

## Simulation of Large Networks on Smaller Networks

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Parallel algorithms are normally designed for execution on networks of  $N$  processors, with  $N$  depending on the size of the problem to be solved. In practice there will be a varying problem size but a fixed network size. In Fishburn and Finkel (*IEEE Trans. Comput.* **31** (1982), 288-295), the notion of network emulation was proposed, to obtain a structure preserving simulation of large networks on smaller networks. We present a detailed analysis of the possible emulations for some important classes of networks, namely: the shuffle-exchange network, the cube network, the ring network, and the 2-dimensional grid. We also study the possibility of cross-emulations, and characterize the networks that can be emulated at all on a given network using some class of emulation functions. © 1986 Academic Press, Inc.

### 1. INTRODUCTION

Parallel algorithms are normally designed for execution on a suitable network of  $N$  processors (viewed as SIMD- or MIMD-machine; cf. van Leeuwen, 1985), with  $N$  depending on the size of the problem to be solved. In practice  $N$  will be large and varying, whereas processor networks will be small and fixed. The resulting disparity between algorithm design and implementation must be resolved by simulating a network of some size  $N$  on a fixed and smaller size network of a similar or different kind, in a structure preserving and efficient manner. Notions of simulation are well understood in, e.g., automata theory (see Herman, 1971), and suitable analogs can be brought to bear on networks of processors. In this paper we study a notion of simulation, termed: emulation, that was recently proposed by Fishburn and Finkel (1982).

DEFINITION. Let  $G = (V_G, E_G)$  and  $H = (V_H, E_H)$  be networks of

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processors (graphs). We say that  $G$  can be emulated on  $H$  if there exists a function  $f: V_G \rightarrow V_H$  such that for every edge  $(g, g') \in E_G$ :  $f(g) = f(g')$  or  $(f(g), f(g')) \in E_H$ . The function  $f$  is called an emulation function, or in short, an emulation of  $G$  on  $H$ .

Clearly, emulation between networks is transitive. We shall only be interested in emulations  $f$  that are "onto."

Let  $f$  be an emulation of  $G$  on  $H$ . Any processor  $h \in V_H$  must actively emulate the processors  $\in f^{-1}(h)$  in  $G$ . When  $g \in f^{-1}(h)$  communicates information to a neighboring processor  $g'$ , then  $h$  must communicate the corresponding information "internally" when it emulates  $g'$  itself or to a neighboring processor  $h' = f(g')$  in  $H$  otherwise. If all processors act synchronously in  $G$ , then the emulation will be slowed by a factor proportional to  $\max_{h \in V_H} |f^{-1}(h)|$ . For a set  $A$ , we use  $|A|$  to denote the cardinality of  $A$ .

**DEFINITION.** Let  $G$ ,  $H$ , and  $f$  be as above. The emulation  $f$  is said to be (computationally) uniform if for all  $h, h' \in V_H$ :  $|f^{-1}(h)| = |f^{-1}(h')|$ .

Every uniform emulation  $f$  has associated with it a fixed constant  $c$ , called: the computation factor, such that for all  $h \in V_H$ :  $|f^{-1}(h)| = c$ . It means that every processor of  $H$  emulates the same number of processors of  $G$ . Again, uniform emulation between networks is transitive. When  $G$  can be uniformly emulated on  $H$  and  $H$  can be uniformly emulated on  $G$ , then  $G$  and  $H$  are necessarily isomorphic. (Thus uniform emulation establishes a partial ordering of networks.) For graphs  $A, B$  let  $A[B]$  denote the composition of  $A$  and  $B$  (cf. Harary, 1969).

**LEMMA 1.1.**  $G$  can be uniformly emulated on  $H$  if and only if there exists a graph  $G'$  such that  $G$  is a spanning subgraph of  $H[G']$ .

*Proof.*  $\Rightarrow$  Let  $f$  be a uniform emulation of  $G$  on  $H$  with computation factor  $c$ . The sets  $\{f^{-1}(h)\}$ ,  $h \in H$ , partition  $G$  into blocks of size  $c$ . Let  $G'$  be any graph on  $c$  nodes such that the induced subgraph of every block (in  $G$ ) is contained in  $G'$ . Next observe that for any two nodes  $g \in f^{-1}(h)$  and  $g' \in f^{-1}(h')$  of  $G$ :  $(g, g') \in E_G \Rightarrow h = h'$  (and the edge is in  $G'$ ) or  $(h, h') \in E_H$ . It follows that  $G$  is a spanning subgraph of  $H[G']$ .

$\Leftarrow$  From the definition of composition (cf. Harary, 1969), by projection on  $H$ . ■

$G'$  can always be chosen to be equal to  $K_c$ , the complete graph on  $c$  nodes. When  $G$  is uniformly emulated on  $H$ , then  $H$  can be viewed as a "factor" of  $G$  (and, in particular:  $|V_H| \mid |V_G|$ ). When  $|V_G| = |V_H|$ , then  $G$  can be uniformly emulated on  $H$  if and only if  $G$  is isomorphic to a subgraph of  $H$ .

With this observation it is not hard to show that the general UNIFORM NETWORK EMULATION problem is *NP*-complete (cf. Garey and Johnson, 1979. Reduce from HAMILTONIAN CIRCUIT. Let  $H$  be an instance of HAMILTONIAN CIRCUIT and let  $G$  be a ring with  $|V_H|$  nodes.) Another useful property is the following.

**DEFINITION.** For directed graphs  $G=(V, E)$  let  $G^R$  be the (directed) graph obtained from  $G$  by reversing the direction of the edges, i.e.,  $G^R=(V, E^R)$  with  $E^R=\{(g', g) \mid (g, g') \in E\}$ .

**LEMMA 1.2.**  *$f$  is a (uniform) emulation of  $G$  on  $H$  if and only if  $f$  is a (uniform) emulation of  $G^R$  on  $H^R$ .*

*Proof.*  $\Rightarrow$  Let  $f$  be an emulation of  $G$  on  $H$ . It means that for every edge  $(g, g') \in E_G: f(g)=f(g')$  or  $(f(g), f(g')) \in E_H$ . Thus, by simple translation, we have for every edge  $(g', g) \in E_{G^R}: f(g')=f(g)$  or  $(f(g'), f(g)) \in E_{H^R}$ . Hence  $f$  is an emulation of  $G^R$  on  $H^R$ .

$\Leftarrow$  By a similar argument, observing that  $(G^R)^R=G$  for all graphs  $G$ . Finally we note that uniformity is preserved in the constructions. ■

The relevant question is whether (large) networks of some class  $C$  can be uniformly emulated by networks of a smaller size within the same class  $C$ . Fishburn and Finkel (1982) answered this question affirmatively for the following classes of processor networks: the shuffle-exchange network, the grid-connected network, the  $n$ -dimensional cube, the plus-minus network, the binary lens, and the cube-connected cycles. (For definitions of these networks, see Fishburn and Finkel, 1982.) In this paper we shall take a more fundamental approach and develop a detailed analysis of all possible emulations in selected classes of networks: the shuffle-exchange network, the  $n$ -dimensional cube, the ring, and the 2-dimensional grid. The results will be presented in Sections 2–4. In Section 5 we consider the question of emulating networks of some class  $C$  on (smaller) networks of some class  $C'$ . In Section 6 we show that there is a natural way to describe the networks that can be emulated on a given network  $H$ , using a set of permissible emulations. The results lead to interesting characterisations of all networks considered in terms of their emulated behaviour. Some proofs are deferred to Appendix A and B.

## 2. EMULATIONS OF THE SHUFFLE-EXCHANGE NETWORK

Let  $S_n$  denote the shuffle-exchange network with  $2^n$  nodes. Our main result will be that there are exactly 6 different uniform emulations of  $S_n$  on  $S_{n-1}$ . We also show that there are at least  $2 \cdot 2^{2^k} - 2^{2^{k-1}}$  uniform emulations

of  $S_n$  on  $S_{n-k}$  ( $k \geq 1$ ). In Section 2.1 we give some preliminary definitions and results, in Section 2.2 we give the analysis leading to our main result. The proof of the main theorem is deferred to Appendix A. In Section 2.3 we discuss the uniform emulation of  $S_n$  on  $S_{n-k}$  in general and argue that the results hold for the uniform emulation of the inverse shuffle-exchange network as well.

### 2.1. Preliminaries

The shuffle-exchange network was proposed initially by Stone (1971), and has been successfully used as the interconnection network underlying a variety of parallel processing algorithms. The nodes are given  $n$ -bit addresses in the range  $0 \cdots 2^n - 1$ , and there is an edge from node  $b$  to node  $c$  if and only if  $b$  can be “shuffled” (move leading bit to tail position) and “exchanged” (flip the tail bit) into  $c$ . Computations proceed by iterating through the network some  $n$  or more times, in a synchronised manner. We use the following notations.

- $\frac{0}{1}$ : a bit that can be 0 or 1,
- $\bar{x}$ : the complement of bit  $x$  ( $\bar{0} = 1, \bar{1} = 0$ ),
- $\alpha \equiv \beta$ : the “equivalence” test on bits  
( $0 \equiv 0 = 1; 0 \equiv 1 = 0; 1 \equiv 0 = 0; 1 \equiv 1 = 1$ ),
- $b$ : the  $n$ -bit address  $b_1 \cdots b_n$ ,
- $b|_i$ :  $b_1 \cdots b_i$  (truncation after the  $i$ th bit),
- ${}_i b$ :  $b_i \cdots b_n$  (truncation “before” the  $i$ th bit),
- $(b)_i$ :  $b_i$  (the  $i$ th bit).

For functions  $f$  defined on  $n$ -bit numbers  $b$  we use:

$$f_i(b): (f(b))_i \quad (\text{projection on the } i\text{th bit}).$$

We use  $b, c, \dots$ , to denote full addresses and  $x, y, \dots$ , to denote segments of bits. Individual bits are denoted  $\alpha, \beta, \dots$

**DEFINITION.** The shuffle-exchange network is the graph  $S_n = (V_n, E_n)$  with  $V_n = \{(b_1 \cdots b_n) \mid \forall 1 \leq i \leq n \ b_i = \frac{0}{1}\}$  and  $E_n = \{(b, c) \mid b, c \in V_n \text{ and } \forall 2 \leq i \leq n \ b_i = c_{i-1}\}$ . The inverse shuffle-exchange network is the graph  $\tilde{S}_n = (V_n, \tilde{E}_n)$  with  $\tilde{E}_n = \{(b, c) \mid b, c \in V_n \text{ and } \forall 2 \leq i \leq n \ b_{i-1} = c_i\}$ .

It follows that in  $S_n$  a node  $b_1 \cdots b_n$  is connected to  $b_2 \cdots b_n 0$  and  $b_2 \cdots b_n 1$ , in  $\tilde{S}_n$  to  $0b_1 \cdots b_{n-1}$  and  $1b_1 \cdots b_{n-1}$ . The fact that  $S_n$  can be (uniformly) emulated on  $S_{n-1}$  and, hence, on every  $S_{n-k}$  ( $k \geq 1$ ) derives from the following observation, using Lemma 1.1 (cf. Fishburn and Finkel (1982), Theorem 2).

**LEMMA 2.1.**  $S_n$  is a spanning subgraph of  $S_{n-1}[\bar{K}_2]$ , for  $n \geq 2$ .

*Proof.* Consider the mapping  $h: S_n \rightarrow S_{n-1}[\bar{K}_2]$  defined by  $h(b_1 \cdots b_n) = \langle b_1 \cdots b_{n-1}, b_n \rangle$ , which clearly is 1-to-1 and onto on the set of nodes. Let  $(b, c) \in E_n$  with  $c = b_2 \cdots b_n \frac{0}{1}$  (necessarily). Then  $(b_1 \cdots b_{n-1}, b_2 \cdots b_n) \in E_n$ , hence  $h(b)$  and  $h(c)$  are adjacent in  $S_{n-1}[\bar{K}_2]$ . Thus  $S_n$  is isomorphic to a spanning subgraph of  $S_{n-1}[\bar{K}_2]$ . ■

**COROLLARY 2.2.**  $S_n$  is a spanning subgraph of  $S_{n-k}[\bar{K}_{2^k}]$ , for every  $1 \leq k < n$ .

By using a mapping  $h$  defined by  $h(b_1 \cdots b_n) = \langle b_{n-1} \cdots b_1, b_n \rangle$ , a similar argument shows that  $S_n$  is a spanning subgraph of  $\tilde{S}_{n-1}[K_2]$  and (hence) that  $S_n$  can be uniformly emulated on  $S_{n-1}$  and any smaller inverse shuffle-exchange network. Clearly  $\tilde{S}_n \cong S_n$ .

**LEMMA 2.3.**  $f$  is an emulation of  $S_n$  on  $S_{n-k}$  if and only if for all  $x \in (\frac{0}{1})^{n-1}$ ,  $y \in (\frac{0}{1})^{n-k-1}$  and  $\alpha, \beta \in (\frac{0}{1})$ : if  $f(\alpha x) = \beta y$  then  $(f(x0) = \beta y \vee f(x0) = y \frac{0}{1})$  and  $(f(x1) = \beta y \vee f(x1) = y \frac{0}{1})$ .

(The proof follows straight from the definitions involved.) For a mapping  $f$ , define its “companion”  $\bar{f}$  by  $\bar{f}_i(b) = \overline{f_i(b)}$  for all  $1 \leq i \leq n$ .

**LEMMA 2.4.** If  $f$  is an emulation of  $S_n$  on  $S_{n-k}$ , then so is  $\bar{f}$ .

*Proof.* Immediate from Lemma 2.3. ■

## 2.2. Uniform Emulations of $S_n$ on $S_{n-1}$

The uniform emulations of  $S_n$  on  $S_{n-1}$  will be shown to be “step-simulating” in a very precise sense. Our aim will be to characterize all step-simulations of  $S_n$  on  $S_{n-1}$ , and to derive from it all uniform emulations.

**DEFINITION.** A mapping  $g: S_n \rightarrow S_{n-1}$  is called step-simulating (or: a “step-simulation” of  $S_n$  on  $S_{n-1}$ ) if and only if for all  $x \in (\frac{0}{1})^{n-1}$ ,  $y \in (\frac{0}{1})^{n-2}$  and  $\alpha, \beta \in (\frac{0}{1})$ : if  $g(\alpha x) = \beta y$ , then  $g(x0) = y \frac{0}{1}$  and  $g(x1) = y \frac{0}{1}$ .

**LEMMA 2.5.** Every step-simulation  $g$  of  $S_n$  on  $S_{n-1}$  is an emulation.

*Proof.* Immediate (c.f. Lemma 2.3). ■

We shall call a step-simulation “uniform” when it is uniform as an emulation. When  $g$  is a step-simulation, then so is  $\bar{g}$ .

**LEMMA 2.6.** A mapping  $g: S_n \rightarrow S_{n-1}$  is step-simulating if and only if for all  $x \in (\frac{0}{1})^{n-1}$ ,  $y \in (\frac{0}{1})^{n-2}$  and  $\alpha, \beta \in (\frac{0}{1})$ : if  $g(\alpha x) = \beta y$  then  $g(0x) = \frac{0}{1}y$  and  $g(1x) = \frac{0}{1}y$ .

*Proof.* By verifying equivalence with the definition of step-simulation. (Use the string character of  $x$  and  $y$ .) ■

Lemma 2.6 can be interpreted as stating that the (uniform) step-simulations of  $S_n$  on  $S_{n-1}$  act at the same time as (uniform) step-simulations of  $\tilde{S}_n$  on  $\tilde{S}_{n-1}$ . Note the following useful properties of step-simulations  $g$ :

$$\begin{aligned} g(b_1 \cdots b_{n-1} 0)|_{n-2=2} &= g(0b_1 \cdots b_{n-1}), \\ g(b_1 \cdots b_{n-1} 1)|_{n-2=2} &= g(1b_1 \cdots b_{n-1}). \end{aligned}$$

We shall now aim for a characterisation of the possible step-simulations and uniform step-simulations of  $S_n$  on  $S_{n-1}$ .

DEFINITION. For  $n \geq 3$ , define the operators  $\Pi^n: [V_n \rightarrow V_{n-1}] \rightarrow [V_{n-1} \rightarrow V_{n-2}]$  and  $T^n: [V_{n-1} \rightarrow V_{n-2}] \rightarrow [V_n \rightarrow V_{n-1}]$  as follows:

$$\begin{aligned} \Pi^n(g)(b_1 \cdots b_{n-1}) &= g(b_1 \cdots b_{n-1} 0)|_{n-2}, \\ T^n(h)(b_1 \cdots b_n) &= h(b_1 \cdots b_{n-1}) \cdot h_{n-2}(b_2 \cdots b_n). \end{aligned}$$

THEOREM 2.7. For  $n \geq 3$ ,

- (i) if  $g$  is a step-simulation of  $S_n$  on  $S_{n-1}$ , then  $\Pi^n(g)$  is a step-simulation of  $S_{n-1}$  on  $S_{n-2}$ ,
- (ii) if  $h$  is a step-simulation of  $S_{n-1}$  on  $S_{n-2}$ , then  $T^n(h)$  is a step-simulation of  $S_n$  on  $S_{n-1}$ ,
- (iii) restricted to step-simulations,  $\Pi^n$  and  $T^n$  are inverses,
- (iv) restricted to step-simulations,  $\Pi^n$  preserves uniformity.

*Proof.* (i) Verify the condition of Lemma 2.6:  $\Pi^n(g)(x\alpha) = y\beta \Rightarrow g(x\alpha 0) = y\beta \frac{0}{1}$  (definition of  $\Pi^n$ )  $\Rightarrow g(0x\alpha) = \frac{0}{1}y\beta$  and  $g(1x\alpha) = \frac{0}{1}y\beta \Rightarrow g(0x0) = \frac{0}{1}y\frac{0}{1}$  and  $g(1x0) = \frac{0}{1}y\frac{0}{1}$  (by shifting right and then left)  $\Rightarrow \Pi^n(g)(0x) = \frac{0}{1}y$  and  $\Pi^n(g)(1x) = \frac{0}{1}y$ .

(ii) Similarly  $T^n(h)(x\gamma a) = y\delta\beta \Rightarrow h(x\gamma) = y\delta \Rightarrow T^n(h)(0x\gamma) = h(0x) \cdot h_{n-2}(x\gamma) = \frac{0}{1}y\delta$  and  $T^n(h)(1x\gamma) = h(1x) \cdot h_{n-2}(x\gamma) = \frac{0}{1}y\delta$ .

(iii) Let  $g$  be a step-simulation of  $S_n$  on  $S_{n-1}$ . Then  $T^n \circ \Pi^n(g)(\gamma x \delta) = \Pi^n(g)(\gamma x) \cdot \Pi^n(g)_{n-2}(x\delta) = g(\gamma x 0)|_{n-2} \cdot g_{n-2}(x\delta 0) = g(\gamma x \delta)|_{n-2} \cdot g_{n-1}(\gamma x \delta) = g(\gamma x \delta)$  for all  $\gamma, x, \delta$ . Hence  $T^n \circ \Pi^n = \text{id}$ . Conversely, let  $h$  be a step-simulation of  $S_{n-1}$  on  $S_{n-2}$ . Then  $\Pi^n \circ T^n(h)(\gamma x) = T^n(h)(\gamma x 0)|_{n-2} = (h(\gamma x) \cdot h_{n-2}(x 0))|_{n-2} = h(\gamma x)$  for all  $\gamma, x$ . Hence also  $\Pi^n \circ T^n = \text{id}$ . It follows that  $\Pi^n$  and  $T^n$  are inverses to one another when considered as operators on step-simulations.

(iv) Let  $g$  be a uniform step-simulation of  $S_n$  on  $S_{n-1}$ . Suppose  $\Pi^n(g)$  is not uniform. Then there must be a  $y \in V_{n-2}$  such that  $|\Pi^n(g)^{-1}(y)| > 2$ . Let  $x^{(1)}, x^{(2)}, x^{(3)}$  be distinct elements of  $\Pi^n(g)^{-1}(y)$ . It follows that  $g(x^{(1)}0), g(x^{(2)}0), g(x^{(3)}0) \in \{y0, y1\}$ . Because  $g$  is step-simulating we have, in fact:  $g(x^{(1)}0), g(x^{(1)}1), g(x^{(2)}0), g(x^{(2)}1), g(x^{(3)}0), g(x^{(3)}1) \in \{y0, y1\}$  and hence  $|g^{-1}(y0)| \geq 3$  or  $|g^{-1}(y1)| \geq 3$ . This contradicts the uniformity of  $g$ . ■

Theorem 2.7(i)–(iii) shows that there is a 1-to-1 onto correspondence between the step-simulations of  $S_n$  on  $S_{n-1}$  and the step-simulations of  $S_{n-1}$  on  $S_{n-2}$ , for  $n \geq 3$ . Theorem 2.7(iv) does not quite show that this correspondence holds for the subclass of uniform step-simulations, but in the next theorem we will show that it is the case.

THEOREM 2.8. For  $n \geq 2$ ,

- (i) there are exactly 16 possible step-simulations of  $S_n$  on  $S_{n-1}$ .
- (ii) There are exactly 6 possible uniform step-simulations of  $S_n$  on  $S_{n-1}$  (see Table I).

*Proof.* (i) By theorem 2.7(i)–(iii) the number of step-simulations of  $S_n$  on  $S_{n-1}$  is equal to the number of step-simulations of  $S_{n-1}$  on  $S_{n-2}$ , for  $n \geq 3$  (because  $\Pi^n$  is bijective). By induction this number is equal to the number of step-simulations of  $S_2$  on  $S_1$ . Clearly every mapping  $\in [V_2 \rightarrow V_1]$  is step-simulating. There are exactly  $2^4 = 16$  mappings in this set.

(ii) There are exactly  $\binom{4}{2} = 6$  mappings  $\in [V_2 \rightarrow V_1]$  that are uniform and step-simulating. By theorem 2.7(i)–(iv) the number of uniform step-simulations of  $S_n$  on  $S_{n-1}$  ( $n \geq 3$ ) is not larger than the number of uniform step-simulations of  $S_{n-1}$  on  $S_{n-2}$  and thus, by induction, not larger than 6. On the other hand at least 6 uniform step-simulations of  $S_n$  on  $S_{n-1}$  can be explicitly given, see Table I. (The verification of the mappings is immediate from the definition.) ■

TABLE I

Listing of the 6 Possible Uniform Step-Simulations of the Shuffle-Exchange Network with  $2^n$  Nodes on the Shuffle-Exchange Network with  $2^{n-1}$  Nodes

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$f_1:$	$f_1(b_1 \cdots b_n) = b_1 \cdots b_{n-1}$	
$\bar{f}_1:$	$\bar{f}_1(b_1 \cdots b_n) = \bar{b}_1 \cdots \bar{b}_{n-1}$	
$f_2:$	$f_2(b_1 \cdots b_n) = b_2 \cdots b_n$	
$\bar{f}_2:$	$\bar{f}_2(b_1 \cdots b_n) = \bar{b}_2 \cdots \bar{b}_n$	
$f_3:$	$f_3(b_1 \cdots b_n) = c_1 \cdots c_{n-1}$	with $c_i = (b_i \equiv b_{i+1}), 1 \leq i \leq n-1$
$\bar{f}_3:$	$\bar{f}_3(b_1 \cdots b_n) = \bar{c}_1 \cdots \bar{c}_{n-1}$	with $c_i = (b_i \equiv b_{i+1}), 1 \leq i \leq n-1$

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The remaining problem is to determine whether any other uniform emulation of  $S_n$  on  $S_{n-1}$  exists. Our main result is the following.

**THEOREM 2.9.** (Characterisation theorem). *Every uniform emulation of  $S_n$  on  $S_{n-1}$  is step-simulating, and thus equal to one of the mappings listed in Table I.*

The proof of Theorem 2.9 is long and tedious, and is given in Appendix A.

### 2.3. Uniform Emulations of $S_n$ on $S_{n-k}$

We will extend the notion of “step-simulation” of  $S_n$  on  $S_{n-k}$ , in order to attempt a characterisation of the uniform emulations in general. We show that the step-simulations of  $S_n$  on  $S_{n-k}$  (which are not all uniform) can again be characterized in terms of the step-simulations of  $S_{k+1}$  on  $S_1$  (cf. Theorem 2.8). It remains an open question whether all uniform emulations of  $S_n$  on  $S_{n-k}$  are step-simulating, and thus whether a suitable analogue of Theorem 2.9 holds for  $k \geq 1$ . We show that there are at least  $2 \cdot 2^k - 2^{k-1}$  uniform step-simulations of  $S_n$  on  $S_{n-k}$ . We also discuss the uniform emulations of  $\tilde{S}_n$  on  $\tilde{S}_{n-k}$ .

**DEFINITION.** A mapping  $g: S_n \rightarrow S_{n-k}$  is called *step-simulating* (or: a “step-simulation” of  $S_n$  on  $S_{n-k}$ ) if for all  $x \in (\frac{0}{1})^{n-1}$ ,  $y \in (\frac{0}{1})^{n-k-1}$  and  $\alpha, \beta \in \frac{0}{1}$ : if  $g(\alpha x) = \beta y$  then  $g(x \frac{0}{1}) = y \frac{0}{1}$ .

Every step-simulation clearly is an emulation (verify Lemma 2.3) and also the following analogue of Lemma 2.6 holds.

**LEMMA 2.10.** *A mapping  $g: S_n \rightarrow S_{n-k}$  is step-simulating if and only if for all  $x \in (\frac{0}{1})^{n-1}$ ,  $y \in (\frac{0}{1})^{n-k-1}$  and  $\alpha, \beta \in (\frac{0}{1})$ : if  $g(x\alpha) = y\beta$  then  $g(0x) = \frac{0}{1}y$  and  $g(1x) = \frac{0}{1}y$ .*

We now aim for a characterization of all step-simulations of  $S_n$  on  $S_{n-k}$ .

**DEFINITION.** For  $n \geq k+2$ , define the operators  $\Pi^{n,k}: [V_n \rightarrow V_{n-k}] \rightarrow [V_{n-1} \rightarrow V_{n-k-1}]$  and  $T^{n,k}: [V_{n-1} \rightarrow V_{n-k-1}] \rightarrow [V_n \rightarrow V_{n-k}]$  as follows:

$$\begin{aligned} \Pi^{n,k}(g)(b_1 \cdots b_{n-1}) &= g(b_1 \cdots b_{n-1} 0)|_{n-k-1}, \\ T^{n,k}(h)(b_1 \cdots b_n) &= h(b_1 \cdots b_{n-1}) \cdot h_{n-k-1}(b_2 \cdots b_n). \end{aligned}$$

**THEOREM 2.11.** *For  $n \geq k+2$ ,*

- (i) *If  $g$  is a step-simulation of  $S_n$  on  $S_{n-k}$ , then  $\Pi^{n,k}(g)$  is a step-simulation of  $S_{n-1}$  on  $S_{n-k-1}$ .*
- (ii) *If  $h$  is a step-simulation of  $S_{n-1}$  on  $S_{n-k-1}$ , then  $T^{n,k}(h)$  is a step-simulation of  $S_n$  on  $S_{n-k}$ .*



- (iii) *Restricted to step-simulations,  $\Pi^{n,k}$  and  $T^{n,k}$  are inverses.*
- (iv) *Restricted to step-simulations,  $\Pi^{n,k}$  preserves uniformity.*

*Proof.* The proof is virtually the same as for Theorem 2.7 and therefore left to the reader. ■

We conclude the following results (cf. Theorem 2.8):

**THEOREM 2.12.** *For  $n \geq k + 2$ ,*

- (i) *There is a bijection from the set of step-simulations of  $S_n$  on  $S_{n-k}$  to the set of step-simulations of  $S_{n-1}$  on  $S_{n-k-1}$  and (hence) to the set of step-simulations of  $S_{k+1}$  on  $S_1$ .*
- (ii) *There is an injection from the set of uniform step-simulations of  $S_n$  on  $S_{n-k}$  to the set of uniform step-simulations of  $S_{n-1}$  on  $S_{n-k-1}$  and (hence) to the set of uniform step-simulations of  $S_{k+1}$  on  $S_1$ .*

Theorem 2.12 is important, as it characterizes the step-simulations of  $S_n$  on  $S_{n-k}$ . Clearly every mapping  $\in [V_{k+1} \rightarrow V_1]$  is step-simulating, and thus there are precisely  $2^{2^{k+1}}$  step-simulations of  $S_n$  on  $S_{n-k}$ .

**COROLLARY 2.13.** *For  $n \geq 1$ ,  $S_n$  has precisely 2 graph-isomorphisms.*

*Proof.* Every isomorphism of  $S_n$  must be step-simulating. By Theorem 2.12(i) the step-simulations of  $S_n$  on  $S_n$  are in 1-to-1 correspondence to the step-simulations of  $S_1$  on  $S_1$ . There are four mappings of this kind and thus precisely four step-simulations of  $S_n$  on  $S_n$ :  $g_1(b_1 \cdots b_n) = b_1 \cdots b_n$ ,  $g_2(b_1 \cdots b_n) = \bar{b}_1 \cdots \bar{b}_n$ ,  $g_3(b_1 \cdots b_n) = 0 \cdots 0$ ,  $g_4(b_1 \cdots b_n) = 1 \cdots 1$ . Clearly, only  $g_1$  and  $g_2$  are isomorphisms. ■

The 1-to-1 correspondence referred to in Theorem 2.12(i) can be made explicit as follows. Given a step-simulation  $g$  of  $S_n$  on  $S_{n-k}$ , the uniquely corresponding step-simulation  $\tilde{g}$  of  $S_{k+1}$  on  $S_1$  is defined by the formula  $\tilde{g}(b_1 \cdots b_{k+1}) = g(b_1 \cdots b_{k+1} 0 \cdots 0)|_1$ . Conversely, given a step-simulation  $h$  of  $S_{k+1}$  on  $S_1$ , the uniquely corresponding step-simulation  $\hat{h}$  of  $S_n$  on  $S_{n-k}$  is defined by  $\hat{h}(b_1 \cdots b_n) = h(b_1 \cdots b_{k+1}) \cdot h(b_2 \cdots b_{k+2}) \cdots h(b_{n-k} \cdots b_n)$ . While the correspondence  $g \rightarrow \tilde{g}$  preserves uniformity (cf. Theorem 2.11(iv)), it does not induce a bijection from the uniform step-simulations of  $S_n$  on  $S_{n-k}$  to the uniform step-simulations of  $S_{k+1}$  to  $S_1$  for  $k > 1$ . The existence of such a bijection for  $k = 1$  (cf. Theorem 2.8(ii)) was the key to the complete characterisation of the uniform step-simulations of  $S_n$  on  $S_{n-1}$  and of the uniform emulations of  $S_n$  on  $S_{n-1}$  (cf. Theorem 2.9). A similar characterisation of the uniform step-simulations and of the uniform emulations of  $S_n$  on  $S_{n-k}$  for  $k > 1$  remains a challenging open problem. We can characterize a large class of uniform step-simulations.

**THEOREM 2.14.** *Let  $n \geq k+1$ , and let  $g$  be a step-simulation of  $S_n$  on  $S_{n-k}$ .*

(i) *if  $\tilde{g}(b_1 \cdots b_{k+1}) = \overline{\tilde{g}(\bar{b}_1 b_2 \cdots b_{k+1})}$  for all  $b_1 \cdots b_{k+1} \in (\frac{0}{1})^{k+1}$ , then  $g$  is uniform.*

(ii) *if  $\tilde{g}(b_1 \cdots b_{k+1}) = \overline{\tilde{g}(b_1 \cdots b_k \bar{b}_{k+1})}$  for all  $b_1 \cdots b_{k+1} \in (\frac{0}{1})^{k+1}$ , then  $g$  is uniform.*

*Proof.* We only prove (i) as the proof of (ii) is similar. Induct on  $n$ . For  $n = k+1$ , observe from the assumption that of every pair  $b_1 \cdots b_{k+1}$ ,  $\bar{b}_1 b_2 \cdots b_{k+1}$   $\tilde{g}$  will map one to  $0 \in V_1$  and one to  $1 \in V_1$ . Thus  $g = \tilde{g}$  is uniform. Assume it holds up to  $n-1 \geq k+1$ . Let  $g$  be a step-simulation of  $S_n$  on  $S_{n-k}$  for which the constant on  $\tilde{g}$  is satisfied. Let  $g'$  be the uniquely corresponding step-simulation of  $S_{n-1}$  on  $S_{n-k-1}$  (cf. Theorem 2.12(i)) defined by the formula  $g'(b_1 \cdots b_{n-1}) = g(b_1 \cdots b_{n-1} 0)|_{n-1}$ . Observe that for all  $b_0 \cdots b_{n-1} \in (\frac{0}{1})^n$ :  $g(b_0 b_1 \cdots b_{n-1}) = \tilde{g}(b_0 b_1 \cdots b_k) \cdot \tilde{g}(b_1 \cdots b_{k+1}) \cdots \tilde{g}(b_{n-k-2} \cdots b_{n-1})$  and likewise for  $g'(b_1 \cdots b_{n-1})$ , hence  $g(b_0 b_1 \cdots b_{n-1}) = \tilde{g}(b_0 b_1 \cdots b_k) \cdot g'(b_1 \cdots b_{n-1})$ . Since  $\tilde{g}' = \tilde{g}$ , it follows by induction that  $g'$  is uniform. Thus for every  $c_1 \cdots c_{n-k-1} \in (\frac{0}{1})^{n-k-1}$ :  $|(g')^{-1}(c_1 \cdots c_{n-k-1})| = 2^k$ . Let  $b_1 \cdots b_{n-1} \in (g')^{-1}(c_1 \cdots c_{n-k-1})$ . By assumption it follows that of the pair  $0b_1 \cdots b_k, 1b_1 \cdots b_k$   $\tilde{g}$  will map one to  $0 \in V_1$  and one to  $1 \in V_1$ , and thus  $g$  will map one of the strings  $0b_1 \cdots b_{n-1}, 1b_1 \cdots b_{n-1}$  to  $0c_1 \cdots c_{n-k-1}$  and the other to  $1c_1 \cdots c_{n-k-1}$ . It follows that for all  $c_0 c_1 \cdots c_{n-k-1} \in (\frac{0}{1})^{n-k}$ :  $|g^{-1}(c_0 c_1 \cdots c_{n-k-1})| = |(g')^{-1}(c_1 \cdots c_{n-k-1})| = 2^k$ , which implies that  $g$  is uniform. This completes the inductive argument. ■

**THEOREM 2.15.** *For  $n \geq k+1$ , there are at least  $2 \cdot 2^{2^k} - 2^{2^{k-1}}$  uniform step-simulations of  $S_n$  on  $S_{n-k}$ .*

*Proof.* For  $k=1$  the result follows from Theorem 2.8(ii). For  $k > 1$  we use the characterisation from Theorem 2.14. By induction on  $k$  one easily derives that there exist  $2^{2^k}$  functions  $\tilde{g}: V_{k+1} \rightarrow V_1$  that satisfy the constraint  $\tilde{g}(b_1 \cdots b_{k+1}) = \overline{\tilde{g}(\bar{b}_1 b_2 \cdots b_{k+1})}$ ,  $2^{2^k}$  functions  $\tilde{g}: V_{k+1} \rightarrow V_1$  that satisfy the constraint  $\tilde{g}(b_1 \cdots b_{k+1}) = \overline{\tilde{g}(b_1 \cdots b_k \bar{b}_{k+1})}$ , and  $2^{2^{k-1}}$  functions  $\tilde{g}$  that satisfy both constraints simultaneously. Using the unique correspondence of  $\tilde{g}$  and  $g$ , the given bound follows. ■

By Lemma 1.2 every uniform emulation  $f: S_n \rightarrow S_{n-k}$  ( $n, k \geq 1$ ) also is a uniform emulation of  $\tilde{S}_n$  on  $\tilde{S}_{n-k}$ , and conversely. (Note that  $\tilde{S}_n = (S_n)^R$ .) It follows that all results concerning the uniform emulations of  $S_n$  on  $S_{n-k}$  hold ipso facto for the uniform emulations of  $\tilde{S}_n$  on  $\tilde{S}_{n-k}$ .

### 3. EMULATIONS OF THE CUBE NETWORK

Let  $C_n$  denote the cube network with  $2^n$  nodes. Our main result will be a complete characterisation of the uniform emulations of  $C_n$  on  $C_{n-1}$ , in terms of the uniform emulations of  $C_3$  on  $C_2$ . This Section will be devoted to various auxiliary results and the proof of the main theorem. The argument depends on a crucial lemma (Theorem 3.5) whose lengthy proof is deferred to Appendix B.

The cube network with  $2^n$  nodes (also called an  $n$ -cube) has perhaps been the first proposal ever for processor interconnection. The nodes in the network again are given  $n$ -bit addresses in the range  $0 \cdots 2^n - 1$ , and there is an edge from node  $b$  to node  $c$  if and only if  $c$  is obtained by flipping precisely one bit in  $b$ . Information can be routed from a source  $b$  to a destination  $c$  in at most  $n$  steps, by flipping the bits  $b_i$  to the corresponding bits  $c_i$  in some order. Since nodes thus have degree  $n$ , the cube network is considered practical only for small values of  $n$ . We use  $b, c, \dots$ , to denote full addresses and  $x, y, \dots$ , to denote segments of bits. The  $i$ th bit of an address  $b$  is denoted by  $b_i$  ( $1 \leq i \leq n$ ). For  $|x| = |y|$ , let  $d(x, y)$  be the Hamming distance between the bitstrings  $x$  and  $y$ , i.e., the number of bit-positions in which  $x$  and  $y$  differ (see, e.g., Deo (1974), Sect. 12-5).

**DEFINITION.** The cube network (or  $n$ -cube) is the graph  $C_n = (V_n, E_n)$  with  $V_n = \{(b_1 \cdots b_n) \mid \forall 1 \leq i \leq n \ b_i = \frac{0}{1}\}$  and  $E_n = \{(b, c) \mid b, c \in V_n \text{ and } d(b, c) = 1\}$ .

The fact that  $C_n$  can be (uniformly) emulated on every  $C_{n-k}$  for  $k \geq 1$  easily derives from the following observation, using Lemma 1.1.

**PROPOSITION 3.1.** For  $k \geq 1$ ,  $C_n$  is isomorphic to a spanning subgraph of  $C_{n-k} [C_k]$ .

*Proof.* One verifies that the mapping  $h: C_n \rightarrow C_{n-k} [C_k]$  defined by the formula  $h(b_1 \cdots b_n) = \langle b_1 \cdots b_{n-k}, b_{n-k+1} \cdots b_n \rangle$  is a subgraph-isomorphism. ■

**LEMMA 3.2.**  $f$  is an emulation of  $C_n$  on  $C_{n-k}$  if and only if for all  $b, c \in V_n$ : if  $d(b, c) = 1$  then  $d(f(b), f(c)) \leq 1$ .

(The proof follows directly from the definition of emulation. Note that the condition can be equivalently written as:  $d(f(b), f(c)) \leq d(b, c)$ .) We shall be interested in characterizing the uniform emulations of  $C_n$  on  $C_{n-1}$ .

The distinguishing feature of  $C_n$  is that it admits many more isomorphisms than, e.g.,  $S_n$  (cf. Corollary 2.13). This immediately has consequences for the characterization of uniform emulations, because of the following fact.

LEMMA 3.3. *Let  $I, I'$  be isomorphisms of  $C_n, C_{n-1}$ , respectively. For every  $f$ , if  $f$  is a uniform emulation of  $C_n$  on  $C_{n-1}$  then so is  $I' \circ f \circ I$  (and conversely).*

(The easy proof of Lemma 3.3 is left as an exercise.) The isomorphisms of  $C_n$  can be characterized. For permutations  $\Pi$  let  $I_\Pi$  be the isomorphism defined by  $I_\Pi(b_1 \cdots b_n) = b_{\pi(1)} \cdots b_{\pi(n)}$ , and for index sets  $J \subseteq \{1, \dots, n\}$  let  $I_J$  be the isomorphism defined by

$$(I_J)_i(b_1 \cdots b_n) = \begin{cases} \bar{b}_i & \text{if } i \in J \\ b_i & \text{otherwise} \end{cases}$$

for  $1 \leq i \leq n$ . Thus,  $I_J$  flips the bits in the positions with index in  $J$ .

THEOREM 3.4.  *$I$  is an isomorphism of  $C_n$  if and only if there are  $J, \Pi$  such that  $I = I_J \circ I_\Pi$ .*

*Proof.* The “if”-part is obvious. To prove the “only-if”-part, proceed as follows. Consider  $I(0 \cdots 0)$  and choose  $J$  such that  $i \in J$  if and only if  $I_i(0 \cdots 0) = 1$ . Furthermore choose  $\Pi$  such that if the  $i$ th bit of  $0 \cdots 0$  is flipped, then so is the  $\Pi(i)$ th bit of  $I(0 \cdots 0)$ . Observe that such a permutation  $\Pi$  must exist. Define the weight  $w(b)$  of a bitstring  $b$  as the number of non-zero bits in  $b$ . We prove by induction on  $w(b)$  that for all  $b \in V_n$ :  $I_J^{-1} \circ I = I_\Pi$ . For  $w(b) \leq 1$  it holds: observe that  $I_J^{-1} \circ I(0 \cdots 0) = (0 \cdots 0)$  and that if the  $i$ th bit of  $0 \cdots 0$  is flipped, then so is the  $\Pi(i)$ th bit of  $I_J^{-1} \circ I(0 \cdots 0)$ . Suppose it holds for all  $b$  with  $w(b) \leq m$  for some  $m \geq 1$ . Consider  $b \in V_n$  with  $w(b) = m + 1$  and choose  $c, c' \in V_n$  of weight  $m$ , with  $c \neq c'$  and  $d(b, c) = d(b, c') = 1$ . Suppose  $b$  is obtained from  $c, c'$  by flipping the  $i$ th,  $j$ th bit from 0 to 1, respectively, for some  $i \neq j$ . By induction  $I_J^{-1} \circ I(c) = I_\Pi(c)$  and  $I_J^{-1} \circ I(c') = I_\Pi(c')$  and clearly  $I_\Pi(c)$  and  $I_\Pi(c')$  differ in the  $\Pi(i)$ th and  $\Pi(j)$ th position. If  $I_J^{-1} \circ I(b)$  is obtained from  $I_\Pi(c)$  by flipping a bit in a position  $\notin \{\Pi(i), \Pi(j)\}$  then it will have a distance  $\geq 2$  from  $I_\Pi(c')$ . Contradiction. Suppose  $I_J^{-1} \circ I(b)$  is obtained from  $I_\Pi(c)$  by flipping the  $\Pi(j)$ th bit. Clearly  $c_j = 1$ . Let  $c''$  be the string obtained from  $c$  by setting the  $j$ th bit to 0.  $I_\Pi(c'')$  is obtained from  $I_\Pi(c)$  by flipping the  $\Pi(j)$ th bit, so  $I_J^{-1} \circ I(b) = I_\Pi(c'')$ . It follows that  $w(c'') = m - 1$  and (hence)  $b \neq c''$  and (by induction)  $I_J^{-1} \circ I(b) = I_\Pi(c'') = I_J^{-1} \circ I(c'')$ , contradicting that  $I_J^{-1} \circ I$  is 1-to-1. Thus  $I_J^{-1} \circ I(b)$  is obtained from  $I_\Pi(c)$  by flipping the  $\Pi(i)$ th bit and thus  $I_J^{-1} \circ I(b) = I_\Pi(b)$ . This completes the induction. We conclude that  $I_J^{-1} \circ I = I_\Pi$ , or  $I = I_J \circ I_\Pi$ . ■

Viewing  $C_n$  as the  $n$ -dimensional unit cube brings the analysis of emulations into the realm of combinatorial topology.

DEFINITION. For  $0 \leq m \leq n$ , an  $m$ -face of  $C_n$  is any subgraph (subcube) of  $2^m$  nodes of  $C_n$  that have identical bits in  $n - m$  corresponding positions.

Crucial for the characterization of uniform emulations is the following result, the proof of which is deferred to Appendix B.

THEOREM 3.5. (Topological reduction theorem). *Let  $n \geq 4$ , and let  $f$  be a uniform emulation of  $C_n$  on  $C_{n-1}$ . Then there exists an  $(n-1)$ -face  $A$  of  $C_n$  such that  $f(A)$  is an  $(n-2)$ -face of  $C_{n-1}$ .*

DEFINITION. For mappings  $g: V_3 \rightarrow V_2$ , let  $\hat{g}: V_n \rightarrow V_{n-1}$  be the mapping defined by  $\hat{g}(b_1 \cdots b_n) = g(b_1 b_2 b_3) b_4 \cdots b_n$  ( $n \geq 4$ ).

THEOREM 3.6. (Characterization theorem). *For  $n \geq 3$ ,  $f$  is a uniform emulation of  $C_n$  on  $C_{n-1}$  if and only if there are isomorphisms  $I$  and  $I'$  of  $C_n$  and  $C_{n-1}$ , respectively, and a uniform emulation  $g$  of  $C_3$  on  $C_2$  such that  $f = I' \circ \hat{g} \circ I$ .*

*Proof.* The "if"-part is obvious. For the "only if"-part we induct on  $n$ . The characterization is obvious for  $n = 3$ . Assume it holds up to  $n - 1 \geq 3$ , and consider a uniform emulation  $f$  of  $C_n$  on  $C_{n-1}$ . By Theorem 3.5 there is an  $(n-1)$ -face  $A$  of  $C_n$  such that  $f(A)$  is an  $(n-2)$ -face of  $C_{n-1}$ . Up to isomorphisms of  $C_n$  and  $C_{n-1}$  we may assume that  $A$  is determined by elements  $b$  that have identical  $b_n$  and that  $f(A)$  is determined by elements  $c$  that have identical  $c_{n-1}$ . Because of uniformity no elements of the complementary face  $A^c$  (i.e., the elements with bit  $b_n$  flipped) can be mapped into  $f(A)$ . It follows that  $A^c$  is mapped to  $f(A)^c$  (i.e., the elements of  $f(A)$  with bit  $c_{n-1}$  flipped) and, because  $f$  emulates, that  $f(b_1 \cdots b_{n-1} b_n)$  and  $f(b_1 \cdots b_{n-1} \bar{b}_n)$  are equal in the first  $n-2$  bits for all  $b_1 \cdots b_{n-1} \in V_n$ . It follows that, restricted to  $A \cong V_{n-1}$ ,  $f$  reduces to a mapping  $f'$  depending on  $b_1 \cdots b_{n-1}$  only and  $f(b_1 \cdots b_n) = f'(b_1 \cdots b_{n-1}) b_n$  for all  $b_1 \cdots b_n \in V_n$  or  $f(b_1 \cdots b_n) = f'(b_1 \cdots b_{n-1}) \bar{b}_n$  for all  $b_1 \cdots b_n \in V_n$ . Up to another isomorphism of  $C_{n-1}$  we can assume the former. As a mapping from  $A \cong V_{n-1}$  to  $f(A) \cong V_{n-2}$ ,  $f'$  is seen to act as a uniform emulation of  $C_{n-1}$  on  $C_{n-2}$ . The induction hypothesis now applies to obtain the desired form for  $f$ . ■

The characterisation of Theorem 3.6 is complete once the uniform emulations of  $C_3$  on  $C_2$  are explicitly given. Clearly there are many that are similar, by Lemma 3.3. Characterized by the smallest  $m$  such that an  $m$ -face is mapped to an  $(m-1)$ -face, only three essentially different uniform emulations of  $C_3$  on  $C_2$  can arise. The different types are given in Fig. 1.

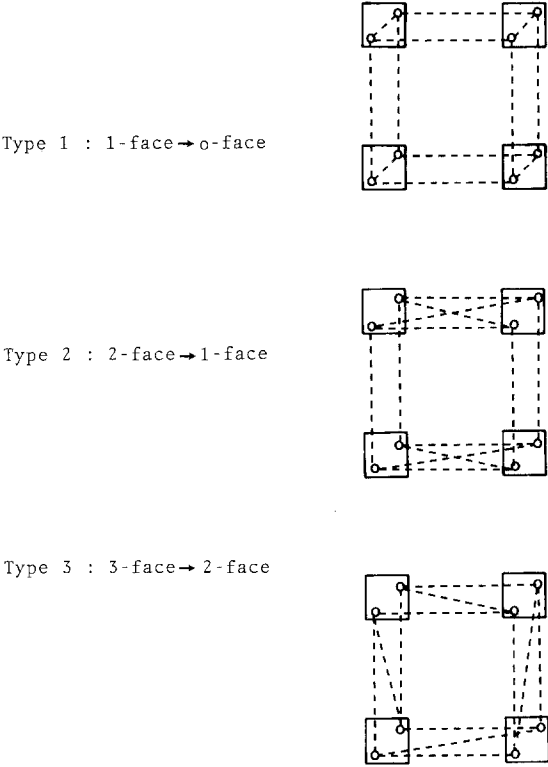


FIGURE 1

It is open whether a similar, complete characterisation can be given of the uniform emulations of  $C_n$  on  $C_{n-k}$  for  $k > 1$ .

4. EMULATIONS OF THE RING  
AND THE TWO-DIMENSIONAL GRID NETWORK

Throughout this section let  $n$  be even, unless stated otherwise. Let  $R_n$  be the ring network with  $n$  nodes, and let  $GR_n$  be the  $n \times n$  grid network (with  $n^2$  nodes) with wrap-around connections. In Section 4.1 we give a complete characterization of the uniform emulations of  $R_n$  on  $R_{n/2}$ . In Section 4.2 we show that the number of emulations of  $GR_n$  on  $GR_{n/2}$  is at least exponential in  $n$ .

4.1. Uniform Emulations of  $R_n$  on  $R_{n/2}$

The ring network is important in practice (cf. Tanenbaum [11]), but hardly occurs as an interconnection network for multiprocessor algorithms.

Indeed the analysis in this Section only prepares for the later study of  $GR_n$ , because  $GR_n \cong R_n \times R_n$ . The nodes of  $R_n$  are named  $0, 1, \dots, n-1$  in consecutive order.

DEFINITION. The ring network (or  $n$ -ring) is the graph  $R_n = (V_n, E_n)$  with  $V_n = \{i \mid i \in N \text{ and } 0 \leq i \leq n-1\}$  and  $E_n = \{(i, i \pm 1) \mid i \in V_n\}$ , where “+” is the addition modulo  $n$ .

By “wrapping” it around  $R_{n/2}$  twice, it follows that  $R_n$  can be uniformly emulated on  $R_{n/2}$ . Our aim will be to characterize all possible uniform emulations of  $R_n$  on  $R_{n/2}$ .

It will be helpful to view  $R_n$  (hence  $R_{n/2}$ ) as a subdivision of the unit circle  $S^1$  in the plane. Clearly, every emulation of  $R_n$  on  $R_{n/2}$  induces a continuous mapping from  $S^1$  to itself. It is well known that such mappings can be characterized by their topological degree or “winding number.” The winding number indicates the number of times the image of  $S^1$  is wrapped

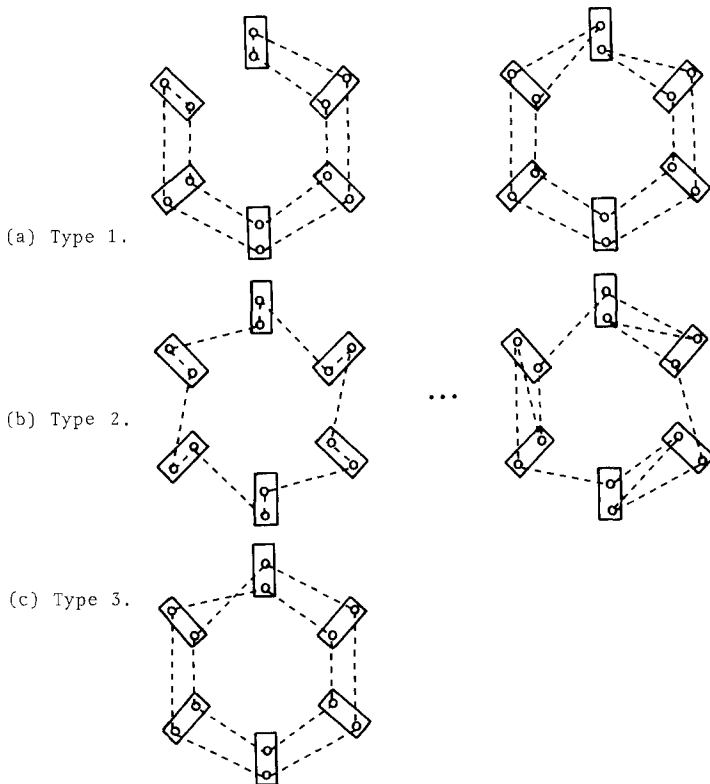


FIGURE 2

around the unit circle when  $S^1$  is traversed once. By analogy we can speak of the winding number of an emulation.

**PROPOSITION 4.1.** *The winding number of an emulation of  $R_n$  on  $R_{n/2}$  is  $-2, -1, 0, +1$ , or  $+2$ .*

*Proof.* Let  $f$  be an emulation of  $R_n$  on  $R_{n/2}$ . If the image of  $R_n$  wraps around  $R_{n/2}$  3 times or more, then the  $n$  nodes of  $R_n$  are mapped to a trajectory of at least  $\frac{3}{2}n$  consecutive points on  $R_{n/2}$ . This is impossible. ■

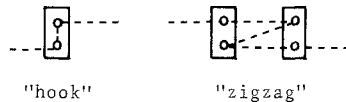
It is relatively straightforward to classify the possible uniform emulations of  $R_n$  on  $R_{n/2}$  by their (positive) winding number.

**Case I.** Winding number = 0.

If  $f(R_n)$  cannot make a full turn around  $R_{n/2}$  then the condition of uniformity forces  $f$  to be one of the two forms suggested in Fig. 2a. We shall refer to the emulations as being of “type 1.”

**Case II.** Winding number = 1.

One verifies that  $f(R)$  must be composed of a number of “hooks” and “zigzags”:



Conversely, any combination of hooks and zigzags defines a uniform emulation of  $R_n$  on  $R_{n/2}$  with winding number 1. We shall refer to the emulations of this kind as being of “type 2”. Figure 2b shows two extreme examples of emulations of type 2.

**Case III.** Winding number = 2.

$f(R_n)$  is necessarily of the kind suggested in Fig. 2c. We shall refer to the emulations of this kind as being of “type 3.”

We conclude the following:

**THEOREM 4.2.** (Characterization theorem). *For  $n$  even,  $f$  is a uniform emulation of  $R_n$  on  $R_{n/2}$  if and only if it is of type 1, type 2, or type 3.*

**COROLLARY 4.3.** *The number of different uniform emulations of  $R_n$  on  $R_{n/2}$  is exponential in  $n$ .*

*Proof.* (Two emulations  $f$  and  $g$  are said to be “different” if  $g$  cannot be obtained by a rotational shift of  $f$ .) Clearly the number of uniform emulations of  $R_n$  on  $R_{n/2}$  of type 2 is exponential in  $n$ . ■



#### 4.2. Uniform Emulations of $GR_n$ on $GR_{n/2}$

The two-dimensional grid (or mesh) has been used as a processor interconnection network, and extensive studies have been made of algorithms to be executed on a grid (e.g., Nassimi and Sahni, 1980). We use a version of the grid with "wrap-around" connections along the boundaries, which gives the nodes a uniform neighbourhood structure. The nodes of  $GR_n$  are named by their plane coordinates  $(i, j)$  with  $0 \leq i, j \leq n-1$ .

**DEFINITION.** The two-dimensional grid network is the graph  $GR_n = (V_n, E_n)$  with  $V_n = \{(i, j) \mid i, j \in N \text{ and } 0 \leq i, j \leq n-1\}$  and  $E_n = \{((i, j), (i', j')) \mid (i, j), (i', j') \in V_n \text{ and } (i = i' \wedge j = j' \pm 1) \text{ or } (i = i' \pm 1 \wedge j = j')\}$ , where "+" is the addition modulo  $n$ .

By "folding"  $GR_n$ , it follows that  $GR_n$  can be uniformly emulated on  $GR_{n/2}$ . Every uniform emulation of  $GR_n$  on  $GR_{n/2}$  has computation factor 4. The classification of the uniform emulations is presently open, but some useful observations can be made.

As  $GR_n \cong R_n \times R_n$ , it can effectively be viewed as a torus. Let  $n \geq 10$  and let  $f$  be a uniform emulation of  $GR_n$  on  $GR_{n/2}$ . Every cycle with 4 nodes, i.e., a "square" in  $GR_n$  must be mapped on  $GR_{n/2}$  by  $f$  in one of the ways shown in Fig. 3.

From this one easily derives that  $f$  induces a continuous mapping of the torus to itself. Again the notion of topological degree (winding number) can be introduced, as expounded in homology theory. Let  $GR_n$  be "spanned" by the oriented cycles  $\mathbf{a} \cong \{(0, j) \mid 0 \leq j \leq n-1\}$  and  $\mathbf{b} \cong \{(i, 0) \mid$

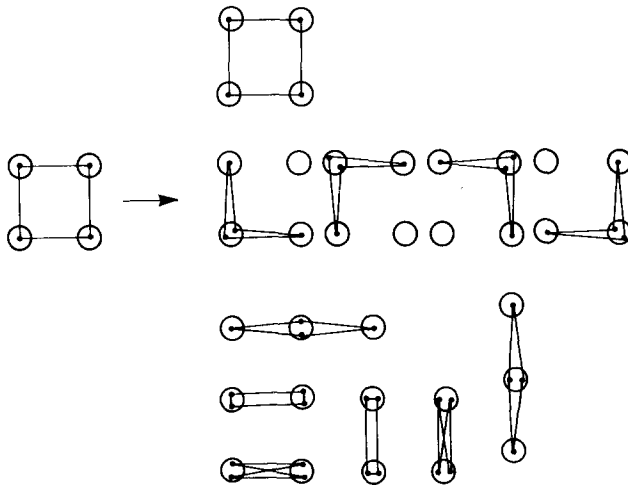


FIGURE 3

$0 \leq i \leq n-1$ }. A closed curve  $C$  can be classified by the pair  $(k, l)$ , where  $k$  is the number of times  $C$  is wrapped in the **a**-direction and  $l$  is the number of times  $C$  is wrapped in the **b**-direction. One can now classify (uniform) emulations by the topological degrees of  $f(\mathbf{a})$  and  $f(\mathbf{b})$ , considered as closed curves on the torus  $GR_{n/2}$ . The underlying reason is the following fact from homology theory: if closed curves  $C, C'$  on the torus have equal topological degree and  $f$  is continuous, then  $f(C)$  and  $f(C')$  have equal topological degree on the torus also.

For  $n \leq 8$ , the same analysis does not necessarily hold. In Table II we given an example of a uniform emulation of  $GR_6$  on  $GR_3$  for which  $f(\mathbf{a}) = f(\{(0, j) \mid 0 \leq j \leq 5\})$  has topological degree  $(1, 1)$  and  $f(\{(1, j) \mid 0 \leq j \leq 5\})$  has topological degree  $(-1, 1)$ . (Hence  $f$  cannot induce a continuous mapping of the torus to itself.)

**PROPOSITION 4.4.** *Let  $f$  be an emulation of  $GR_n$  on  $GR_{n/2}$ . The topological degree  $(k, l)$  of  $f(\mathbf{a})$  and  $f(\mathbf{b})$  satisfies  $|k| + |l| \leq 2$ .*

(The proof follows by observing that the  $n$  points of **a** or **b** can be mapped to a trajectory of at most  $n$  points on  $GR_{n/2}$ .)

**THEOREM 4.5.** *The number of uniform emulations of  $GR_n$  on  $GR_{n/2}$  is at least exponential in  $n$ .*

*Proof.* Let  $g, h$  be uniform emulations of  $R_n$  on  $R_{n/2}$ . Clearly the mapping  $f$  defined by  $f(i, j) = (g(i), h(j))$  is a uniform emulation of  $GR_n$  on  $GR_{n/2}$ . By Corollary 4.3 at least exponentially many uniform emulations are obtained. ■

For the uniform emulations  $f$  defined in the proof of Theorem 4.5, the topological degrees of  $f(\mathbf{a})$  and  $f(\mathbf{b})$  are of the form  $(k, 0)$  and  $(0, l)$ , respectively. Table III shows an example of a uniform emulation  $f$  of  $GR_8$  on  $GR_4$

TABLE II  
A Uniform Emulation of  $GR_6$  on  $GR_3$   
that Does Not Induce a Continuous Mapping  
of the Torus to Itself

23	32	03	13	35	44
34	43	30	31	53	55
01	10	02	14	15	25
24	42	20	41	51	52
00	11	04	12	05	45
22	33	21	40	50	54

TABLE III  
A Uniform Emulation of  $GR_8$  on  $GR_4$  that Is Not  
the Direct Product of Two Uniform Emulations of  $R_8$  on  $R_4$

07	32	24	33	14	25	06	15
43	76	60	77	50	61	42	51
22	31	12	23	04	13	05	30
66	75	56	67	40	57	41	74
10	21	02	11	03	36	20	37
54	65	46	55	47	72	64	73
00	17	01	34	26	35	16	27
44	53	45	70	62	71	52	63

for which  $f(\mathbf{a})$  has topological degree  $(1, 1)$  and  $f(\mathbf{b})$  has topological degree  $(1, -1)$ . (The example can easily be generalized to obtain uniform emulations  $f$  of  $GR_n$  on  $GR_{n/2}$  for which  $f(\mathbf{a})$  has topological degree  $(1, 1)$  and  $f(\mathbf{b})$  has topological degree  $(1, -1)$ , for all even  $n \geq 6$ .)

Similar results can be obtained for the  $d$ -dimensional analogue of  $GR_n$ . Let  $GR_n^d$  be the  $d$ -dimensional grid network (with wrap-around) with size  $n$  in each dimension, i.e., a "hypertorus" with  $n^d$  nodes.

**THEOREM 4.6.** *The number of uniform emulations of  $GR_n^d$  on  $GR_{n/2}^d$  is at least exponential in  $dn$ .*

The proof is a straightforward extension of the argument used for Theorem 4.5.

## 5. CROSS EMULATIONS

By cross-emulation we refer to the emulation of a network  $G$  belonging to some class  $C_1$  on a network  $H$  belonging to a different class  $C_2$ . The question of cross-emulating  $G$  on  $H$  can be important if algorithms must be transported from one type of interconnection network to another. We only consider situations with  $|G| = |H|$ , which means that the resulting uniform (cross-) emulations will necessarily have computation factor 1. Several results of Parker (1980) concerning the "topological" equivalence of some common types of multi-stage networks are easily put into this framework. We only consider cross-emulations between  $S_n$ ,  $C_n$ ,  $R_n$ , and  $GR_n$  (as defined in Sect. 2-4).

In a number of cases the existence of cross-emulations is impossible by degree arguments. For example  $S_n$ ,  $C_n$ , and  $GR_n$  cannot be emulated on a ring network of the same number of nodes.  $C_n$  and  $GR_n$  cannot be

emulated on a shuffle-exchange network with a corresponding number of nodes, and neither can  $S_n$  be cross-emulated on the grid network of an equal number of nodes.

PROPOSITION 5.1. *For  $n \geq 2$ ,  $S_n$  cannot be uniformly emulated on  $C_n$ .*

*Proof.* Suppose there was a uniform emulation  $f$  of  $S_n$  on  $C_n$ . Clearly  $f(00^{n-1})$ ,  $f(10^{n-1})$ , and  $f(0^{n-1}1)$  must be adjacent to one another in  $C_n$ , as the arguments are in  $S_n$ . Thus  $C_n$  contains a triangle. Contradiction. ■

On the positive side, consider  $GR_{2^n}$  (with  $2^{2^n}$  nodes).

THEOREM 5.2. *For  $n \geq 1$ ,  $GR_{2^n}$  can be uniformly emulated on  $C_{2^n}$ .*

*Proof.* We prove the following, slightly stronger claim: for every  $m, n \geq 1$  there is a uniform emulation of the  $2^m \times 2^n$  grid network (with wrap-around connections) on  $C_{m+n}$ . Putting  $m = n$  proves the theorem. To prove the claim, induct on  $m$  and  $n$ . For  $m = n = 1$  the result is immediate. Assume the claim holds for some  $m, n \geq 1$ . Let  $f$  be a uniform emulation of the  $2^m \times 2^n$  grid network on  $C_{m+n}$ . Consider the  $2^{m+1} \times 2^n$  grid network, and map it to  $C_{m+n+1}$  using the mapping  $f'$  defined by

$$f'(i, j) = \begin{cases} 0 \cdot f(i, j) & \text{if } 0 \leq i < 2^m, \\ 1 \cdot f(2^{m+1} - i - 1, j) & \text{otherwise } (2^m \leq i < 2^{m+1}). \end{cases}$$

One easily verifies that  $f'$  is a uniform emulation. Likewise the  $2^m \times 2^{n+1}$  grid network can be uniformly emulated on  $C_{m+n+1}$ . This completes the inductive argument. ■

By a degree argument it easily follows that, conversely,  $C_{2^n}$  can be uniformly emulated on  $GR_{2^n}$  only for  $n = 2$ .

THEOREM 5.3. *For values of  $n$  as indicated,  $R_n$  can be emulated on the following networks:*

- (i) for  $n = k^2$ , on  $GR_k$ .
- (ii) for  $n = 2^k$ , on  $S_k$  and on  $C_k$ .

*Proof.* (The results are equivalent to claiming that  $GR_k$ ,  $S_k$ , and  $C_k$  are hamiltonian.)

- (i) Left as an exercise.

(ii) By the existence of binary de Bruijn sequences (de Bruijn, 1946) it follows that every  $S_k$  has a hamiltonian circuit. To obtain the result for  $C_k$ , write  $k = k_1 + k_2$ . As the result is obvious for  $k = 1$ , we may assume that  $k_{1,2} \geq 1$ . It is easy to show that  $R_n$  can be uniformly emulated on the

$2^{k_1} \times 2^{k_2}$  grid network (with wrap-around connections). In the proof of Theorem 5.2 it was shown that the  $2^{k_1} \times 2^{k_2}$  grid network can be uniformly emulated on  $C_{k_1+k_2} = C_k$ . By transitivity the result follows. ■

Observe that every uniform emulation of the ring of  $2^k$  elements on  $C_k$  corresponds to a Gray code of length  $k$  (cf. Reingold, Nievergelt, and Deo, 1977).

## 6. DEFINING NETWORKS BY EMULATION

Every network  $H = (V_H, E_H)$  can act as a "host" under emulation for many different, larger networks. If we restrict the class of admissible (uniform) mappings that should act as emulations, then the set of graphs  $G$  that can be emulated on  $H$  will likely be restricted also. Our main result will be that  $S_n$ ,  $C_n$ ,  $R_n$ , and  $GR_n$  are uniquely defined by their emulations on  $S_{n-1}$ ,  $C_{n-1}$ ,  $R_{n/2}$ , and  $GR_{n/2}$  respectively. In Section 6.1 we derive some general results on defining networks by admissible sets of emulations. In Section 6.2 we prove the main results.

### 6.1. General Characterizations

Let  $H = (V_H, E_H)$  be a given host network,  $V$  a set of nodes with  $|V| \geq |V_H|$ , and  $F$  a collection of functions from  $V$  onto  $V_H$ .

**DEFINITION.** A network  $G = (V, E)$  is said to be  $F$ -emulated on  $H$  if every  $f \in F$  is an emulation of  $G$  on  $H$ .

Our aim will be to characterize all networks  $G$  that are  $F$ -emulated on  $H$ , given  $F$  and  $H$ . We assume  $H$  and  $V$  to be fixed, and  $F$  to be variable.

**DEFINITION.**  $E_F = \{(v, v') \mid v, v' \in V, v \neq v' \text{ and } \forall f \in F: f(v) = f(v') \text{ or } (f(v), f(v')) \in E_H\}$ .

**THEOREM 6.1.** (Characterization theorem).  $G$  is  $F$ -emulated on  $H$  if and only if  $G$  is a spanning subgraph of  $(V, E_F)$ .

*Proof.* Let  $G = (V, E)$  be  $F$ -emulated on  $H$ , and let  $(v, v') \in E$ . By definition (of emulation) we have that for every  $f \in F$ :  $f(v) = f(v')$  or  $(f(v), f(v')) \in E_H$ . Thus  $(v, v') \in E_F$ . It follows that  $E \subseteq E_F$ , and  $G$  is a spanning subgraph of  $(V, E_F)$ . Conversely, it is clear that  $(V, E_F)$  is  $F$ -emulated on  $H$  by definition. Clearly, every spanning subgraph is  $F$ -emulated on  $H$  also. ■

It follows that  $(V, E_F)$  is the maximal graph that can be  $F$ -emulated on  $H$ . Define  $f: V \rightarrow V_H$  to be uniform if for all  $h \in V_H$ :  $|f^{-1}(h)| = c$ , for some constant  $c = |V|/|V_H|$  (the computation factor).

**THEOREM 6.2.** *Let  $f, f': V \rightarrow V_H$  be uniform functions. Then  $(V, E_{\{f\}})$  and  $(V, E_{\{f'\}})$  are isomorphic graphs.*

*Proof.* Since  $f, f': V \rightarrow V_H$  are uniform (and thus map equal sized piles of elements onto every node of  $H$ ) there exists a permutation  $\Pi: V \rightarrow V$  such that for all  $v \in V: f(v) = f'(\Pi(v))$ . One easily verifies that  $\Pi$  is, in fact, an isomorphism of  $(V, E_{\{f\}})$  and  $(V, E_{\{f'\}})$ . ■

We derive a further result to characterize  $(V, E_{\{f\}})$  when  $f$  is uniform. Let  $c$  be as defined above.

**LEMMA 6.3.** *Let  $f: V \rightarrow V_H$  be uniform. Let  $d_{\text{out}}$  and  $d_{\text{in}}$  be the maximum out-degree and the maximum in-degree of the nodes in  $H$ , respectively. (If  $H$  is undirected, let  $d_{\text{out}} = d_{\text{in}}$  be the maximum degree in  $H$ .)*

- (i) *the maximum out-degree in  $(V, E_{\{f\}})$  equals  $(d_{\text{out}} + 1) \cdot c - 1$ .*
- (ii) *the maximum in-degree in  $(V, E_{\{f\}})$  equals  $(d_{\text{in}} + 1) \cdot c - 1$ .*
- (iii) *If  $G$  and  $H$  are undirected graphs, then  $|E_{\{f\}}| = \frac{1}{2}c(c-1)|V_H| + c^2 \cdot |E_H|$ .*
- (iv) *If  $G$  and  $H$  are directed graphs, then  $|E_{\{f\}}| = c(c-1)|V_H| + c^2 \cdot |E_H|$ .*

*Proof.* (i) Consider any node  $v \in V$ . By uniformity there are precisely  $c - 1$  nodes  $v' \neq v$  with  $f(v) = f(v')$ , which thus accounts for  $c - 1$  outgoing edges with this property. Next there are at most  $d_{\text{out}} \cdot c$  nodes  $v'$  with  $(f(v), f(v')) \in E_H$ . This accounts for a maximal out-degree of  $c + 1 - d_{\text{out}} = (d_{\text{out}} + 1)c - 1$ . By choosing  $v$  such that  $f(v)$  has maximum out-degree, it is clear that the bound is attained.

(ii) Similar to (i).

(iii)  $E_H$  contains  $|V_H| \cdot \frac{1}{2}c(c-1)$  edges  $(v, v')$  with  $f(v) = f(v')$ , because  $c$  nodes of  $V$  are mapped to every  $h \in V_H$ . Every edge  $(h, h') \in E_H$  accounts for  $c^2$  edges  $(v, v')$  with  $f(v) = h$  and  $f(v') = h'$ . By definition  $E_f$  contains no other edges than the ones that were distinguished.

(iv) Similar to (iii). ■

Lemma 6.3 will be useful later because, whenever  $f \in F$  and  $G$  is  $F$ -emulated on  $H$ , then  $G$  is a spanning subgraph of  $(V, E_{\{f\}})$ .

## 6.2. Characterization of the Shuffle-Exchange, the Cube, the Ring and the Grid Networks by Emulation

We use the definitions and results concerning  $S_n$ ,  $C_n$ ,  $R_n$ , and  $GR_n$  as presented in Sections 2–4. First we consider  $S_n$ , the shuffle-exchange graph

on  $2^n$  nodes. From Table I we recall the following uniform emulations of  $S_n$  on  $S_{n-1}$ ,

$$f_1: f_1(b_1 \cdots b_n) = b_1 \cdots b_{n-1},$$

$$f_2: f_2(b_1 \cdots b_n) = b_2 \cdots b_n,$$

$$f_3: f_3(b_1 \cdots b_n) = c_1 \cdots c_{n-1} \quad \text{with} \quad c_i = (b_i \equiv b_{i+1}) \text{ for } 1 \leq i \leq n-1.$$

We show that  $S_n$  is uniquely characterized by these three emulations on  $S_{n-1}$ . Let  $V \cong V_n (= \binom{0}{1}^n)$ ,  $n \geq 2$ .

**THEOREM 6.4.**  $(V, E_{\{f_1, f_2, f_3\}}) = S_n$ .

*Proof.* Clearly  $S_n$  is a spanning subgraph of  $(V, E_{\{f_1, f_2, f_3\}})$ , by definition (or Theorem 6.1). We show that every edge of  $(V, E_{\{f_1, f_2, f_3\}})$  must be an edge of  $S_n = (V_n, E_n)$ . Consider any edge  $(b_1 \cdots b_n, c_1 \cdots c_n) \in E_{\{f_1, f_2, f_3\}}$ . We distinguish the following cases:

(a)  $f_i(b_1 \cdots b_n) = f_i(c_1 \cdots c_n)$  for  $1 \leq i \leq 2$ . It follows that  $b_1 \cdots b_{n-1} = c_1 \cdots c_{n-1}$  and  $b_2 \cdots b_n = c_2 \cdots c_n$  and (hence)  $b_1 \cdots b_n = c_1 \cdots c_n$ , contradicting that we had an edge between distinct points.

(b)  $f_i(b_1 \cdots b_n) = f_i(c_1 \cdots c_n)$  for  $i = 1, 3$  (and  $(f_2(b_1 \cdots b_n), f_2(c_1 \cdots c_n)) \in E_{n-1}$ ). It follows that  $b_1 \cdots b_{n-1} = c_1 \cdots c_{n-1}$  and  $(b_{n-1} \equiv b_n) = (c_{n-1} \equiv c_n)$ , so  $b_1 \cdots b_n = c_1 \cdots c_n$ , a contradiction.

(c)  $(f_1(b_1 \cdots b_n), f_1(c_1 \cdots c_n)) \in E_{n-1}$  and  $f_2(b_1 \cdots b_n) = f_2(c_1 \cdots c_n)$ . It follows that  $b_2 \cdots b_{n-1} \alpha = c_1 \cdots c_{n-1}$  and  $b_2 \cdots b_n = c_2 \cdots c_n$  for some  $\alpha$ , hence  $b_1 \cdots b_n = b_1 \alpha^{n-1}$  and  $c_1 \cdots c_n = \alpha^n$ . Clearly  $(b_1 \alpha^{n-1}, \alpha^n) \in E_n$ .

(d)  $f_1(b_1 \cdots b_n) = f_1(c_1 \cdots c_n)$  and  $(f_i(b_1 \cdots b_n), f_i(c_1 \cdots c_n)) \in E_{n-1}$  for  $i = 2, 3$ . It follows that  $b_1 \cdots b_{n-1} = c_1 \cdots c_{n-1}$  and  $b_3 \cdots b_n \alpha = c_2 \cdots c_n$  for some  $\alpha$ , hence  $b_1 \cdots b_n = b_1 b_n^{n-1}$  and  $c_1 \cdots c_n = b_1 b_n^{n-2} \alpha$ . Now  $(f_3(b_1 b_n^{n-1}), f_3(b_1 b_n^{n-2} \alpha)) \in E_{n-1}$  implies  $b_1 = b_n$ , and clearly  $b_1 b_n^{n-1}, b_1 b_n^{n-2} \alpha \in E_n$ .

(e)  $(f_i(b_1 \cdots b_n), f_i(c_1 \cdots c_n)) \in E_{n-1}$  for  $i = 1, 2$ . It follows that  $b_2 \cdots b_{n-1} \alpha = c_1 \cdots c_{n-1}$  and  $b_3 \cdots b_n \beta = c_2 \cdots c_n$  for suitable  $\alpha$  and  $\beta$ , hence  $b_1 \cdots b_n = b_1 c_1 \cdots c_{n-1}$ . Clearly  $(b_1 c_1 \cdots c_{n-1}, c_1 \cdots c_n) \in E_n$ . ■

(It can be verified that no subset of  $\{f_1, f_2, f_3\}$  is sufficient to characterize  $S_n$ .) Next consider  $C_n$ , the cube network on  $2^n$  nodes. We select the following uniform emulations of  $C_n$  on  $C_{n-1}$ :

$$f_1: f_1(b_1 \cdots b_n) = b_1 \cdots b_{n-1}$$

$$f_4: f_4(b_1 \cdots b_n) = (b_1 \equiv b_2) b_3 \cdots b_n$$

**THEOREM 6.5.** For  $n \geq 3$ ,  $(V, E_{\{f_1, f_4\}}) = C_n$ .

*Proof.* Clearly  $C_n$  is a spanning subgraph of  $(V, E_{\{f_1, f_4\}})$ . Consider any edge  $(b_1 \cdots b_n, c_1 \cdots c_n) \in E_{\{f_1, f_4\}}$ . We distinguish the following cases:

(a)  $f_1(b_1 \cdots b_n) = f_1(c_1 \cdots c_n)$ . It follows that  $b_1 \cdots b_{n-1} = c_1 \cdots c_{n-1}$ , and either  $b_1 \cdots b_n = c_1 \cdots c_n$  (a contradiction) or  $b_1 \cdots b_n = c_1 \cdots c_{n-1} \bar{c}_n$ . It follows that  $(b_1 \cdots b_n, c_1 \cdots c_n) \in E_n$ .

(b)  $(f_1(b_1 \cdots b_n), f_1(c_1 \cdots c_n)) \in E_{n-1}$  and  $f_4(b_1 \cdots b_n) = f_4(c_1 \cdots c_n)$ . It follows that  $d(b_1 \cdots b_{n-1}, c_1 \cdots c_{n-1}) = 1$  and  $b_1 \cdots b_n = b_1 b_2 c_3 \cdots c_n$  with  $(b_1 \equiv b_2) = (c_1 \equiv c_2)$ . It follows that  $b_n = c_n$ , and thus  $(b_1 \cdots b_n, c_1 \cdots c_n) \in E_n$ .

(c)  $(f_1(b_1 \cdots b_n), f_1(c_1 \cdots c_n)) \in E_{n-1}$  and  $(f_4(b_1 \cdots b_n), f_4(c_1 \cdots c_n)) \in E_{n-1}$ . It follows that  $d(b_1 \cdots b_{n-1}, c_1 \cdots c_{n-1}) = 1$  and  $d(\alpha b_3 \cdots b_n, \beta c_3 \cdots c_n) = 1$ , with  $\alpha = (b_1 \equiv b_2)$  and  $\beta = (c_1 \equiv c_2)$ . If  $\alpha = \beta$  then necessarily  $b_1 b_2 = c_1 c_2$  and  $d(b_1 \cdots b_n, c_1 \cdots c_n) = 1$  thus  $(b_1 \cdots b_n, c_1 \cdots c_n) \in E_n$ . If  $\alpha \neq \beta$  then  $b_3 \cdots b_n = c_3 \cdots c_n$  and (hence)  $d(b_1 b_2, c_1 c_2) = 1$ . Clearly it follows that  $(b_1 \cdots b_n, c_1 \cdots c_n) \in E_n$ .

We conclude that every edge of  $(V, E_{\{f_1, f_4\}})$  also is an edge of  $C_n$ . ■

Theorem 6.5 is “minimal” in the sense that  $C_n$  cannot be characterized from  $C_{n-1}$  by means of just one uniform emulation.

**PROPOSITION 6.6.** There does not exist a uniform emulation  $f$  of  $C_n$  on  $C_{n-1}$  such that  $(V, E_{\{f\}}) = C_n$ .

*Proof.* Observe that (the undirected graph)  $C_{n-1}$  has  $2^{n-1}$  nodes and  $\frac{1}{2}2^{n-1}(n-1)$  edges. Suppose a uniform mapping  $f: V \rightarrow V_{n-1}$  exists with  $(V, E_{\{f\}}) = C_n$ . By Lemma 6.3(iii)  $(V, E_{\{f\}})$  must have  $n \cdot 2^n - 2^{n-1}$  edges ( $c = 2$ ), which is more than  $C_n$  can have. ■

Consider  $R_n$ , the ring on  $n$  nodes. Define the following uniform emulations of  $R_n$  on  $R_{n/2}$  ( $n$  even):

$$\begin{aligned} g_1: g_1(i) &= \lfloor \frac{i}{2} \rfloor, \\ g_2: g_2(i) &= i \bmod n/2. \end{aligned}$$

**THEOREM 6.7.** For  $n > 8$ ,  $(V, E_{\{g_1, g_2\}}) = R_n$ .

*Proof.* Clearly  $R_n$  is a spanning subgraph of  $(V, E_{\{g_1, g_2\}})$ . Consider any edge  $(i, j) \in E_{\{g_1, g_2\}}$ . If  $g_1(i) = g_1(j)$  then  $|i - j| = 1$  and  $(i, j) \in E_n$ . If  $(g_1(i), g_1(j)) \in E_{n/2}$ , then we may assume without loss of generality that  $\lfloor i/2 \rfloor = \lfloor j/2 \rfloor + 1 \pmod{n/2}$ . It follows that  $i \equiv j + 2 + \delta_i - \delta_j \pmod{n}$ , with  $\delta_i$  and  $\delta_j$  Kronecker  $\delta$ 's. Now, in addition,  $g_2(i) = g_2(j)$  or  $(g_2(i), g_2(j)) \in E_{n/2}$ . If  $g_2(i) = g_2(j)$  then  $|i - j| \equiv 0 \pmod{n/2}$ , hence  $2 + \delta_i - \delta_j \equiv 0 \pmod{n/2}$  and,



by the assumption on  $n$ , necessarily  $2 + \delta_i - \delta_j = 0$  and  $i = j$ . Contradiction. If  $(g_2(i), g_2(j)) \in E_{n/2}$  then  $i \equiv j \pm 1 \pmod{n/2}$ , hence  $2 + \delta_i - \delta_j \equiv \pm 1 \pmod{n/2}$ . Since  $n > 8$ , we have  $2 + \delta_i - \delta_j = 1$  and  $i \equiv j + 1 \pmod{n}$ . Thus  $(i, j) \in E_n$ . We conclude that every edge of  $(V, E_{\{g_1, g_2\}})$  is an edge of  $R_n$ . ■

Finally consider  $GR_n$ , the grid network on  $n^2$  nodes. Define the following uniform emulations of  $GR_n$  on  $GR_{n/2}$  ( $n$  even):

$$\begin{aligned} h_1: h_1(i, j) &= (\lfloor \frac{i}{2} \rfloor, \lfloor \frac{j}{2} \rfloor), \\ h_2: h_2(i, j) &= (i \bmod n/2, j \bmod n/2). \end{aligned}$$

THEOREM 6.8. For  $n > 8$ ,  $(V, E_{\{h_1, h_2\}}) = GR_n$ .

*Proof.* Similar to the proof of theorem 6.7. ■

#### APPENDIX A: THE PROOF OF THE CHARACTERISATION THEOREM FOR THE UNIFORM EMULATIONS OF $S_n$ ON $S_{n-1}$ (THEOREM 2.9)

We use the notations and terminology of Section 2. Our aim is to prove the following result:

THEOREM 2.9. (Characterisation theorem). *Every uniform emulation of  $S_n$  on  $S_{n-1}$  is step-simulating, and thus equal to one of the mappings listed in Table I.*

The proof is based on the lemma below and a subsequent analysis of cases. Some further notational conventions will be helpful to deal with the elements of  $(\frac{0}{1})^n$  and similar sets as strings:

- [0]: zero or one occurrence of bit 0 (i.e., “empty” or “0”),
- [1]: zero or one occurrence of bit 1 (i.e., “empty” or “1”),
- (01)\*: zero or more repetitions of the string 01 (as required),
- (10)\*: zero or more repetitions of the string 10 (as required).

The length ( $n$ ) of a bitstring will always be clear from the context, and is usually not given by separate indices. For example, the notation  $(01)^*[0]$  for  $n$  odd will denote the string  $(01)^{\lfloor n/2 \rfloor}0$ . For  $n$  even it will denote the string  $(01)^{n/2}$ . Assume  $n > 2$ .

For the proof of Theorem 2.9, assume that there exists a uniform emulation  $f$  of  $S_n$  on  $S_{n-1}$  that is not step-simulating. It follows that there must be an  $x \in (\frac{0}{1})^{n-1}$ ,  $y \in (\frac{0}{1})^{n-3}$  and  $\alpha, \beta, \gamma, \delta \in (\frac{0}{1})$  such that  $f(\alpha x) = \beta y \delta$

and  $f(x\gamma) = \beta y \delta$ , with  $\beta y \neq y \delta$  (cf. Lemmas 2.3 and 2.6). We will fix the notation throughout the remainder of this section.

*Claim 2.9.1.* Under the assumption stated, one of the following situations must hold

- (i)  $x = 0^{n-1}$  and  $(\alpha = 0 \vee \gamma = 0)$ ,
- (ii)  $x = 1^{n-1}$  and  $(\alpha = 1 \vee \gamma = 1)$ ,
- (iii)  $\beta y \delta = (01)^*[0]$ ,
- (iv)  $\beta y \delta = (10)^*[1]$ .

*Proof.* In addition to  $f(\alpha x) = f(x\gamma) = \beta y \delta$  we must have:  $(f(\bar{\alpha}x) = \beta y \delta \vee f(\bar{\alpha}x) = \frac{0}{1}\beta y)$  and  $(f(x\bar{\gamma}) = \beta y \delta \vee f(x\bar{\gamma}) = y\delta \frac{0}{1})$ , from the emulation property. Because  $f$  is uniform, only two nodes can be mapped to  $\beta y \delta$ . The following situations can be distinguished:

(a)  $f(\bar{\alpha}x) = f(\alpha x) = \beta y \delta$ . Because  $f(x\gamma) = \beta y \delta$  also, we have  $x\gamma = \bar{\alpha}x$  ( $\Rightarrow x = 0^{n-1}$  and  $\alpha = 1$  and  $\gamma = 0$ , or  $x = 1^{n-1}$  and  $\alpha = 0$  and  $\gamma = 1$ ) or  $x\gamma = \alpha x$  ( $\Rightarrow x = 0^{n-1}$  and  $\alpha = 0$  and  $\gamma = 0$ , or  $x = 1^{n-1}$  and  $\alpha = 1$  and  $\gamma = 1$ ).

(b)  $f(x\bar{\gamma}) = f(x\gamma) = \beta y \delta$ . Now also  $f(\alpha x) = \beta y \delta$ , and the same cases as under (a) result.

(c)  $f(\bar{\alpha}x) = \frac{0}{1}\beta y$  and  $f(x\bar{\gamma}) = y\delta \frac{0}{1}$ . Clearly  $f(x\bar{\gamma}) = \frac{0}{1}\beta y$  or  $f(x\bar{\gamma}) = \beta y \delta$ , hence  $\frac{0}{1}\beta y = y\delta \frac{0}{1}$  or  $\beta y \delta = y\delta \frac{0}{1}$ . Because  $\beta y \neq y \delta$  only the former case can arise:  $\frac{0}{1}\beta y = y\delta \frac{0}{1}$ . It follows that  $\beta y \delta = (01)^*[0]$  or  $\beta y \delta = (10)^*[1]$ . (The "solutions"  $\beta y \delta = 0^{n-1}$  and  $\beta y \delta = 1^{n-1}$  are not valid, because it would yield  $\beta y = y \delta$ .) ■

We now obtain the basic step for the further case analysis.

*LEMMA 2.9.2.* Under the assumption stated, one of the following six cases must hold:

- (I)  $f((01)^*[0]) = 0^{n-1}$ ,  $f((10)^*[1]) = 0^{n-1}$ ,
- (II)  $f((01)^*[0]) = 1^{n-1}$ ,  $f((10)^*[1]) = 1^{n-1}$ ,
- (III)  $f((01)^*[0]) = (01)^*[0]$ ,  $f((10)^*[1]) = (01)^*[0]$ ,
- (IV)  $f((01)^*[0]) = (01)^*[0]$ ,  $f((10)^*[1]) = (10)^*[1]$ ,
- (V)  $f((01)^*[0]) = (10)^*[1]$ ,  $f((10)^*[1]) = (01)^*[0]$ ,
- (VI)  $f((01)^*[0]) = (10)^*[1]$ ,  $f((10)^*[1]) = (10)^*[1]$ .

*Proof.* Let  $f((01)^*[0]) = u_1 \cdots u_{n-1}$  and  $f((10)^*[1]) = v_1 \cdots v_{n-1}$ . Because  $(01)^*[0]$  and  $(10)^*[1]$  are adjacent in  $S_n$  and  $f$  is an emulation, the following situations can arise:

(a)  $u_1 \cdots u_{n-1} = v_1 \cdots v_{n-1}$ . Write  $u_1 \cdots u_{n-1} = \beta y \delta$ . (Note that we cannot assume that  $\beta y \neq y \delta$ .) By the analysis under Claim 2.9.1 it follows that

$\frac{0}{1}\beta y = y\delta_1^0$  (hence  $\beta y\delta = 0^{n-1}$ ,  $1^{n-1}$ ,  $(01)^*[0]$ , or  $(10)^*[1]$ ) or  $\beta y\delta = y\delta_1^0$  (hence  $\beta y\delta = 0^{n-1}$  or  $1^{n-1}$ ). This proves cases (I), (II), (III), and (VI).

(b)  $u_1 \cdots u_{n-1} \neq v_1 \cdots v_{n-1}$ , but  $u_1 \cdots u_{n-1} = v_2 \cdots v_{n-1} \frac{0}{1} = \frac{0}{1}v_1 \cdots v_{n-2}$ . It follows that  $v_1 \cdots v_{n-1} = 0^{n-1}$ ,  $1^{n-1}$ ,  $(01)^*[0]$ ,  $(10)^*[1]$  but only for the latter two cases can  $u_1 \cdots u_{n-1}$  be chosen to satisfy the constraint (namely  $u_1 \cdots u_{n-1} = (10)^*[1]$ ,  $(01)^*[0]$ , respectively). This proves cases (IV) and (V). ■

We proceed by analysing the cases of Lemma 2.9.2 and showing that in each case a contradiction must arise. (Recall the assumption that  $f$  is uniform and not step-simulating.)

*Case I.*  $f((01)^*[0]) = f((10)^*[1]) = 0^{n-1}$ .

We show that this forces  $f$  to be equal to  $f_3$ , one of the six step-simulations listed in Table I.

*Claim 2.9.3.* For  $1 \leq i \leq n-1$  and  $b \in (\frac{0}{1})^n$ ,  $f_i(b_1 \cdots b) = (b_i \equiv b_{i+1})$ .

*Proof.* Define  $B_i^n = \{b_1 \cdots b_n \mid \forall j: 1 \leq j \leq i-1 \Rightarrow b_j \neq b_{j+1}\} \subseteq (\frac{0}{1})^n$  and  $C_{i-1}^{n-1} = \{c_1 \cdots c_{n-1} \mid \forall j: 1 \leq j \leq i-1 \Rightarrow c_j = 0\} \subseteq (\frac{0}{1})^{n-1}$ . Note that  $B_n^n = \{(01)^*[0], (10)^*[1]\}$  and  $C_{n-1}^{n-1} = \{0^{n-1}\}$ , and hence that  $f(B_n^n) = C_{n-1}^{n-1}$  and  $f^{-1}(C_{n-1}^{n-1}) = B_n^n$  (by uniformity). We claim that for all  $1 \leq i \leq n$ ,  $f(B_i^n) = C_{i-1}^{n-1}$  and  $f^{-1}(C_{i-1}^{n-1}) = B_i^n$ . For a proof, use downward induction starting with  $i = n$ , for which the claim clearly holds. Suppose it holds for some  $i \geq 1$ . Consider any  $b_1 \cdots b_n \in B_{i-1}^n$ . It follows that  $\bar{b}_1 b_1 \cdots b_{n-1} \in B_i^n$  and thus that  $f(\bar{b}_1 b_1 \cdots b_{n-1}) \in C_{i-1}^{n-1}$ . Since  $f$  is an emulation, we must have  $f(b_1 \cdots b_n) \in C_{i-1}^{n-1}$  or  $f(b_1 \cdots b_n) \in C_{i-2}^{n-1}$ . In either case  $f(b_1 \cdots b_n) \in C_{i-2}^{n-1}$ , and we have  $f(B_{i-1}^n) \subseteq C_{i-2}^{n-1}$ . Because  $|B_{i-1}^n| = 2|C_{i-2}^{n-1}|$  and  $f$  is uniform, we have in fact  $f(B_{i-1}^n) = C_{i-2}^{n-1}$  and ipso facto  $f^{-1}(C_{i-2}^{n-1}) = B_{i-1}^n$ . This completes the inductive argument.

We immediately conclude (take  $i = 2$ ) that for all  $x \in (\frac{0}{1})^{n-2}$ ,  $f_1(01x) = 0$  and  $f_1(10x) = 0$ . Because of uniformity this forces  $f_1(00x) = f_1(11x) = 1$  for all  $x \in (\frac{0}{1})^{n-2}$ . Define  $\tilde{B}_i^n = \{b_1 \cdots b_n \mid b_n \cdots b_1 \in B_i^n\}$  and  $\tilde{C}_{i-1}^{n-1} = \{c_1 \cdots c_{n-1} \mid c_{n-1} \cdots c_1 \in C_{i-1}^{n-1}\}$ . As before one shows that for all  $1 \leq i \leq n$ ,  $f(\tilde{B}_i^n) = \tilde{C}_{i-1}^{n-1}$  and  $f^{-1}(\tilde{C}_{i-1}^{n-1}) = \tilde{B}_i^n$ . We now argue by downward induction on  $i$  that for all  $x \in \tilde{B}_i^n$ ,  $f(x) = f_3(x)$  (with  $f_3$  as in Table I). For  $i = n$  we have  $\tilde{B}_n^n = \{(01)^*[0], (10)^*[1]\}$  and  $f((01)^*[0]) = f((10)^*[1]) = 0^{n-1}$ , which indeed coincides with  $f_3$ . Suppose it holds for some  $i \geq 1$ . Consider any  $x_1 \cdots x_{n-i+1}(01)^*[0] \in \tilde{B}_{i-1}^n$ . If  $x_{n-i+1} = 1$  then  $f$  and  $f_3$  coincide on the argument by induction. Let  $x_{n-i+1} = 0$ . Observe that  $f(x_2 \cdots x_{n-i+1}(01)^*[0]) \in f(\tilde{B}_i^n) = \tilde{C}_{i-1}^{n-1}$  and that  $f(x_1 \cdots x_{n-i+1}(01)^*[0]) \in f(\tilde{B}_{i-1}^n \setminus \tilde{B}_i^n) = \tilde{C}_{i-2}^{n-1} \setminus \tilde{C}_{i-1}^{n-1}$ , where the latter holds because  $x_{n-i+1} = 0$  and uniformity of  $f$ . It follows that  $f(x_1 \cdots x_{n-i+1}(01)^*[0]) \neq$

$f(x_2 \cdots x_{n-i+1}(01)^*[0])$  and thus, because  $f$  is an emulation, necessarily  $f(x_1 \cdots x_{n-i+1}(01)^*[0]) = \frac{0}{1} \cdot f(x_2 \cdots x_{n-i+1}(01)^*[0])|_{n-2}$ . Using the inductive assertion it follows that  $f_i(x_1 \cdots x_{n-i+1}(01)^*[0]) = (f_3)_i(x_1 \cdots x_{n-i+1}(01)^*[0])$  for all  $2 \leq i \leq n-1$ . At the beginning of this paragraph we showed that this must hold also for  $i=1$ . Thus  $f$  and  $f_3$  coincide on  $\tilde{B}_{i-1}^n$ , which completes the inductive argument. Because  $\tilde{B}_1^n = V_n$  this shows that  $f$  and  $f_3$  coincide for all arguments, which proves the claim. ■

Because  $f$  was assumed not to be step-simulating, Claim 2.9.3 clearly proves that Case I is contradictory.

*Case II.*  $f((01)^*[0]) = f((10)^*[1]) = 1^{n-1}$ .

The proof of Claim 2.9.3 can be completely dualized to show that in this case  $f$  must be equal to  $f_3$ , another one of the six step-simulations listed in Table I. Because  $f$  was assumed not to be step-simulating, this case is also contradictory.

*Case III.*  $f((01)^*[0]) = f((10)^*[1]) = (01)^*[0]$ .

We show that for  $n > 2$  no emulation  $f$  of  $S_n$  on  $S_{n-1}$  with this property exists. Suppose on the contrary that an  $f$  does exist. We derive a contradiction as follows.

First let  $n$  be odd, which implies that the assumption turns into  $f((01)^*0) = f((10)^*1) = (01)^*$ . Since  $f$  is uniform no other nodes can be mapped to  $(01)^*$ , and we necessarily obtain:  $f(00(10)^*1) = \frac{0}{1}(01)^*0$ ,  $f(1(01)^*00) = 1(01)^*\frac{0}{1}$ ,  $f(11(01)^*0) \in \{1(01)^*\frac{0}{1}, 11(01)^*\}$ ,  $f(0(10)^*11) \in \{\frac{0}{1}(01)^*0, (01)^*00\}$ . Observing that necessarily  $(f(11(01)^*0), f((10)^*1)) \in E_{n-1}$  and  $(f((10)^*1), f(0(10)^*11)) \in E_{n-1}$  it follows that  $f(11(01)^*0) = 1(01)^*\frac{0}{1}$  and  $f(0(10)^*11) = \frac{0}{1}(01)^*0$ , and the emulation property now forces that  $f(1(01)^*00) = f(11(01)^*0) = 1(01)^*\frac{0}{1}$  and  $f(00(10)^*1) = f(0(10)^*11) = \frac{0}{1}(01)^*0$ . From the assumption one easily derives  $f(11(01)^*0) = \frac{0}{1}(01)^*0$  and  $f(0(10)^*11) = 1(01)^*\frac{0}{1}$ , thus forcing all four nodes to be mapped to  $1(01)^*0$ . This contradicts uniformity.

For  $n$  even we have  $f((01)^*) = f((10)^*) = (01)^*0$ . By uniformity again no other nodes are mapped to  $(01)^*0$ , and we necessarily obtain:  $f(00(10)^*) = \frac{0}{1}(01)^*$ ,  $f((01)^*00) = (10)^*\frac{0}{1}$ ,  $f((10)^*11) = (10)^*\frac{0}{1}$ ,  $f(11(01)^*) = \frac{0}{1}(01)^*$ . The emulation property forces  $f(00(10)^*)$  and  $f(01)^*00$  to be adjacent in  $S_{n-1}$  (impossible) or equal, hence  $f(00(10)^*) = f(01)^*00 = 1(01)^*$ . By the same argument  $f((10)^*11) = f(11(01)^*) = 1(01)^*$ . Thus four nodes are mapped to  $1(01)^*$ , contradicting uniformity.

*Case VI.*  $f((01)^*[0]) = (01)^*[0]$ ,  $f((10)^*[1]) = (10)^*[1]$ .

A more tedious argument is required to show that in this case again every uniform emulation  $f$  that satisfies the constraint must be step-simulating, contrary to our basic assumption.

First let  $n=3$ , which turns the constraint into  $f(010)=01$  and  $f(101)=10$ . We show that  $f$  must be equal to the step-simulations  $f_1$  or  $f_2$  from Table I. By emulation  $f(001) \in \{01, 00, 10\}$ ,  $f(100) \in \{01, 10, 11\}$ ,  $f(011) \in \{10, 00, 01\}$ ,  $f(110) \in \{10, 01, 11\}$ . Uniformity is heavily used in the following further analysis:

(a) Suppose  $f(001)=f(010)=01$ . Then  $f(100)=\overset{0}{1}0=1\overset{0}{1}$ , hence  $f(100)=f(101)=10$ . It follows that  $f(011)=1\overset{0}{1}=0\overset{0}{1}$ , contradiction.

(b) Suppose  $f(001)=00$ . It follows that  $f(100)=10$  ( $=f(101)$ ) and  $f(000) \in \{00, 10\}$ , hence  $f(000)=f(001)=00$  and thus  $f(011)=01$  ( $=f(010)$ ) and  $f(110)=11$ . Necessarily  $f(111)=11$ , and  $f$  is proved to coincide with  $f_1$  from Table I.

(c) Suppose  $f(001)=f(101)=10$ . It follows that  $f(100) \in \{01, 11\}$ . If  $f(100)=f(010)=01$ , then  $f(110)=00$  and this is impossible. Thus  $f(100)=11$  and necessarily  $f(110)=01$  ( $=f(010)$ ) and  $f(011)=00$ . It follows by emulation that  $f(000)=f(100)=11$ , and  $f(111)=00$ . This proves  $f$  equal to  $f_2$  from Table I.

Now let  $n \geq 4$ . We shall first derive a number of auxiliary facts that are needed later.

*Claim 2.9.4.* For  $n \geq 4$ ,  $f(0^n) \in \{0^{n-1}, 1^{n-1}\}$  and  $f(1^n) \in \{0^{n-1}, 1^{n-1}\}$ .

*Proof.* We only consider  $f(0^n)$ , as the argument for  $f(1^n)$  is similar. Let  $f(0^n)=u_1 \cdots u_{n-1}$ . Then  $f(10^{n-1}) \in \{u_1 \cdots u_{n-1}, \overset{0}{1}u_1 \cdots u_{n-2}\}$  and  $f(0^{n-1}1) \in \{u_1 \cdots u_{n-1}, u_2 \cdots u_{n-1}\overset{0}{1}\}$ . The following cases can arise:

(a)  $f(10^{n-1})=f(0^n)=u_1 \cdots u_{n-1}$ . Because of uniformity we must have that  $f(0^{n-1}1)=u_2 \cdots u_{n-1}\overset{0}{1} \neq u_1 \cdots u_{n-1}$ , and also  $f(010^{n-2})=\overset{0}{1}u_1 \cdots u_{n-2}$  and  $f(10^{n-2}1) \in \{\overset{0}{1}u_1 \cdots u_{n-2}, u_1 \cdots u_{n-2}\overset{0}{1}\}$ . If  $f(10^{n-2}1)=f(010^{n-2})=\overset{0}{1}u_1 \cdots u_{n-2}$  then by uniformity  $f(0^{n-2}10)=u_1 \cdots u_{n-2}\overset{0}{1}$  and  $f(0^{n-2}11)=u_1 \cdots u_{n-2}\overset{0}{1}$ , hence  $f(0^{n-2}10)=f(0^{n-2}11)=u_1 \cdots u_{n-2}\bar{u}_{n-1}$ . Thus  $f(0^{n-1}1)=u_2 \cdots u_{n-1}\overset{0}{1}=\overset{0}{1}u_1 \cdots u_{n-2}$  and necessarily  $u=0^{n-1}$  or  $u=1^{n-1}$ . In either case uniformity is contradicted. Thus  $f(10^{n-2}1)=u_1 \cdots u_{n-2}\overset{0}{1}$ , which implies in fact that  $f(10^{n-2}1)=u_1 \cdots u_{n-2}\bar{u}_{n-1}$  and hence  $f(0^{n-2}10) \in \{u_1 \cdots u_{n-2}\bar{u}_{n-1}, u_2 \cdots u_{n-2}\bar{u}_{n-1}\overset{0}{1}\}$ . If  $f(0^{n-2}10)=f(10^{n-2}1)=u_1 \cdots u_{n-2}\bar{u}_{n-1}$  then  $f(0^{n-1}1)=u_2 \cdots u_{n-1}\overset{0}{1}=\overset{0}{1}u_1 \cdots u_{n-2}$  and necessarily  $u=0^{n-1}$  or  $u=1^{n-1}$ . In either case uniformity is contradicted again. Thus  $f(10^{n-2}1)=u_1 \cdots u_{n-2}\bar{u}_{n-1}$  and  $f(0^{n-2}10)=u_2 \cdots u_{n-2}\bar{u}_{n-1}\overset{0}{1}$ , and thus  $f(0^{n-1}1)=u_2 \cdots u_{n-1}\overset{0}{1}=\overset{0}{1}u_2 \cdots u_{n-2}\bar{u}_{n-1}$ . It follows that  $u_2 \cdots u_{n-1}=\alpha^{n-2}$  (with  $\alpha=0$  or  $\alpha=1$ ) and  $f(0^{n-1}1)=\alpha^{n-2}\bar{\alpha}$ . If  $u_1=\alpha$  then we are finished. Thus assume that  $u_1=\bar{\alpha}$ , hence  $f(0^n)=f(10^{n-1})=\bar{\alpha}\alpha^{n-2}$ .

Consider  $b_1 \cdots b_n \in f^{-1}(\alpha^{n-1})$ , thus  $f(b_1 \cdots b_n) = \alpha^{n-1}$ . Because of uniformity it follows that  $f(0b_1 \cdots b_{n-1}) = f(1b_1 \cdots b_{n-1}) = \alpha^{n-1}$  (using that  $b_1 \cdots b_{n-1} \neq 0^{n-1}$ ), and likewise  $f(01b_1 \cdots b_{n-2}) = f(11b_1 \cdots b_{n-2}) = \alpha^{n-1}$  and, provided  $b_1 \cdots b_{n-2} \neq 0^{n-2}$ , also  $f(00b_1 \cdots b_{n-2}) = \alpha^{n-1}$ . This shows that at least 3 nodes are mapped to  $\alpha^{n-1}$ , contradicting uniformity.

(b)  $f(0^{n-1}1) = f(0^n) = u_1 \cdots u_{n-1}$ . The argument is analogous to case (a) by ‘reversing’ the orientation of the strings.

(c)  $f(10^{n-1}) = \frac{0}{1}u_1 \cdots u_{n-2}$  and  $f(0^{n-1}1) = u_2 \cdots u_{n-1} \frac{0}{1}$ . If  $f(10^{n-1}) = f(0^{n-1}1)$  then necessarily  $u_1 \cdots u_{n-1} = (\alpha\beta)^*[\alpha]$ . Because of uniformity  $\alpha = \beta$  (otherwise one of  $(01)^*[0]$  and  $(01)^*[1]$  would be mapped to  $(\alpha\beta)^*[\alpha]$  too), and thus  $u_1 \cdots u_{n-1} = 0^{n-1}$  or  $u_1 \cdots u_{n-1} = 1^{n-1}$ . It follows that  $f(10^{n-1}) = f(0^{n-1}1) = f(0^n)$ , contradicting uniformity. Thus  $f(10^{n-1}) \neq f(0^{n-1}1)$  and by emulation necessarily  $f(0^{n-1}1) = u_2 \cdots u_{n-1} \frac{0}{1} = u_1 \cdots u_{n-2} \frac{0}{1}$ , hence  $u_1 \cdots u_{n-1} = 0^{n-1}$  or  $u_1 \cdots u_{n-1} = 1^{n-1}$ . ■

(The condition  $n \geq 4$  was used in case (a), to make sure that  $0^{n-2}10 \neq 010^{n-2}$  and (hence)  $10^{n-2}1 \neq (10)^*[1]$  and  $0^{n-2}10 \neq (01)^*[0]$ .) Next observe from  $f((01)^*[0]) = (01)^*[0]$  that  $f(00(10)^*[1]) \in \{(01)^*[0], (10)^*[1], 00(10)^*[1]\}$  and from  $f((10)^*[1]) = (10)^*[1]$  that  $f(11(01)^*[0]) \in \{(01)^*[0], (10)^*[1], 11(01)^*[0]\}$ .

We tackle a particular combination first, because it will be central in the remainder of the proof.

*Claim 2.9.5.* For  $n \geq 4$ .

(i) if  $f(00(10)^*[1]) = 00(10)^*[1]$ , then for all  $b_1 \cdots b_{n-3} \in (\frac{0}{1})^{n-3}$  there exist  $c_1 \cdots c_{n-3}, c'_1 \cdots c'_{n-3} \in (\frac{0}{1})^{n-3}$  such that  $f(b_1 \cdots b_{n-3}000) = c_1 \cdots c_{n-3}00$  and  $f(b_1 \cdots b_{n-3}001) = c'_1 \cdots c'_{n-3}00$ .

(ii) if  $f(11(01)^*[0]) = 11(01)^*[0]$ , then for all  $b_1 \cdots b_{n-3} \in (\frac{0}{1})^{n-3}$  there exist  $c_1 \cdots c_{n-3}, c'_1 \cdots c'_{n-3} \in (\frac{0}{1})^{n-3}$  such that  $f(b_1 \cdots b_{n-3}110) = c_1 \cdots c_{n-3}11$  and  $f(b_1 \cdots b_{n-3}111) = c'_1 \cdots c'_{n-3}11$ .

*Proof.* We only prove (i), as (ii) is similar. First we induct on  $i$  to show that for all  $b_1 \cdots b_i \in (\frac{0}{1})^i$  there exist a  $c_1 \cdots c_i \in (\frac{0}{1})^i$  with  $f(b_1 \cdots b_i 001(01)^*[0]) = c_1 \cdots c_i 00(10)^*[1]$ . Since  $f(001(01)^*[0]) = 00(10)^*[1]$  by assumption, we have for  $i=1$ :  $f(b_1 001(01)^*[0]) \in \{001(01)^*[0], \frac{0}{1}00(10)^*[1]\}$ . If  $f(b_1 001(01)^*[0]) = f(001(01)^*[0]) = 00(10)^*[1]$  then one easily verifies that Claim 2.9.1 is contradicted. (Use  $\alpha = b_1$ ,  $x = 001(01)^*[0]$ ,  $\gamma = 0$  or 1.) Thus  $f(b_1 001(01)^*[0]) = c_1 00(10)^*[1]$ , for some  $c_1 \in \frac{0}{1}$ . Suppose it holds for some  $i$ ,  $1 \leq i \leq n-3$ . Consider  $f(b_1 \cdots b_{i+1} 001(01)^*[0])$ . By induction there exists a  $c_2 \cdots c_{i+1} \in (\frac{0}{1})^i$  such that  $f(b_2 \cdots b_{i+1} 001(01)^*[0]) = c_2 \cdots c_{i+1} 00(10)^*[1]$ , and thus  $f(b_1 b_2 \cdots b_{i+1} 001(01)^*[0]) \in \{c_2 \cdots c_{i+1} 00(10)^*[1], \frac{0}{1}c_2 \cdots c_{i+1} 00(10)^*[1]\}$ . If  $f(b_1 b_2 \cdots b_{i+1} 001(01)^*[0]) = f(b_2 \cdots b_{i+1} 001(01)^*[0]) =$

$c_2 \cdots c_{i+1} 00(10)^*[1]$ , then one easily verifies again that Claim 2.9.1 is contradicted. Thus  $f(b_1 \cdots b_{i+1} 001(01)^*[0]) = c_1 c_2 \cdots c_{i+1} 00(10)^*[1]$ , for some  $c_1 \in \frac{0}{1}$ . This completes the inductive argument. We conclude in particular (take  $i = n-3$ ) that for every  $b_1 \cdots b_{n-3} \in (\frac{0}{1})^{n-3}$  there exists a  $c_1 \cdots c_{n-3} \in (\frac{0}{1})^{n-3}$  such that  $f(b_1 \cdots b_{n-3} 001) = c_1 \cdots c_{n-3} 00$ .

Next consider  $f(b_1 \cdots b_{n-3} 000)$ . Since  $f(b_2 \cdots b_{n-3} 0001) = c_1 \cdots c_{n-3} 00$  for suitable  $c_1 \cdots c_{n-3} \in (\frac{0}{1})^{n-3}$ , it follows that  $f(b_1 b_2 \cdots b_{n-3} 000) \in \{c_1 \cdots c_{n-3} 00, \frac{0}{1} c_1 \cdots c_{n-3} 0\}$ . If  $b_1 \cdots b_{n-3} = 0^{n-3}$ , then necessarily  $f(b_1 \cdots b_{n-3} 000) = 0^{n-1}$  by Claim 2.9.4 and the form claimed under (i) holds. Thus let  $b_1 \cdots b_{n-3} \neq 0^{n-3}$ . If  $f(b_1 \cdots b_{n-3} 000) = c_1 \cdots c_{n-3} 00$ , then the form claimed under (i) holds too. Hence let  $f(b_1 \cdots b_{n-3} 000) = \beta c_1 \cdots c_{n-3} 0$  and, consequently,  $f(b_0 b_1 \cdots b_{n-3} 00) \in \{\beta c_1 \cdots c_{n-3} 0, \frac{0}{1} \beta c_1 \cdots c_{n-3}\}$  for some  $\beta \in \frac{0}{1}$  and  $b_0 \in \frac{0}{1}$ . (Note that necessarily  $b_0 b_1 \cdots b_{n-3} 00 \neq b_1 \cdots b_{n-3} 000$ .) If  $f(b_0 \cdots b_{n-3} 00) = f(b_1 \cdots b_{n-3} 000) = \beta c_1 \cdots c_{n-3} 0$ , then it follows from Claim 2.9.1 that  $\beta c_1 \cdots c_{n-3} 0 \in \{0^{n-1}, (01)^*[0], (10)^*[1]\}$ . In each of the three cases uniformity is contradicted. (Note that  $0^n \in f^{-1}(0^{n-1})$  by Claim 2.9.4 in this case, and  $(01)^*[0] \in f^{-1}((01)^*[0])$  and  $(10)^*[1] \in f^{-1}((10)^*[1])$ .) Thus  $f(b_0 \cdots b_{n-3} 00) = \frac{0}{1} \beta c_1 \cdots c_{n-3}$ . Since  $f(b_1 \cdots b_{n-3} 001) = c'_1 \cdots c'_{n-3} 00$  for suitable  $c'_1 \cdots c'_{n-3} \in (\frac{0}{1})^{n-3}$ , it follows that  $f(b_0 b_1 \cdots b_{n-3} 00) \in \{c'_1 \cdots c'_{n-3} 00, \frac{0}{1} c'_1 \cdots c'_{n-3} 0\}$  and thus ends with a "0." Hence  $c_{n-3} = 0$ , and  $f(b_1 \cdots b_{n-3} 000) = \frac{0}{1} c_1 \cdots c_{n-4} 00$  as claimed. ■

We now begin our case analysis.

*Claim 2.9.6.* For  $n \geq 4$ , the case  $f(00(10)^*[1]) = 00(10)^*[1]$  and  $f(11(01)^*[0]) = 11(01)^*[0]$  is contradictory.

*Proof.* By Claim 2.9.5 the  $2^{n-2}$  strings of  $(\frac{0}{1})^{n-3} 000 \cup (\frac{0}{1})^{n-3} 001$  are mapped to the  $2^{n-3}$  strings of  $(\frac{0}{1})^{n-3} 00$ . By uniformity it follows that no other strings can be mapped to  $(\frac{0}{1})^{n-3} 00$ . Likewise no other strings than the elements of  $(\frac{0}{1})^{n-3} 110 \cup (\frac{0}{1})^{n-3} 111$  are mapped to  $(\frac{0}{1})^{n-3} 11$ . Let  $b_1 \cdots b_{n-3} \in (\frac{0}{1})^{n-3}$ . By Claim 2.9.5 we have  $f(b_2 \cdots b_{n-3} 0110) = c_1 \cdots c_{n-3} 11$  and  $f(b_2 \cdots b_{n-3} 1000) = c'_1 \cdots c'_{n-3} 00$  and, consequently,  $f(b_1 \cdots b_{n-3} 011) \in \{c_1 \cdots c_{n-3} 11, \frac{0}{1} c_1 \cdots c_{n-3} 1\}$  and  $f(b_1 \cdots b_{n-3} 100) \in \{c'_1 \cdots c'_{n-3} 00, \frac{0}{1} c'_1 \cdots c'_{n-3} 0\}$ . The cases that  $f(b_1 \cdots b_{n-3} 011) = f(b_2 \cdots b_{n-3} 0110) = c_1 \cdots c_{n-3} 11$  or  $f(b_1 \cdots b_{n-3} 100) = f(b_2 \cdots b_{n-3} 1000) = c'_1 \cdots c'_{n-3} 00$  clash with Claim 2.9.1. Thus  $f(b_1 \cdots b_{n-3} 011) = \frac{0}{1} c_1 \cdots c_{n-3} 1$  and  $f(b_1 \cdots b_{n-3} 100) = \frac{0}{1} c'_1 \cdots c'_{n-3} 0$  and, since neither one can "end" with 00 or 11, we have in fact that  $f(b_1 \cdots b_{n-3} 011) = d_1 \cdots d_{n-3} 01$  and  $f(b_1 \cdots b_{n-3} 100) = d'_1 \cdots d'_{n-3} 10$  (for suitable  $d_1 \cdots d_{n-3}$  and  $d'_1 \cdots d'_{n-3}$ ). By a very similar argument one now shows that  $f(b_1 \cdots b_{n-3} 010) = e_1 \cdots e_{n-3} 01$  and  $f(b_1 \cdots b_{n-3} 101) = e'_1 \cdots e'_{n-3} 10$ , for suitable  $e_1 \cdots e_{n-3}$  and  $e'_1 \cdots e'_{n-3}$ . (It follows that  $f$  "resembles"  $f_1$  of Table I).

As  $f$  was assumed not to be step-simulating, there must be  $x \in (\frac{0}{1})^{n-1}$  and  $y \in (\frac{0}{1})^{n-3}$  and  $\alpha, \beta, \gamma, \delta \in (\frac{0}{1})$  such that  $f(\alpha x) = f(x\gamma) = \beta y \delta$  and  $\beta y \neq y \delta$ . By Claim 2.9.1 one of the strings  $\alpha x$ ,  $x\gamma$  is  $0^n$  or  $1^n$  and hence (by Claim 2.9.4)  $\beta y \delta \in \{0^n, 1^n\}$ , or  $\beta y \delta \in \{(01)^*[0], (10)^*[1]\}$ . In the former case the condition  $\beta y \neq y \delta$  is violated. In the latter case we necessarily have  $\beta y \delta = \beta y' \delta \delta$  (for a suitable  $y'$ ) and hence, by our earlier analysis, necessarily  $\alpha x = \alpha x' \delta \delta \frac{0}{1}$  for suitable  $x'$ . It follows that  $x\gamma = x' \delta \delta \frac{0}{1} \gamma$  and thus  $f(x\gamma)$  ends in  $\delta \frac{0}{1}$ , contradicting that it equals  $\beta y \delta$  and thus ends in  $\delta \delta$ . ■

Next let  $f(00(10)^*[1]) = (01)^*[0]$  and  $f(11(01)^*[0]) = 11(01)^*[0]$ . Observe that  $f(00(10)^*[1]) = f((01)^*[0]) = (01)^*[0]$  and thus by uniformity,  $f(100(10)^*[1]) = \frac{0}{1}(01)^*[0]$ .

*Claim 2.9.7.* For  $n \geq 4$ , the case  $f(00(10)^*[1]) = (01)^*[0]$  and  $f(11(01)^*[0]) = 11(01)^*[0]$  is contradictory.

*Proof.* We distinguish two further cases.

(a)  $f(100(10)^*[1]) = 0(01)^*[0]$ . As in the proof of Claim 2.9.5 one shows by induction that for all  $1 \leq i \leq n-3$  and  $b_1 \cdots b_i \in (\frac{0}{1})^i$  there exists  $c_1 \cdots c_i \in (\frac{0}{1})^i$  such that  $f(b_1 \cdots b_i 100(10)^*[1]) = c_1 \cdots c_i 0(01)^*[0]$ . Thus for  $b_1 \cdots b_{n-3} \in (\frac{0}{1})^{n-3}$  we have  $f(b_1 \cdots b_{n-3} 100) = c_1 \cdots c_{n-3} 00$ . It also follows that for every  $b_0 \in (\frac{0}{1})$   $f(b_0 b_1 \cdots b_{n-3} 10) = \frac{0}{1} c_1 \cdots c_{n-3} 0$ . For  $b_{n-3} = 1$  this contradicts Claim 2.9.5(ii).

(b)  $f(100(10)^*[1]) = 1(01)^*[0] = f((10)^*[1])$ . By uniformity one must have  $f(0100(10)^*[1]) = 11(01)^*[0]$ . By induction one shows that for all  $1 \leq i \leq n-4$  and  $b_1 \cdots b_i \in (\frac{0}{1})^i$  there exists  $c_1 \cdots c_i \in (\frac{0}{1})^i$  with  $f(b_1 \cdots b_i 0100(10)^*[1]) = c_1 \cdots c_i 11(01)^*[0]$ . Thus for  $i = n-4$  we have  $f(b_1 \cdots b_{n-4} 0100) = c_1 \cdots c_{n-4} 110$ , and it follows that also for every  $b_0 \in \frac{0}{1}$  that  $f(b_0 b_1 \cdots b_{n-4} 010) = \frac{0}{1} c_1 \cdots c_{n-4} 11$ . (For if  $f(b_0 b_1 \cdots b_{n-4} 010) = f(b_1 \cdots b_{n-4} 0100) = c_1 \cdots c_{n-4} 110$ , one easily derives a contradiction with Claim 2.9.1.) By Claim 2.9.5 and a uniformity argument (cf. the proof of Claim 2.9.6), no other strings than the elements of  $(\frac{0}{1})^{n-3} 110 \cup (\frac{0}{1})^{n-3} 111$  can be mapped to  $(\frac{0}{1})^{n-3} 11$ . This contradicts the assertion for  $f(b_0 b_1 \cdots b_{n-4} 010)$ . ■

By a similar argument the following cases are proved contradictory as well:  $f(00(10)^*[1]) = (10)^*[1]$  and  $f(11(01)^*[0]) = 11(01)^*[0]$ ,  $f(00(10)^*[1]) = 00(10)^*[1]$  and  $f(11(01)^*[0]) = (01)^*[0]$ , and  $f(00(10)^*[1]) = 00(10)^*[1]$  and  $f(11(01)^*[0]) = (10)^*[1]$ .

*Claim 2.9.8.* For  $n \geq 4$ , the case  $f(00(10)^*[1]) = (01)^*[0]$  and  $f(11(01)^*[0]) = (10)^*[1]$  is contradictory.

*Proof.* By uniformity (recall that  $f((01)^*[0]) = (01)^*[0]$  and  $f((10)^*[1]) = (10)^*[1]$ ) we necessarily have  $f([1](01)^*00) = [1](01)^*00$ ,



and also  $f([0](10)*11) = [0](10)*11$ . Thus we have a situation similar to the one considered in Claims 2.9.5 and 2.9.6, with the orientation of the strings involved “reversed.” Clearly a contradiction is again derived. ■

The case  $f(00(10)*[1]) = (10)*[1]$  and  $f(11(01)*[0]) = (01)*[0]$  is proved contradictory in the same way. By noting that the cases  $f(00(10)*[1]) = f(11(01)*[0]) = (01)*[0]$  and  $f(00(10)*[1]) = f(11(01)*[0]) = (10)*[1]$  cannot occur because of uniformity, the case analysis is complete.

Case V.  $f((01)*[0]) = (10)*[1]$ ,  $f((10)*[1]) = (01)*[0]$ .

This case is “dual” to Case IV, which was shown to be contradictory.

Case VI.  $f((01)*[0]) = f((10)*[1]) = (10)*[1]$ .

This case is “dual” to Case III, which was shown to be contradictory.

This completes the proof of Theorem 2.9. ■

## APPENDIX B: THE PROOF OF THE TOPOLOGICAL THEOREM FOR EMULATIONS OF $C_n$ ON $C_{n-1}$ (THEOREM 3.5)

We use the notations and terminology of Section 3. Our aim is to prove the following result.

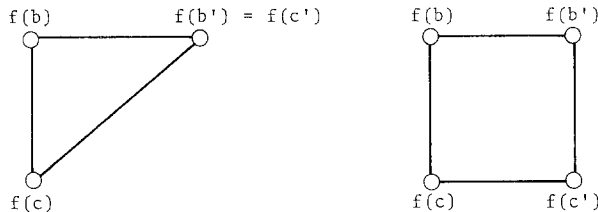
**THEOREM 3.5.** (Topological reduction theorem). *Let  $n \geq 4$ , and let  $f$  be a uniform emulation of  $C_n$  on  $C_{n-1}$ . Then there exists an  $(n-1)$ -face  $A$  of  $C_n$  such that  $f(A)$  is an  $(n-2)$ -face of  $C_{n-1}$ .*

The proof proceeds by way of contradiction. Let  $n \geq 4$ , and let  $f$  be a uniform emulation of  $C_n$  on  $C_{n-1}$ . Suppose that there does not exist an  $(n-1)$ -face  $A$  of  $C_n$  such that  $f(A)$  is an  $(n-2)$ -face of  $C_{n-1}$ .

*Claim 3.5.1.* For every  $k$  with  $1 \leq k \leq n-1$ , there does not exist a  $k$ -face  $A$  of  $C_n$  such that  $f(A)$  is a  $(k-1)$ -face of  $C_{n-1}$ .

*Proof.* Without loss of generality let  $k < n-1$ . Suppose the claim is false. Let  $k$  be the largest integer  $\in 1 \cdots n-2$  for which there exists a  $k$ -face  $A$  of  $C_n$  such that  $f(A)$  is a  $(k-1)$ -face of  $C_{n-1}$ . Without loss of generality we may assume that the elements of  $A$  have identical bits in the last  $n-k$  positions, hence  $A = \{x\alpha u \mid x \in (\frac{0}{1})^k\}$  for certain  $\alpha \in (\frac{0}{1})$  and  $u \in (\frac{0}{1})^{n-k-1}$ . Consider the  $(k+1)$ -face  $A' = \{x\frac{0}{1}u \mid x \in (\frac{0}{1})^k\} = \{x\alpha u \mid x \in (\frac{0}{1})^k\} \cup \{x\bar{\alpha}u \mid x \in (\frac{0}{1})^k\}$ . For every  $b = x\alpha u \in A$ , let  $b' = x\bar{\alpha}u$ . Because of uniformity no elements  $b'$  can be mapped into  $f(A)$ . It follows that  $f(b')$  is obtained from  $f(b)$  by flipping one bit. We claim that for all  $b, c \in A$  one has

$f(b) - f(b') = f(c) - f(c')$ , where “ $-$ ” denotes the component-wise subtraction, i.e.,  $(b_1 \cdots b_k) - (c_1 \cdots c_k) = (b_1 - c_1 \cdots b_k - c_k) \in \{-1, 0, 1\}^k$ . It is sufficient to prove this for pairs  $b, c \in A$  with  $d(b, c) = 1$ . Note that  $f(b'), f(c') \notin f(A)$ . Suppose  $f(b') = f(c')$ . If  $f(b) = f(c)$ , then  $f(b) - f(b') = f(c) - f(c')$  and we are finished. If  $f(b) \neq f(c)$ , then necessarily  $d(f(b), f(c)) = 1$  and  $(f(b), f(c))$  is an edge of  $C_{n-1}$  (in fact, of  $f(A)$ ). However, both  $f(b)$  and  $f(c)$  are connected to  $f(b') = f(c') \notin f(A)$  too. It follows that  $C_{n-1}$  contains a triangle, which is impossible. Next suppose  $f(b') \neq f(c')$ . If  $f(b) = f(c)$ , then one easily argues again that  $C_{n-1}$  contains a triangle, and a contradiction arises. If  $f(b) \neq f(c)$ , then the nodes must form a 4-cycle and hence (necessarily) a 2-face of  $C_{n-1}$ . It follows that  $f(b) - f(b') = f(c) - f(c')$ .



Using the claim we now argue that  $f(A')$  is a  $k$ -face of  $C_{n-1}$ . Note that  $A' = A \cup \{b' \mid b \in A\}$ . Fix a  $b \in A$  and assume that  $f(b')$  is obtained from  $f(b)$  by flipping the  $i$ th bit, where  $i$  belongs to the bit-positions with fixed values for face  $f(A)$ . For arbitrary  $c \in A$ , the identity  $f(b) - f(b') = f(c) - f(c')$  forces that  $f(c')$  is obtained from  $f(c)$  by flipping exactly the same  $i$ th bit. Thus  $f(A') \supset f(A)$  is a  $k$ -face of  $C_{n-1}$ . This contradicts that  $k$  was the largest integer for which a face of  $C_n$  with this property exists. ■

We shall now prove a number of results that will eventually contradict Claim 3.5.1, which thus proves that our initial assumption was false.

**DEFINITION.** For  $1 \leq k \leq n-1$ , a  $k$ -face  $A$  of  $C_n$  is called *stable* if  $f(A)$  is a  $k$ -face of  $C_{n-1}$ .

**Claim 3.5.2.** There exists a 2-face  $A$  of  $C_n$  that is stable.

*Proof.* Consider the 2-face  $A = \{x00 \cdots 0 \mid x \in (\frac{0}{1})^2\}$ . Suppose  $A$  is not stable, i.e.,  $f(A)$  is not a 2-face of  $C_{n-1}$ . By uniformity  $f(A)$  contains at least 2 elements, but by Claim 3.5.1 it can not be a 1-face. It follows that  $f(A)$  contains precisely 3 elements. Observing adjacencies, the following two cases can arise:

(a)  $f(0000 \cdots 0) = f(1100 \cdots 0)$  and  $f(0100 \cdots 0) \neq f(1000 \cdots 0)$ . Consider  $f(0010 \cdots 0)$ . By uniformity it cannot be equal to  $f(0000 \cdots 0)$  and

$f(1100 \cdots 0)$ . If  $f(0010 \cdots 0) = f(0100 \cdots 0)$ , then either  $f(1010 \cdots 0) = f(1000 \cdots 0)$  or  $f(1010 \cdots 0)$  is different from  $f(0000 \cdots 0)$ ,  $f(0010 \cdots 0)$ , and  $f(1000 \cdots 0)$ . In the former case  $f(0000 \cdots 0)$ ,  $f(0100 \cdots 0)$  and  $f(1000 \cdots 0)$  will form a triangle, which is impossible in  $C_{n-1}$ . In the latter case one verifies that  $B = \{\alpha 0 \beta 0 \cdots 0 \mid \alpha, \beta \in \binom{0}{1}\}$  is a stable 2-face of  $C_n$ . If  $f(0010 \cdots 00) = f(1000 \cdots 0)$ , then a similar argument shows that  $B' = \{0 \alpha \beta 0 \cdots 0 \mid \alpha, \beta \in \binom{0}{1}\}$  must be a stable 2-face again. If  $f(0010 \cdots 0) \neq f(0100 \cdots 0)$  and  $\neq f(1000 \cdots 0)$ , then observe the following. If  $f(1010 \cdots 0)$  would coincide with either  $f(0100 \cdots 0)$ ,  $f(1000 \cdots 0)$ , or  $f(0010 \cdots 0)$ , then triangles are formed in  $C_{n-1}$ . Contradiction. Thus  $f(1010 \cdots 0)$  is different from all these, and one verifies again that  $B$  is a stable 2-face.

(b)  $f(0000 \cdots 0) \neq f(1100 \cdots 0)$  and  $f(0100 \cdots 0) = f(1000 \cdots 0)$ . Consider  $f(1010 \cdots 0)$  and distinguish cases as under (a). Once again triangles in  $C_{n-1}$  are formed (contradiction), or  $B = \{\alpha 0 \beta 0 \cdots 0 \mid \alpha, \beta \in \binom{0}{1}\}$  or  $B' = \{0 \alpha \beta 0 \cdots 0 \mid \alpha, \beta \in \binom{0}{1}\}$  is proved a stable 2-face of  $C_n$ .

The cases " $f(0000 \cdots 0) = f(0100 \cdots 0)$  and  $f(1100 \cdots 0) \neq f(0100 \cdots 0)$ " and alike cannot arise, because it would lead to 1-faces being mapped to 0-faces (points), contradicting Claim 3.5.1. ■

The proof of Claim 3.5.2 shows, in fact, that either  $\binom{0}{1} 2^{0^{n-2}}$ ,  $\binom{0}{1} 0 \binom{0}{1} 0^{n-3}$  or  $0 \binom{0}{1} 2^{0^{n-3}}$  must be a stable 2-face of  $C_n$ .

*Claim 3.5.3.* For every  $k$  with  $2 \leq k \leq n-2$ , there exists a  $k$ -face  $A$  of  $C_n$  that is stable.

*Proof.* We induct on  $k$ . The case  $k=2$  follows by Claim 3.5.2. Assume it holds up to some  $k$  with  $2 \leq k < n-2$ . Let  $A$  be a stable  $k$ -face of  $C_n$ . Without loss of generality we can let  $A = \{x \alpha u \mid x \in \binom{0}{1}^k\}$  for some  $\alpha \in \binom{0}{1}$  and  $u \in \binom{0}{1}^{n-k-1}$ . Let  $A' = \{x \bar{\alpha} u \mid x \in \binom{0}{1}^k\}$  (a  $k$ -face), and for every  $b = x \alpha u \in A$  let  $b' = x \bar{\alpha} u \in A'$ . We show that there must exist a stable  $(k+1)$ -face.

Suppose first that  $f(A) \cap f(A') = \emptyset$ . As in the proof of Claim 3.5.1 one shows that for all  $b, c \in A$ :  $f(b) - f(b') = f(c) - f(c')$ . Now note that  $f(A)$  is a  $k$ -face of  $C_{n-1}$ . As in the proof of Claim 3.5.1 one shows that for all  $b \in A$   $f(b')$  is obtained from  $f(b)$  by flipping the same bit (in a position with fixed value for the elements of  $f(A)$ ). Thus  $f(A')$  is a  $k$ -face of  $C_{n-1}$  too, and one easily verifies that  $A \cup A' = \{y u \mid y \in \binom{0}{1}^{k+1}\}$  is a stable  $(k+1)$ -face of  $C_n$ .

Suppose next that  $f(A) \cap f(A') \neq \emptyset$ . If  $f(A) = f(A')$ , then  $A \cup A'$  is a  $(k+1)$ -face of  $C_n$  whose image is a  $k$ -face (namely,  $f(A)$ ) of  $C_{n-1}$  and a contradiction with Claim 3.5.1 arises. Thus  $f(A) \neq f(A')$ , and it easily follows that  $b', c' \in A'$  must exist with  $d(b', c') = 1$  and  $f(b') \notin f(A)$  and  $f(c') \in f(A)$ . (We assume that  $b, c$  are the corresponding nodes in  $A$ .) Let  $\mathbf{b} \neq \mathbf{c}$  be any other node  $\in A$  adjacent to  $b$ , and let  $\mathbf{c} \in A$  be obtained from  $c$  by flipping the same bit (as the one flipped to obtain  $\mathbf{b}$  from  $b$ ). We now

claim: (i)  $f(b) = f(c')$ , (ii)  $f(\mathbf{b}') \notin f(A)$ , and (iii)  $f(\mathbf{c}') \in f(A)$ . For the proof, observe the following.

(i) Suppose  $f(b) = f(c')$ , and consider  $f(c)$ . If  $f(c) = f(c')$  then  $f(b)$ ,  $f(b')$ , and  $f(c)$  form a triangle in  $C_{n-1}$  (by observing adjacencies). Contradiction. If  $f(c) \neq f(c')$ , then note that also  $f(b) \neq f(c)$  (because  $f$  is necessarily 1-to-1 as a mapping from  $k$ -face  $A$  onto  $k$ -face  $f(A)$ ). Thus  $f(b)$ ,  $f(b')$ ,  $f(c')$ , and  $f(c)$  form a 4-cycle, hence a 2-face of  $C_{n-1}$ . But with  $f(b)$ ,  $f(c')$ , and  $f(c)$  belonging to  $f(A)$  the entire 2-face must belong to  $C_{n-1}$ , hence  $f(b') \in f(A)$ . Contradiction. We conclude  $f(b) = f(c')$ .

(ii) Suppose  $f(\mathbf{b}')$ . By uniformity  $f(\mathbf{b}') \neq f(b) = f(c')$ . If  $f(\mathbf{b}') = f(\mathbf{b})$  then  $f(b)$ ,  $f(b')$ , and  $f(\mathbf{b}')$  form a triangle in  $C_{n-1}$ . Contradiction. If  $f(\mathbf{b}') \neq f(\mathbf{b})$  then  $f(b)$ ,  $f(b')$ ,  $f(\mathbf{b})$ , and  $f(\mathbf{b}')$  form a 4-cycle, hence a 2-face of  $C_{n-1}$  with three nodes in the  $k$ -face  $f(A)$ . It follows that also  $f(b') \in f(A)$ . Contradiction. We conclude  $f(\mathbf{b}') \notin f(A)$ .

(iii) Note that  $f(\mathbf{c}) \neq f(\mathbf{b})$ , (else a contradiction with Claim 3.5.1 arises, so  $f(\mathbf{c})$  is adjacent to  $f(\mathbf{b})$ , and  $f(\mathbf{b})$  is adjacent to  $f(b) = f(c')$ . Hence the distance between  $f(c')$  and  $f(\mathbf{c})$  is 2.  $f(\mathbf{c}')$  must be adjacent to  $f(c') \in f(A)$  and  $f(\mathbf{c}) \in f(A)$ , hence  $f(c') \in f(A)$ .

From the claim we derive that  $\mathbf{b}', \mathbf{c}'$  is a pair exactly like  $b', c'$  and the argument can be repeated. In this way we can let  $b'$  range over all of  $A'$ , and obtain that  $f(A')$  must be a  $k$ -face of  $C_{n-1}$  and  $f(A) \cap f(A')$  is a  $(k-1)$ -face (because nodes are paired in adjacent couples with one mapped to  $f(A) \cap f(A')$  and the other to  $f(A') - f(A)$ ). Now consider two more  $k$ -faces  $A'', A'''$  adjacent (parallel) to  $A$  obtained, say, by flipping the first and second bit of  $u$  respectively. (Note that  $|u| \geq 2$ , because  $k < n-2$ .) Either  $f(A) \cap f(A'') = \emptyset$  or  $f(A) \cap f(A''') = \emptyset$  and we would be finished by the first part of the proof, or both  $f(A) \cap f(A'') \neq \emptyset$  and  $f(A) \cap f(A''') \neq \emptyset$ . In the latter case one derives the same conclusion for  $f(A'')$  and  $f(A''')$  as for  $f(A')$ . It follows that  $f^{-1}(f(A))$  contains at least  $2^{k-1}$  elements of each  $A', A'',$  and  $A'''$ , thus at least  $2\frac{1}{2} \cdot 2^{k-1}$  elements in all. This contradicts the uniformity of  $f$ .

This completes the induction argument. ■

We now derive a contradiction as follows. By Claim 3.5.3 there exists a  $(n-2)$ -face  $A$  of  $C_n$  that is stable. Without loss of generality we can let  $A = \{x00 \mid x \in (\frac{0}{1})^{n-2}\}$ . Let  $A' = \{x10 \mid x \in (\frac{0}{1})^{n-2}\}$ ,  $A'' = \{x01 \mid x \in (\frac{0}{1})^{n-2}\}$ , and  $A''' = \{x11 \mid x \in (\frac{0}{1})^{n-2}\}$ . From the proof of Claim 3.5.3 one derives that  $A', A'',$  and  $A'''$  must be stable  $(n-2)$ -faces of  $C_n$  as well, and that the  $f$ -images of adjacent (parallel) faces are either disjoint or intersect (pairwise) in a  $(n-3)$ -face. We distinguish the following cases for the pairwise intersections:

(a)  $f(A) \cap f(A')$  is an  $(n-3)$ -face,  $f(A) \cap f(A'')$  is an  $(n-3)$ -face. If  $f(A') \cap f(A''') = \emptyset$  or  $f(A'') \cap f(A''') = \emptyset$ , then  $A' \cup A''' = \{x1\frac{0}{1} \mid x \in (\frac{0}{1})^{n-2}\}$  or  $A'' \cup A''' = \{x\frac{0}{1}1 \mid x \in (\frac{0}{1})^{n-2}\}$  is stable  $(n-1)$ -face (as is its one parallel face  $A \cup A''$  or  $A \cup A'$ , resp.) and either  $f(A)$  and  $f(A'')$  or  $f(A)$  and  $f(A')$  must be disjoint, respectively. Contradiction. We conclude that  $f(A') \cap f(A''')$  and  $f(A'') \cap f(A''')$  both are  $(n-3)$ -faces too, in this case. Let  $b = x000 \in A$  and  $c' = y10 \in A'$  be such that  $f(b) = f(c') \in f(A) \cap f(A')$ . Without loss of generality let  $f(A) = (\frac{0}{1})^{n-3}(\frac{0}{1})\beta$  and  $f(A') = (\frac{0}{1})^{n-3}\alpha(\frac{0}{1})$ . Because  $f \upharpoonright A$  and  $f \upharpoonright A'$  act like isomorphisms of  $C_{n-2}$  Theorem 3.4 applies, and there must be literals  $l_i$  and  $l'_i$  corresponding to  $b_i$  ( $1 \leq i \leq n-2$ ) and permutations  $\Pi$  and  $\Pi'$  such that  $f(b_1 \cdots b_{n-2}00) = l_{\Pi(1)} \cdots l_{\Pi(n-3)} l_{\Pi(n-2)} \beta$  and  $f(b_1 \cdots b_{n-2}10) = l'_{\Pi'(1)} \cdots l'_{\Pi'(n-3)} \alpha l'_{\Pi'(n-2)}$ . By letting the argument  $b_1 \cdots b_{n-2}$  range over  $(\frac{0}{1})^{n-2}$  and observing that  $f(b_1 \cdots b_{n-2}00)$  and  $f(b_1 \cdots b_{n-2}10)$  must have distance  $\leq 1$ , one easily concludes that  $\Pi = \Pi'$  and  $l_{\Pi(i)} = l'_{\Pi(i)}$  for  $1 \leq i \leq n-3$ . If  $f(x00) = f(y10)$  then necessarily  $x = y$  or  $d(x, y) = 1$ . Now let  $b' = x10 \in A'$ ,  $b'' = x01 \in A''$ ,  $b''' = x11 \in A'''$ , and let  $c = y00 \in A$ ,  $c'' = y01 \in A''$ ,  $c''' = y11 \in A'''$ . If  $x = y$ , then one obtains that the 1-face of  $C_n$  spanned by  $b$  and  $c'$  is mapped to a 0-face (a point), contradicting Claim 3.5.1 for  $k = 1$ . If  $d(x, y) = 1$ , then  $b$  and  $c$  are adjacent and likewise are their primed companions. By a similar analysis of  $f(A') \cap f(A''')$  and alike, one shows that necessarily:  $f(b') = f(c''')$ ,  $f(b'') = f(c)$ , and  $f(b''') = f(c'')$ . It follows that the 3-face of  $C_n$  spanned by  $b, b', b'', b''', c, c', c'', c'''$  is mapped to a 2-face of  $C_{n-1}$ . (The case that more  $f$ -value coincide is excluded by uniformity.) This contradicts Claim 3.5.1 for  $k = 3$ .

(b)  $f(A) \cap f(A')$  is an  $(n-3)$ -face,  $f(A) \cap f(A'') = \emptyset$ . If  $f(A') \cap f(A''') = \emptyset$  is an  $(n-3)$ -face, then one can use the argument under case (a) and derive a contradiction. Thus let  $f(A') \cap f(A''') = \emptyset$ . It follows that both  $A \cup A'' = \{x0(\frac{0}{1}) \mid x \in (\frac{0}{1})^{n-2}\}$  and  $A' \cup A''' = \{x1(\frac{0}{1}) \mid x \in (\frac{0}{1})^{n-2}\}$  are stable  $(n-1)$ -faces, thus their images each span  $C_{n-1}$ . It follows that  $f(A) \cap f(A''')$  and  $f(A'') \cap f(A''')$  cannot be empty, and thus must be  $(n-3)$ -faces. Now a similar argument as given under case (a) applies to derive a contradiction.

(c)  $f(A) \cap f(A') = \emptyset$ ,  $f(A) \cap f(A'') = \emptyset$ . We may assume that  $f(A') \cap f(A''') = \emptyset$  and  $f(A'') \cap f(A''') = \emptyset$ , otherwise analyses similar to cases (a) and (b) apply. It follows that  $f(A) = f(A''')$  and  $f(A') = f(A'')$ , and the sets are complementary  $(n-2)$ -faces of  $C_{n-1}$ . Consider  $b = x00 \in A$ ,  $b' = x10 \in A'$ ,  $b'' = x01 \in A''$ , and  $b''' = x11 \in A'''$ . Note that there is exactly one node in the  $(n-2)$ -face  $f(A)$  that is adjacent to  $f(b') \notin f(A)$ . Hence  $f(b) = f(b''')$ . With a similar argument one shows  $f(b') = f(b'')$ . It follows that the 2-face of  $C_n$  spanned by  $b, b', b'', b'''$  is mapped to a 1-face. Contradiction with Claim 3.5.1.

This ends the proof of Theorem 3.5. ■

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