

Critique of ‘Elements of Quantum Probability’
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Abstract. We analyse the thesis of Kümmerer and Maassen (1996) that classical probability is unable to model the the stochastic nature of the Aspect experiment, in which violation of Bell’s inequality was experimentally demonstrated. According to these authors the experiment shows the need to introduce the extension of classical probability known as Quantum Probability. We show that their argument depends on hidden assumptions and a highly restrictive view of the scope of classical probability. A careful probabilistic analysis shows, on the contrary, that it is classical deterministic physical thinking which cannot cope with the Aspect experiment and therefore needs revision. The ulterior aim of the paper is to help mathematical statisticians and probabilists to find their way into the fascinating world of quantum probability (thus: the same aim as that of Kümmerer and Maassen) by dismantling the bamboo curtain between ordinary and quantum probability which over the years has been built up as physicists and pure mathematicians have repeated to one another Feynman’s famous dictum ‘quantum probability is a different kind of probability’.

Introduction

This paper is meant to be read together with the first part of Kümmerer and Maassen (1996), hereafter referred to as KM. I take it that the reader is now familiar with KM’s description of a card game, following local tradition referred to here as ‘the Bell spel’ (*spel* is the Dutch word for a game), in which two players separated from one another are allowed to share some source of randomisation in an attempt to coordinate their responses to independent separated random stimuli. KM show that using traditional sources of randomisation like throwing dice, tossing coins, or looking at the weather, cannot permit them to improve their performance above a level set by the Bell (1964) inequality. However if they instead make their responses according to the random transmission or absorption in polarization filters of two photons emitted by a Calcium atom, they are able to beat this bound.

According to KM, and they are not alone in making such claims, this graphic example shows that in quantum physics random phenomena occur which cannot be modelled using classical probability (on which the Bell inequality is supposedly based). They claim

that quantum probability, an extension of the classical Kolmogorov set-up, does yield a satisfactory model.

In the paper I will argue that KM have adopted a too narrow view of classical probability. By insisting on a concrete physical interpretation of certain elements of the classical probability model, they put constraints on the modelling which cannot be satisfied in the Aspect experiment (the physical experiment which inspired the more fanciful Bell spel). Classical probabilistic reasoning therefore allows one to draw interesting physical conclusions from the Bell spel. The concrete physical interpretation which KM requires actually corresponds to a classical *deterministic* physical picture of reality, and we learn that this deterministic physical picture is untenable.

The main ideas of this paper echo Bell (1964). Our mathematical analysis is close to the analysis of Fine (1982). A particularly careful dissection of the concept of locality and the impact of the Aspect experiment is given in Maudlin (1994). My aim in presenting this critique of KM is not to deter readers from entering into the wonderful world of quantum probability but precisely the opposite aim, to help them by removing what is certainly experienced as an insurmountable barrier to newcomers and which, in my opinion, is in origin just a communication problem between physicists and mathematicians. I do not deny that the mathematical model of quantum probability is the perfect vehicle to describe the Aspect experiment and similar phenomena, nor that as a mathematical structure it can be seen as an extension of classical probability theory. My quarrel is purely with the claim that random phenomena occur here, outside of the ambit of classical probability. That widely made claim has over the years done a great deal of harm, isolating quantum probability from the attention of ordinary probabilists and statisticians. I agree with Biane (1995; p. 4) that the usual introductions to the theory of non-commutative random variables throw ordinary probabilists into perplexity, and some demythification is urgently needed.

The paper starts with a brief summary of my interpretation of classical probability. Then I analyse the Bell spel, and finally give a number of conclusions on the nature of quantum probability. A 'critique' is by definition negative, but elsewhere (Gill, 1995a,b) I offer a constructive presentation of quantum probability in which from the start a careful and consistent interpretation is given to the objects of the theory and in which quantum and classical probability exist in symbiosis, not in opposition. Also despite the criticism here I would like to emphasise my indebtedness to the authors of KM whose lectures and whose paper raised my interest (or frustration) to such a level that I abandoned all other research projects for more than a year in order to find out how it could possibly be true that real world random phenomena can contradict the axioms of Kolmogorov.

Classical Probability

Classical (as opposed to quantum) probability theory obtained its foundations and links into modern mathematics in precisely the same period which saw the discovery of quantum mechanics and its own formulation as an axiomatic mathematical theory (quantum probability). It seems possible that quantum probability has got hard-wired into it a view of ‘classical’ probability which is indeed classical as opposed to modern; let us say, nineteenth century probability. In this view probability is part of physics; probability spaces do not exist and random variables are objects in physics, not mathematics. From physical models are derived properties of probability distributions and it is probability distributions (in those days, probability densities or mass functions) which can be further studied within mathematics. When the mathematical formulation of probability theory is as vague as that, it is no wonder that misunderstandings can arise and be perpetuated for generations since conflicts have to be resolved by reference to reality (physical experiments), not to mathematical models.

By classical probability I will refer in the rest of this paper to the modern-day probability theory ushered in by the Kolmogorov (1933) axiomatization. I will not repeat the Kolmogorov axioms here but rather discuss their interpretation. It is the claim of KM that a classical probability model, by which they presumably think of a single probability space (Ω, \mathcal{F}, P) , cannot accommodate the Aspect experiment. This claim depends on the one-to-one correspondence which we make between elements in the description of the Bell spel, and mathematical objects connected to (Ω, \mathcal{F}, P) . I will show that several classical probability models apply to the Bell spel. (In fact I will claim that in general a quantum probability model can always be embedded in a classical probability space, the opposite to what is usually stated in introductions to quantum probability!). Since I am to give an interpretation of the elements of a classical probability model I must commit myself; here I take an unashamedly naive frequentist viewpoint.

The usual space (Ω, \mathcal{F}, P) is in modern probability (but also for Kolmogorov himself) a model for an *experiment* which, at least conceptually, can be repeated many times. The space Ω is a list of all possible *outcomes* ω of the experiment. Each time the experiment is carried out the goddess Fortuna selects a point ω in Ω . She is directed by the probability measure P in the sense that in a long, long series of repetitions of the experiment, she will choose an outcome $\omega \in A \subseteq \Omega$ in a fraction $P(A)$ of the time. Here $A \in \mathcal{F}$ is just a particular collection of outcomes ω ; it is called an *event*. When we say ‘the event A happens’ we mean: the outcome ω lies in A , i.e., satisfies the property defining elements of A .

The Kolmogorov axioms therefore attempt no more than to reflect the arithmetical fact that if in a sequence of repetitions of an experiment, two events never happen simultaneously, then the relative frequency with which either of the events happen is the sum of their separate relative frequencies. In other words, by this interpretation of probability the Kolmogorov axioms do not say much more than one plus one equals two (I realise that this is a point of discussion in quantum physics; however I claim that any experimentalist who actually counts occurrences of different events in actually carried out experiments does heavily rely on the fact that one plus one is two).

We go on to interpret conditional probabilities as relative frequencies within subsequences, leading to the formula $P(A | B) = P(A \cap B)/P(B)$ when the denominator is positive. We define independence as equality of conditional and unconditional probabilities. A random variable X is simply a numerical function of the possible outcomes ω of the experiment. Its expectation becomes its mean value in many repetitions of the experiment, and so on.

One may prefer a subjective interpretation of probability. One may point to apparent circularities in the above description though I believe they are not present if one carefully distinguishes two levels: the ordinary language of practical experience of casinos, insurance companies, experimental physicists; and the language of pure mathematics. It is of course a beautiful and confusing fact that laws of large numbers play a role both as a motivating principle in constructing the mathematical model, and as a main result inside the resulting model. This is possible because we imitate one of the interpretational principles—many repetitions of the same model—with a mathematical structure—product spaces—inside the model. Apparently the modelling achieves a nice consistency. Whatever controversies and paradoxes there may be in the foundations of classical probability, I emphasize that they equally infect quantum probability since, as I believe most physicists agree, quantum probability theory also describes or predicts probabilities in the sense of: relative frequencies in many many repetitions.

Now, when applying classical probability theory to the real world one might in principle have to construct the space Ω in a mathematical sense; describe the σ -algebra of events \mathcal{F} ; specify the probability measure P . Moreover one should make some one-to-one correspondence between items in the mathematical world (ω, A, X, \dots) and the objects in the real world one wants to model. In practice one does not and cannot give a complete mathematical specification of the underlying probability space. Rather one works several layers higher, perhaps identifying outcomes of some random variables in the model to actually available statistical data, assuming structural relations between various random variables and assuming various distributional properties. Typically not much thought is given to the specification and interpretation of ω and Ω . In fact, it is usually only when mathematical or empirical inconsistencies turn up that one makes a more careful analysis of the underlying stochastic model, in an attempt to identify hidden (and inconsistent) assumptions or false arguments. However let me reiterate and reemphasize that the outcomes ω are no more than *labels* of possible different outcomes of the experiment modelled by (Ω, \mathcal{F}, P) . In practice it may be common and it may be convenient to identify the outcomes with physical objects; for instance, when accurately measuring the height and weight of a US army recruit one might take ω as the name of a particular person. But it does not have to be so: ω is just the label of an outcome of an experiment, not a pre-existing physical object. I believe that this is one of the sources of confusion in the great quantum debate: physicists are used to associate every mathematical object with something real; in fact, they use mathematical language to describe reality so this way of thinking is engrained in their discipline. Mathematicians however, especially those who make it their business to apply various kinds of pure mathematics to various kinds of real world problems, are trained precisely to distinguish between mathematical model and real world.

The Bell spel classically modelled

In the Bell spel two players are repeatedly (separately, simultaneously and independently) dealt a random playing card. They then each separately say ‘yes’ or ‘no’. Their aim is to say the same word more frequently when they both have a red card than when any of the other three cases occur, together (red/black, black/red, black/black). They are allowed to let their statement (yes/no) depend on the colour of their own card, and on common, perhaps random, information, generated independently of the two cards (e.g., the results of tossing dice or coins or of looking out of the window at the weather). In advance, they may agree on whatever kind of strategy to follow they like. The only thing they are not allowed to do is to communicate information about their own card to one another: they must simultaneously, and at a distance, decide on saying yes or no based on the colour of just their own playing card and the auxiliary common random information.

We will make and analyse several classical probability models of the Bell spel. In the paper KM it is shown that the players’ aim cannot be satisfied when using ordinary randomness. However if they let their yes/no statement depend on whether or not each of a pair of photons passes through a polarization filter with setting (orientation) determined by their respective card’s colour, then they are able to win the game.

The quantum experiment which in principle can be repeated many times consists of drawing two playing cards, the simultaneous emission of two photons from an excited Calcium atom, and their transmission or absorption in two polarization filters which have been set in orientations according to the colours (red/black) of the two playing cards. Next to this we consider classical experiments in which instead of the excited Calcium atom, the photons and the filters we take perhaps some dice, coins, or whatever; toss them, write down the results, and let each player use some rule depending on her own card and these auxiliary results to come to a yes/no answer.

Let us first write down and analyse the classical version of the game. Let A and B be two 0 – 1 valued random variables representing the colours of the two playing cards. They are modelled by independent Bernoulli ($\frac{1}{2}$) variables. Let Z be the auxiliary randomization which the players may share in their playing strategies. It is a random element taking values in an arbitrary space; it is independent of A and B . Let X and Y be two 0 – 1 valued random variables representing the statements of the two players. The players’ strategies consist of two functions which determine X and Y from Z together with A and B respectively; say $X = g(A, Z)$, $Y = h(B, Z)$. The players’ aim is to specify Z and the functions g and h in order to achieve that

$$\begin{aligned} &P(X = Y \mid A = 0, B = 0) > \\ &P(X = Y \mid A = 0, B = 1) + P(X = Y \mid A = 1, B = 0) + P(X = Y \mid A = 1, B = 1). \end{aligned}$$

In fact, in KM the conditional ‘|’ is replaced by the conjunction ‘&’ throughout, but since we are given that the four marginal probabilities $P(A = a, B = b)$, $a, b = 0, 1$, are all equal (and equal to $\frac{1}{4}$) that is exactly the same.

We show this is impossible. Since A, B, Z are random variables defined on the same probability space we can define next to X and Y (what the players actually say) some further random variables representing what the players would have said, had their cards

had each possible colour. Let $X_0 = g(0, Z)$, $X_1 = g(1, Z)$, $Y_0 = h(0, Z)$, $Y_1 = h(1, Z)$; in this notation, we have $X = X_A$ and $Y = Y_B$. The random variables (note the special order I will write them in) X_0, Y_1, X_1, Y_0 are all 0 – 1 valued and hence if within each subsequent pair (X_0, Y_1) , (Y_1, X_1) , (X_1, Y_0) the values are different, the value taken by the quadruple (X_0, Y_1, X_1, Y_0) has to be either $(1, 0, 1, 0)$ or $(0, 1, 0, 1)$. Thus

$$X_0 \neq Y_1 \ \& \ Y_1 \neq X_1 \ \& \ X_1 \neq Y_0 \ \Rightarrow \ X_0 \neq Y_0$$

or equivalently

$$X_0 = Y_0 \ \Rightarrow \ X_0 = Y_1 \ \text{or} \ Y_1 = X_1 \ \text{or} \ X_1 = Y_0.$$

Hence

$$P(X_0 = Y_0) \leq P(X_0 = Y_1) + P(Y_1 = X_1) + P(X_1 = Y_0).$$

But using independence of Z from A, B ,

$$\begin{aligned} P(X_a = Y_b) &= P(g(a, Z) = h(b, Z)) \\ &= P(g(a, Z) = h(b, Z) \mid A = a, B = b) \\ &= P(g(A, Z) = h(B, Z) \mid A = a, B = b) \\ &= P(X = Y \mid A = a, B = b), \end{aligned}$$

or in other words,

$$\begin{aligned} P(X = Y \mid A = 0, B = 0) &\leq \\ P(X = Y \mid A = 0, B = 1) &+ P(X = Y \mid A = 1, B = 0) + P(X = Y \mid A = 1, B = 1). \end{aligned}$$

Note that this derivation of Bell's inequality makes it very clear that the inequality is not a deep result from classical probability theory (classical probabilists have never heard of it!), but the trivial probabilistic consequence of the logical or combinatorial fact that for any $x_0, y_1, x_1, y_0 \in \{0, 1\}$:

$$x_0 \neq y_1 \ \& \ y_1 \neq x_1 \ \& \ x_1 \neq y_0 \ \Rightarrow \ x_0 \neq y_0.$$

We could apply this basic fact to our probabilistic model because the model entailed that 'what each player would have said if her card had had the other colour' was defined independently of the colour of the other player's card.

Now I turn to the quantum experiment. I postpone the quantum analogue of the auxiliary randomisation Z for the moment, though of course its putative existence lies at the heart of the matter. However, we can already start by saying that the experiment involves 0 – 1 valued random variables A, B, X , and Y where now A and B represent simultaneously the colours of the two players' cards and the orientations of two polarization filters. X and Y represent simultaneously the statements of the two players and whether or not each of the two photons passes through its respective filter. We agree that A and B are independent Bernoulli ($\frac{1}{2}$) variables. We also agree on the joint distribution of X and Y given A and B , as follows (this is where quantum physics comes in). If the two filters

had fixed orientations α and β respectively, and letting $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ be 0 – 1 variables describing the photons' behaviours in that case (0 for transmission, 1 for absorption), then quantum physics predicts (and experiment confirms) that

$$P(X_{\alpha,\beta} = 0, Y_{\alpha,\beta} = 0) = P(X_{\alpha,\beta} = 1, Y_{\alpha,\beta} = 1) = \frac{1}{2} \sin^2(\alpha - \beta),$$

$$P(X_{\alpha,\beta} = 0, Y_{\alpha,\beta} = 1) = P(X_{\alpha,\beta} = 1, Y_{\alpha,\beta} = 0) = \frac{1}{2} \cos^2(\alpha - \beta).$$

Specifying now angles $\alpha_0 = 0, \alpha_1 = \pi/3, \beta_0 = \pi/2, \beta_1 = \pi/6$ the classical probabilistic description of KM's Bell spel is finished by defining $X = X_{\alpha_A, \beta_B}, Y = Y_{\alpha_A, \beta_B}$. Summarizing, A and B are independent Bernoulli($\frac{1}{2}$); conditional on $A = a, B = b$, the pair $X_{\alpha_a, \beta_b}, Y_{\alpha_a, \beta_b}$ has the joint distribution just stated; and $X = X_{\alpha_A, \beta_B}, Y = Y_{\alpha_A, \beta_B}$.

The probability space carrying all these random variables carries alongside the playing cards A and B and the player's responses X and Y another eight random variables describing what the players would have responded under each possible combination of cards:

$$\begin{aligned} X_{\alpha_0, \beta_0}, Y_{\alpha_0, \beta_0}, \\ X_{\alpha_0, \beta_1}, Y_{\alpha_0, \beta_1}, \\ X_{\alpha_1, \beta_0}, Y_{\alpha_1, \beta_0}, \\ X_{\alpha_1, \beta_1}, Y_{\alpha_1, \beta_1}. \end{aligned}$$

Since we have defined $X = X_{\alpha_A, \beta_B}, Y = Y_{\alpha_A, \beta_B}$, we need just ten of these twelve 0 – 1-valued variables, so we could work with an abstract sample space Ω containing $2^{10} = 1024$ points, conceptual outcomes, ω . We only observe four of the variables, namely A, B, X, Y so we can reduce the sample space to just $2^4 = 16$ distinguishable outcomes. Their probabilities have been specified and will violate Bell's inequality. The point to be made is that: a classical sample space suffices perfectly to describe the experiment which is actually carried out; Bell's inequality is not true; the probabilities which we assign to points of the sample space were derived from quantum physics. The larger sample space with 1024 outcomes includes a modelling of 'what would have happened if ...' as well as modelling what actually did happen. It is not forbidden in probability theory to work with a larger sample space than strictly speaking necessary.

Now as we saw before Bell's inequality followed in classical probability not just from the mere existence of the four random variables A, B, X, Y but from a further assumption, which had nothing to do with the requirements of classical probability, but from physical modelling: namely that $X = g(A, Z), Y = h(B, Z)$, where g and h are some functions and Z a random element independent of A, B . The fact that Bell's inequality is violated implies not that Kolmogorov's axioms of probability fail, but that the physical modelling assumption is untenable. To investigate the place of such an assumption in our quantum experiment we must think more deeply about where the randomness of the photons' behaviour at the two filters comes from.

Let us look back at our classical model for the quantum experiment. Our mathematical model included random variables X_{α_0, β_0} and X_{α_0, β_1} , each describing the (potential) behaviour of one of the two photons at a filter set at the orientation α_0 , while the other photon, somewhere far away, is encountering a filter in the orientation β_0 or β_1 . One could

argue on physical grounds (note: not mathematical grounds!) that the behaviour of the first photon should not depend on the setting of the second filter. In other words, the two random variables X_{α_0, β_0} and X_{α_0, β_1} are identical. The same argument applied to three other pairs of our eight ‘hidden variables’ reduces the eight to four, say $X_{\alpha_0} = X_{\alpha_0, \beta_0} = X_{\alpha_0, \beta_1}$; etc. Now we have Bell’s inequality for the four random variables $X_{\alpha_a}, Y_{\beta_b}, a, b = 0, 1$. But Bell’s inequality is violated. Conclusion: in any model which includes random variables describing *each* photon’s behaviour under *each* pair of filter settings, it cannot be true that $X_{\alpha_0, \beta_0} = X_{\alpha_0, \beta_1}$.

Let us give another analysis of the Bell spel, more closely tied to our model of the classical experiment. It is clear that the outcomes of the filter experiments at the two locations are dependent of one another. But the two measurements are made on two photons which were created in a single atomic event, so it could be that the dependence which we see is caused by the fact that the two photons share some property Z^* which may vary in repetitions of the experiment. In other words, if we were able to restrict attention to a subsequence of outcomes in which Z^* continually took the same value, the behaviour of the two photons at the two filters would become independent. We allow the two photons still to exhibit random behaviour at the two filters. After all, the filter is also built of a large number of particles which also exhibit quantum behaviour if looked at separately. Let us suppose therefore that conditional on $Z^* = z^*$, and conditional on the filter settings $A = a$ and $B = b$, the two photons pass the two filters *independently* of one another with probabilities which depend only on Z^* and the setting of the relevant filter a or b . Note that this assumption is a *physical* assumption; it is not dictated by mathematical demands on the model coming from our choice to work within the usual framework of probability theory. Translated into mathematics, the assumption becomes

$$P(X = x, Y = y \mid Z^* = z^*, A = a, B = b) = p(x \mid a, z^*)q(y \mid b, z^*)$$

where $p(x \mid a, z^*) = P(X = x \mid A = a, Z^* = z^*)$, etc. Now given A and B and Z^* , let U and V be independent uniform $[0, 1]$ random variables. Using the notation $1\{\dots\}$ to denote an indicator random variable, an elementary calculation shows that the quintuple $(A, B, Z^*, 1\{U \leq p(1 \mid A, Z^*)\}, 1\{V \leq q(1 \mid B, Z^*)\})$ has the same joint distribution as (A, B, Z^*, X, Y) . This follows since given $A = a, B = b, Z^* = z^*$, the two indicator variables $1\{U \leq p(1 \mid A, Z^*)\}, 1\{V \leq q(1 \mid B, Z^*)\}$ are independent each with probabilities $p(1 \mid a, z^*)$ and $q(1 \mid b, z^*)$ to take the value 1, just like X and Y .

Now write $Z = (Z^*, U, V)$. We have shown that (in distribution) X and Y are functions $g(A, Z), h(B, Z)$, where Z is independent of A and B ; specifically,

$$g(a, z) = g(a, (z^*, u, v)) = 1\{u \leq p(1 \mid a, z^*)\},$$

and similarly for $h(b, z)$. Thus our physical modelling of the two-photon system has led to it satisfying the same model as a classical randomization device, leading to Bell’s inequality again. The correct conclusion is that our physical assumptions must be false (not the rules of probability theory).

Discussion

We have shown that the Aspect experiment can be perfectly well modelled using classical probability theory. In fact, the way we have done this could be followed in any quantum probability model: use quantum mechanics to derive the joint probability distribution of measurements of any set of compatible observables; now form the product space containing as independent components all the possible sets of measurements which could have been taken.

Admittedly this is a very clumsy construction. In fact it forms a pretty simplistic but physically implausible hidden variables model: it describes in a deterministic way what would have been the outcome of any particular set of measurements on the quantum system. Consequently, as the violation of Bell's inequality shows, this hidden variables model will violate *physical* (not mathematical) requirements of locality. But note: this unfortunate state of affairs follows from the wish to incorporate deterministic physical reasoning into the model, not from the demands of classical probability theory. If one wishes to avoid hidden variables one simply reduces the sample space to contain just the random variables which are actually measured. Now we have a minimal classical probability model about which no complaints can be made.

Now I have not tried to hide the fact that we need quantum physics to write down the probabilities which go into building our models. I could have used slightly different words and said: not quantum physics, but quantum probability. In other words it remains true that quantum probability as further developed in the later sections of KM is a beautiful and useful mathematical model for quantum phenomena. In my opinion it does not replace classical probability, but rather it coordinates the classical probability models which can be written down for any particular set of measurements which one would care to make on a quantum system. From the point of view of modern mathematical statistics, in which the notion of 'statistical experiment' is an established term for certain families of probability spaces, one might better talk of 'Quantum Experiments' than 'Quantum Probability' (cf. also design of experiments, convergence of experiments, ...).

Careful reading of KM shows where they fall into the trap: repeatedly they try to identify ω with a particular physical object, e.g., a photon. In other words their finding that classical probability is not adequate for quantum phenomena is really a finding that classical deterministic thinking is inadequate: quantum phenomena are more truly random than any other phenomena one can imagine. We all believe that the toss of a coin is ruled by completely deterministic laws; however the passage of a photon through a polarization filter is intrinsically truly random (unless one likes to adopt hidden variables models exhibiting severely non-local behaviour). In other words again: God does not throw ordinary dice.

The fact remains that the players in the Bell spel can achieve results unachievable with classical randomization devices. Does this mean that we do have here 'action at a distance'? Are they using the dependence of the behaviour of one photon on the setting of a distant filter to communicate with one another faster than the speed of light? My interpretation of the phenomenon is that the coupled state of the two photons has certainly allowed the two players to coordinate their actions at a distance, in a way which they could also have done if they had been able to exchange information faster than the speed of light. Of course if they had been able to do the latter, they could have achieved even

more spectacular results (like always saying the same answer). Coordination at a distance is not the same as action at a distance. In the actual Aspect experiment filter settings were used so that the predicted violation of Bell's inequality would be as extreme as it possibly can be under standard quantum theory. If empirically an even more severe violation of Bell's inequality had been observed, also standard quantum probability would have been discredited.

From the point of view of the quantum system one could say that the two-photon system reacts to probing at distant locations as a whole; its reaction to the different measurements exhibits a dependence over great distances which cannot be explained in a classical way.

From a pure mathematical point of view one may say that it is true that Quantum Probability Theory is a different kind of probability-theory to ordinary probability theory. Mathematically it is a genuine extension. Drawing analogies between random variables and observables, between probability measures and states, between events and Hilbert subspaces, we see that the axioms of quantum probability are weaker than those of classical probability. In a mathematical sense the one is an extension of the other. This analogy may be useful in many ways, but it is also dangerous since it simply is not true that the notion of an observable makes that of a random variable superfluous: measurement of an observable yields a random variable so we have to talk about both. Another danger is that pure mathematical analogy may lead one to define new quantum probability concepts by mathematical analogy, without there being any physical *raison d'être* for the new concepts; e.g., define quantum conditional expectation $g(X) = E(Y | X)$ by minimizing $E(Y - g(X))^2$, when the difference between two incompatible observables does not have any physical interpretation at all.

An extreme example supposing to show that quantum probability is a different kind of probability was offered recently in Malley and Hornstein (1993). The example has been used in the literature for this purpose for a long time and it is amazing to me to still see it being used, since the conflict with standard stochastic modelling principles is so blatant. The two-slit experiment as discussed by Feynman (1951) as well as in the Feynman lectures (part III) is also the same example in different clothing. KM have dressed up the example in more impressive clothing but really the example is the same.

Malley and Hornstein (1993) considered a photon passing through two polarization filters aligned at right angles so that no photon passing the first also passes the second. Now insert a third filter at 45° between the other two and suddenly photons do pass all three filters. Apparently this is an example of three events A , B , C (passage of the three filters) such that $P(A \cap B \cap C) > P(A \cap C)$; the latter probability being in fact zero. This example strikingly illustrates the trap. We are talking about two different experiments, one with two filters, one with three. The two experiments share two events 'the photon passes the first filter' and 'the photon passes the last filter'. Obviously these events have a lot in common from a physical point of view. However a priori there is no reason to suppose that they can be modelled by the same event in the same probability space. Further physical assumptions are needed before we can identify P , Ω , A , and C in the two experiments. If we do that but must accept that $P(A \cap B \cap C) > P(A \cap C)$ we have not violated classical

probability ($1 + 1 = 2$) but we have proved that the physical assumptions we made are not all true.

I hope that this criticism will help newcomers to quantum probability to be able to penetrate the smokescreen of mistaken interpretations and confused arguments in order to appreciate what a beautiful source it is of fascinating (classical) stochastic models, and a field full of rewards for ordinary probabilists and statisticians looking for fresh challenges. I hope it will also encourage researchers within the field to be more critical of received wisdom concerning the nature of quantum probability, and more careful to give consistent and appealing physical interpretations to their mathematical models so as to aid rather than abet newcomers attempting to appreciate their work.

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