ be a diamond scheme. Then there exist a d-dimensional lattice L $\subseteq \mathbb{Z}^d$ with basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d\}$ and d commuting permutations $\lambda_1, \lambda_2, \ldots, \lambda_d \in S_M$, such that $\forall \vec{x} \in \mathbb{Z}^d\ \forall 1 \le i \le d\ s(\vec{x}+\vec{x}_i) = \lambda_i(s(\vec{x}))$.

Consider an arbitrary $\vec{a} \in \mathbb{Z}^d$. Then $s_L^{\vec{a}}: \mathbb{Z}^d \rightarrow A_{\vec{a}}$. $A_{\vec{a}} \subseteq \{0, 1, ..., M-1\}$, and $s_L^{\vec{a}}(\vec{x}+\vec{e}_i) = \lambda_i(s_L^{\vec{a}}(\vec{x}))$, with $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_d\}$ the orthonormal basis of $\mathbb{Z}^d$. So given $\vec{y} \in \mathbb{Z}^d$,

$\vec{y}$ arbitrary, we can define $\sigma_{\vec{y}} = \lambda_1^{y_1}\lambda_2^{y_2}....\lambda_d^{y_d}$, such that for all $\vec{x} \in \mathbb{Z}^d$ $s_L^{\vec{a}}(\vec{x}+\vec{y}) = \sigma_{\vec{y}}(s_L^{\vec{a}}(\vec{x}))$. From lemma 2.7 it follows that $s_L^{\vec{a}}$ is a periodic skewing scheme. Consider Ran $(s_L^{\vec{a}})$ and Ran $(s_L^{\vec{b}})$, for $\vec{a}, \vec{b} \in \mathbb{Z}^d$.

Then Ran $(s_L^{\vec{a}}) = \{\lambda_1^{i_1}\lambda_2^{i_2} \cdots \lambda_d^{i_d} s(\vec{a}) \mid i_1, i_2, ..., i_d \in \mathbb{Z}\}$ and

Ran $(s_L^{\vec{b}}) = \{\lambda_1^{i_1}\lambda_2^{i_2} \cdots \lambda_d^{i_d} s(\vec{b}) \mid i_1, i_2, ..., i_d \in \mathbb{Z}\}$

Suppose Ran $(s_L^{\vec{a}}) \cap$ Ran $(s_L^{\vec{b}}) \neq \emptyset$. Then $s(\vec{b}) \in$ Ran $(s_L^{\vec{a}})$ and Ran $(s_L^{\vec{a}})$ = Ran $(s_L^{\vec{b}})$. So there exist $i_1^*, i_2^*, ..., i_d^* \in \mathbb{Z}$ such that $s(\vec{b}) = \lambda_1^{i_1^*} . \lambda_2^{i_2^*} . \cdots . \lambda_d^{i_d^*} s(\vec{a})$. Take $\vec{y} = (i_1^*, i_2^*, ..., i_d^*) \in \mathbb{Z}^d$. Then for all $\vec{x} \in \mathbb{Z}^d$, $\vec{x} = (x_1, x_2, \ldots, x_d)\}$,

$$
\begin{aligned}
s_L^{\vec{b}}(\vec{x}) = (s_L^{\vec{b}})(x_1, x_2, \ldots, x_d) &= s((\eta_L^{\vec{b}})^{-1}(x_1, x_2, \ldots, x_d)) \\
&= s(\vec{b} + x_1\vec{x}_1 + x_2\vec{x}_2 + \ldots + x_d\vec{x}_d) \\
&= \lambda_1^{x_1}\lambda_2^{x_2} \cdots \lambda_d^{x_d} s(\vec{b}) \\
&= \lambda_1^{x_1+i_1^*}\lambda_2^{x_2+i_2^*} \cdots \lambda_d^{x_d+i_d^*} s(\vec{a}) \\
&= s(\vec{a} + (x_1+i_1^*)\vec{x}_1 + (x_2+i_2^*)\vec{x}_2 + \ldots + (x_d+i_d^*)\vec{x}_d) \\
&= s((\eta_L^{\vec{a}})^{-1}(x_1+i_1^*, x_2+i_2^*, \ldots, x_d+i_d^*)) \\
&= s_L^{\vec{a}}(\vec{x}+\vec{y}).
\end{aligned}
$$

($\Leftarrow$) Let s : $\mathbb{Z}^d \to \{0, 1, ..., M-1\}$ be a multi-periodic skewing scheme, which satisfies the constraints of the theorem. Consider $a \in \{0, 1, ..., M-1\}$. From the periodicity of each $s_L^{\vec{a}}$ and from the constraints it follows that for all $\vec{x}, \vec{y} \in s^{-1}(a)$ and $\vec{z} \in L$: $s(\vec{x}+\vec{z}) = s(\vec{y}+\vec{z})$. Define $\lambda_1, \lambda_2, \ldots, \lambda_d \in S_M$ by $\lambda_i(a) = s(\vec{x}+\vec{x}_i)$, for some $\vec{x} \in s^{-1}(a)$. The soundness of this definition follows from the previous statement. And furthermore it is obvious that for all $\vec{x} \in \mathbb{Z}^d$, $1 \le i \le d$, $s(\vec{x}+\vec{x}_i) = \lambda_i(s(\vec{x}))$. □

Actually the set of permutations $\{\lambda_1^{i_1}\lambda_2^{i_2} \cdots \lambda_d^{i_d} \mid i_1, i_2, \ldots, i_d \in \mathbb{Z}\}$ is a subgroup of $S_M$. And, hence, Ran $(s_L^{\vec{c}}) = \{\lambda_1^{i_1}\lambda_2^{i_2} \cdots \lambda_d^{i_d} s(\vec{c}) \mid i_1, i_2, \ldots, i_d \in \mathbb{Z}\}$ forms an orbit of the set $\{0,1,...,M-1\}$, for arbitrary $\vec{c} \in \mathbb{Z}^d$. From elementary group theory it is known that the set of orbits form a partition of the set $\{0,1,...,M-1\}$.

**Corollary 2.9.** If s is a periodic skewing scheme, then s is a diamond scheme.

**Proof:**

From proposition 2.5 it follows that s is multi-periodic and s obviously satisfies the conditions of theorem 2.8. □

In the remainder of this section we shall examine the definition of the multi-periodic skewing schemes more thoroughly. We could for instance, extend these skewing schemes as well, by requiring that the $s_L^{\vec{a}}$'s are not strictly periodic, but e.g. multi-periodic themselves. Call these schemes multi-multi-periodic.

**Definition** 2.10. A skewing scheme s : $\mathbb{Z}^d \rightarrow \{0,1,...,M-1\}$ is called *multi-multi-periodic* if there exists a lattice L $\subseteq \mathbb{Z}^d$ (the underlying lattice of s) such that for all $\vec{a} \in \mathbb{Z}^d$ $s_L^{\vec{a}}$ is periodic, or, for all $\vec{a} \in \mathbb{Z}^d$ $s_L^{\vec{a}}$ is multi-multi-periodic and all the $s_L^{\vec{a}}$'s have the same underlying lattice.

Although it is not immediately obvious, we do not achieve anything with this extension. Every multi-multi-periodic scheme is just mult-periodic.

**Lemma** 2.11. Let s : $\mathbb{Z}^d \rightarrow \{0,1,...,M-1\}$ be a skewing scheme. Then s is multi-multi-periodic iff s is multi-periodic.
**Proof:**

Let s : $\mathbb{Z}^d \rightarrow \{0,1,...,M-1\}$ be multi-multi-periodic (write ${}^0s$ for s). Then $\exists$ t≥0, such that for arbitrary $\vec{a} \in \mathbb{Z}^d$ ${}^0s$, ${}^0s_L^{\vec{a}}$ (write ${}^1s$), ${}^1s_L^{\vec{a}}$ (write ${}^2s$),..., ${}^ts_L^{\vec{a}}$ (write ${}^{t+1}s$) are multi-multi-periodic but not periodic, and ${}^{t+1}s_L^{\vec{a}}$ is periodic. Let $L_0,L_1, \ldots ,L_t$ be lattices $\subseteq \mathbb{Z}^d$ such that $L_i$ is the underlying lattice of ${}^is$, and let for each i $\{\vec{x}_1^{(i)}, \vec{x}_2^{(i)}, ..., \vec{x}_d^{(i)}\}$ be a basis of $L_i$. Define the d×d-matrices $A_0,A_1, \ldots ,A_t$ by

$$A_0 = (\vec{x}_1\vec{x}_2 \cdots \vec{x}_d)$$

and for all $1 \le i \le t$

$$A_i = (A_{i-1}\vec{x}_1^{(i)} \; A_{i-1}\vec{x}_2^{(i)} \cdots A_{i-1}\vec{x}_d^{(i)})$$

Define the lattice L, with basis $\{\vec{x}_1,\vec{x}_2, \ldots ,\vec{x}_d\}$, by $\vec{x}_j = A_t\vec{x}_j^{(t)}$, then from definition 2.10 follows that for all $\vec{a} \in \mathbb{Z}^d$ $s_L^{\vec{a}}$ is periodic, which ends the proof. □

Lemma 2.11 will be of use in the next section, where we shall study the L-validity of multi-periodic skewing schemes.

## 3. Multi-periodic skewing with a minimum number of memory banks.

In this section we show that the multi-periodic skewing schemes lend themselves quite well for the L-valid skewing of one ore more templates T. We first need some notions.

**Definition** 3.1. Given a collection of templates C=$\{T_1, T_2, \ldots, T_t\}$ and a lattice L $\subseteq \mathbb{Z}^d$, let $\mu_L$(C) (respectively $\mu_L^p$(C), $\mu_L^{mp}$(C)) be the minimum number M, such that there exists an arbitrary (resp. periodic, multi-periodic) skewing scheme s: $\mathbb{Z}^d \to \{0,1,\ldots,M-1\}$ which is L-valid for each $T_i \in$ C.

$\lambda_L^{mp}$(C) is the minimum number δ, such that there exists a multi-periodic skewing scheme $s: \mathbb{Z}^d \to \{0,1,\ldots,\mu_L^{mp}(C)\}$, that is L-valid for C and has an underlying lattice L′ such that Δ(L′)=δ.

Note that for a periodic skewing scheme s: $\mathbb{Z}^d \to \{0,1,\ldots,M-1\}$ the determinant of the underlying lattice L is equal to M [5].

The reason why we are interested in the number $\lambda_L^{mp}$(C) is that this number is a measure for the representation-costs, as we shall see in section 4. An important means for determining the number $\mu_{\mathbb{Z}^d}^p$ (C) is the fundamental domain of a lattice [5].

**Definition** 3.2. Given a lattice L $\subseteq \mathbb{Z}^d$. A *fundamental domain* F of L is any (viz. connected) set $\subseteq \mathbb{Z}^d$ such that

1) no two points of F are equivalent mod L, and

2) every point $\vec{x} \in \mathbb{Z}^d$ is equivalent mod L to a point of F.

(Thus, F has exactly one point from every equivalent class mod L, F forms an embedding of $\mathbb{Z}^d$/L, and |F|=Δ(L).) The following two facts are of interest.

**Fact** 3.3. [5]. Given a template T and a periodic skewing scheme s with underlying lattice L. Let F be a fundamental domain of L. Then s is valid for T iff for every $\vec{x} \in$ F there exists at most one element $\vec{y} \in$ T such that $\vec{x} \equiv_L \vec{y}$.

**Fact** 3.4. [5]. $\mu_{\mathbb{Z}^d}^p$(C), C arbitrary, can be computed in time polynomial in N and k, with N=max $\{|\vec{x}|, \vec{x} \in T_i \in C\}$.

First we show that the multi-periodic skewing schemes are stronger than the periodic schemes in the sense that one may be able to skew collections of templates in fewer memory banks with the former.

**Lemma** 3.5. For all $d$>0 there exists a collection $C$ of templates such that $\mu_{\mathbb{Z}^d}^p(C) > \mu_{\mathbb{Z}^d}^{mp}(C)$.

**Proof:**

Let C consist of one template T, defined by T={(0,0,...,0), (2,0,...,0)}. Then $\mu_{\mathbb{Z}^d}^p(C) = 3$ and $\mu_{\mathbb{Z}^d}^{mp}(C) = 2$. □

For the one-dimensional case this lemma can even be strengthened.

**Theorem 3.6.** Given C=$\{T_1,T_2, \ldots ,T_t\}$, with all $T_i \subseteq \mathbb{Z}$.
Then $\mu_{\mathbb{Z}}^{mp}$ (C) = $\mu_{\mathbb{Z}}$ (C).
**Proof:**

Let s : $\mathbb{Z} \rightarrow \{0, 1, ..., \mu_{\mathbb{Z}}(C)-1\}$ be an arbitrary skewing scheme valid for C, and let a = $\max_{k,l \in T_i} |k-l|$. Consider the a-tuples $t_0,t_1,t_2,...$ with $t_i=(s(i),s(i+1),..., s(i+a-1))$. By the pidgeon-hole principle there must exist $p$ and $q$ such that $t_p = t_q$. Define the skewing scheme s′: $\mathbb{Z} \rightarrow \{0, 1, ..., \mu_{\mathbb{Z}}(C)-1\}$ by $s'(i) = s(p+i(\bmod q-p))$. Then $s'$ is valid for $C$ also. Furthermore $s'$ is multi-periodic with an underlying lattice defined by the basis $\{(q-p)\}$. □

So for the one-dimensional case the minimum number of memory banks $\mu_{\mathbb{Z}}(C)$ may be achieved by a multi-periodic skewing scheme, although the determinant of the underlying lattice of s is exponential in $\mu_{\mathbb{Z}}(C)$. The question , whether this holds for higher dimensions also, seems to be more complicated and is left open. The following theorem gives a bound on the number of memory banks needed for L-validly skewing a collection of templates.

**Theorem 3.7.** Given a collection of templates C=$\{T_1,...,T_t\}$, $T_i \subseteq \mathbb{Z}^d$ (1≤i≤t), a d-dimensional lattice $L \subseteq Z_d$ and a partition $B=\{B_1,B_2, \ldots ,B_r\}$ of a fundamental domain of $L$. Let $B_k=\{\vec{a}_1^k,\vec{a}_2^k, \cdots \vec{a}_{r_k}^k\}$ for all $1 \le k \le r$. Then there exists a multi-periodic skewing scheme $s$, with underlying lattice $L$, such that $s$ is L-valid for $C$ and $s$ uses $M^*$ memory banks. Where

$$M^* = \sum_{1 \le k \le r} \min_{\substack{\vec{x}_1,\vec{x}_2,\cdots \\ \cdots\vec{x}_{|B_k|} \in \mathbb{Z}^d}} \mu_{\mathbb{Z}^d}^{p}\Big(\{\bigcup_i (T_1^{\vec{a}_i^k}+\vec{x}_i),\bigcup_i (T_2^{\vec{a}_i^k}+\vec{x}_i),...,\bigcup_i (T_t^{\vec{a}_i^k}+\vec{x}_i)\}\ (=C_k)\Big),$$

with $T_i^{\vec{a}} = \eta_L^{\vec{a}}(T_i \cap (\vec{a}+L))$.
**Proof:**

Let for each $B_k$, $1 \le k \le r$, $\vec{x}_1$, $\vec{x}_2$ ,..., $\vec{x}_{|B_k|} \in \mathbb{Z}^d$ be arbitrary and $s_k$: $\mathbb{Z}^d \rightarrow \{0, 1, ..., M_k-1\}$ be a periodic skewing scheme which is valid for $C_k$. Construct the skewing scheme $s : \mathbb{Z}^d \rightarrow \{0, 1,..., \sum_j M_j-1\}$, by defining for each $k$, $1 \le k \le r$,

$s \restriction \{\vec{a}+L \mid \vec{a} \in B_k\}$ by $s(\vec{y}) = s(\vec{a}_i^k+\vec{x})$ $(\vec{x} \in L) = s_k(\eta_L^{\vec{a}_i^k}(\vec{y})-\vec{x}_i)+ \sum_{1 \le j<k} M_j$.

Then for each $k,l$ $(k \ne l)$: $\{\vec{a}+L \mid a \in B_k\} \cap \{\vec{a}+L \mid \vec{a} \in B_l\} = \emptyset$ and

Ran $(s \restriction \{\vec{a}+L \mid \vec{a} \in B_k\})$ ∩ Ran $(s \restriction \{\vec{a}+L \mid \vec{a} \in B_l\}) = \emptyset$.

Thus $s$ is defined sound and between the sets $\{\vec{a}+L \mid \vec{a} \in B_k\}$ and $\{\vec{a}+L \mid \vec{a} \in B_l\}$ "no conflicts" can occur. This means that there is no instance $T_j(\vec{x})$ of some template $T_j$ such that $\exists\, \vec{p} \in \{\vec{a}+L \mid \vec{a} \in B_k\}$, $\vec{q} \in \{\vec{a}+L \mid \vec{a} \in B_l\}$ with $\vec{p}, \vec{q} \in T_j(\vec{x})$ and $s(\vec{p}) = s(\vec{q})$. That "no conflicts" can occur on each set $\{\vec{a}+L \mid \vec{a} \in B_k\}$ itself, follows from the fact that $s_k$ is valid for $C_k$. □

**Theorem 3.8.** The number $M^*$ can be computed in time polynomial in $b$, $N$ and $t$, with $N = \max_{T_i \in C} |T_i|$, $b = \max_k \binom{N^2}{|B_k|} . |B_k|!$

**Proof:**

From fact 3.3 follows, that we do not have to consider for each $B_k$ all possible choices of the points $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_{|B_k|}$, but only those which belong to a fundamental domain of the underlying lattice of the periodic skewing scheme $s_k$ concerned. Together with fact 3.4 this gives the desired result. □

Thus when we take a partition $B = \{B_1, B_2, \ldots, B_r\}$ such that for all $1 \le i \le r$ $|B_i| \le c$, for some constant c, then the number M* can be computed in time polynomial in N and t. Because of the premisses, that are made in section 1, we may assume that for each template $T_i$ and $\vec{a} \in F$, F a fundamental domain of L, $|T_i \cap (\vec{a} + L)|$ is small. With respect to this the following theorem and corollary are interesting.

**Theorem 3.9.** Given a template $T \subseteq \mathbb{Z}_d$ and a lattice $L \subseteq \mathbb{Z}_d$, with basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d\}$ and a fundamental domain $F$. If for all $\vec{a} \in F$ there exists a $\vec{b} \in \mathbb{Z}^d$ such that $T \cap (\vec{a}+L) \subseteq \{\vec{b}, \vec{b} \pm \vec{x}_1, \vec{b} \pm \vec{x}_2, \ldots, \vec{b} \pm \vec{x}_d\}$, but $T \cap (\vec{a}+L) \neq \{ \vec{b}+\vec{x}_i, \vec{b}-\vec{x}_i \mid i \in I_{\vec{a}}\}$ for some $I_{\vec{a}} \subseteq \{1,2,\ldots,d\}$, then there exists a multi-periodic skewing scheme $s$ with underlying lattice $L$, such that $s$ is $L$-valid for $T$ and $s$ uses $|T|$ memory banks.

**Proof:**

Let $T$ satisfy the condition. Without loss of generality we assume that for all $1 \le i \le d$ $\vec{b}+\vec{x}_i \in T \cap (\vec{a}+L)$ or $\vec{b}-\vec{x}_i \in T \cap (\vec{a}+L)$. Then we have that for $\vec{a} \in F$, $\vec{a}$ arbitrary,

$$\eta_L^{\vec{a}}(T \cap (\vec{a}+L)) \subseteq \{(c_1, c_2, \ldots, c_d), (c_1 \pm 1, c_2, \ldots, c_d), (c_1, c_2 \pm 1, c_3, \ldots, c_d), \ldots, (c_1, c_2, \ldots, c_{d-1}, c_d \pm 1)\},$$

for some $(c_1, c_2, \ldots, c_d) \in \mathbb{Z}^d$, and

$$|\eta_L^{\vec{a}}(T \cap (\vec{a}+L))| \le \begin{cases} 2d+1 \text{ if } (c_1, c_2, \ldots, c_d) \in \eta_L^{\vec{a}}(T \cap (\vec{a}+L)) \\ 2d-1 \text{ if } (c_1, c_2, \ldots, c_d) \notin \eta_L^{\vec{a}}(T \cap (\vec{a}+L)). \end{cases}$$

From fact 3.3 and theorem 3.7 follows that we only need to prove that for an arbitrary template T′, with $T' \subseteq \{(0,\ldots,0), (\pm 1,0,\ldots,0), (0,\pm 1,0,\ldots,0), \ldots, (0,\ldots,0,\pm 1)\}$ $(=X^d)$,

$$|T'| \le \begin{cases} 2d+1 \text{ if } (0,\ldots,0) \in T' \\ 2d-1 \text{ if } (0,\ldots,0) \notin T', \end{cases}$$

and with T′ d-dimensional, wich means that for all $1 \le i \le d$ : $(\underbrace{0,\ldots,0}_{i-1}, 1, \underbrace{0,\ldots,0}_{d-i}) \in T'$ or $(\underbrace{0,\ldots,0}_{i-1}, -1, \underbrace{0,\ldots,0}_{d-i}) \in T'$ there exists a periodic skewing scheme $s' : \mathbb{Z}^d \to \{0,1,\ldots,|T'|-1\}$, which is valid for T′. We shall prove a slightly stronger version of this statement.

**Claim 3.9.1.** For all $T' \subseteq X^d$, T′ d-dimensional, is valid that

if $(0,...,0) \in T'$ and $|T'|=2d+1$ or if $(0,...,0) \notin T'$ and $|T'|=2d-1$
then for all $s,t \in \mathbb{Z}$ there exist points $\vec{x}_1,...,\vec{x}_t,\vec{y}_1,...,\vec{y}_t,\vec{z}_1,...,\vec{z}_s \in \mathbb{Z}^d$, such that for all $1 \le j \le t$ $\exists\, \vec{q}_j \in \mathbb{Z}^d$ with $\vec{x}_j - \vec{q}_j = \vec{q}_j - \vec{y}_j$ and such that there exists a periodic skewing scheme $s : \mathbb{Z}^d \rightarrow \{0,1,...,|T'|+2t+s-1\}$ which is valid for $T' \cup \{\vec{x}_i\}_{1 \le i \le t} \cup \{\vec{y}_i\}_{1 \le i \le t} \cup \{\vec{z}_i\}_{1 \le i \le s}$,
else for all $s \in \mathbb{Z}$ there exists points $\vec{z}_1,\vec{z}_2,...,\vec{z}_s \in \mathbb{Z}^d$, such that there exists a periodic skewing scheme $s: \mathbb{Z}^d \rightarrow \{0,1,...,|T'|+s-1\}$ which is valid for $T' \cup \{\vec{z}_i\}_{1 \le i \le s}$.

**Proof:**

The proof is done by induction.

For d=1 the templates to consider are:

,  and .

It can be verified that these templates meet the conditions. Let for d=k the claim be valid and let $T' \subseteq X^{k+1}$, $T'$ (k+1)-dimensional. Then we have two cases.

Case 1. $(0,...,0) \in T'$ and $|T'| = 2(k+1)+1$ or
$(0,...,0) \notin T'$ and $|T'| = 2(k+1)-1$.

Case 2. $(0,...,0) \in T'$ and $|T'| \le 2(k+1)$ or
$(0,...,0) \notin T'$ and $|T'| \le 2(k+1)-2$.

Consider case 1. Let $s,t \in \mathbb{Z}$ be arbitrary.

If $(0,...,0) \in T'$ and $|T'|=2(k+1)+1$, then there exists a j, $1 \le j \le k+1$, such that $|\{(\underbrace{0,...,0}_{i-1}, x, \underbrace{0,...,0}_{k-i+1}) \mid x \in \mathbb{Z}\} \cap T'| = 3$. Thus there exists a $T'' \subseteq X^k$, $T''$ k-dimensional, $(0,...,0) \in T''$, and $|T''|=2k+1$, such that

$$T' = \{(p_1,p_2,...,p_{i-1},0,p_{i+1},...,p_{k+1}) \mid (p_1,p_2,...,p_{i-1},p_{i+1},...,p_k) \in T''\} \cup \{(\underbrace{0,...,0}_{i},1,\underbrace{0,...,0}_{k-i+1}), (\underbrace{0,...,0}_{i}, -1, \underbrace{0,...,0}_{k-i+1})\}.$$

Then from the induction hypothesis follows that there exists points $\vec{x}_1,...,\vec{x}_{t+1}$, $\vec{y}_1,...,\vec{y}_{t+1}$, $\vec{z}_1,...,\vec{z}_s \in \mathbb{Z}^d$, such that for all $1 \le j \le t+1$ there exists a $\vec{q}_j$ with $\vec{x}_j - \vec{q}_j = \vec{q}_j - \vec{y}_j$ (**) and such that there exists a periodic skewing scheme $s : \mathbb{Z}^d \rightarrow \{0,1,...,|T''|+2(t+1)+s-1\}$ which is valid for $\bar{T}''=T'' \cup \{\vec{x}_i\}_{1 \le i \le t+1} \cup \{\vec{y}_i\}_{1 \le i \le t+1} \cup \{\vec{z}_i\}_{1 \le i \le s}$ (*).

Consider now

$\bar{T}'=\{(p_1,p_2,...,p_{i-1},0,p_{i+1},...,p_{k+1}) \mid (p_1,p_2,...,p_{i-1},p_{i+1},...,p_{k+1}) \in \bar{T}''-\{\vec{x}_{t+1},\vec{y}_{t+1}\}\}$.

An analogous argument as in the proof of theorem 3.7 together with (*) provides the existence of a multi-periodic skewing scheme $s' : \mathbb{Z}^{k+1} \rightarrow \{0, 1, ..., |T'|+2t+s-1\}$ which is valid for $\bar{T}'$. With the use of (**) it turns out that $s'$ is periodic as well. The case that $(0,...,0) \notin T'$ and $|T'|=2(k+1)-1$ can be handled analogously.

Consider case 2. Let $s \in \mathbb{Z}$ be arbitrary.

If $(0,...,0) \in T'$ and $|T'| \le 2(k+1)$, then there exists a $1 \le i \le k+1$ such that $|\{(\underbrace{0,...,0}_{i-1}, x, \underbrace{0,...,0}_{k-i+1}) \mid x \in \mathbb{Z}\} \subseteq T'| = 2$. Thus there exists a $T'' \subseteq X^k$, $T''$ k-dimensional, such that

$$T' = \{(p_1,p_2,\ldots,p_{i-1}, 0, p_{i+1},\ldots,p_{k+1}) \mid (p_1,\ldots, p_{i-1},p_{i+1},\ldots,p_{k+1}) \in T''\} \cup \begin{cases} \{(\underbrace{0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{k-i+1})\}, \text{ or} \\ \{(\underbrace{0,\ldots,0}_{i-1},-1,\underbrace{0,\ldots,0}_{k-i+1})\}. \end{cases}$$

Now a similar argument as in the above can be used to show the existence of points $\vec{z}_1,\ldots,\vec{z}_s \in \mathbb{Z}_d$ and a periodic skewing scheme s : $\mathbb{Z}_d \rightarrow \{0, 1, \ldots, |T'|+s-1\}$ which is valid for $T' \cup \{\vec{z}_i\}_{1\le i\le s}$.

The case that $(0,\ldots,0) \notin T'$ and $|T'| \le 2(k+1)-2$ can be handled analogously. □

As a direct consequence we have:

**Corollary** 3.10. Given a template $T \subseteq \mathbb{Z}^d$ and a lattice $L \subseteq \mathbb{Z}^d$, with basis $\{\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_d\}$ and fundamental domain F. Let for all $\vec{a} \in F$ there exists a $\vec{b} \in \mathbb{Z}^d$, such that $T \cap (\vec{a}+L) \subseteq \{\vec{b}, \vec{b}\pm\vec{x}_1, \vec{b}\pm\vec{x}_2,\ldots,\vec{b}\pm\vec{x}_d\}$. If $\vec{a}_1,\vec{a}_2,\ldots,\vec{a}_d$ are the points of F such that for all i, $1\le i\le s$ $T \cap (\vec{a}_i+L) = \{\vec{b}+\vec{x}_i,\vec{b}-\vec{x}_i \mid i\in I_{\vec{a}_i}\}$, for some $I_{\vec{a}_i} \subseteq \{1,2, \ldots,d\}$, and if for $i_1,i_2,\ldots,i_t \in \{1,2, \ldots,d\}$ we have for all i, $1\le i\le s$, there exists a $i_j$ such that $i_j \in I_{\vec{a}_i}$, then there exists a multi-periodic skewing scheme s: $\mathbb{Z}^d \rightarrow \{0,1,\ldots,|T|-1\}$, which is L-valid for T and has an underlying lattice $L^1$ with basis

$$\{\vec{x}_1,\ldots,\vec{x}_{i_1-1},2\vec{x}_{i_1},\vec{x}_{i_1+1},\ldots,\vec{x}_{i_2-1},2\vec{x}_{i_2},\vec{x}_{i_2+1},\ldots\ldots,\vec{x}_{i_t-1},2\vec{x}_{i_t},\vec{x}_{i_t+1},\ldots,\vec{x}_d\}.$$

Jalby et. al. [2] have given a weaker and slightly different version of theorem 3.9 and corollary 3.10. They have proven that for templates $T \subseteq \mathbb{Z}^2$, with the property that for all $\vec{a} \in F$ there exists a $\vec{b} \in \mathbb{Z}^2$, such that $T \cap (\vec{a}+L) \subseteq \{\vec{b}, \vec{b}+\vec{x}_1, \vec{b}+\vec{x}_2, \vec{b}+\vec{x}_1+\vec{x}_2\}$, there exists a diamond scheme which uses $|T|$ memory banks.

Theorem 3.7 does not always yield a multi-periodic skewing scheme which uses the minimum number of memory banks, $\mu_L(C)$, as is shown in the next lemma.

**Lemma** 3.11. There exists a collection of templates C, and a lattice L, such that $M^* > \mu_L(C)$.
**Proof:**

Take the collection C, consisting of only one template $T=\{(0,0),(1,0),(3,0),(5,0),(0,1),(6,1)\} \subseteq \mathbb{Z}^2$ and let L be the lattice with basis $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\}$. Then $M^*=8$ as can be verified, while $\mu_L(C)=6$ (see figure 4). □

... 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 ...

... 3 4 5 6 1 2 5 6 1 2 3 4 3 4 5 6 1 2 5 6 1 2 3 4 ...

1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6

3 4 5 6 1 2 5 6 1 2 3 4 3 4 5 6 1 2 5 6 1 2 3 4

1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6

3 4 5 6 1 2 5 6 1 2 3 4 3 4 5 6 1 2 5 6 1 2 3 4

1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6

...③4 5 6 1 2⑤6 1 2 3 4 3 4 5 6 1 2 5 6 1 2 3 4 ...

...①②3④5⑥1 2 3 4 5 6 1 2 3 4 5 6 1 2 3 4 5 6 ...

Figure 4

Note that the multi-periodic skewing schemes as constructed in theorem 3.7 and theorem 3.9 are all diamond schemes by theorem 2.8. This is not a coincidence, because this subset of the multi-periodic skewing schemes has more structure than the general multi-periodic schemes, and for this reason the diamond schemes can be handled more easily in these cases. However, in general the diamond schemes are not as strong as the multi-periodic skewing schemes, as shown by the following lemma.

**Lemma 3.12.** There exists a collection of templates C and a lattice L such that: if s is a diamond scheme that is L-valid for C and uses $\mu_L^{mp}(C)$ memory banks, then the determinant of the underlying lattice of s $\geq 2.\lambda_L^{mp}(C)$.

**Proof:**

Take the same C as in lemma 3.11. Then the skewing scheme as denoted in figure 4 is a multi-periodic skewing scheme, which uses 6 memory banks and has an underlying lattice with basis $\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \}$. The best possible diamond scheme which uses 6 memory banks has an underlying lattice with basis $\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 12 \end{pmatrix} \}$. □

## 4. A simple representation of multi-periodic skewing schemes.

From classical lattice theory we know that every lattice $L \subseteq \mathbb{Z}^d$ has a fundamental domain, which is "box-like" in the following sense.

**Lemma** 4.1. [6]. Given a d-dimensional lattice $L \subseteq \mathbb{Z}^d$, there exists a basis $U = \{\vec{u}_1,...,\vec{u}_d\}$ of $\mathbb{Z}^d$ with $|det(\vec{u}_1,...,\vec{u}_d)| = 1$ and $s_1, s_2, ..., s_d \in \mathbb{Z}$ such that

$F^* = \{(x_1,x_2,...,x_d)_U \dagger \mid x_i \in \{0,1,...,s_i\}$ for all $1 \le i \le d\ \}$

is a fundamental domain of L.

Using this lemma the following theorem can be shown.

**Theorem** 4.2. [6]. Let s: $\mathbb{Z}^d \to \{0,1,...,M-1\}$ be a periodic skewing scheme and let L be its underlying lattice. Then there exists a map

$\alpha : \mathbb{Z}^d \to \{(x_1,...,x_d) \mid x_i \in \{0,1,...,s_i\}$ for all $1 \le i \le d\ \}$ (=B)

and a map

$t : B \to \{0,1,...,M-1\}$,

such that $s = t \circ \alpha$ and $\alpha$ is given by an expression of the type

$\alpha(\vec{i}) = \alpha(i_1,i_2,...,i_d) = (L_1(\vec{i})\ mod\ s_1,\ L_2(\vec{i})\ mod\ s_2, ...,\ L_d(\vec{i})\ mod\ s_d)$,

with $L_k(\vec{i}) = \lambda_{k,1}.i_1+...+\lambda_{k,d}.i_d$, $\lambda_{k,j} \in \mathbb{Z}$, for all $1 \le j \le d$.

Consider an arbitrary multi-periodic skewing scheme s, and let L be its underlying lattice. Because for each $\vec{a}_U \in F^*$ $S_L^{\vec{a}_U}$ is periodic there exists a map $\alpha^{\vec{a}_U}$ and $t^{\vec{a}_U}$ such that $t^{\vec{a}_U} \circ \alpha^{\vec{a}_U} = s_L^{\vec{a}_U}$. Thus we can state the following theorem.

**Theorem** 4.3. Let s be a multi-periodic skewing scheme and L be its underlying lattice. Let $\{\vec{u}_1,\vec{u}_2,...,\vec{u}_d\}$ and $s_1,s_2,...,s_d$ be as in lemma 4.1. Then for all $\vec{x} \in \mathbb{Z}^d$

$s(\vec{x}) = t^{(i_1 mod\ s_1,...,i_d mod\ s_d)_U} \circ s^{(i_1 mod\ s_1,...,i_d mod\ s_d)_U}(i_1\ over\ s_1, \cdots i_d\ over\ s_d)$,

with $(i_1,i_2,...,i_d) = (\vec{u}_1\ \vec{u}_2 \cdots \vec{u}_d)^{-1}(\vec{x})$.

**Proof:**

Immediate. □

Concluding we can say that a multi-periodic skewing scheme s: $\mathbb{Z}^d \to \{0,1,...,M-1\}$, with underlying lattice L can be represented in the amount of space of approximatly $|F^*| \cdot (d^2+M) = \Delta(L) \cdot (d^2+M)$ memory locations.

(†) $(.., \cdots ,..)_U$ denotes a point with respect to U.

## 5. References.


[1] Budnik, P. and D.J. Kuck, "The organization and use of parallel memories", IEEE Trans. Comput., vol. C-20, pp. 1566-1569, 1971.

[2] Jalby, W., J.-M. Frailong and J. Lenfant, "Diamond schemes: an organization of parallel memories for efficient array processing", Rapports de Recherche, N°342, INRIA, Centre de Rocquencourt, 1984.

[3] Lawrie, D.H., "Access and alignment in an array processor", IEEE Trans. Comput., vol. C-24, pp. 1145-1155, 1975.

[4] Shapiro, H.D., "Theoretical limitations on the use of parallel memories", IEEE Trans. Comput., vol C-27, pp. 241-248, 1978.

[5] Wijshoff, H.A.G. and J. van Leeuwen, "Periodic storage schemes with a minimum number of memory banks", Techn. Report RUU-CS-83-4, Dept. of Computer Science, University of Utrecht, Utrecht, 1983.

[6] Wijshoff, H.A.G. and J. van Leeuwen, "The structure of periodic storage schemes for parallel memories", IEEE Trans. Comput., vol. C-34, pp. 501-505, 1985.