Induced representations and the Langlands classification

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Abstract. In these lecture notes[1] we discuss the concept of induction and some of its applications to the representation theory of a real semisimple Lie group. In particular, we give an introduction to parabolic induction, Bruhat theory, the asymptotic behavior of matrix coefficients, the subrepresentation theorem, characterization of discrete series and tempered representations, and, finally, the Langlands classification of irreducible admissible representations.

1 Induced representations

1.1 Homogeneous vector bundles

The process of induction allows us to create representations of a Lie group, starting from representations of a subgroup.

We start by recalling the notion of an associated vector bundle. Let G be a Lie group, H a closed subgroup, and (ξ, V) a finite dimensional continuous representation of H (here and in the following, representation spaces are always assumed to be complex linear). The group H acts on the product $G \times V$ by

$$h \cdot (g, v) = (gh^{-1}, \xi(h)v),$$

turning $G \times V$ into a principal fiber bundle with structure group H. The associated quotient space $G \times_H V := G \times V/H$ is a smooth manifold. Projection onto the first coordinate induces a smooth map:

$$p: G \times_H V \to G/H. \tag{1}$$

Now p is a fiber bundle; for each $g \in G$ the map the map $v \mapsto (g, v)$ induces a bijection φ_g from V onto the fiber $p^{-1}(gH)$ over gH. The requirement that every φ_g be linear determines a unique structure of (complex) vector bundle on the fiber bundle (1). This vector bundle is said to be associated with the representation ξ; we shall also denote it by $\mathcal{V} := G \times_H V$.

The natural action of G on $G \times V$ by left multiplication on the first coordinate induces a smooth action of G on $\mathcal{V}$. In this way $p: \mathcal{V} \to G/H$ becomes a homogeneous vector bundle. Here we recall that a homogeneous vector bundle over G/H is a vector bundle $q: \mathcal{W} \to G/H$ together with a smooth action of G on $\mathcal{W}$ such that for each $g \in G$ the following two conditions are fulfilled:

[1] This material was presented in a series of 5 lectures in the 'Instructional Conference on Representation Theory and Automorphic Forms', Edinburgh, March 1996.

(a) the following diagram commutes:

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{g\cdot} & \mathcal{W} \\ q\downarrow & & \downarrow q \\ G/H & \xrightarrow{l_g} & G/H \end{array}$$

(in particular $g\cdot$ maps each fiber $\mathcal{W}_x := q^{-1}(x)$ onto the fiber $\mathcal{W}_{gx}$);

(b) for every $x \in G/H$, the map $g\cdot : \mathcal{W}_x \to \mathcal{W}_{gx}$ is linear.

Any homogeneous vector bundle $q: \mathcal{W} \to G/H$ is associated with a continuous representation of H. Indeed the fiber $V := q^{-1}(eH)$ is invariant under the action of H; we thus obtain a continuous representation ξ of H in V. The smooth G-map $G \times V \to \mathcal{W}$, $(g, v) \mapsto g \cdot v$ factorizes to an isomorphism of the associated homogeneous vector bundle $\mathcal{V} = G \times_H V$ onto $\mathcal{W}$.

It follows from the above that the category of continuous finite dimensional representations of H is equivalent to the category of G-homogeneous vector bundles on G/H. The equivalence is established by the above construction of the associated vector bundle, its inverse by restriction to the fiber above the origin eH of G/H.

Let $\xi, \mathcal{V}$ be as above, then by $C(\mathcal{V})$, respectively $C^\infty(\mathcal{V})$, we denote the space of continuous, respectively smooth, sections of $\mathcal{V}$. The group G has a natural representation π in $C(\mathcal{V})$, given by the rule:

$$[\pi(g)s](x) = g \cdot [s(g^{-1}x)],$$

for $s \in C(\mathcal{V})$, $x \in G/H$, $g \in G$. The representation π is called the representation of G induced from the representation ξ of H; it is denoted by

$$\pi = \mathrm{ind}_H^G(\xi).$$

Note that the space $C(\mathcal{V})$, equipped with the topology of uniform convergence on compact sets, is a Fréchet space. The induced representation is continuous for this topology. Depending on the context it is sometimes convenient to work with a different representation space. For instance the action of G on the space of smooth sections $C^\infty(\mathcal{V})$ is a continuous representation $\pi_\circ$ of G in a Fréchet space as well. Moreover, $\pi_\circ$ is the restriction of π to the G-invariant subspace $C^\infty(\mathcal{V})$ of $C(\mathcal{V})$. By density of this subspace, the representation π is completely determined by $\pi_\circ$. In this sense we are justified to also call $\pi_\circ$ the induced representation.

1.2 The induced picture

For the purpose of representation theory it is often convenient to realize the induced representation $\mathrm{ind}_H^G(\xi)$ on a space of vector-valued functions rather than sections of a bundle.

We identify V with the fiber of $\mathcal{V}$ above eH via the linear isomorphism induced by the map $v \mapsto (e, v)$. By $C(G, V)$ we denote the space of continuous functions $G \to V$. Given a section $s \in C(\mathcal{V})$ we define the function $\varphi = \varphi_s \in C(G, V)$ by $\varphi(g) = g^{-1} \cdot s(gH)$. Then φ transforms according to the rule

$$\varphi(gh) = \xi(h)^{-1}\varphi(g) \qquad (g \in G,\ h \in H).$$

The space of functions $\varphi \in C(G, V)$ transforming according to the above rule is denoted by $C(G, \xi) = C(G, H, \xi)$. Let R denote the representation of H on $C(G)$ by right translation. Then via the natural identification $C(G, V) \simeq C(G) \otimes V$ we have an isomorphism

$$C(G, \xi) \simeq [C(G) \otimes V_\xi]^H,$$

where the superscript H indicates that the subspace of invariants for the representation $R \otimes \xi$ of H has been taken.

It is easily seen that the map $s \mapsto \varphi_s$ is a topological linear isomorphism from $C(\mathcal{V})$ onto $C(G, \xi)$ (equipped with the natural structure of a Fréchet space). Indeed, if $\varphi \in C(G, \xi)$, then the associated section $s = s_\varphi$ is given by

$$s_\varphi(gH) = [(g, \varphi(g))]$$

(note that this definition is unambiguous by the transformation property of φ).

By transference under the isomorphism $C(\mathcal{V}) \simeq C(V, \xi)$ we may realize the representation $\pi = \mathrm{ind}_H^G(\xi)$ of G in $C(G, \xi)$. It is then given by:

$$[\pi(g)\varphi](x) = \varphi(g^{-1}x) \qquad (g, x \in G).$$

In future references we shall call this realization of $\mathrm{ind}_H^G(\pi)$ the 'induced picture' (to distinguish it from the 'geometric picture'). One of the advantages of the induced picture is that it allows a straightforward generalization to infinite dimensional representations ξ.

Remark 1.1. In the above induced picture the representation space is characterized by means of transformation properties from the right. Equivalently one may of course use transformation properties from the left. Indeed, let ${}'C(G, \xi)$ denote the space of continuous functions $\varphi: G \to \mathcal{H}_\xi$ transforming according to the rule

$$\varphi(hx) = \xi(h)\varphi(x) \qquad (x \in G,\ h \in H).$$

Moreover, let ${}'\pi$ be the representation of G in ${}'C(G, \xi)$ coming from the right regular action. If $\varphi \in C(G, \xi)$, then the function ${}'\varphi: x \mapsto \varphi(x^{-1})$ belongs to ${}'C(G, \xi)$, and the map $\varphi \mapsto {}'\varphi$ defines an equivalence of the reprentations π and ${}'\pi$.

1.3 Frobenius reciprocity

Let H be a closed subgroup of G, and let ξ be a continuous (not necessarily unitary) representation of H in a Hilbert space $V = V_\xi$. We drop the assumption that V is finite dimensional and define $C(G, \xi)$ and $\mathrm{ind}_H^G(\xi)$ by the formulas of the induced picture. The following result is known as Frobenius reciprocity.

Lemma 1.2. (Frobenius reciprocity). *Let (δ, V_δ) be a finite dimensional continuous representation of G. Then the map $\varphi: T \mapsto \mathrm{ev}_e \circ T$ defines a natural isomorphism:*

$$\mathrm{Hom}_G(V_\delta, \mathrm{ind}_H^G(\xi)) \simeq \mathrm{Hom}_H(V_\delta, V_\xi). \tag{2}$$

Proof. If T belongs to the space on the left, put $\varphi(T) = \mathrm{ev}_e \circ T$. Then $\varphi(T): V_\delta \to V$ is a linear map, which is readily seen to be H-equivariant. By G-equivariance, if $v \in V_\delta$, $g \in G$ then $T(v)(g) = T(\delta(g)^{-1}v)(e) = \varphi(T)(\delta(g)^{-1})$, from which the injectivity of φ follows. If S belongs to the space on the right then the map $T: V_\delta \to C(G, V)$, defined by $T(v)(g) = S(\delta(g^{-1})v)$ is readily checked to belong to the space on the left-hand side of (2), and $S = \varphi(T)$. Hence φ is surjective as well. $\square$

1.4 The bundle of densities

If V is an n-dimensional real linear space, then a density on V is a map $\omega: V^n \to \mathbb{C}$ transforming according to the rule:

$$T^*\omega := \omega \circ T^n = |\det T|\,\omega \qquad (T \in \mathrm{End}(V)).$$

In these notes the (complex linear) space of densities on V is denoted by $\mathcal{D}V$.

If φ is a linear isomorphism from V onto a real linear space W, then the map $\varphi^*: \omega \mapsto \omega \circ \varphi^n$ is a linear isomorphism $\mathcal{D}W \to \mathcal{D}V$ of the associated spaces of densities. The space $\mathcal{D}V$ is one dimensional; in fact, if $v_1, \ldots, v_n$ is a basis of V then the map $T \mapsto T(v_1, \ldots, v_n)$ is a linear isomorphism from $\mathcal{D}V$ onto $\mathbb{C}$.

If X is a smooth manifold, then by T_xX we denote the tangent space of X at a point x. By a well known procedure we may define the bundle $\mathcal{D}TX$ of densities on X; it is a complex line bundle with fiber $(\mathcal{D}TX)_x \simeq \mathcal{D}(T_xX)$. If φ is a diffeomorphism of X onto a manifold Y, then we define the map $\varphi^*: C(\mathcal{D}TY) \to C(\mathcal{D}TX)$ by $(\varphi^*\omega)(x) = D\varphi(x)^*\omega(\varphi(x))$.

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. The density $\lambda \in \mathcal{D}\mathbb{R}^n$ given by $\lambda(e_1, \ldots, e_n) = 1$ is called the standard density on $\mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be an open subset. Then by triviality of the tangent bundle $TU \simeq U \times \mathbb{R}^n$, the map $f \mapsto f\lambda$ defines a linear isomorphism from $C^\infty(U)$ onto $C^\infty(\mathcal{D}TU)$. If $f \in C_c(U)$ we define the integral

$$\int_U f\lambda := \int_{\mathbb{R}^n} f(x)\, dx,$$

where dx denotes normalized Lebesgue measure. If φ is a diffeomorphism from U onto a second open subset $V \subset \mathbb{R}^n$, then, for $g \in C_c(V)$, we have $\varphi^*(g\lambda)(x) = g(\varphi(x))|\det D\varphi(x)|\lambda(\varphi(x))$. Thus, by the substitution of variables theorem:

$$\int_U \varphi^*\omega = \int_V \omega \qquad (\omega \in C_c\mathcal{D}TV). \tag{3}$$

This observation allows us to extend the notion of integral to any compactly supported continuous density on any smooth manifold. The extension involves reduction to charts by using partitions of unity, exactly as in the definition of integration of differential forms of top dimension. Note that integration of forms is oriented, whereas the present integration of densities is non-oriented.

The following result is a consequence of these definitions.

Proposition 1.3. *Let $\varphi: U \to V$ be a diffeomorphism of C^∞-manifolds. Then* (3).

Half densities. If V is an n-dimensional real linear space and $\alpha \in \mathbb{C}$ a complex number, then an α-density on V is a function $\nu: V^n \to \mathbb{C}$ transforming according to the rule $\nu \circ T^n = |\det T|^\alpha \nu$, for every $T \in \mathrm{End}(V)$. The space of α-densities on V is denoted by $\mathcal{D}^\alpha V$. Thus $\mathcal{D}V = \mathcal{D}^1 V$. The elements of $\mathcal{D}^{1/2}V$ are called half densities on V.

The product of two densities is a density; multiplication induces a linear isomorphism from $\mathcal{D}^\alpha V \otimes \mathcal{D}^\beta V$ onto $\mathcal{D}^{\alpha+\beta}V$ $(\alpha, \beta \in \mathbb{C})$. Note that 0-densities are constant functions, hence $\mathcal{D}^0 V \simeq \mathbb{C}$. The natural isomorphism $\mathcal{D}^{-\alpha}V \otimes \mathcal{D}^\alpha V \simeq \mathbb{C}$ induces a natural identification $(\mathcal{D}^\alpha V)^* \simeq \mathcal{D}^{-\alpha}V$.

If X is a manifold, then the bundle of α-densities on X is denoted by $\mathcal{D}^\alpha TX$. Fiberwise multiplication induces, for $\alpha, \beta \in \mathbb{C}$, an isomorphism $\mathcal{D}^\alpha TX \otimes \mathcal{D}^\beta TX \simeq \mathcal{D}^{\alpha+\beta}TX$ of vector bundles. The dual vector bundle $(\mathcal{D}^\alpha TX)^*$ is naturally isomorphic to $\mathcal{D}^{-\alpha}TX$.

Densities and generalized sections. If X is a smooth manifold, and $p: \mathcal{V} \to X$ a vector bundle on X, then by $C^\infty(\mathcal{V})$ we denote the space of smooth sections of $\mathcal{V}$. This space is equipped with a Fréchet topology in the usual way. The space of compactly supported smooth sections of $\mathcal{V}$ is denoted by $C_c^\infty(\mathcal{V})$. It is equipped with the structure of a complete locally convex Hausdorff space in the usual way.

The space $C^{-\infty}(\mathcal{V})$ of generalized sections of $\mathcal{V}$ is defined by

$$C^{-\infty}(\mathcal{V}) := C_c^\infty(\mathcal{V}^* \otimes \mathcal{D}TX)'.$$

This definition has the effect that it allows a natural embedding $\iota: C^\infty(\mathcal{V}) \hookrightarrow C^{-\infty}(\mathcal{V})$. Indeed, let $\langle \cdot\,, \cdot \rangle$ denote the (pointwise defined) natural bilinear map $C^\infty(\mathcal{V}) \times C_c^\infty(\mathcal{V}^* \otimes \mathcal{D}TX) \to C_c^\infty \mathcal{D}TX$, then the embedding ι is given by:

$$\iota(s): \sigma \mapsto \int_X \langle s\,, \sigma \rangle.$$

Here we adopt the convention to denote the linear dual of a linear space and the dual of a vector bundle by a star. The topological linear dual of a topological linear space is denoted by a prime.

A special case of the above is the definition of the space $C^{-\infty}(X)$ of $\mathbb{C}$-valued generalized functions on X by

$$C^{-\infty}(X) = C_c^\infty(\mathcal{D}TX)'.$$

Since $(\mathcal{D}^{1/2}TX)^* \otimes \mathcal{D}TX \simeq \mathcal{D}^{1/2}TX$ naturally, it follows from the above that:

$$C^{-\infty}(\mathcal{V} \otimes \mathcal{D}^{1/2}TX) \simeq C_c^\infty(\mathcal{V}^* \otimes \mathcal{D}^{1/2}TX)'. \tag{4}$$

1.5 Densities on G/H

Let H be a closed subgroup of a Lie group G. The tangent bundle $T(G/H)$ is a G-homogeneous vector bundle on G/H; the action of G on the space $CT(G/H)$ of continuous vector fields on G/H is given by:

$$g \cdot v(x) = Dl_g(x) v(g^{-1}x).$$

Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H respectively. Here we adopt the convention that Lie groups are denoted by Roman capitals, their Lie algebras by the corresponding Gothic lower case letters. The projection $G \to G/H$ induces a natural isomorphism $\mathfrak{g}/\mathfrak{h} \simeq T_{eH}(G/H)$. Accordingly, $T(G/H)$ is the homogeneous bundle associated with the representation $\xi = \mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}$ of H on $\mathfrak{g}/\mathfrak{h}$ defined by $\xi(h)(X + \mathfrak{h}) = \mathrm{Ad}(h)X + \mathfrak{h}$.

In a similar way we see that the bundle of densities $\mathcal{D}T(G/H)$ is G-homogeneous; the action of G on the associated space of continuous densities is given by $g \cdot \omega = l_g^{-1*}\omega$, for $\omega \in C\mathcal{D}T(G/H)$. This is the homogeneous bundle associated with the character δ of H given by $\delta(h) = |\mathrm{det}\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)^{*-1}|$, hence

$$\delta(h) = |\mathrm{det}\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)|^{-1} \qquad (h \in H). \tag{5}$$

For $\alpha \in \mathbb{C}$, the bundle of α-densities on G/H is homogeneous as well; it is associated with the character δ^α.

Generalized sections. If (σ, V) is a finite dimensional continuous representation of H, let $\mathcal{V}$ denote the associated vector bundle. Then $\mathcal{V} \otimes \mathcal{D}^{1/2}T(G/H)$ is naturally isomorphic to the homogeneous bundle associated with the tensor product representation of H in $V \otimes \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h})$. The isomorphism (4) then naturally corresponds to an isomorphism:

$$C^{-\infty}(G, \sigma \otimes \delta^{1/2}) \simeq C_c^{\infty}(G, \sigma^{\vee} \otimes \delta^{1/2})', \tag{6}$$

where $\sigma^{\vee}$ denotes the representation contragredient to σ.

Densities and measures. Let H be a closed subgroup of the Lie group G. If Ω is a continuous density on G/H then the map $\mu_\Omega \colon C_c(G/H) \to \mathbb{C}$ defined by

$$\mu_\Omega(f) = \int_{G/H} f\,\Omega \qquad (f \in C_c(G/H))$$

is continuous linear, hence defines a Radon measure on G/H. It follows from Proposition 1.3 that

$$\mu_{l_g^{*-1}\Omega} = \mu_\Omega \circ l_g^*.$$

One now readily sees that $\Omega \mapsto \mu_\Omega$ defines a linear isomorphism from the space $C(\mathcal{D}T(G/H))^G$ of G-invariant densities on G/H onto the space of G-invariant Radon measures on G/H.

On the other hand, the map $\Omega \mapsto \Omega(eH)$ is an isomorphism from $C(\mathcal{D}T(G/H))^G$ onto $[\mathcal{D}(\mathfrak{g}/\mathfrak{h})]^H$. We thus see that there exists a natural isomorphism from $[\mathcal{D}(\mathfrak{g}/\mathfrak{h})]^H$ onto the space of G-invariant Radon measures on G/H. In particular, the latter space is non-trivial if and only if the character (5) is identically 1.

1.6 Normalized induction

Let H be a closed subgroup of the Lie group G, and assume that the quotient space G/H is compact. Let ξ be a (not necessarily unitary) representation of H in a (possibly infinite dimensional) Hilbert space V (we denote the inner product by $\langle \cdot\,, \cdot \rangle_\xi$; inner products on complex Hilbert spaces are always assumed to be skew-linear in the second variable).

The induced representation $\pi = \mathrm{ind}_H^G(\xi)$ is said to be unitarizable if the representation space $C(G, \xi)$ allows a pre-Hilbert structure such that π extends to a unitary representation in the associated Hilbert completion.

Unitarity of the representation ξ does not necessarily imply unitarizability of $\mathrm{ind}_H^G(\xi)$. However, as we will see, by twisting with half densities we may normalize the induction so that unitarity is preserved.

Let $\xi \otimes \delta^{1/2}$ denote the tensor product representation of H in $V \otimes \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h})$. We now observe that $(\lambda_1, \lambda_2) \mapsto \lambda_1 \bar{\lambda}_2$ defines a sesquilinear map from $\mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h}) \times \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h})$ onto $\mathcal{D}(\mathfrak{g}/\mathfrak{h})$. Given $v_1, v_2 \in V$, $\lambda_1, \lambda_2 \in \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h})$ we define $(v_1 \otimes \lambda_1, v_2 \otimes \lambda_2) = \langle v_1\,, v_2 \rangle_\xi \lambda_1 \bar{\lambda}_2$ and extend this to a sesquilinear pairing $V \otimes \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h}) \times V \otimes \mathcal{D}^{1/2}(\mathfrak{g}/\mathfrak{h}) \to \mathcal{D}(\mathfrak{g}/\mathfrak{h})$.

Now assume that ξ is unitary. If $\varphi, \psi \in C(G, \xi \otimes \delta^{1/2})$, then the function $(\varphi, \psi) \colon g \mapsto (\varphi(g), \psi(g))$ belongs to $C(G, \delta)$. It may therefore be canonically identified with a density on G/H, which in turn may be integrated. We put:

$$\langle \varphi\,, \psi \rangle \colon = \int_{G/H} (\varphi, \psi) \qquad (\varphi, \psi \in C(G, \xi \otimes \delta^{1/2})). \tag{7}$$

Lemma 1.4. *Let ξ be a unitary representation of H. Then the sesqui-linear pairing (7) defines a pre-Hilbert structure on $C(G, \xi \otimes \delta^{1/2})$, which is G-equivariant. The induced representation $\mathrm{ind}_H^G(\xi \otimes \delta^{1/2})$ extends to a unitary representation in the associated Holbert completion. In particular it is unitarizable.*

Proof. We denote the induced representation by π. One readily verifies that the given pairing defines a pre-Hilbert structure. If $\varphi, \psi \in C(G, \xi \otimes \delta^{1/2})$, then $(\pi(g)\varphi, \pi(g)\psi) = (l_g^{-1*}\varphi)(l_g^{-1*}\psi) = l_g^{-1*}(\varphi, \psi)$. The integral of the latter density over G/H equals the integral of (φ, ψ) over G/H, by Proposition 1.3, whence the equivariance of the pre-Hilbert structure.

It follows that for every $g \in G$ the map $\pi(g)$ extends uniquely to a unitary endomorphism of the Hilbert completion $\mathcal{H}$ of $C(G, \xi \otimes \delta^{1/2})$. One readily sees that the pre-Hilbert structure is a continuous form on $C(G, \xi \otimes \delta^{1/2})$; hence the latter space embeds continuously into $\mathcal{H}$. From this and the unitarity of each $\pi(g)$ $(g \in G)$ it easily follows that the extension of π is a continuous representation of G in $\mathcal{H}$. $\square$

In view of the above result the representation $\mathrm{ind}_H^G(\xi \otimes \delta^{1/2})$ is said to be obtained from ξ by *normalized* induction. In these notes this induced representation will also be denoted by $\mathrm{Ind}_H^G(\xi)$ (in fact, the same notation for $\mathrm{ind}_H^G(\xi \otimes \delta^{1/2})$ will be used if ξ is not unitary).

2 Parabolically induced representations

2.1 Basic notions

From now on we assume that G is a real reductive group of Harish-Chandra's class. For purposes of induction this class is more convenient than the slightly smaller class of connected semisimple groups with finite center.

Let K be a maximal compact subgroup of G, and θ the associated Cartan involution. The infinitesimal involution of the Lie algebra $\mathfrak{g}$ of G associated with θ is denoted by the same symbol; the associated Cartan decomposition is denoted by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Thus $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces for θ for the eigenvalues 1 and -1 respectively. We fix a non-degenerate bilinear form B on $\mathfrak{g}$, which is $\mathrm{Ad}(G)$- and θ-invariant. Then $\mathfrak{k} \perp \mathfrak{p}$ relative to B.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\Sigma \subset \mathfrak{a}^*$ be the (possibly non-reduced) root system of $\mathfrak{a}$ in $\mathfrak{g}$. Its Weyl group W is naturally isomorphic with $N_K(\mathfrak{a})/M$, where $N_K(\mathfrak{a})$, M denote the normalizer respectively the centralizer of $\mathfrak{a}$ in K.

We fix a positive system Σ^+ for Σ and denote the associated system of simple roots by Δ. Let $\mathfrak{n}$ be the sum of the root spaces $\mathfrak{g}_\alpha$ $(\alpha \in \Sigma^+)$, and let $N = \exp \mathfrak{n}$, $A = \exp \mathfrak{a}$. We recall that G decomposes according to the Iwasawa decomposition

$$G = KAN;$$

here the product map $K \times A \times N \to G$ is a diffeomorphism.

Every parabolic subgroup of G is K-conjugate to a standard parabolic subgroup (relative to Σ^+). We recall that the standard parabolic subgroups are in one-to-one correspondence with the collection of subsets of Δ. For $F \subset \Delta$, the associated standard parabolic subgroup Q_F can be described as follows. Let $\mathfrak{a}_F$ be the intersection of the root spaces $\ker \alpha$ $(\alpha \in F)$, and let M_{1F} be the centralizer of $\mathfrak{a}_F$ in G. Furthermore, let $\mathfrak{n}_F$ be the sum of the root spaces $\mathfrak{g}_\alpha$ $(\alpha \in \Sigma^+ \setminus \mathrm{span}(F))$, and let $N_F = \exp \mathfrak{n}_F$. Then $Q_F = M_{1F} N_F$.

The group M_{1F} is stable under θ, hence decomposes as $M_{1F} = K_F \exp(\mathfrak{m}_{1F} \cap \mathfrak{p})$, where $K_F{:}= K \cap M_{1F}$. Let $\mathfrak{m}_F$ denote the B-orthocomplement of $\mathfrak{a}_F$ in $\mathfrak{m}_{1F}$; it is the Lie algebra of the group $M_F = K_F \exp(\mathfrak{m}_F \cap \mathfrak{p})$. The latter group is again of Harish-Chandra's class and has compact center: this allows induction on the dimension of G as a method of proof. Note that $M_{1F} = M_F A_F$. Hence

$$Q_F = M_F A_F N_F;$$

this is the Langlands decomposition of Q_F. Note that $A_\emptyset = A$, $N_\emptyset = N$; moreover, $M_\emptyset$ equals the centralizer M of $\mathfrak{a}$ in K. Hence the minimal standard parabolic subgroup of G is given by $Q_\emptyset = Q = MAN$.

We recall that exp maps $\mathfrak{a}$, the Lie algebra of A, diffeomorphically onto A; the inverse of $\exp{:}\,\mathfrak{a} \to A$ is denoted by $\log$. Moreover, given $\lambda \in \mathfrak{a}^*_{\mathbb{C}}{:}= \mathrm{Hom}(\mathfrak{a}, \mathbb{C})$, $a \in A$ we write

$$a^\lambda{:}= e^{\lambda \log a}.$$

The (complexified) bilinear form B naturally defines a linear isomorphism of the complexification $\mathfrak{g}_{\mathbb{C}}$ with its dual $\mathfrak{g}^*_{\mathbb{C}}$, by which we shall identify. Accordingly we identify $\mathfrak{a}^*_{F\mathbb{C}}$ with a linear subspace of $\mathfrak{a}^*_{\mathbb{C}}$.

If σ is a continuous representation of M_F in a Hilbert space $\mathcal{H}_\sigma$, and $\lambda \in \mathfrak{a}^*_{F\mathbb{C}}$, then a representation $\sigma \otimes \lambda \otimes 1$ of Q_F in $\mathcal{H}_\sigma$ is defined by

$$(\sigma \otimes \lambda \otimes 1)(man) = a^\lambda \sigma(m) \qquad (m \in M_F,\ a \in A_F,\ n \in N_F)$$

(this is indeed a representation since M_F centralizes A_F and $M_F A_F$ normalizes N_F).

Normalized induction from Q_F to G involves the function $\delta_F{:}\,Q_F \to \mathbb{C}$ defined by

$$\delta_F(man) = |\det \mathrm{Ad}_{\mathfrak{g}/\mathfrak{q}_F}(man)|^{-1} \qquad (m \in M_F,\ a \in A_F,\ n \in N_F).$$

Now $\mathrm{Ad}(m)$ and $\mathrm{Ad}(n)$ act by determinant 1 on $\mathfrak{g}$ and $\mathfrak{q}_F$, and $\mathrm{Ad}(a)$ preserves the spaces $\mathfrak{q}_F$ and $\bar{\mathfrak{n}}_F = \theta\mathfrak{n}_F$. Note that $\bar{\mathfrak{n}}_F$ is the sum of the root spaces $\mathfrak{g}_{-\alpha}$ ($\alpha \in \Sigma^+ \setminus \mathrm{span}(F)$). Thus $\mathfrak{g} = \bar{\mathfrak{n}}_F \oplus \mathfrak{q}_F$ as a linear space, and it follows that $\mathrm{Ad}(a)$ acts by determinant $\det[\mathrm{Ad}(a)|\bar{\mathfrak{n}}_F]$ on the quotient $\mathfrak{g}/\mathfrak{q}_F$. Hence

$$\delta_F(man) = a^{2\rho_F},$$

where $\rho_F \in \mathfrak{a}^*_F$ is defined by $\rho_F(X){:}= \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|\mathfrak{n}_F)$.

We define the representation $\pi_{\sigma,\lambda}$ of G by

$$\pi_{\sigma,\lambda}{:}= \mathrm{ind}_{Q_F}^G(\sigma \otimes (\lambda + \rho_F) \otimes 1).$$

The underlying representation space

$$C(G, \sigma, \lambda){:}= C(G, \sigma \otimes (\lambda + \rho_F) \otimes 1)$$

is defined as in 1.2. Thus it consists of the continuous functions $\varphi{:}\,G \to \mathcal{H}_\sigma$ transforming according to the rule

$$\varphi(xman) = a^{-\lambda - \rho_F}\, \sigma(m)^{-1}\, \varphi(x),$$

for $x \in G$ and $(m, a, n) \in M_F \times A_F \times N_F$. The action is by left translation.

If σ is unitary and $\lambda \in i\mathfrak{a}^*_F$, then $\xi_{\sigma,\lambda}{:}= \sigma \otimes \lambda \otimes 1$ is a unitary representation of Q_F and $\xi_{\sigma,\lambda} \otimes \delta_F^{1/2} \simeq \sigma \otimes (\lambda + \rho_F) \otimes 1$. Hence $\pi_{\sigma,\lambda}$ is the representation normally induced from $\xi_{\sigma,\lambda}$. It follows that $\pi_{\sigma,\lambda}$ is unitarizable when σ is unitary and λ is imaginary. Accordingly we write, also when λ is not imaginary:

$$\pi_{\sigma,\lambda} = \mathrm{Ind}_{Q_F}^{G}(\sigma \otimes \lambda \otimes 1).$$

Now assume that σ is unitary, but $\lambda \in \mathfrak{a}_{F\mathbb{C}}^*$ general. Let dk be the normalized Haar measure on K. If $\varphi \in C(G, \sigma, \lambda)$ and $\psi \in C(G, \sigma, -\bar{\lambda})$, we define

$$\langle \varphi\,,\, \psi \rangle := \int_K (\varphi(k), \psi(k))_\sigma \, dk; \tag{8}$$

here $(\cdot, \cdot)_\sigma$ denotes the inner product of $\mathcal{H}_\sigma$.

Lemma 2.1. *Let σ be a unitary representation of M_F and let $\lambda \in \mathfrak{a}_{F\mathbb{C}}^*$. Then (8) defines a G-equivariant non-degenerate sesquilinear pairing*

$$C(G, \sigma, \lambda) \times C(G, \sigma, -\bar{\lambda}) \to \mathbb{C}.$$

In particular, if λ is imaginary then the pairing defines a G-equivariant pre-Hilbert structure on $C(G, \sigma, \lambda)$. The representation $\pi_{\sigma,\lambda}$ extends to a unitary representation in the associated Hilbert completion.

Proof. From the Iwasawa decomposition $G = KAN$ it follows that $G/Q_F \simeq K/K \cap Q_F = K/K_F$. Hence restriction to K induces a continuous linear isomorphism

$$C(G, \sigma, \lambda) \xrightarrow{\simeq} C(K, K_F, \sigma_F),$$

where σ_F denotes the restriction of σ to K_F. The latter space is the representation space of the induced representation $\mathrm{ind}_{K_F}^{K}(\sigma_F)$. It now follows straightforwardly that the pairing is non-degenerate, sesquilinear and K-equivariant; the G-equivariance remains to be established.

Let $\omega \in \mathcal{D}(\mathfrak{k}/\mathfrak{k}_F) \simeq \mathcal{D}(\mathfrak{g}/\mathfrak{q}_F)$ be the density corresponding to the normalized K-invariant measure dkK_F on K/K_F (see 1.5). If $f: G \to \mathbb{C}$ is a continuous function transforming according to the character δ_F of Q_F on the right, i.e. $f \in C(G, \delta_F)$, then $f \otimes \omega \in C(G, \mathcal{D}(\mathfrak{g}/\mathfrak{q}_F))$ may be identified canonically with a density on G/Q_F. Its integral equals

$$\int_{G/Q_F} f \otimes \omega = \int_{K/K_F} f(k)\, dkK_F = \int_K f(k)\, dk.$$

If $\varphi \in C(G, \sigma, \lambda)$, $\psi \in C(G, \sigma, -\bar{\lambda})$ and $g \in G$, then the function $f = (\varphi, \psi)_\sigma$, defined by $f(x) = (\varphi(x), \psi(x))_\sigma$, belongs to $C(G, \delta_F)$, and so does $L_g f: x \mapsto f(g^{-1}x)$. Hence by the above observation we obtain:

$$\begin{aligned}
\langle \pi_{\sigma,\lambda}(g)\varphi\,,\, \pi_{\sigma,-\bar{\lambda}}(g)\psi \rangle &= \int_{G/Q_F} L_g f \otimes \omega \\
&= \int_{G/Q_F} l_g^{*-1}(f \otimes \omega) \\
&= \int_{G/Q_F} f \otimes \omega = \langle \varphi\,,\, \psi \rangle
\end{aligned}$$

We conclude that the pairing (8) is G-equivariant. If λ is imaginary, the pairing defines a (continuous) pre-Hilbert structure on $C(G, \sigma, \lambda)$ which is G-equivariant. It follows that $\pi_{\sigma,\lambda}$ extends to a continuous unitary representation of G in the Hilbert completion of $C(G, \sigma, \lambda)$ (see also the argument at the end of the proof of Lemma 1.4). $\square$

2.2 The three pictures for the induced representation

The induced picture. We assume that σ is a unitary representation. If $\lambda \in \mathfrak{a}_{F\mathbb{C}}^*$ then we equip $C(G, \sigma, \lambda)$ (the space for $\pi_{\sigma,\lambda}$ in the induced picture) with the pre-Hilbert structure defined by

$$(\varphi_1, \varphi_2) = \int_K (\varphi_1(k), \varphi_2(k))_\sigma \, dk$$

for $\varphi_j \in C(G, \sigma, \lambda)$. By $\mathcal{H}_{\sigma,\lambda}$ we denote the completion of this pre-Hilbert space. The representation $\pi_{\sigma,\lambda}$ has a unique extension to a continuous representation of G in $\mathcal{H}_{\sigma,\lambda}$ which we denote by the same symbol. Indeed, for imaginary λ this follows from Lemma 2.1 (and then the extension is unitary); for general λ it is best seen in the compact picture discussed below.

Alternatively $\mathcal{H}_{\sigma,\lambda}$ may be described as the space of measurable (almost everywhere defined) functions $\varphi: G \to \mathcal{H}_\sigma$ such that

(a) $\varphi(xman) = a^{-\lambda-\rho_F}\sigma(m)^{-1}\varphi(x) \qquad (x \in G,\ m \in M_F,\ a \in A_F,\ n \in N_F);$

(b) $\varphi|K \in L^2(K, \mathcal{H}_\xi)$.

In this picture the induced representation is given by the formula

$$[\pi_{\sigma,\lambda}(g)\varphi](x) = \varphi(g^{-1}x) \qquad (x, g \in G).$$

The compact picture. We denote by $L^2(K, \mathcal{H}_\sigma)$ the space of $\mathcal{H}_\sigma$-valued L^2-functions relative to the Haar measure dk. Then restriction to K induces a surjective isometry

$$\mathcal{H}_{\sigma,\lambda} \xrightarrow{\simeq} L^2(K, \sigma_F), \tag{9}$$

where $L^2(K, \sigma_F)$ denotes the Hilbert space of functions $\varphi \in L^2(K, \mathcal{H}_\sigma)$ transforming according to the rule $\varphi(km) = \sigma(m)^{-1}\varphi(k)$ for $k \in K$, $m \in K_F$.

By transference under the isometry (9), the induced representation $\pi_{\sigma,\lambda}$ may be realized in the Hilbert space $L^2(K, \sigma_F)$, which has the advantage that it is independent of λ. We call this realization of the induced representation the 'compact picture.' It may described as follows.

The multiplication map $K \times \exp(\mathfrak{m}_F \cap \mathfrak{p}) \times A_F \times N_F \to G$ is a diffeomorphism. Accordingly we may define analytic maps κ_F, μ_F, H_F, ν_F from G to K, $\exp(\mathfrak{m}_F \cap \mathfrak{p})$, $\mathfrak{a}_F$, N_F respectively such that, for all $x \in G$,

$$x = \kappa_F(x)\mu_F(x)\exp H_F(x)\nu_F(x).$$

If $\varphi \in \mathcal{H}_{\sigma,\lambda}$ then for $x \in G$, $k \in K$ we have

$$\varphi(x^{-1}k) = \varphi(\kappa_F(x^{-1}k)\mu_F(x^{-1}k)\exp H_F(x^{-1}k)),$$

and hence in the compact picture the representation $\pi_{\sigma,\lambda}$ is described by

$$[\pi_{\sigma,\lambda}(x)\varphi](k) = e^{(-\lambda-\rho_F)H_F(x^{-1}k)}\,\sigma(\mu_F(x^{-1}k))^{-1}\,\varphi(\kappa_F(x^{-1}k)).$$

The non-compact picture. It is known that the inclusion $\bar{N}_F \to G$ induces a diffeomorphism j from $\bar{N}_F$ onto an open dense subset of G/Q_F. Let Ω be the K-invariant density on $G/Q_F \simeq K/K_F$ corresponding to the normalized K-invariant Radon measure on K/K_F (see 1.5). Let λ be the $\bar{N}_F$-invariant density on $\bar{N}_F$ determined by $\lambda(e) = j^*(\Omega)(e)$. Then

$$\lambda(\bar{n}) = Dl_{\bar{n}}(e)^{*-1}j^*(\Omega)(e).$$

Let $d\bar{n}$ be the Haar measure on $\bar{N}_F$ corresponding to the density λ (use 1.5 with $\bar{N}_F$, $\{e\}$ in place of G, H). Then

Lemma 2.2. $j^*(d\bar{k}) = e^{-2\rho_F H_F(\bar{n})}\, d\bar{n}$.

Proof. Let $\bar{n} \in \bar{N}_F$ and put $t(\bar{n}) = \mu_F(\bar{n}) \exp H_F(\bar{n})\nu_F(\bar{n})$. Then by K-invariance of Ω it follows that $l_{\bar{n}}^*\Omega = l_{t(\bar{n})}^*\Omega$. Now $l_{t(\bar{n})}$ preserves the origin of G/Q_F; its tangent map at the origin is the isomorphism of $\mathfrak{g}/\mathfrak{q}_F$ induced by $\mathrm{Ad}(t(\bar{n}))$, hence has determinant $\exp(-2\rho_F H_F(\bar{n}))$. It follows that

$$l_{\bar{n}}^*\Omega(e) = e^{-2\rho_F H_F(\bar{n})}\Omega(e).$$

Hence

$$\begin{aligned} j^*(\Omega)(\bar{n}) &= Dl_{\bar{n}}(e)^{*-1}[l_{\bar{n}}^* j^*\Omega(e)] \\ &= Dl_{\bar{n}}(e)^{*-1} j^*(l_{\bar{n}}^*\Omega)(e) \\ &= e^{-2\rho_F H_F(\bar{n})}\, \lambda(\bar{n}) \end{aligned}$$

The equality between the densities in the first and last member in the above display corresponds to the equality of the measures in the assertion of the lemma. □

It follows from the lemma that, for $\varphi_1, \varphi_2 \in C(G, \sigma, \lambda)$:

$$\begin{aligned} (\varphi_1, \varphi_2) &= \int_{K/K_F} (\varphi_1(k), \varphi_2(k))_\sigma \; dkK_F \\ &= \int_{\bar{N}_F} (\varphi_1(\kappa_F(\bar{n}), \varphi_2(\kappa_F(\bar{n}))_\sigma \, e^{-2\rho_F H(\bar{n})}\; d\bar{n} \\ &= \int_{\bar{N}_F} (\varphi_1(\bar{n}), \varphi_2(\bar{n}))_\sigma \; e^{-2\mathrm{Re}\,\lambda H_F(\bar{n})}\; d\bar{n}. \end{aligned} \tag{10}$$

Hence restriction to $\bar{N}_F$ induces an isometry

$$\mathcal{H}_{\sigma,\lambda} \simeq L^2(\bar{N}_F, \mathcal{H}_\sigma, e^{-2\mathrm{Re}\,\lambda H(\bar{n})}\, d\bar{n}).$$

In the latter Hilbert space ('the non-compact picture') the representation $\pi_{\sigma,\lambda}$ can be realized as follows. Let the analytic maps $\bar{n}_F, m_F, a_F, n_F$ from $\bar{N}_F Q_F$ to $\bar{N}_F, M_F, A_F, N_F$, respectively, be defined by

$$y = \bar{n}_F(y) m_F(y) a_F(y) n_F(y).$$

Then the representation $\pi_{\sigma,\lambda}$ is given by:

$$\pi_{\sigma,\lambda}(x)\varphi(\bar{n}) = a_F(x^{-1}\bar{n})^{-\lambda-\rho_F}\sigma(m_F(x^{-1}\bar{n}))^{-1}\varphi(\bar{n}_F(x^{-1}\bar{n})),$$

for $\varphi \in L^2(\bar{N}_F, \mathcal{H}_\sigma, e^{-2\mathrm{Re}\,\lambda H(\bar{n})}\, d\bar{n})$, $x \in G$, $\bar{n} \in \bar{N}_F$.

Remark 2.3. For later purposes we also describe the sesquilinear pairing of Lemma 2.1 in the non-compact picture. If $\varphi_1 \in C(G, \sigma, \lambda)$ and $\varphi_2 \in C(G, \sigma, -\bar{\lambda})$, then by the same substitution of variables $k = \kappa_F(\bar{n})$ as in the array of equations (10) we obtain:

$$\langle \varphi_1\, ,\, \varphi_2 \rangle = \int_{\bar{N}_F} (\varphi_1(\bar{n}), \varphi_2(\bar{n}))_\sigma \; d\bar{n}.$$

2.3 K-finite vectors

If π is a continuous representation of G in a locally convex space V, we denote by V^∞ the space of C^∞-vectors, and by V_K the space of K-finite vectors of the representation. If (δ, V_δ) is an irreducible finite dimensional representation of K, we denote by $V(\delta)$ the space of vectors in V_K that are K-isotypical of type δ. We recall that the map $(T, v) \mapsto T(v)$ induces a natural isomorphism

$$\mathrm{Hom}_K(V_\delta, V) \otimes V_\delta \xrightarrow{\simeq} V(\delta) \tag{11}$$

of K-modules; here K acts on the tensor product by $I \otimes \delta$. Let $Q_F = M_F A_F N_F$ be the standard parabolic subgroup determined by a set $F \subset \Delta$. The group M_F is of Harish-Chandra's class; $K_F = K \cap M_F$ is a maximal compact subgroup of M_F. Let $(\sigma, \mathcal{H}_\sigma)$ be a unitary representation of M_F.

Lemma 2.4. *If σ is admissible (for M_F, K_F), then for every $\lambda \in \mathfrak{a}_{F\mathbb{C}}^*$ the induced representation $\mathrm{Ind}_{Q_F}^G(\sigma \otimes \lambda \otimes 1)$ is admissible for G, K.*

Proof. Let (δ, V_δ) be a finite dimensional irreducible representation of K. Then we must show that $\dim \mathcal{H}_{\sigma,\lambda} < \infty$. In view of (11) this is equivalent to $\dim \mathrm{Hom}_K(V_\delta, \mathcal{H}_{\sigma,\lambda}) < \infty$. By the compact picture the K-module $\mathcal{H}_{\sigma,\lambda}$ is isomorphic to $L^2(K, \sigma_F)$, the representation space for $\mathrm{Ind}_{K_F}^{M_F}(\sigma_F)$. By Frobenius reciprocity (Lemma 1.2) we have

$$\mathrm{Hom}_K(V_\delta, L^2(K, \sigma_F)) \simeq \mathrm{Hom}_{K_F}(V_\delta, \mathcal{H}_\sigma),$$

and the latter space is finite dimensional, since σ is admissible. □

2.4 The infinitesimal character

Let $\mathcal{Z}(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$ of the complexication $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$. In this section we investigate the action of $\mathcal{Z}(\mathfrak{g})$ on parabolically induced representations.

If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, then by $I(\mathfrak{h})$ we denote the space of Weyl group invariants in $S(\mathfrak{h})$, the symmetric algebra of $\mathfrak{h}_\mathbb{C}$. Moreover, by $\gamma = \gamma_\mathfrak{h}^\mathfrak{g}$ we denote the canonical (Harish-Chandra) isomorphism from $\mathcal{Z}(\mathfrak{g})$ onto $I(\mathfrak{h})$.

A continuous representation (π, V) of G is said to have infinitesimal character $\Lambda \in \mathfrak{h}_\mathbb{C}^*$, if for all $v \in V^\infty$:

$$\pi(Z)v = \gamma_\mathfrak{h}^\mathfrak{g}(Z, \Lambda) \qquad (Z \in \mathcal{Z}(\mathfrak{g})).$$

In the following we let $F \subset \Delta$ and assume that $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_F$. Since $\mathfrak{m}_{1F} = \mathfrak{m}_F + \mathfrak{a}_F$ is the centralizer of $\mathfrak{a}_F$ in $\mathfrak{g}$, it follows that

$$\mathfrak{h} = \mathfrak{h}_{M_F} \oplus \mathfrak{a}_F, \tag{12}$$

where $\mathfrak{h}_{M_F} := \mathfrak{h} \cap \mathfrak{m}_F$ is a Cartan subalgebra of $\mathfrak{m}_F$. As mentioned before we use the bilinear form B to identify the dual spaces $\mathfrak{h}_{M_F}^*$ and $\mathfrak{a}_F^*$ (as well as their complexifications) with subspaces of $\mathfrak{h}_\mathbb{C}^*$. Since the decomposition (12) is B-perpendicular, $\mathfrak{h}_{M_F}^*$ corresponds to the subspace of functionals in $\mathfrak{h}^*$ which vanish on $\mathfrak{a}_F$; similarly, $\mathfrak{a}_F^*$ corresponds to the space of functionals in $\mathfrak{h}^*$ which vanish on $\mathfrak{h}_{M_F}$.

Lemma 2.5. *Let $(\sigma, \mathcal{H}_\sigma)$ be a unitary representation of M_F with infinitesimal character $\Lambda_\sigma \in \mathfrak{h}^*_{M_F\mathbb{C}}$. Then for every $\lambda \in \mathfrak{a}^*_{F\mathbb{C}}$ the induced representation $\mathrm{Ind}_{Q_F}^G(\sigma \otimes \lambda \otimes 1)$ has infinitesimal character $\Lambda_\sigma + \lambda$.*

Proof. Let $\varphi \in C^\infty(G, \sigma, \lambda)$. Then $\varphi(e)$ is a smooth vector in $\mathcal{H}_\sigma$, hence for $Z_M \in \mathcal{Z}(\mathfrak{m})$ we have $\sigma(Z_M)\varphi(e) = \gamma^{\mathfrak{m}}_{\mathfrak{h}_{M_F}}(Z_M, \Lambda_\sigma)\varphi(e)$.

Let $Z \in \mathcal{Z}(\mathfrak{g})$. Then

$$[\pi_{\sigma,\lambda}(Z)\varphi](e) = [L_Z\varphi](e) = [R_{Z^\vee}\varphi](e), \tag{13}$$

where $Z \mapsto Z^\vee$ denotes the anti-automorphism of $U(\mathfrak{g})$ induced by the anti-automorphism $X \mapsto -X$ of $\mathfrak{g}$. There exists a unique $Z_0 \in U(\mathfrak{m}_F + \mathfrak{a}_F)$ such that $Z \simeq Z_0 \mod \mathfrak{n}_F U(\mathfrak{g})$. The element Z_0 belongs to $\mathcal{Z}(\mathfrak{m}_{1F})$. It follows from the characterization of the Harish-Chandra isomorphisms of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{m}_{1F}, \mathfrak{h})$ in terms of the root structure that that for every $\mu \in \mathfrak{h}^*_{\mathbb{C}}$:

$$\gamma^{\mathfrak{m}_{1F}}_{\mathfrak{h}}(Z_0, \mu + \rho_F) = \gamma^{\mathfrak{g}}_{\mathfrak{h}}(Z, \mu)$$

From the Q_F-behavior of φ on the right it now follows that the right-hand side of (13) equals

$$[\sigma \otimes (\lambda + \rho_F)](Z_0)\, \varphi(e) = \gamma^{\mathfrak{m}_F + \mathfrak{a}_F}_{\mathfrak{h}}(Z_0, \Lambda_\sigma + \lambda + \rho_F)\varphi(e) = \gamma^{\mathfrak{g}}_{\mathfrak{h}}(Z, \Lambda_\sigma + \lambda)\varphi(e).$$

Hence $\pi_{\sigma,\lambda}(Z)\varphi = \gamma^{\mathfrak{g}}_{\mathfrak{h}}(Z, \Lambda_\sigma + \lambda)\varphi$ at the identity element; since G centralizes $\mathcal{Z}(\mathfrak{g})$ the identity holds at every point of G. $\square$

2.5 Irreducibility

The following result on the irreducibility of parabolically induced representations is due to F. Bruhat [1] for a minimal parabolic subgroup, and to Harish-Chandra [6] in general. A proof of the general result following the original ideas of Bruhat can be found in the recent paper [12].

Theorem 2.6. (Bruhat, Harish-Chandra). *Let σ be an irreducible unitary representation of M_F, and let $\lambda \in i\mathfrak{a}^*_F$. Assume that*

(a) *σ has real infinitesimal character (i.e. $\langle \Lambda_\sigma\,, \alpha\rangle \in \mathbb{R}$ for every $(\mathfrak{m}_{1F\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$-root α);*

(b) *$\langle \lambda\,, \beta\rangle \neq 0$ for every $\mathfrak{a}_F$-weight β in $\mathfrak{n}_F$.*

Then the induced representation $\mathrm{Ind}_{Q_F}^G(\sigma \otimes \lambda \otimes 1)$ is irreducible.

Proof. (ideas). We sketch the ideas of Bruhat's proof in the case that $F = \emptyset$; then $Q_F = Q = MAN$ is minimal. Since M is compact, σ is finite dimensional; this simplifies the functional analysis involved in the argument.

Let σ, λ fulfill the hypotheses of the theorem. Since $\pi_{\sigma,\lambda}$ is unitary it suffices to show, that the space $\mathrm{End}_G(\mathcal{H}_{\sigma,\lambda})$ of continuous self-intertwining operators of $\pi_{\sigma,\lambda}$ consists of the scalar multiples of the identity. Let $T \in \mathrm{End}_G(\mathcal{H}_{\sigma,\lambda})$. Then by equivariance T maps the space $\mathcal{H}^\infty_{\sigma,\lambda}$ of smooth vectors continuously and equivariantly into itself. Now $\mathcal{H}^\infty_{\sigma,\lambda} = C^\infty(G, \sigma, \lambda)$, hence the evaluation map $\mathrm{ev}_e \colon \varphi \mapsto \varphi(e)$ is a continuous linear map from $C^\infty(G, \sigma, \lambda)$ to $\mathcal{H}_\sigma$. Put $u_T = \mathrm{ev}_e \circ T$, then

$$u_T \in [(\mathcal{H}^\infty_{\sigma,\lambda})' \otimes \mathcal{H}_\sigma]^Q \tag{14}$$

where the prime indicates that the topological linear dual has been taken, and where the superscript Q indicates that the space of invariants for the tensor product of $\pi'_{\sigma,\lambda}|Q$ and $\sigma\otimes(\lambda+\rho_F)\otimes 1$ has been taken.

The G-module $\mathcal{H}^\infty_{\sigma,\lambda}$ is isomorphic to the space of smooth sections in the G-homogeneous vector bundle associated with the representation $\sigma\otimes(\lambda+\rho_F)\otimes 1$ of Q in $\mathcal{H}_\sigma$; accordingly its linear topological dual may be indentified with the G-module of generalized sections in the bundle associated to the representation $\sigma^\vee\otimes(-\lambda+\rho_F)\otimes 1$ (see (6)). We denote the latter space by $C^{-\infty}(G,\sigma^\vee,-\lambda)$. The projection $p\colon G\to G/Q$ induces a natural embedding of $C^{-\infty}(G,\sigma^\vee,-\lambda)$ into $C^{-\infty}(G)\otimes\mathcal{H}_\sigma$, with image the space of Q-invariants for the tensor product $R\otimes[\sigma^\vee\otimes(-\lambda+\rho_F)\otimes 1]$. Thus we see that the space in (14) is naturally isomorphic to

$$[C^{-\infty}(G)\otimes(\mathcal{H}^*_\sigma\otimes\mathcal{H}_\sigma)]^{Q\times Q}, \tag{15}$$

where superscript $Q\times Q$ indicates the subspace of invariants for the following action of $Q\times Q$. The action of $Q\times Q$ on $C^{-\infty}(G)$ is $L\hat{\otimes}R$, the exterior tensor product of the left- and the right regular actions. The action of $Q\times Q$ on $\mathcal{H}^*_\sigma\otimes\mathcal{H}_\sigma$ is by the exterior tensor product $[\sigma^\vee\otimes(-\lambda+\rho_F)]\hat{\otimes}[\sigma\otimes(\lambda+\rho_F)]$. Finally, in (15) the tensor product of these two actions of $Q\times Q$ has been taken.

It follows from the above that $\operatorname{supp} u_T$ is a union of double cosets for the $Q\times Q$-action on G. Thus the *Bruhat decomposition* comes into play. Let $W=N_K(\mathfrak{a})/M$, where $N_K(\mathfrak{a})$ denotes the normalizer of $\mathfrak{a}$ in K. Then W is naturally isomorphic to the Weyl group of the root system Σ. We recall that

$$G=\bigcup_{s\in W} QsQ \qquad \text{(disjoint union)}.$$

There is a unique open double coset, which is dense in G; it corresponds to the longest element in W (relative to Σ^+).

Suppose that $s\in W$ is such that QsQ is maximal among the $Q\times Q$-orbits in $\operatorname{supp} u_T$. First we assume that QsQ is open, that is, s is the longest Weyl group element. The generalized function u_T restricts to a smooth $Q\times Q$-invariant $\mathcal{H}^*_\sigma\otimes\mathcal{H}_\sigma$-valued function on this open orbit; its value $u_T(s)$ at s must be fixed under the stabilizer $\operatorname{Stab}(s)$ of s in $Q\times Q$. The latter group equals:

$$\operatorname{Stab}(s)=\{(q_1,q_2)\in Q\times Q\mid q_1 s q_2^{-1}=s\}=\{ma, s^{-1}mas)\mid m\in M,\ a\in A\}.$$

Hence

$$(\mathcal{H}^*_\sigma\otimes\mathcal{H}_\sigma)^{\operatorname{Stab}(s)}\simeq \operatorname{Hom}_{MA}(\sigma\otimes(\lambda-\rho_F), s\sigma\otimes s(\lambda+\rho_F)).$$

The latter space is trivial because $\lambda\neq s\lambda$ by the regularity assumption on λ. It follows that $u_T(s)=0$, hence u_T is supported by the lower dimensional $Q\times Q$-orbits.

Now assume that QsQ is not open. Then by an analysis in the same spirit as above, but with the additional complication that transversal derivates to the orbit QsQ have to be taken into account it follows again that $\sigma\sim s\sigma$ and $\lambda=s\lambda$. Because of the condition on λ this can only happen when $s=1$. It follows from this that u_T has to be supported at e hence is a derivative of a Dirac function with coefficients in $\operatorname{End}(\mathcal{H}_\sigma)$. By a further analysis one can show that u_T must have order 0 and is equal to $\delta_e\otimes A_T$, with δ_e a Dirac function in e and A_T an element of the space $\operatorname{Hom}_M(\mathcal{H}_\sigma)$. The latter space is one dimensional by the irreducibility of σ.

Finally, it follows from the above that the map $T\mapsto A_T$, $\operatorname{End}_G(\mathcal{H}_{\sigma,\lambda})\to\operatorname{Hom}_M(\mathcal{H}_\sigma)$ is injective. Hence $\operatorname{End}_G(\mathcal{H}_{\sigma,\lambda})$ is one dimensional. $\square$

3 Asymptotic behavior of matrix coefficients

3.1 Matrix coefficients

Let π be a continuous representation of G in a locally convex Hausdorff space V. By a matrix coefficient of π we mean a function of the form

$$m_{v,v'}\colon\, x \mapsto \langle \pi(x)v\,,\, v'\rangle,\; G \to \mathbb{C}$$

where $v \in V$ and where v' belongs to V', the topological linear dual of V. The following lemma is easily verified.

Lemma 3.1. *Let $v' \in V'$.*

(a) *The map $v \mapsto m_{v,v'}$ intertwines π with the right regular action R of G.*

(b) *If $v \in V^\infty$ then $m_{v',v} \in C^\infty(G)$.*

(c) *The map $v \mapsto m_{v,v'}$, $V^\infty \to C^\infty(G)$ intertwines the $U(\mathfrak{g})$-actions induced by π and R.*

If the representation π has an infinitesimal character $\Lambda \in \mathfrak{h}_{\mathbb{C}}^*$, then it follows from the above that every matrix coefficient $m = m_{v,v'}$, with v a smooth vector, is a function in $C^\infty(G)$ satisfying the following system of differential equations:

$$R_Z\, m = \gamma(Z, \Lambda)m \qquad (Z \in \mathcal{Z}(\mathfrak{g})); \tag{16}$$

here R denotes the right regular representation of $U(\mathfrak{g})$ in $C^\infty(G)$.

Example 3.2. With $Q_F = M_F A_F N_F$ as in Section 2.1, let σ be an irreducible unitary representation of M, and $\lambda \in \mathfrak{a}_{F\mathbb{C}}^*$. Then the sesquilinear pairing $\mathcal{H}_{\sigma,\lambda} \times \mathcal{H}_{\sigma,-\bar\lambda} \to \mathbb{C}$ defined by (8) is non-degenerate, hence induces an anti-linear embedding $\mathcal{H}_{\sigma,-\bar\lambda} \subset \mathcal{H}'_{\sigma,\lambda}$. For elements $\varphi \in \mathcal{H}_{\sigma,\lambda}$, $\psi \in \mathcal{H}_{\sigma,-\bar\lambda}$ the corresponding matrix coefficient of $\pi_{\sigma,\lambda}$ is given by

$$m_{\varphi,\psi}(x) = \int_K \langle \varphi(x^{-1}k)\,,\, \psi(k)\rangle\, dk \qquad (x \in G).$$

If φ (or ψ) is a smooth vector, then the matrix coefficient $m_{\varphi,\psi}$ is a smooth function on G. Moreover, by (16) and Lemma 2.5 it satisfies the following system of differential equations

$$R_Z m_{\varphi,\psi} = \gamma(Z, \Lambda_\sigma + \lambda)m_{\varphi,\psi} \qquad (Z \in \mathcal{Z}(\mathfrak{g}));$$

here Λ_σ denotes the infinitesimal character of σ.

Example 3.3. Let notation be as in the above example, but now assume that $F = \emptyset$, i.e. Q_F is the minimal standard parabolic subgroup $Q = MAN$. Then M is compact, hence σ is finite dimensional. The representations $\pi_{\sigma,\lambda}$ are said to belong to the minimal principal series. From the compact picture one sees that $\pi_{\sigma,\lambda}$ has a K-fixed vector if and only if $\sigma = 1$. The representations $\pi_\lambda = \pi_{1,\lambda}$ constitute the spherical principal series. The space of K-fixed (or spherical) vectors in $\mathcal{H}_\lambda = \mathcal{H}_{1,\lambda}$ is equal to $\mathbb{C}\mathbf{1}_\lambda$, where $\mathbf{1}_\lambda$ is determined by $\mathbf{1}_\lambda|K = 1$. In the induced picture the vector $\mathbf{1}_\lambda$ is the function $G \to \mathbb{C}$ described by:

$$\mathbf{1}_\lambda(x) = \mathbf{1}_\lambda(\kappa(x)\exp H(x)\nu(x)) = e^{(-\lambda-\rho)H(x)} \qquad (x \in G),$$

where we have suppressed the index $F = \emptyset$ in the notation.

By equivariance of the pairing (8), the matrix coeffient $\varphi_\lambda := m_{\mathbf{1}_\lambda, \mathbf{1}_{-\bar{\lambda}}}$ is given by the formula

$$\varphi_\lambda(x) = \langle \mathbf{1}_\lambda \, , \, \pi_{-\bar{\lambda}}(x^{-1})\mathbf{1}_{-\bar{\lambda}} \rangle = \int_K e^{(\lambda - \rho)H(xk)} \, dk \qquad (x \in G).$$

This is Harish-Chandra's formula for the elementary (or zonal) spherical function associated with the Riemannian symmetric space G/K. The function φ_λ is a bi-K-invariant smooth function on G satisfying a system of differential equations of the form (16), coming from the action of the center of $U(\mathfrak{g})$.

Note also that $U(\mathfrak{g})^K$, the algebra of K-invariants in $U(\mathfrak{g})$ preserves the space $(\mathcal{H}_\lambda)^K = \mathbb{C}\mathbf{1}_\lambda$. Hence there exists an algebra homomorphism $\chi_\lambda : U(\mathfrak{g})^K \to \mathbb{C}$ such that

$$X\mathbf{1}_\lambda = \chi_\lambda(X)\mathbf{1}_\lambda \qquad (X \in U(\mathfrak{g})^K).$$

By the equivariance stated in Lemma 3.1 (c) it now follows that φ_λ satisfies the system of differential equations

$$R_X \varphi_\lambda = \chi_\lambda(X)\varphi_\lambda \qquad (X \in U(\mathfrak{g})^K).$$

The space $C^\infty(G/K)$ of C^∞-functions on G/K is canonically isomorphic with the space of right K-invariant C^∞-functions on G. Accordingly, if $X \in U(\mathfrak{g})^K$, then R_X acts on $C^\infty(G/K)$ as an element from $\mathbb{D}(G/K)$, the algebra of left G-invariant differential operators on G/K. It is known that $X \mapsto R_X$ is a surjective algebra homomorphism from $U(\mathfrak{g})^K$ onto $\mathbb{D}(G/K)$, with kernel $U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$. Hence it induces an algebra isomorphism from $U(\mathfrak{g})^K / U(\mathfrak{g})^K \cap U(\mathfrak{g})\mathfrak{k}$ onto $\mathbb{D}(G/H)$. We thus see that the zonal spherical function φ_λ is a simultaneous eigenfunction for the algebra $\mathbb{D}(G/K)$.

The function $\Xi := \varphi_0$, given by the formula

$$\Xi(x) = \int_K e^{-\rho H(xk)} \, dk \qquad (x \in G)$$

plays a fundamental role in harmonic analysis on G.

3.2 A cofinite ideal

Let $(\pi, \tilde{V})$ be an admissible representation of G, such that the associated $(\mathfrak{g}, K)$-module $V := \tilde{V}_K$ is finitely generated. The dual $(\mathfrak{g}, K)$-module $V' := (V^*)_K$ is readily seen to be admissible and naturally isomorphic to $[\tilde{V}']_K$ (it is also finitely generated, but this is not so obvious; see e.g. [17], Lemma 4.3.2).

The goal of this section is to describe the asymptotic behavior, for $v \in V$, $v' \in V'$, of the matrix coeffient $m_{v,v'}(x)$, as x tends to infinity in G.

Since v and v' are K-finite, the matrix coefficient $m_{v,v'}$ is (real) analytic and behaves finitely under the actions of K from the left and from the right. In view of the Cartan decomposition

$$G = K \operatorname{cl}(A^+) K \tag{17}$$

it is therefore sufficient to study the behavior of $m_{v,v'}(a)$ as $a \to \infty$ in $\operatorname{cl}(A^+)$. Here $\operatorname{cl}(A^+)$ denotes the closure of $A^+ = \exp(\mathfrak{a}^+)$, where $\mathfrak{a}^+$ is the open positive Weyl chamber in $\mathfrak{a}$. We recall that if $x \in G$, then $x \in KaK$ with $a \in \operatorname{cl}(A^+)$ uniquely determined.

The matrix coefficient's behavior along $\mathrm{cl}\,(A^+)$ will turn out to be severely restricted by a system of differential equations it satisfies. Define the ideal I of $\mathcal{Z}(\mathfrak{g})$ by

$$I = \{Z \in \mathcal{Z}(\mathfrak{g}) \mid \pi(Z) = 0 \text{ on } V\}.$$

Then it follows from the equivariance formulated in Lemma 3.1 (c) that

$$R_Z\, m_{v,v'} = 0 \qquad (Z \in I). \tag{18}$$

An ideal $\mathcal{I}$ of an algebra $\mathcal{A}$ over $\mathbb{C}$ is said to be cofinite if $\mathcal{A}/\mathcal{I}$ is a finite dimensional complex vector space. The following result expresses that the system (18) is large.

Lemma 3.4. *The ideal I is cofinite in the algebra $\mathcal{Z}(\mathfrak{g})$.*

Proof. For $\vartheta \subset \widehat{K}$ a finite subset, the finite dimensional space $V(\vartheta) := \oplus_{\delta\in\vartheta} V(\delta)$ is invariant for the action of $\mathcal{Z}(\mathfrak{g})$; let $\nu = \nu_\vartheta : \mathcal{Z}(\mathfrak{g}) \to \mathrm{End}(V(\vartheta))$ be the induced homomorphism of algebras. Then $\ker\nu$ is an ideal of $\mathcal{Z}(\mathfrak{g})$, containing I, and of finite codimension at most $\dim \mathrm{End}(V(\vartheta))$.

Since V is finitely generated as a $(\mathfrak{g}, K)$-module, we may fix a finite set $\vartheta \subset \widehat{K}$ such that $U(\mathfrak{g})V(\vartheta) = V$. Since $\mathcal{Z}(\mathfrak{g})$ is central it follows that $\ker\nu = I$; hence $I = \ker\nu$ and we see that I is cofinite. $\square$

Remark 3.5. If V has an infinitesimal character $\Lambda \in \mathfrak{h}^*_{\mathbb{C}}$, then the associated ideal I is the kernel of the character $\gamma(\cdot, \Lambda)$ of $\mathcal{Z}(\mathfrak{g})$, hence of codimension 1. In this case, (18) is a system of eigenequations.

3.3 Spherical functions

In view of its K-finiteness, the restriction of the function $m_{v,v'}$ satisfies a system of differential equations on the group A (which is diffeomorphic to the vector space $\mathfrak{a}$). We arrive at this system essentially by applying the method of separation of variables. For this it is more convenient to work with spherical than with K-finite functions.

Let τ be a (continuous) representation of $K \times K$ on a finite dimensional complex linear space E. We agree to write $\tau_1(k_1)v\tau_2(k_2) = \tau(k_1, k_2^{-1})v$ for $v \in V$, $k_1, k_2 \in K$. A continuous function $\psi: G \to E$ is said to be τ-spherical if it transforms according to the rule

$$\psi(k_1 x k_2) = \tau(k_1)\psi(x)\tau(k_2) \qquad (x \in G,\ k_1, k_2 \in K).$$

The space of all such functions is denoted by $C(G, \tau)$, the space of all analytic τ-spherical functions by $\mathcal{A}(G, \tau)$; the spaces $C^\infty(G, \tau), C^\infty_c(G, \tau)$ are defined similarly. Note that

$$C(G, \tau) \simeq (C(G) \otimes E)^{K\times K};$$

where $K \times K$ acts on $C(G)$ by the left times right action.

The K-finite matrix coeffient $m_{v,v'}$ is expressible in terms of a spherical function as follows. Let $\vartheta \subset \widehat{K}$ be a finite subset containing the K-types occurring in v and v'. Define the representation τ of $K \times K$ on $E := \mathrm{End}(V(\vartheta))$ by $\tau(k_1, k_2)A = \pi(k_1) \circ A \circ \pi(k_2)^{-1}$. Let $\iota_\vartheta : V(\vartheta) \to V$

be the inclusion map and $P_\vartheta : V \to V(\vartheta)$ the K-equivariant projection map. Then the function $\varphi : G \to E$ defined by

$$\varphi(x) = P_\vartheta \circ \pi(x) \circ \iota_\vartheta$$

is τ-spherical. Moreover, let $\eta = \eta_{v,v'}$ be the linear functional on E defined by $\eta(A) = \langle Av\,,\, v'\rangle$. Then

$$m_{v,v'} = \eta \circ \varphi.$$

Note that the function φ belongs to the space $\mathcal{A}(G,\tau)$ and satisfies the system (18) as well. We denote the space of all such functions by $\mathcal{A}(G,\tau,I)$ and proceed by studying the asymptotic behavior of its elements at infinity.

3.4 The radial differential equations

The restriction $\mathrm{Res}\,\varphi$ to A^+ of a function $\varphi \in C^\infty(G,\tau)$ has values in the space

$$E^M := \{v \in E \mid v = \tau(m)\, v\, \tau(m)^{-1} \quad (\forall m \in M)\};$$

where M denotes the centralizer of A in K. Indeed, this follows from the observation $\varphi(a) = \varphi(mam^{-1}) = \tau_1(m)\varphi(a)\tau_2(m)^{-1}$, for $a \in A,\; m \in M$.

The map $(k_1, k_2, a) \mapsto k_1 a k_2$ induces a diffeomorphism of $K \times_M K \times A^+$ onto an open subset of G. Hence if $f \in C_c^\infty(A^+, E^M)$, then there exists a unique function in $C_c^\infty(G,\tau)$, denoted $\mathrm{Lift}\, f$, whose restriction to A^+ is f.

If $Z \in \mathcal{Z}(\mathfrak{g})$, then the operator

$$\Pi_\tau(Z) := \mathrm{Res} \circ R_Z \circ \mathrm{Lift} \tag{19}$$

from $C_c^\infty(A^+, E^M)$ to $C^\infty(A^+, E^M)$ is readily seen to be continous linear and support preserving. Hence it is a differential operator on A^+ with smooth $\mathrm{End}(E^M)$-valued coefficients. We denote the algebra of all such operators by $\mathcal{D}^\infty$. One readily verifies that $Z \mapsto \Pi_\tau(Z)$ is a homomorphism of algebras from $\mathcal{Z}(\mathfrak{g})$ to $\mathcal{D}^\infty$. The differential operator (19) is called the τ-radial component of Z.

Let $S(\mathfrak{a})$ denote the symmetric algebra of $\mathfrak{a}_\mathbb{C}$. The right regular representation of A in $C^\infty(A)$ induces an isomorphism of $S(\mathfrak{a})$ onto the algebra of invariant differential operators on A. Accordingly we identify elements of $S(\mathfrak{a})$ with differential operators; in particular if $H \in \mathfrak{a}$, $f \in C^\infty(A)$, then $Hf(a) = d/dt(f(a \exp tH))_{t=0}$. The above identification induces a linear isomorphism

$$C^\infty(A^+) \otimes \mathrm{End}(E^M) \otimes S(\mathfrak{a}) \simeq \mathcal{D}^\infty, \tag{20}$$

by which we shall identify from now on. An element $c \otimes L \otimes v$ of (20) thus acts on $C^\infty(A^+, E^M)$ according to the formula:

$$[(c \otimes L \otimes v)f](a) := c(a)L(vf(a)).$$

In particular, the tensor product on the left-hand side of (20) is equipped with the structure of an algebra. Its multiplication law is readily determined by using Leibniz' rule for differentiation.

Let $\mathcal{R}$ be the ring of functions $A^+ \to \mathbb{C}$ generated by 1 and the functions $a \mapsto a^{-\alpha}$, $a \mapsto (1 - a^{-2\beta})^{-1}$ $(\alpha, \beta \in \Sigma^+)$. This ring is stable under differentiation by elements from $S(\mathfrak{a})$. Hence

$$\mathcal{D} := \mathcal{R} \otimes \mathrm{End}(E^M) \otimes S(\mathfrak{a})$$

defines a subalgebra of $\mathcal{D}^\infty$. The following result can be proved by a computation in $U(\mathfrak{g})$:

Lemma 3.6. *The map Π_τ is an algebra homomorphism from $\mathcal{Z}(\mathfrak{g})$ into $\mathcal{D}$.*

It follows from the above discussion that if $\varphi \in \mathcal{A}(G, \tau, I)$, then its restriction $f = \operatorname{Res}\varphi$ to A^+ is a smooth function $A^+ \to E^M$ which satisfies the following system of radial differential equations:

$$\Pi_\tau(D)f = 0 \qquad (Z \in I). \tag{21}$$

Example 3.7. We consider the group $G = \mathrm{SL}\,(2, \mathbb{R})$. As a basis of its Lie algebra we take the following standard $sl(2)$-triple:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Let $U := X - Y$, and put $k_\varphi = \exp(\varphi U)$. Then

$$k_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

and we see that $K = \exp(\mathbb{R}U) = \mathrm{SO}\,(2)$ is a maximal compact subgroup of G. Put $\mathfrak{a} = \mathbb{R}H$. Then $\Sigma = \{\alpha, -\alpha\}$, where α is determined by $\alpha(H) = 2$. Fix $\Sigma^+ = \{\alpha\}$, and put $H_\alpha = \frac{1}{2}H$. Note that $\mathfrak{n} = \mathbb{R}X$ and $\bar{\mathfrak{n}} = \mathbb{R}Y$. The center $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$ is the polynomial algebra generated by the following element C (which is two times the Casimir):

$$C = H_\alpha^2 + H_\alpha + YX.$$

Using the identity $Y = \varphi_1(a)U^a + \varphi_2(a)U$, where $U^a = \mathrm{Ad}(a^{-1})U$ and

$$\varphi_1(a) = \frac{-a^{-\alpha}}{(1 - a^{-2\alpha})^2}, \quad \varphi_2(a) = \frac{a^{-2\alpha}}{(1 - a^{-2\alpha})^2},$$

we obtain, for every $a \in A^+$:

$$C = H_\alpha^2 + \frac{1 + a^{-2\alpha}}{1 - a^{-2\alpha}}\, H_\alpha + \frac{a^{-2\alpha}}{(1 - a^{-2\alpha})^2}\,([U^a]^2 + U^2) - \frac{a^{-\alpha}(1 + a^{-2\alpha})}{(1 - a^{-2\alpha})^2}\, U^a U. \tag{22}$$

Let the representation $\tau = (\tau_1, \tau_2)$ of $K \times K$ in $E = \mathbb{C}$ be defined by $\tau_1(k_\varphi) = e^{in\varphi}, \quad \tau_2(k_\varphi) = e^{im\varphi}$. Then for the associated representations of $\mathfrak{k}$ we have: $\tau_1(U) = n,\ \tau_2(U) = m$. Hence from (22) we see that the radial component of C is given by

$$\Pi_\tau(C) = H_\alpha^2 + \frac{1 + a^{-2\alpha}}{1 - a^{-2\alpha}}\, H_\alpha + (n^2 + m^2)\frac{a^{-2\alpha}}{(1 - a^{-2\alpha})^2} + nm\frac{a^{-\alpha}(1 + a^{-2\alpha})}{(1 - a^{-2\alpha})^2}.$$

Here H_α is identified with a first order differential operator on A via the right regular representation, in the usual way.

The system of radial differential equations is cofinite in the following sense:

Proposition 3.8. *Let J be the left ideal of $\mathcal{D}$ generated by $\Pi_\tau(I)$. Then $\mathcal{D}/J$ is finitely generated as a left module over $\mathcal{R} \otimes \mathrm{End}(E^M)$.*

The proof of this proposition, which we shall not give here, relies on the cofiniteness of the ideal I and on the following lemma, that will be useful at a later stage as well.

Lemma 3.9. *There exists a finite dimensional complex linear subspace $\mathcal{E} \subset U(\mathfrak{a})$ such that*

$$U(\mathfrak{g}) = U(\bar{\mathfrak{n}})\,\mathcal{E}\,\mathcal{Z}(\mathfrak{g})\,U(\mathfrak{k}).$$

In the following we write Δ for the collection of simple roots in Σ^+, and assume that Δ is a basis of $\mathfrak{a}^*$ (this assumption, which is equivalent to the assumption that G has a compact center, is only made to ease the notations). The basis of $\mathfrak{a}$ dual to Δ is denoted by $(H_\alpha \mid \alpha \in \Delta)$. Its elements may be viewed as differential operators on A, in the fashion described above.

As an immediate consequence of Proposition 3.8 we obtain:

Corollary 3.10. *There exist finitely many operators $D_1 = 1, D_2, \dots, D_n \in \mathcal{D}$ and functions $g_\alpha \in \mathcal{R} \otimes \mathrm{End}((E^M)^n)$ $(\alpha \in \Delta)$ such that the function*

$$F := \begin{pmatrix} D_1 f \\ \vdots \\ D_n f \end{pmatrix} \quad \text{satisfies} \quad H_\alpha F = g_\alpha F \qquad (\alpha \in \Delta). \tag{23}$$

Example 3.11. We return to the situation discussed in Example 3.7, and assume that I is an ideal of codimension one in $\mathcal{Z}(\mathfrak{g})$; then it is generated by $C - s$, for some $s \in \mathbb{C}$. Under this assumption the system of radial differential equations consists of one eigenequation: $\Pi_\tau(C)f = s\,f$. Now $E^M = \mathbb{C}$, and the assertion of the above corollary is valid with $n = 2$ and $D_1 = 1,\ D_2 = H_\alpha$. This corresponds to the usual reduction of a second order differential equation to a system of two first order differential equations.

Remark 3.12. The results of the present section are essentially due to Harish-Chandra [4], but his results remained unpublished for a long time. In [3] it was observed that the system (23) is of the regular singular type at infinity: this allowed a simplification of Harish-Chandra's original theory. Our presentation of the theory follows [3] rather closely.

That the system (23) is of the regular singular type at infinity is seen by using the coordinates $z_\alpha = a^{-\alpha}$ $(\alpha \in \Delta)$ on A^+. More precisely, define the map $\underline{z}$ from A^+ onto the Δ-fold Cartesian product of intervals $]0,1[^\Delta$ by

$$\underline{z}(a)_\alpha = a^{-\alpha}.$$

Then $\underline{z}$ is an analytic bijection. We denote its inverse by $\underline{a}$. In the following a function f on A^+ will be identified with the corresponding function $\tilde{f} := f \circ \underline{a}$ on $]0,1[^\Delta$. Note that every function $g \in \mathcal{R}$ corresponds to a real analytic function on $]0,1[^\Delta$, which has a unique extension to a holomorphic function on D^Δ; here D denotes the complex unit disc $\{z \in \mathbb{C} \mid |z| < 1\}$. Thus we may view $\mathcal{R}$ as a subring of $\mathcal{O}(D^\Delta)$, the ring of holomorphic functions $D^\Delta \to \mathbb{C}$. In the coordinates z_α the system (23) becomes:

$$z_\alpha \frac{\partial}{\partial z_\alpha} F(z) = g_\alpha(z) F(z) \qquad (\alpha \in \Delta). \tag{24}$$

Here the functions g_α belong to $\mathcal{O}(D^\Delta) \otimes \mathrm{End}((E^M)^n)$; the system has regular singularities of the simple type along the coordinate hyperplanes $z_\alpha = 0$.

Example 3.13. We return to the situation of Example 3.11. The radial component takes the following form in the variable $z = a^{-\alpha}$:

$$\Pi_\tau(C) = (z\frac{d}{dz})^2 + \frac{1+z^2}{1-z^2}\, z\frac{d}{dz} + \frac{(n^2+m^2)\, z^2 + nm\, z\,(1+z^2)}{(1-z^2)^2};$$

this operator is of the regular singular type in $z = 0$. The reduction of the eigenequation $\Pi_\tau(C)f = sf$ to a first order system now takes the form $F = (f, z\, df/dz)$, as in the classical theory.

By a variation on the monodromy arguments of the classical one variable theory the following lemma can be proved. If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, we write $\lambda_\alpha := \lambda(H_\alpha)$; thus $\lambda = \sum_{\alpha\in\Delta} \lambda_\alpha \alpha$. Moreover, if $m \in \mathbb{N}\Delta$ $\xi \in \mathfrak{a}_{\mathbb{C}}^*$, we define multi-valued holomorphic functions on $(D^*)^\Delta$ by the formulas:

$$(\log z)^m := \prod_{\alpha\in\Delta} (\log z_\alpha)^{m_\alpha}, \qquad z^\xi := \prod_{\alpha\in\Delta} (z_\alpha)^{\xi_\alpha}.$$

Lemma 3.14. *Every solution of (24) is a multi-valued holomorphic functions on $(D^*)^\Delta$ of the form*

$$\sum_{\xi,m} (\log z)^m\, z^\xi\, F_{\xi,m}(z), \tag{25}$$

where (ξ, m) ranges over a finite subset of $\mathfrak{a}_{\mathbb{C}}^* \times \mathbb{N}\Delta$, and where the $F_{\xi,m}$ are $(E^M)^n$-valued holomorphic functions on D^Δ.

If $m \in \mathbb{N}\Delta$, we write $|m| = \sum_{\alpha\in\Delta} m_\alpha$ and

$$(\log a)^m = \prod_{\alpha\in\Delta} \alpha(\log a)^{m_\alpha};$$

thus $z \mapsto \log \underline{a}(z)^m$ a branch over $]0,1[^\Delta$ of the multi-valued holomorphic function $z \mapsto (\log z)^m$. Similarly, $z \mapsto \underline{a}(z)^{-\xi}$ is a branch of $z \mapsto z^\xi$, for $\xi \in \mathfrak{a}_{\mathbb{C}}^*$.

Proposition 3.15. *There exists a finite set $X \subset \mathfrak{a}_{\mathbb{C}}^*$ and a $d \in \mathbb{N}$ such that every $\varphi \in \mathcal{A}(G, \tau, I)$ admits an absolutely convergent expansion of the form:*

$$\varphi(a) = \sum_{\substack{\xi\in X-\mathbb{N}\Delta \\ |m|\le d}} (\log a)^m a^\xi c_{\xi,m} \qquad (a \in A^+) \tag{26}$$

with uniquely determined coefficients $c_{\xi,m} \in E^M$. (Here $X - \mathbb{N}\Delta$ denotes the collection of elements $\xi - \mu$ ($\xi \in X$, $\mu \in \mathbb{N}\Delta$).)

Proof. As before we denote the restriction of φ to A^+ by f. In the z variables the function $F = (D_1 f, \dots, D_n f)$ has an expression of the form (25). In particular its first component $D_1 f = f$ does. Expanding the holomorphic functions $F_{\xi,m}$ into power series around 0 and rewriting the resulting series in terms of functions of the form $(\log a)^m a^\xi$ on A^+ one obtains existence of the above expansion for f.

We also give a sketch of the argument that establises uniqueness of the expansion; it is in the spirit of [3]. In the following we put $c_{\xi,m} = 0$ when $\xi \notin X - \mathbb{N}\Delta$ or $|m| > d$. There exists a set $S \subset \mathfrak{a}_{\mathbb{C}}^*$ such that $X - \mathbb{N}\Delta \subset S - \mathbb{N}\Delta$ and for $s, s' \in S$ we have $s - s' \in \mathbb{Z}\Delta \Rightarrow s = s'$. For

$s \in S$ put $f_{s,m}(a) = \sum_{\mu \in \mathbb{N}\Delta} c_{s-\mu,m} a^{-\mu}$. Then the series for $f_{s,m}$ converges absolutely on A^+, and hence the corresponding power series $\sum_\mu z^\mu c_{s-\mu,m}$ converges absolutely on D^Δ. It follows that the $f_{s,m}$ may be viewed as holomorphic functions on D^Δ. Moreover, in the coordinates z the function f is given by the finite sum

$$f(z) = \sum_{\substack{s \in S \\ |m| \le d}} (\log z)^m z^{-s} f_{s,m}(z) \tag{27}$$

on $]0,1[^\Delta$, with real valued branches for the occurring multi-valued functions. The function $F = (D_1 f, \ldots, D_n f)$ satisfies the system (24). It follows that F, hence $f = F_1$ admits a multi-valued analytic extension to $(\Delta^*)^n$; by analytic continuation the expression (27) holds on $(\mathcal{D}^*)^n$ as well. From the monodromy behavior around the coordinate axes $z_\alpha = 0$ it now follows that an expression like (27) is uniquely determined (once S is prescribed). It follows that the coefficients $c_{\xi,m}$ are unique. $\square$

Remark 3.16. It the above proof the series (26) is rewritten as (27). The occurring functions $f_{s,m}$ are holomorphic on $\mathcal{O}(D^\Delta)$, hence admit power series expansions on D^Δ. It follows from this that the series in (26) converges in a much stronger sense than stated in the lemma. In particular the convergence allows term by term application of differential operators from $\mathcal{O}(D^\Delta) \otimes S(\mathfrak{a})$.

Let $\varphi \in \mathcal{A}(G, \tau, I)$; an element $\xi \in X - \mathbb{N}\Delta$ for which there exists an $m \in \mathbb{N}\Delta$ such that $c_{\xi,m} \neq 0$ is called an exponent for φ (along A^+). The set of all exponents of φ is denoted by $\mathcal{E}(\varphi)$. Let the partial ordering $\preceq$ on $\mathfrak{a}^*_\mathbb{C}$ be defined by

$$\xi_1 \preceq \xi_2 \iff \xi_2 - \xi_1 \in \mathbb{N}\Delta.$$

The $\preceq$-maximal elements in $\mathcal{E}(\varphi)$ are called the leading exponents of φ; the set of these is denoted by $\mathcal{E}_L(\varphi)$.

Theorem 3.17. *There exists a finite set $\mathcal{E}_I \subset \mathfrak{a}^*_\mathbb{C}$, only depending on the cofinite ideal I, such that $\mathcal{E}_L(\varphi) \subset \mathcal{E}_I$, for every τ and all $\varphi \in \mathcal{A}(G, \tau, I)$.*

Proof. (idea). There exists a system of polynomial equations, only depending on I, which is the appropriate analogue of the classical indicial equation. The (finite) set of solutions for this system determines $\mathcal{E}_I$. $\square$

3.5 The subrepresentation theorem

Let $(\pi, \tilde{V}), V, V'$ be as in the beginning of Section 3.2. The asymptotic theory of the previous sections implies the following.

Corollary 3.18. *There exist unique bilinear maps $c_{\xi,m}: V \times V' \to \mathbb{C}$ $(\xi \in \mathcal{E}_I - \mathbb{N}\Delta, m \in \mathbb{N}^\Delta)$ such that, for $v \in V, v' \in V'$:*

$$m_{v,v'}(a) = \sum_{\xi,m} c_{\xi,m}(v, v')\, a^\xi \,(\log a)^m \qquad (a \in A^+)$$

with absolutely converging series.

To $v \in V$ we may now associate the set of exponents

$$\mathcal{E}(v) := \{\xi \in \mathcal{E}_I - \mathbb{N}\Delta \mid \exists m \in \mathbb{N}\Delta\ \exists v' \in V' \colon c_{\xi,m}(v, v') \neq 0\}.$$

The union $\mathcal{E}(V) := \cup_{v \in V} \mathcal{E}(v)$ of these sets is called the set of exponents of the Harish-Chandra module V; its $\preceq$-maximal elements are called the leading exponents of V. The set of these is denoted by $\mathcal{E}_L(V)$.

Lemma 3.19. *Let $v' \in V'$. Then for $\xi \in \mathcal{E}_L(V)$ the map $\gamma \colon v \mapsto \sum_m c_{\xi,m}(v, v') a^\xi (\log a)^m$ factors to a non-trivial $\mathfrak{a}$-module homomorphism $V/\bar{\mathfrak{n}}V \to C^\infty(A)$.*

Proof. If $v \in V$, $X \in \mathfrak{g}$, then

$$m_{Xv,v'}(a) = \left.\frac{d}{dt}\right|_{t=0} m_{v,v'}(a \exp tX) = m_{v,-Ad(a)Xv'}(a). \tag{28}$$

If $\alpha \in \Sigma^+$, $X \in \mathfrak{g}_{-\alpha}$, then $\mathrm{Ad}(a)X_\alpha = a^{-\alpha} X_\alpha$, so $m_{Xv,v'}(a) = -a^{-\alpha} m_{v,Xv'}$. Hence if $X \in \bar{\mathfrak{n}}$, no exponent $\xi \in \mathcal{E}_L(V)$ occurs in $m_{Xv,v'}$. Therefore each map $v \mapsto c_{\xi,m}(v, v')$ is zero on $\bar{\mathfrak{n}}V$. From (28) and the fact that term by term differentiations are allowed (Remark 3.16) it follows that the map γ is an $\mathfrak{a}$-module homomorphism. □

The group M normalizes $\bar{\mathfrak{n}}$, hence $V/\bar{\mathfrak{n}}V$ is a $(\mathfrak{m} + \mathfrak{a}, M)$-module.

Lemma 3.20. *The $(\mathfrak{m} + \mathfrak{a}, M)$-module $V/\bar{\mathfrak{n}}V$ is non-trivial and finite dimensional.*

Proof. The non-triviality follows from Lemma 3.19, the finite dimensionality from Lemma 3.9. □

The following result is due to Casselman. Let $\bar{Q} = MA\bar{N}$ be the minimal parabolic subgroup of G opposite to $Q = MAN$.

Theorem 3.21. (Subrepresentation theorem). *Let V be irreducible. Then there exist $\sigma \in \widehat{M}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ such that V occurs as a $(\mathfrak{g}, K)$-submodule of $\mathrm{Ind}_{\bar{Q}}^G(\sigma \otimes \lambda \otimes 1)_K$.*

Proof. Fix an irreducible quotient H of the finite dimensional non-trivial $(\mathfrak{m} + \mathfrak{a}, M)$-module $V/\bar{\mathfrak{n}}V$. Let $\sigma \otimes (\lambda - \rho_Q)$ be the associated representation of MA. Then

$$\mathrm{Hom}_{\mathfrak{m}+\mathfrak{a},M}(V/\bar{\mathfrak{n}}V, \sigma \otimes (\lambda - \rho_Q)) \neq 0,$$

hence by the Frobenius theorem formulated below, there exists a non-trivial $(\mathfrak{g}, K)$-module homomorphism T from V into $\mathrm{Ind}_{\bar{Q}}^G(\sigma \otimes \lambda \otimes 1)_K$. Since V is irreducible, $\ker T = 0$. □

Lemma 3.22. (Frobenius reciprocity). *Let σ be an irreducible representation of M, and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Let V be a $(\mathfrak{g}, K)$-module. Then*

$$\mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{\bar{Q}}^G(\sigma \otimes \lambda \otimes 1)) \simeq \mathrm{Hom}_{\mathfrak{m}+\mathfrak{a},M}(V/\bar{\mathfrak{n}}V, \sigma \otimes (\lambda - \rho_Q)).$$

Proof. One readily checks that $T \mapsto \mathrm{ev}_e \circ T$ provides the isomorphism (see also the proof of Lemma 1.2). □

Remark 3.23. Let V be an irreducible admissible $(\mathfrak{g}, K)$-module as above. It follows from Frobenius reciprocity that the collection of parameters $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ such that $V \hookrightarrow \mathrm{Ind}_Q^G(\sigma \otimes (\lambda - \rho) \otimes 1)$ for some σ is equal to the collection $\mathcal{E}(V, \bar{\mathfrak{n}})$ of $\mathfrak{a}$-weights in $V/\bar{\mathfrak{n}}V$. Moreover, it follows from Lemma 3.19 that

$$\mathcal{E}_L(V) \subset \mathcal{E}(V, \bar{\mathfrak{n}}).$$

This inclusion is proper in general. However, it can be shown that $\mathcal{E}_L(V)$ equals the set of $\preceq$-maximal elements in $\mathcal{E}(V, \bar{\mathfrak{n}})$. Thus we may regard $\mathcal{E}(V, \bar{\mathfrak{n}})$ as a set of algebraic asymptotic exponents associated with the module V.

Remark 3.24. The observation that the $\mathfrak{a}$-weights of $V/\bar{\mathfrak{n}}V$ play a role in the asymptotics of the (matrix coefficients) of the module V is the starting point of another approach to asymptotics, via the theory of Jacquet modules. We refer the reader to [2], [17], [8] for more details.

Remark 3.25. It can be shown that every finitely generated admissible $(\mathfrak{g}, K)$-module V is the module of K-finite vectors for some G-module $\tilde{V}$ (see e.g. [17], 4.2.4). For more information on such 'globalizations' of V we refer the reader to [17], [15].

It is important to note that the K-finite matrix coefficients are independent of the globalization under consideration. See [3] for details.

3.6 Asymptotic behavior along the walls

The asymptotic theory along A^+ described so far is strong enough to obtain uniform estimates for matrix coefficients on subsets of $\mathrm{cl}\,(A^+)$ of the form

$$A^+(R) = \{a \in A \mid a^\alpha \geq R \ \ (\alpha \in \Delta)\},$$

with $R > 1$. In this section we shall briefly indicate how such estimates may be extended to $A^+(1) = \mathrm{cl}\,(A^+)$, by using converging expansions 'along the walls.'

Let $\varphi \in \mathcal{A}(G, \tau, I)$ (see Section 3.3). We will say that a functional $\omega \in \mathfrak{a}^*$ dominates $\mathcal{E}_L(\varphi)$ on $\mathfrak{a}^+$ if for each $\xi \in \mathcal{E}_L(\varphi)$:

$$\mathrm{Re}\,\xi \leq \omega \quad \text{on} \quad \mathfrak{a}^+.$$

One readily sees that the above estimate is equivalent to the collection of estimates

$$(\mathrm{Re}\,\xi)_\alpha \leq \omega_\alpha \qquad (\alpha \in \Delta).$$

Lemma 3.26. *Let $\varphi \in \mathcal{A}(G, \tau, I)$ and assume that ξ dominates $\mathcal{E}_L(\varphi)$. Then there exist constants $d \in \mathbb{N}$, $C > 0$ such that*

$$\varphi(a) \leq C(1 + |\log a|)^d \, a^\xi \qquad (a \in \mathrm{cl}\,(A^+)). \tag{29}$$

Proof. (Some ideas). This result can be established by using asymptotic expansions along *walls* of A^+, using the method of [3], of which we shall only give a sketch.

To $F \subset \Delta$ we associate the wall

$$A_F^+ := \{a \in A \mid a^\alpha = 1 \ \ (\alpha \in F), \quad a^\beta > 1 \ \ (\beta \in \Delta \setminus F)\}.$$

Note that $\mathrm{cl}\,(A^+)$ is the disjoint union of the walls A_F^+ $(F \subset \Delta)$.

By a 'grouping of terms' (see [3]) one may rewrite the expansion (26) of the function φ as an 'expansion along the wall A_F^+.' For every $\epsilon > 0$ this expansion converges uniformly absolutely on the set

$$A^+(F,\epsilon) := \{a \in A \mid 1 \leq a^\alpha \leq 1+\epsilon \;\; (\alpha \in F), \quad a^\beta > 1+\epsilon \;\; (\beta \in \Delta \setminus F)\}.$$

The leading exponents of φ along A_F^+ are elements of $\mathfrak{a}_{F\mathbb{C}}^*$. From the grouping of terms procedure one reads off that on $\mathfrak{a}_F^+$ the leading exponents are dominated by the element $\xi|\mathfrak{a}_F$. From this one obtains, for every $\epsilon > 0$, that the estimate (29) holds on $A^+(F,\epsilon)$ (with a constant C depending on ϵ.) The proof is completed by the observation that for every fixed $\epsilon > 0$ the sets $A^+(F,\epsilon)$ $(F \subset \Delta)$ cover $\mathrm{cl}\,(A^+)$. $\square$

4 Tempered representations

In this section we give several characterizations of tempered representations in terms of their asymptotic exponents.

4.1 The discrete series

Our first goal is to give different characterizations of square integrable representations, i.e. representations whose matrix coefficients belong to the space $L^2(G)$ of functions that are square integrable with respect to a bi-invariant Haar measure on G (which we assume to be fixed from now on).

The left and right regular representations (denoted L and R respectively) of G in $L^2(G)$ are unitary, by invariance of the Haar measure. We state the following lemma without proof.

Lemma 4.1. *If π is an irreducible unitary representation of G then the following conditions are equivalent:*

(a) *π is unitarily equivalent to an irreducible closed subrepresentation of $(R, L^2(G))$.*

(b) *π has a matrix coefficient that belongs to $L^2(G)$.*

(c) *Every matrix coefficient of π belongs to $L^2(G)$.*

If an irreducible unitary representation π of G satisfies any of the above conditions, it is said to belong to the discrete series of G.

Our next goal is to characterize discrete series representations in terms of their leading exponents along A^+. For this we need the following lemma. Let dk be normalized Haar measure on K. We put $m_\alpha = \dim(\mathfrak{g}_\alpha)$, for $\alpha \in \Sigma$, and define the function $J\colon A \to [0,\infty[$ by

$$J(a) = \prod_{\alpha \in \Sigma^+} |a^\alpha - a^{-\alpha}|^{m_\alpha}. \tag{30}$$

Lemma 4.2. *There exists a (unique) choice of Haar measure da on A such that for all $f \in C_c(G)$:*

$$\int_G f(x)dx = \int_K \int_{A^+} \int_K f(k_1 a k_2)\, J(a)\, dk_1 da dk_2,$$

This lemma can be proved by substitution of variables; the function J occurs as a Jacobian.

For $R > 1$ we put

$$A^+(R) = \{a \in A \mid a^\alpha > R \ \ (\alpha \in \Delta)\}.$$

From (30) we readily see that for every $R > 1$ there exists a constant $C_R > 0$ such that

$$C_R\, a^{2\rho} \leq J(a) \leq a^{2\rho} \qquad (a \in A^+(R)). \tag{31}$$

In the following we assume that G has compact center. Then Δ, the collection of simple roots in Σ^+, is a basis of $\mathfrak{a}^*$. Let $\omega_\alpha \in \mathfrak{a}^*$ $(\alpha \in \Delta)$ be the associated fundamental weights, i.e. $2\langle \omega_\alpha\,,\, \beta\rangle / \langle \beta\,,\, \beta\rangle = \delta_{\alpha\beta}$ for $\alpha, \beta \in \Delta$.

Proposition 4.3. *Let V be the Harish-Chandra module of an irreducible unitary representation π. Then the following conditions are equivalent:*

(a) *π belongs to the discrete series of G;*

(b) *each $\xi \in \mathcal{E}_L(V)$ satisfies the estimates*

$$\langle \mathrm{Re}\, \xi + \rho\,,\, \omega_\alpha \rangle < 0 \quad \text{for every} \quad \alpha \in \Delta; \tag{32}$$

(c) *each $\xi \in \mathcal{E}(V, \bar{\mathfrak{n}})$ satisfies the estimates (32).*

Proof. (sketch) It follows from Remark 3.23 that (b) $\Longleftrightarrow$ (c). We shall sketch the proof of (a) $\Longleftrightarrow$ (b).

Assume (a) and let $\xi \in \mathcal{E}_L(V)$. Then $\xi \in \mathcal{E}_L(\varphi)$ for a spherical function φ associated with π as in the end of Section 3.3. Now $\varphi \in L^2(G)$, since π belongs to the discrete series. Fix $R > 1$. Then from Lemma 4.2 and the estimate (31) we see that the function $\psi\colon A^+(R) \to \mathbb{R}$ defined by $\psi(a) = \|\varphi(a)\|^2 a^{2\rho}$ has a bounded L^1-norm $\|\psi\|_1$ with respect to da (note that $\|\varphi\|$ is bi-K-invariant).

Fix $H \in \mathrm{cl}\,(\mathfrak{a}^+)$, $H \neq 0$. Then for every $a \in A^+(R)$ the ray $l_{a,H} = a\exp(\mathbb{R}_+ H)$ is contained in $A^+(R)$. The expansion (26) describes the asymptotic behavior of $\psi(a\exp(tH))$ as $t \to \infty$, locally uniformly in $a \in A^+(R)$. It follows that $\psi(a\exp(tH)) \sim C_a e^{tr}$, where $r = 2\max_{\eta \in \mathcal{E}(\varphi)} \mathrm{Re}\,(\eta + \rho)(H)$, and where $C_a > 0$ is a constant which may be chosen locally independent of a. If $\mathcal{K}$ is a compact subset of A^+ then the L^1 norm of ψ's restriction to the union of the rays $l_{a,H}(a \in \mathcal{K})$ is bounded by $\|\psi\|_1$. This implies that $r < 0$. It follows from the above that $\mathrm{Re}\,\xi + \rho < 0$ on $\mathrm{cl}\,(\mathfrak{a}^+) \setminus \{0\}$. Hence (32).

We have now established the implication (a) $\Rightarrow$ (b) by using the uniform absolute convergence of the series (29) on sets of the form $A^+(R)$, with $R > 1$. To prove the converse implication we need to invoke Lemma 3.26, obtained from 'asymptotics along the walls.' Assume (b). Then we may fix $r > 0$ sufficiently small so that for all $\xi \in \mathcal{E}_L(V)$:

$$\mathrm{Re}\,(\xi + \rho)(H) \leq -r\rho(H) \qquad (H \in \mathrm{cl}\,(\mathfrak{a}^+)).$$

In other words, $-(1 + r)\rho$ dominates $\mathcal{E}_L(V)$. Let φ be any τ-spherical function associated with π as in the end of Section 3.3. Then $\mathcal{E}(\varphi) \subset \mathcal{E}(V)$, hence by Lemma 3.26 it follows that

$$J(a)\|\varphi(a)\|^2 \leq a^{2\rho}\|\varphi(a)\|^2 \leq C(1 + |\log a|)^{2d} a^{-2r\rho} \qquad (a \in A^+),$$

with $C > 0$ a suitable constant. In view of Lemma 4.2 this implies that $\varphi \in L^2(G)$. By Lemma 4.1 we conclude that π belongs to the discrete series.

4.2 Tempered representations

An admissible representation $(\pi, \tilde{V})$ is said to be tempered if its K-finite matrix coefficients belong to $L^{2+\epsilon}(G)$, for every $\epsilon > 0$.

Let V be the Harish-Chandra module associated with π. Then by an asymptotic analysis as in the previous section we conclude that π is tempered if and only if every $\xi \in \mathcal{E}(V)$ satisfies the estimates: $\langle (2+\epsilon)\mathrm{Re}\,\xi + 2\rho\,,\,\omega_\alpha\rangle < 0 \;\; (\alpha \in \Delta)$. From this we readily obtain:

Lemma 4.4. *Let π be an admissible representation with associated Harish-Chandra module V. Then the following conditions are equivalent.*

(a) *π is tempered;*

(b) *every $\xi \in \mathcal{E}(V)$ satisfies the estimates:*

$$\langle \mathrm{Re}\,\xi + \rho\,,\,\omega_\alpha\rangle \leq 0 \qquad (\alpha \in \Delta); \tag{33}$$

(c) *every $\xi \in \mathcal{E}(V, \bar{\mathfrak{n}})$ satisfies the estimates (33).*

4.3 Embedding of tempered representations

The following result, due to Langlands ([13], Lemma 4.10), is the main result of this section. For $F \subset \Delta$ let $\bar{Q}_F = \theta Q_F$ be the parabolic subgroup opposite to the standard parabolic subgroup Q_F.

Theorem 4.5. *Let π be an irreducible tempered representation, with associated Harish-Chandra module V. Then there exists a standard parabolic subgroup Q_F, a discrete series representation σ of M_F and a $\lambda \in i\mathfrak{a}_F^*$, such that the $(\mathfrak{g}, K)$-module V allows an embedding:*

$$V \hookrightarrow \mathrm{Ind}_{\bar{Q}_F}^G(\sigma \otimes \lambda \otimes 1).$$

We shall explain the main ideas that enter the proof of this result. Let us first recall some facts about standard parabolic subgroups, meanwhile fixing notation. If $F \subset \Delta$, let ${}^*\mathfrak{a}_F$ be the B-orthocomplement of $\mathfrak{a}_F$ in $\mathfrak{a}$ (this is a maximal abelian subspace of $\mathfrak{m}_F \cap \mathfrak{p}$). Then $\mathfrak{a} = {}^*\mathfrak{a}_F \oplus \mathfrak{a}_F$. Via B we identify the dual spaces ${}^*\mathfrak{a}_F^*$ and $\mathfrak{a}_F^*$ with subspaces of $\mathfrak{a}^*$; thus ${}^*\mathfrak{a}_F^*$ is the space of linear functionals in $\mathfrak{a}^*$ that vanish on $\mathfrak{a}_F$, and vice versa. These spaces and embeddings are naturally complexified.

The set $\Sigma_F := \mathbb{Z}F \cap \Sigma$ is naturally identified with the system $\Sigma(\mathfrak{m}_F, {}^*\mathfrak{a}_F)$ of restricted roots of ${}^*\mathfrak{a}_F$ in $\mathfrak{m}_F$. We note that

$${}^*\mathfrak{a}_F^* = \mathrm{span}(F), \qquad \mathfrak{a}_F^* = \mathrm{span}\{\omega_\beta \mid \beta \in \Delta \setminus F\}.$$

Let ${}^*\mathfrak{n}_F := \mathfrak{n} \cap \mathfrak{m}_F$. Then $\mathfrak{m} + {}^*\mathfrak{a}_F + {}^*\mathfrak{n}_F$ is a minimal parabolic subalgebra of $\mathfrak{m}_F$; the associated ρ is denoted by ${}^*\rho_F$. Note that $\mathfrak{n} = {}^*\mathfrak{n}_F \oplus \mathfrak{n}_F$ as a direct sum of vector spaces. Since $\mathfrak{a}_F$ centralizes ${}^*\mathfrak{n}_F$, whereas $\mathfrak{m}_F \supset {}^*\mathfrak{a}_F$ acts traceless on $\mathfrak{n}_F$, we have

$$\rho = {}^*\rho_F + \rho_F.$$

Lemma 3.20 has the following generalization which we state without proof.

Lemma 4.6. *Let V be a finitely generated admissible $(\mathfrak{g}, K)$-module. Then $V/\bar{\mathfrak{n}}_F V$ is a non-trivial, finitely generated, admissible $(\mathfrak{m}_{1F}, K_F)$-module.*

As in the proof of the subrepresentation theorem a key role is played by the following generalization of the Frobenius reciprocity result, for σ an admissible representation of M_F, $\lambda \in \mathfrak{a}^*_{F\mathbb{C}}$. Let $V_{\sigma\otimes(\lambda-\rho_F)}$ denote the representation space of σ, equipped with the action of A_F by the character $\lambda - \rho_F$.

Proposition 4.7. (Frobenius reciprocity). *Let V be a $(\mathfrak{g}, K)$-module. Then*

$$\mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{Q_F}^G(\sigma \otimes \lambda \otimes 1) \simeq \mathrm{Hom}_{\mathfrak{m}_{1F},K_F}(V/\bar{\mathfrak{n}}_F V, V_{\sigma\otimes(\lambda-\rho_F)}).$$

Proof. As before the map $T \mapsto \mathrm{ev}_e \circ T$ provides the isomorphism. □

Thus, in order to prove the main result, we must find a subset $F \subset \Delta$ such that $V/\bar{\mathfrak{n}}_F V$ has a quotient that is square integrable. For this we need the following lemmas.

Since $V/\bar{\mathfrak{n}}_F V$ is a finitely generated admissible $(\mathfrak{m}_{1F}, K_F)$-module, the central subalgebra $\mathfrak{a}_F$ of $\mathfrak{m}_{1F}$ acts globally finitely. Hence if $\mu \in \mathfrak{a}^*_{F\mathbb{C}}$, then the associated generalized weight space $(V/\bar{\mathfrak{n}}_F V)_\mu$ is a finitely generated admissible sub $(\mathfrak{m}_{1F}, K_F)$-module of $V/\bar{\mathfrak{n}}_F V$.

Lemma 4.8. *The algebra $\mathfrak{a}_F$ acts (globally) finitely on $V/\bar{\mathfrak{n}}_F V$, with a set of generalized weights equal to $\mathcal{E}(V, \bar{\mathfrak{n}})|\mathfrak{a}_F$. If $\mu \in \mathcal{E}(V, \bar{\mathfrak{n}})|\mathfrak{a}_F$, then*

$$\mathcal{E}((V/\bar{\mathfrak{n}}_F V)_\mu, {}^*\bar{\mathfrak{n}}_F) + \mu \subset \mathcal{E}(V, \bar{\mathfrak{n}}).$$

Proof. From the direct sum decomposition $\bar{\mathfrak{n}} = {}^*\bar{\mathfrak{n}}_F + \bar{\mathfrak{n}}_F$ it follows that

$$0 \to {}^*\bar{\mathfrak{n}}_F(V/\bar{\mathfrak{n}}_F V) \to V/\bar{\mathfrak{n}}_F V \to V/\bar{\mathfrak{n}} V \to 0.$$

is a short exact sequence of $\mathfrak{a}$-modules. The assertion about the weights follows from inspection of this sequence. □

Remark 4.9. The set $\mathcal{E}(V, \bar{\mathfrak{n}})|\mathfrak{a}_F$ governs the asymptotic exponents of V along the wall A_F^+. This is analogous to what was said in Remark 3.23.

Proof of Theorem 4.5. By Frobenius reciprocity it suffices to find $F \subset \Delta$, $\lambda \in i\mathfrak{a}^*_{F\mathbb{C}}$ and an irreducible quotient U of the $(\mathfrak{m}_F, K_F)$-module $(V/\bar{\mathfrak{n}}_F V)_{\lambda-\rho_F}$ which is square integrable (see also the proof of Theorem 3.21). By Lemma 4.1 applied to the group M_F, the requirement that U is square integrable is equivalent to the requirement that every ${}^*\eta \in \mathcal{E}(U, {}^*\bar{\mathfrak{n}}_F)$ satisfies the estimates

$$\langle \mathrm{Re}\, {}^*\eta + {}^*\rho_F \,,\, \omega_\alpha \rangle < 0 \quad (\alpha \in F). \tag{34}$$

For each $\xi \in \mathcal{E}(V, \bar{\mathfrak{n}})$, we define

$$\Delta_\xi = \{\alpha \in \Delta \mid \langle \mathrm{Re}\, \xi + \rho \,,\, \omega_\alpha \rangle < 0\}.$$

Fix $\xi \in \mathcal{E}(V, \bar{\mathfrak{n}})$ such that Δ_ξ has a minimal number of elements, and put $F = \Delta_\xi$. Then

$$\langle \mathrm{Re}\, \xi + \rho \,,\, \omega_\beta \rangle = 0 \qquad (\beta \in \Delta \setminus F),$$

by temperedness of π and minimality of Δ_ξ. The weights ω_β $(\beta \in \Delta \setminus F)$ span $\mathfrak{a}_F^*$; hence if we put $\mu = \xi|\mathfrak{a}_F$, then $\operatorname{Re}\mu + \rho_F = (\operatorname{Re}\xi + \rho_F)|\mathfrak{a}_F = 0$, and we see that $\lambda := \mu + \rho_F$ belongs to $i\mathfrak{a}_F^*$.

It follows from Lemma 4.8 that $(V/\bar{\mathfrak{n}}_F V)_\mu \neq 0$. We claim that every ${}^*\eta \in \mathcal{E}((V/\bar{\mathfrak{n}}_F V)_\mu, {}^*\bar{\mathfrak{n}}_F)$ satisfies the estimates (34). To see this, fix such a weight ${}^*\eta \in {}^*\mathfrak{a}_{F\mathbb{C}}^*$ Then by Lemma 4.8 we have ${}^*\eta + \mu \in \mathcal{E}(V, \bar{\mathfrak{n}})$. Now obviously

$$\operatorname{Re}({}^*\eta + \mu + \rho) = \operatorname{Re}({}^*\eta + \lambda + {}^*\rho_F) = \operatorname{Re}({}^*\eta + {}^*\rho_F), \tag{35}$$

hence $\langle \operatorname{Re}({}^*\eta + \mu + \rho), \omega_\beta \rangle = 0$ for $\beta \in \Delta \setminus F$, and we see that $\Delta_{{}^*\eta+\mu} \subset F$. By minimality of $|F|$ the latter inclusion is actually an equality. In view of (35) this implies the estimates (34) and the claim follows.

We now select any irreducible quotient U of the finitely generated admissible $(\mathfrak{m}_F, K_F)$-module $(V/\bar{\mathfrak{n}}_F V)_\mu$. This quotient satifies (34); hence U is square integrable $\square$

The following result asserts that all induced representations occurring in Theorem 4.5 are tempered.

Proposition 4.10. *If $F \subset \Delta$, σ a discrete series representation of M_F, and $\lambda \in i\mathfrak{a}_F^*$, then*

$$\operatorname{Ind}_{Q_F}^G(\sigma \otimes \lambda \otimes 1) \tag{36}$$

is a tempered representation.

It follows from Theorem 4.5 and the above proposition that the irreducible summands of the induced representations (36) *exhaust* the tempered representations. The *classification* of the irreducible tempered representation has been achieved in [11].

5 The Langlands classification

In this section we describe the classification of the irreducible admissible $(\mathfrak{g}, K)$-modules, which is due to R.P. Langlands ([13]).

By Langlands data we shall mean a triple (Q_F, σ, λ) with $F \subset \Delta$, Q_F the associated standard parabolic subgroup, σ an irreducible tempered representation of M_F, and λ an element of $\mathfrak{a}_{F\mathbb{C}}^*$ satisfying:

$$\langle \operatorname{Re}\lambda\,,\,\alpha\rangle > 0 \qquad (\alpha \in \Delta \setminus F).$$

Here we recall that $\mathfrak{a}_{F\mathbb{C}}^*$ is embedded in $\mathfrak{a}_{\mathbb{C}}^*$, in the fashion described in the text following Theorem 4.5.

Theorem 5.1. (Langlands).

(a) *If (Q_F, σ, λ) are Langlands data, then the $(\mathfrak{g}, K)$-module $\operatorname{Ind}_{Q_F}^G(\sigma \otimes \lambda \otimes 1)_K$ has a unique irreducible quotient $J(Q_F, \sigma, \lambda)$ ('the Langlands quotient').*

(b) *Assume $(Q_{F_j}, \sigma_j, \lambda_j)$ are Langlands data for $j = 1, 2$. If the associated Langlands quotients $J(Q_{F_j}, \sigma_j, \lambda_j)$ $(j = 1, 2)$ are equivalent, then $F_1 = F_2$, the representations σ_1 and σ_2 are equivalent, and $\lambda_1 = \lambda_2$.*

(c) *Every irreducible admissible $(\mathfrak{g}, K)$-module is equivalent to a Langlands quotient.*

Remark 5.2. In [13] (text preceding Lemma 3.13), Langlands defines $J = J(Q_F, \sigma, \lambda)$ as the quotient of $I = \mathrm{Ind}_{Q_F}^G(\sigma \otimes \lambda \otimes 1)_K$ by the kernel of the intertwining operator introduced in the lemmas below (see also Cor. 5.8). In [14] it is observed that J is actually the unique irreducible quotient of I.

The rest of this section is devoted to a sketch of some of the main ideas that enter the proof of this theorem. We start with a crucial lemma. From the description of the non-compact picture in Section 2.2 we recall that $\bar{N}_F$ is equipped with a bi-invariant Haar measure $d\bar{n}$.

Lemma 5.3. *Let (Q_F, σ, λ) be Langlands data. Then for every function $f \in (\mathcal{H}_{Q_F,\sigma,\lambda})_K$ and every $x \in G$ the integral*

$$A(\bar{Q}_F, Q_F, \sigma, \lambda)f(x) := \int_{\bar{N}_F} f(x\bar{n})\, d\bar{n} \tag{37}$$

converges absolutely.

This result is proved by careful estimation of the integrand.

Lemma 5.4. *Let (Q_F, σ, λ) be Langlands data. Then the map $A = A(\bar{Q}_F, Q_F, \sigma, \lambda)$ defined by (37) is a non-zero $(\mathfrak{g}, K)$-map*

$$A\colon (\mathcal{H}_{Q_F,\sigma,\lambda})_K \to (\mathcal{H}_{\bar{Q}_F,\sigma,\lambda})_K. \tag{38}$$

Proof. (sketch). In the compact picture (see Section 2.2) this operator is readily seen to be an integral operator with a non-trivial integral kernel; hence the operator is non-trivial.

The following reasoning can be justified by showing that the occurring integrals all converge absolutely. Let $f \in (\mathcal{H}_{Q_F,\sigma,\lambda})_K$, $x \in G$. Then for $(m, a, \bar{n}_0) \in M_F \times A_F \times \bar{N}_F$ we have:

$$\begin{aligned}
Af(xma\bar{n}_0) &= \int_{\bar{N}_F} f(xma\bar{n}_0\bar{n})\, d\bar{n} \\
&= \int_{\bar{N}_F} f(xma\bar{n}(ma)^{-1}ma)\, d\bar{n} \\
&= a^{2\rho_F} \int_{\bar{N}_F} f(x\bar{n}'ma)\, d\bar{n}' \\
&= a^{-\lambda+\rho_F}\sigma(m)^{-1} \int_{\bar{N}_F} f(x\bar{n})\, d\bar{n} \\
&= a^{-\lambda+\rho_F}\sigma(m)^{-1} Af(x).
\end{aligned}$$

In the above sequence of equations the second equality follows from the left invariance of $d\bar{n}$. The endomorphism $\mathrm{Ad}(ma)$ normalizes $\bar{\mathfrak{n}}_F$, and has determinant $a^{-2\rho_F}$. Hence the third equality follows by the substitution of variables $\bar{n}' = ma\bar{n}(ma)^{-1}$. Finally, the fourth equality follows from the transformation properties of f under the action by Q_F (on the right).

It follows that A maps $(\mathcal{H}_{Q_F,\sigma,\lambda})_K$ into $(\mathcal{H}_{\bar{Q}_F,\sigma,\lambda})_K$. The $(\mathfrak{g}, K)$ actions are on the left; hence formally the $(\mathfrak{g}, K)$-equivariance of A is obvious. □

Remark 5.5. The operator $A(\bar{Q}_F, Q_F, \sigma, \lambda)$ is called the standard intertwining operator from $\mathrm{Ind}_{Q_F}^G(\sigma \otimes \lambda \otimes 1)$ to $\mathrm{Ind}_{\bar{Q}_F}^G(\sigma \otimes \lambda \otimes 1)$. In a well defined sense this operator has a meromorphic continuation in the parameter $\lambda \in \mathfrak{a}_{F\mathbb{C}}^*$. For more information on the standard intertwining operator and its role in harmonic analysis we refer the reader to [10], [7], [16].

The following result is crucial in the proof of the Langlands classification. It relates the intertwining operator to the asymptotic behavior of matrix coefficients. We assume that (Q_F, σ, λ) are Langlands data.

Proposition 5.6. *Let $f \in (\mathcal{H}_{Q_F,\sigma,\lambda})_K$, $g \in (\mathcal{H}_{Q_F,\sigma,-\bar{\lambda}})_K$. Then for $X \in \mathfrak{a}_F^+$:*

$$\lim_{t\to\infty} e^{(-\lambda+\rho_F)(tX)} \langle \pi(\exp tX) f\,,\, g\rangle = \langle [A(\bar{Q}_F, Q_F, \sigma, \lambda) f](e)\,,\, g(e)\rangle_\sigma. \tag{39}$$

Proof. (sketch). Put $a_t = \exp tX$. Then by equivariance of the pairing $\mathcal{H}_{\sigma,\lambda} \times \mathcal{H}_{\sigma,-\bar{\lambda}} \to \mathbb{C}$ and by Remark 2.3 the expression under the limit in (39) may be rewritten as:

$$a_t^{-\lambda+\rho_F} \langle f\,,\, \pi(a_t^{-1}) g\rangle = \int_{\bar{N}_F} \langle f(\bar{n})\,,\, g(a_t \bar{n} a_t^{-1})\rangle_\sigma \, d\bar{n}.$$

If $t \to \infty$, then the integrand on the right-hand side tends (pointwise) to $\langle f(\bar{n})\,,\, g(e)\rangle_\sigma$. The result now follows by an application of the dominated convergence theorem. □

Corollary 5.7. *Suppose that $U \subset (\mathcal{H}_{Q_F,\sigma,\lambda})_K$ is a proper $(\mathfrak{g}, K)$-submodule. Then $U \subset \ker A(\bar{Q}_F, Q_F, \sigma, \lambda)$.*

Proof. The orthocomplement $U^\perp$ of U in $(\mathcal{H}_{Q_F,\sigma,-\bar{\lambda}})_K$ (with respect to the non-degenerate pairing (8)) is a non-trivial $(\mathfrak{g}, K)$-submodule. Let $f \in U$, $g \in U^\perp$. Then for $X \in \mathfrak{a}_F^+$, $t \in \mathbb{R}$ we have $\pi(\exp tX) f \in \bar{U}$ (the closure in $\mathcal{H}_{Q_F,\sigma,\lambda}$), hence $\pi(\exp tX) f \perp g$. By taking the limit for $t \to \infty$ it follows that

$$\langle Af(e)\,,\, g(e)\rangle_\sigma = 0. \tag{40}$$

The map $\mathrm{ev}_e\colon g \mapsto g(e)$ from $(\mathcal{H}_{Q_F,\sigma,-\bar{\lambda}})_K$ to $(\mathcal{H}_\sigma)_{K_F}$ is a homomorphism of $(\mathfrak{m}_F, K_F)$-modules. Its image $\mathcal{H}_0$ is either 0 or $(\mathcal{H}_\sigma)_{K_F}$, by irreducibility of σ. If $g \in U^\perp \setminus \{0\}$, then $g(k) \neq 0$ for some $k \in K$. Now $\pi(k^{-1}) g \in U^\perp$ and $\mathrm{ev}_e(\pi(k^{-1})g) = g(k) \neq 0$. Hence $\mathcal{H}_0$ is non-trivial; it must therefore be equal to $\mathcal{H}_\sigma$. From (40) it now follows that $Af(e) = 0$ for all $f \in U$. By K_F-equivariance of A this implies that $Af = 0$ on K_F, hence on G. Hence $Af = 0$ for all $f \in U$. □

As an immediate consequence of the last corollary we obtain:

Corollary 5.8. *The kernel of the operator (38) is the unique maximal proper submodule of the $(\mathfrak{g}, K)$-module $(\mathcal{H}_{Q_F,\sigma,\lambda})_K$.*

Assertion (a) of Theorem 5.1 is an immediate consequence of this corollary. Assertion (b) follows from a careful analysis of the asymptotic behavior of the matrix coefficients of induced representations.

Corollary 5.9. *The $(\mathfrak{g}, K)$-module $(\mathcal{H}_{\bar{Q}_F,\sigma,\lambda})_K$ has a unique irreducible submodule. The standard intertwining operator $A = A(\bar{Q}_F, Q_F, \sigma, \lambda)$ factorizes to an isomorphism from $J(Q_F, \sigma, \lambda)$ onto this unique irreducible submodule.*

Proof. We first note that $(\bar{Q}_F, \sigma, -\bar{\lambda})$ are Langlands data with respect to the positive system $-\Sigma^+$. By assertion (a) of Theorem 5.1 for these Langlands data, it follows that $(\mathcal{H}_{\bar{Q}_F,\sigma,-\bar{\lambda}})_K$ has a unique irreducible quotient. By non-degeneracy and equivariance of the sesquilinear pairing $\mathcal{H}_{\bar{Q}_F,\sigma,\lambda} \times \mathcal{H}_{\bar{Q}_F,\sigma,-\bar{\lambda}} \to \mathbb{C}$ it follows that $(\mathcal{H}_{\bar{Q}_F,\sigma,\lambda})_K$ has a unique irreducible submodule.

By Corollary 5.8 the intertwining operator A factorizes to an isomorphism from $J(Q_F, \sigma, \lambda)$ onto a submodule of $(\mathcal{H}_{\bar{Q}_F,\sigma,\lambda})_K$. The latter submodule must be irreducible, since $J(Q_F, \sigma, \lambda)$ is. The final assertion now follows. $\square$

Let V be an irreducible admissible $(\mathfrak{g}, K)$-module. Then by an asymptotic analysis in the spirit of the previous section one can establish the existence of Langlands data (Q_F, σ, λ) such that the $(\mathfrak{m}_F, K_F)$-module $V/\bar{\mathfrak{n}}_F V$ has $\sigma \otimes \lambda \otimes 1$ as an irreducible quotient. By Frobenius reciprocity (Proposition 4.7) this implies that

$$\mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{Q_F}^G(\sigma \otimes \lambda \otimes 1)) \neq 0,$$

and by Corollary 5.9 it now follows that

$$V \simeq J(Q_F, \sigma, \lambda).$$

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