Malfliet, R.
Ruijgrok, Th.
1967

Physica 33
607-619

THEORY OF NON-RELATIVISTIC THREE-PARTICLE SCATTERING

by R. MALFLIET and TH. RUIJGROK

Instituut voor theoretische fysica der Rijksuniversiteit, Utrecht, Nederland

Synopsis

A new method, using asymptotically stationary states, is developed to calculate the S-matrix for the scattering of a non-relativistic particle by the bound state of two other particles. For the scattering with breakup of this bound state, we obtain a simplified form of the Faddeev integral equations.

It is then shown that all scattering processes without breakup of the bound state can be simply connected to the process where possible excitation or deexcitation of the bound state but no particle exchange occurs. The scattering amplitude for these processes satisfies a two-particle Lippmann-Schwinger equation with a compact kernel. It is in a form where all quantities have a clear physical meaning, so that it is well suited as a starting point for the application of approximation methods.

§ 1. *Introduction.* Faddeev[1-4]) has shown that the quantum mechanical problem of scattering of a particle by a bound state of two other particles can be formulated mathematically in such a way that the equations have a unique solution. Afterwards many authors[5-8]) have tried to extend Faddeev's work and describe it in more physical terms. The latter, however, was in our opinion not always successfull.

We believe that the method developed in the previous paper[9]) provides a simple way of presenting the physics involved in three-particle scattering There we used the asymptotically stationary (a.s.) states, first introduced by Van Hove[10]), to describe the possible outgoing states, in terms of which the S-matrix could be defined uniquely. The advantage of this method is that S can be considered as a unitary operator which connects the free ingoing and the free outgoing states. This does not hold in the usual treatment, where the products of three plane wave states are taken as the basis of the Hilbert space. Since the a.s. states form an overcomplete set the expansion of the stationary scattering states in terms of these a.s. states is not unique. The scattering amplitudes are therefore not uniquely defined, except on the energy shell (as was shown in ref. 9), where they determine the S-matrix. This property reflects the fact that only at large distances from the reaction region we can tell the probabilities with which certain particle states occur. In the reaction region we cannot distinguish between

free and bound particles. It is therefore only natural to adjust the formulation of the scattering problem to this situation. The purpose of the present paper is to show how, in spite of the indeterminateness of the expansion coefficients, the scattering amplitudes on the energy shell can be calculated uniquely. We find that our equations are simpler and more transparent than the ones derived before. Especially the equations for the elastic amplitudes do not mix the different channels and can be considered as the equations for two-particle scattering.

After the precise definition of the a.s. states (§ 2), we define in section 3 new scattering amplitudes and derive the equations which they satisfy. Section 4 and 5 are then devoted to a more detailed examination of the equations for the inelastic and elastic scattering amplitudes respectively. In section 4 the decoupling procedure of Faddeev has to be used. In the last section we give our conclusions together with a brief comparison with the work of Lovelace[6]).

§ 2. *The asymptotically stationary (a.s.) states and the S-matrix.* We consider a system of spinless particles of three different kinds, 1, 2, and 3. We suppose that we have only two-particle interactions, so that the total hamiltonian is:

$$H = H_0 + \sum_{i=1}^{3} V_i \tag{2.1}$$

With H_0 the total kinetic energy and V_i the potential energy between particles j and $k (i \neq j \neq k \neq i)$.

We introduce the following notation:

$|\{\alpha\}_i\, \alpha_i\rangle$ is a three-particle eigenstate of H_0 with corresponding energy $E(\{\alpha\}_i\, \alpha_i) = \varepsilon_{\{\alpha\}_i} + \varepsilon_{\alpha_i} \cdot \alpha_i$ refers to the i-th particle and $\{\alpha\}_i$ is the set of the other two particles $\{\alpha\}_i = \alpha_j, \alpha_k$ $(i \neq j \neq k \neq i)$. Thus $\varepsilon_{\{\alpha\}_i} = \varepsilon_{\alpha_j} + \varepsilon_{\alpha_k}$ with ε_{α_i} the kinetic energy of the i-th particle.

$|A_i \alpha_i\rangle$ is a three-particle eigenstate of $H_i = H_0 + V_i$ with corresponding energy $\omega(A_i \alpha_i) = \varepsilon_{\alpha_i} + E_{A_i}$. The quantumnumber A_i refers to the boundstate of particles j and k, and E_{A_i} is the energy of this state (kinetic energy-binding energy).

It is easy to see that the states $|\{\alpha\}_i\, \alpha_i\rangle$ and $|A_i \alpha_i\rangle$ $(i = 1, 2, 3)$ are the asymptotically orthonormal states of reference 9. They form an overcomplete set because they will be linearly dependent of each other in the sense that the states $|\{\alpha\}_i\, \alpha_i\rangle$ form a complete set, and the states $|A_i \alpha_i\rangle$ can always be written as a linear combination of the states $|\{\alpha\}_i\, \alpha_i\rangle$:

$$|A_i \alpha_i\rangle = \int_{\{\alpha\}_i} F_i(\{\alpha\}_i\, A_i)\, |\{\alpha\}_i\, \alpha_i\rangle \tag{2.2}$$

The integration is over the variables $\{\alpha\}_i = \alpha_j, \alpha_k$ $(i \neq j \neq k \neq i)$ and the

expansion coefficients $F_i(\{\alpha\}_i A_i)$ satisfy the Schrödinger-equation:

$$[\omega(A_i\alpha_i) - E(\{\alpha\}_i \alpha_i)]\, F_i(\{\alpha\}_i A_i) = \int_{\{\alpha_i'\}} V_i(\{\alpha\}_i | \{\alpha'\}_i)\, F_i(\{\alpha'\}_i A_i) \qquad (2.3)$$

with the notation: $V_i(\{\alpha\}_i | \{\alpha'\}_i) = \langle \{\alpha\}_i | V_i | \{\alpha'\}_i \rangle$.

(We note that the functions $F_i(\{\alpha\}_i A_i)$ are just the boundstate wave functions, which are independent of the variable α_i). Furthermore we have the orthonormality relation:

$$\int_{\{\alpha\}_i} F_i^*(\{\alpha\}_i A_i)\, F_i(\{\alpha\}_i A_i') = \delta_{A_i, A_i'}. \qquad (2.4)$$

We now define a unitary S-matrix in the same way as was done in the previous article[9]). We expand the stationary scattering state $|A_i\alpha_i\rangle^{\pm}$ into our overcomplete states:

$$|A_i\alpha_i\rangle^{\pm} = |A_i\alpha_i\rangle - \int_{\alpha_1'\alpha_2'\alpha_3'} \frac{\tilde{D}_i^{\pm}(\alpha_1'\alpha_2'\alpha_3' | A_i\alpha_i)}{E(\alpha_1'\alpha_2'\alpha_3') - \omega(A_i\alpha_i) \mp i\varepsilon} |\alpha_1'\alpha_2'\alpha_3'\rangle - \\ - \sum_{k=1}^{3} \int_{A_k'\alpha_k'} \frac{C_{ki}^{\pm}(A_k'\alpha_k' | A_i\alpha_i)}{\omega(A_k'\alpha_k') - \omega(A_i\alpha_i) \mp i\varepsilon} |A_k'\alpha_k'\rangle \qquad (2.5)$$

($\int_{A_k'}$ means also a summation over the discrete variables)
($\tilde{D}_i^{\pm}(\alpha_1'\alpha_2'\alpha_3' | A_i\alpha_i)$ and $C_{ki}^{\pm}(A_k'\alpha_k' | A_i\alpha_i)$ being functions of $\omega(A_i\alpha_i)$). The $\pm$ sign refers to two different continuations off the energy-shell in order that the integrals exist (ε being small positive and has to be taken $\varepsilon \downarrow 0$) and also to formulate the unitary relations. We make the important remark that the coefficients $C_{ki}^{\pm}$ and $\tilde{D}_i^{\pm}$ in (2.5) are not uniquely determined because of our overcomplete basic set, but one can show[9]) that on the energy-shell they do have a unique value and this is the only quantity which is of physical importance.

It is the main purpose of this paper to give a procedure by which the on-shell values of the transition amplitudes $C_{ki}^{\pm}$ and $\tilde{D}_i^{+}$ may be calculated. The matrixelements of the S-matrix, which are relevant in this case, are given by (see also ref. 9):

$$S(A_j''\alpha_j'' | A_i\alpha_i) = \delta_{i,j}\, \delta(\alpha_j'' - \alpha_i)\, \delta_{A_j'', A_i} - \\ - 2\pi i C_{ji}^{+}(A_j''\alpha_j'' | A_i\alpha_i)\, \delta(\omega(A_j''\alpha_j'') - \omega(A_i\alpha_i)) \qquad (2.6)$$

$$S(\alpha_1''\alpha_2''\alpha_3'' | A_i\alpha_i) = -2\pi i \tilde{D}_i^{+}(\alpha_1''\alpha_2''\alpha_3'' | A_i\alpha_i)\, \delta(E(\alpha_1''\alpha_2''\alpha_3'') - \omega(A_i\alpha_i)) \qquad (2.7)$$

Formula (2.6) gives the S-matrix for the process in which particle i is scattered by a bound state of the two other particles, and ends up with particle j free and the two remaining particles bound.

C_{ji}^{+} is the transition amplitude for this process. Also, formula (2.7) gives the process in which particle i breaks up the bound state of the other two

particles, giving three free particles in the final state. Thus $\tilde{D}_i^+$ is the corresponding transition amplitude. These are the only possible transitions for our system of three particles.

In ref. 9 it is proved that the S-matrix defined in (2.6) and (2.7) is unitary, which implies the following reciprocity relations:

$$(C_{ji}^+)^* = C_{ij}^- \qquad (\tilde{D}_i^+)^* = \tilde{D}_i^- \tag{2.8}$$

We remark that unitarity means[9]):

$$\begin{aligned} SS^+(A_j''\alpha_j''\,|A_i\alpha_i) = S^+S(A_j''\alpha_j''\,|A_i\alpha_i) = \delta_{i,j}\,\delta(\alpha_j'' - \alpha_i) \times \\ \times\, \delta_{A_j'',A_i} = \sum_{k=1}^{3} \Big[\int_{A_k'\alpha_k'} S^+(A_j''\alpha_j''\,|A_k'\alpha_k')\, S(A_k'\alpha_k'\,|A_i\alpha_i)\Big] + \\ + \int_{\alpha_1'\alpha_2'\alpha_3'} S^+(A_j''\alpha_j''\,|\alpha_1'\alpha_2'\alpha_3')\, S(\alpha_1'\alpha_2'\alpha_3'\,|A_i\alpha_i) \end{aligned} \tag{2.9}$$

§ 3. *Basic integral equations.* We now derive equations which determine completely the transition amplitudes occurring in the expressions (2.6) and (2.7) for the S-matrix elements. To do this we use the Schrödinger equation for the scattering state $|A_i\alpha_i\rangle^+$:

$$H\,|A_i\alpha_i\rangle^+ = \omega(A_i\alpha_i)|A_i\alpha_i\rangle^+ \tag{3.1}$$

Taking the scalar products with the asymptotic orthonormal states $|\alpha_1''\alpha_2''\alpha_3''\rangle$ resp. $|A_j''\alpha_j''\rangle$ gives us:

$$\langle\alpha_1''\alpha_2''\alpha_3''\,|H|\,A_i\alpha_i\rangle^+ = \omega(A_i\alpha_i)\langle\alpha_1''\alpha_2''\alpha_3''\,|A_i\alpha_i\rangle^+ \tag{3.2}$$

$$\langle A_j''\alpha_j''\,|H|\,A_i\alpha_i\rangle^+ = \omega(A_i\alpha_i)\,\langle A_j''\alpha_j''\,|A_i\alpha_i\rangle^+ \tag{3.3}$$

($i = j$ is not excluded)

To proceed further we need some matrixelements of the total hamiltonian between the various asymptotic orthonormal states. It is easy to see that the following relations hold:

$$\langle\alpha_1''\alpha_2''\alpha_3''\,|H|\,\alpha_1'\alpha_2'\alpha_3'\rangle = \prod_{i=1}^{3}\delta(\alpha_i'' - \alpha_i')\,E(\alpha_1'\alpha_2'\alpha_3') + W(\alpha_1''\alpha_2''\alpha_3''|\alpha_1'\alpha_2'\alpha_3') \tag{3.4}$$

with the notation: $W = \sum_{i=1}^{3} V_i$

$$\langle A_i''\alpha_i''\,|H|\,\alpha_1'\alpha_2'\alpha_3'\rangle = \delta(\alpha_i'' - \alpha_i')\,F_i^*(\{\alpha'\}_i\,A_i'')\,\omega(A_i''\alpha_i'') + W_i(A_i''\alpha_i''|\alpha_1'\alpha_2'\alpha_3') \tag{3.5}$$

with: $W_i = \sum_{l,\,l\neq i} V_l$

$$\langle A_j''\alpha_j''\,|H|\,A_i'\alpha_i'\rangle = \omega(A_i'\alpha_i')\,\langle A_j''\alpha_j''\,|A_i'\alpha_i'\rangle + W_i(A_j''\alpha_j''\,|A_i'\alpha_i') \tag{3.6}$$

We are able now to write down explicitely the equations (3.2) and (3.3) using the expression (2.5) for the scattering state $|A_i\alpha_i\rangle^+$. First we define

the following transformation

$$D_i(\alpha_1'\alpha_2'\alpha_3' | A_i\alpha_i) = \tilde{D}_i^+(\alpha_1'\alpha_2'\alpha_3' | A_i\alpha_i) + \\ + [E(\alpha_1'\alpha_2'\alpha_3') - \omega(A_i\alpha_i)] \sum_{k=1}^{3} \int_{A_k'} \frac{C_{ki}^+(A_k'\alpha_k' | A_i\alpha_i)\, F_k(\{\alpha'\}_k A_k')}{\omega(A_k'\alpha_k') - \omega(A_i\alpha_i) - i\varepsilon} \tag{3.7}$$

We do so because we want to construct an equation which determines the transition amplitude $\tilde{D}_i^+$ on the energy shell, so we have to eliminate the other transitionamplitudes. The importance of this transformation lies in the fact that on the energy shell the transition amplitude $\tilde{D}_i^+$ equal sD_i. Thus, when we can calculate D_i, we immediately can write down the expression for the corresponding S-matrixelement (2.7). Using the transformation (3.7) in the expression (2.5) we find from (3.2) after some calculation the following integral equation for D_i:

$$D_i(\alpha_1''\alpha_2''\alpha_3'' | A_i\alpha_i) = W_i(\alpha_1''\alpha_2''\alpha_3'' | A_i\alpha_i) - \\ - \int_{\alpha_1'\alpha_2'\alpha_3'} \frac{D_i(\alpha_1'\alpha_2'\alpha_3' | A_i\alpha_i)}{E(\alpha_1'\alpha_2'\alpha_3') - \omega(A_i\alpha_i) - i\varepsilon} W(\alpha_1''\alpha_2''\alpha_3'' | \alpha_1'\alpha_2'\alpha_3') \tag{3.8}$$

It should be noted that the same equation (3.8) would have been obtained if we had put C_{ki}^+ in (2.5) equal to zero. For this reason and because the states $|\alpha_1\alpha_2\alpha_3\rangle$ form a complete orthonormal set, we expect (3.8) to have a unique solution. The kernel of the integral equation (3.8) is not compact, because of the appearence of different δ-functions in the expression $W(\alpha_1''\alpha_2''\alpha_3'' | \alpha_1'\alpha_2'\alpha_3')$ which cannot be separated out. This is a well-known difficulty which we shall treat later on(see § 4). We will now attack equation (3.3) by defining the following transformation:

$$D_{ji}(\alpha_1'\alpha_2'\alpha_3' | A_i\alpha_i) = \tilde{D}_i^+(\alpha_1'\alpha_2'\alpha_3' | A_i\alpha_i) + \\ + [E(\alpha_1'\alpha_2'\alpha_3') - \omega(A_i\alpha_i)] \sum_{k \neq j} \int_{A_k'} \frac{C_{ki}^+(A_k'\alpha_k' | A_i\alpha_i)\, F_k(\{\alpha'\}_k A_k')}{\omega(A_k'\alpha_k') - \omega(A_i\alpha_i) - i\varepsilon} \tag{3.9}$$

Again we want to eliminate the irrelevant transition amplitudes, viz. $\tilde{D}_i^+(\alpha_1'\alpha_2'\alpha_3' | A_i\alpha_i)$ and $C_{ki}^+(A_k'\alpha_k' | A_i\alpha_i)$ with $k \neq j$). Using (3.9) in formula (2.5) we get:

$$|A_i\alpha_i\rangle^+ = |A_i\alpha_i\rangle - \int_{\alpha_1'\alpha_2'\alpha_3'} \frac{D_{ji}(\alpha_1'\alpha_2'\alpha_3' | A_i\alpha_i)}{E(\alpha_1'\alpha_2'\alpha_3') - \omega(A_i\alpha_i) - i\varepsilon} |\alpha_1'\alpha_2'\alpha_3'\rangle \\ - \int_{A_j'\alpha_j'} \frac{C_{ji}^+(A_j'\alpha_j' | A_i\alpha_i)}{\omega(A_j'\alpha_j') - \omega(A_i\alpha_i) - i\varepsilon} |A_j'\alpha_j'\rangle \tag{3.10}$$

Analogous to the transformation (3.7) we define:

$$\bar{C}_{ji}(A_j''\alpha_j''\,|\,A_i\alpha_i) = C_{ji}^{+}(A_j''\alpha_j''\,|\,A_i\alpha_i) + \\ + [\omega(A_j''\alpha_j'') - \omega(A_i\alpha_i)] \int_{\{\alpha'\}_j} \frac{D_{ji}(\{\alpha'\}_j\,\alpha_j''\,|\,A_i\alpha_i)\,F_j^{*}(\{\alpha'\}_j\,A_j'')}{E(\{\alpha'\}_j\,\alpha_j'') - \omega(A_i\alpha_i) - i\varepsilon} \tag{3.11}$$

Using the transformations (3.10) and (3.11) in equation (3.3) we get the equation:

$$\bar{C}_{ji}(A_j''\alpha_j''\,|\,A_i\alpha_i) = W_i(A_j''\alpha_j''\,|\,A_i\alpha_i) - \\ - \int_{A_j'\alpha_j'} \frac{\bar{C}_{ji}(A_j'\alpha_j'\,|\,A_i\alpha_i)}{\omega(A_j'\alpha_j') - \omega(A_i\alpha_i) - i\varepsilon}\, W_j(A_j''\alpha_j''\,|\,A_j'\alpha_j') - T_{ji}(A_j''\alpha_j''\,|\,A_i\alpha_i) \tag{3.12}$$

Where $T_{ji}(A_j''\alpha_j''\,|\,A_i\alpha_i)$ is an abbreviation for the following expression:

$$T_{ji}(A_j''\alpha_j''\,|\,A_i\alpha_i) = \int_{\alpha_1'\alpha_2'\alpha_3'} \frac{D_{ji}(\alpha_1'\alpha_2'\alpha_3'\,|\,A_i\alpha_i)}{E(\alpha_1'\alpha_2'\alpha_3') - \omega(A_i\alpha_i) - i\varepsilon} \Big\{ W_j(A_j''\alpha_j''\,|\,\alpha_1'\alpha_2'\alpha_3') \\ - \int_{A_j'} W_j(A_j''\alpha_j''\,|\,A_j'\alpha_j')\,F_j^{*}(\{\alpha'\}_j\,A_j') \Big\} \tag{3.13}$$

The expression between curly brackets, and therefore the T_{ji} itself, does not vanish because the two particle boundstates do not form a complete set for the two-particle problem. It is clear from equation (3.11) that on the energyshell the transition amplitude C_{ji}^{+} equals $\bar{C}_{ji}$. In order to construct our S-matrix element (2.6) uniquely it is therefore sufficient to show that (3.12) has only one solution. This is done by proving that T_{ji} is uniquely determined. For that purpose we note that, using (3.7) and (3.9), the following relation holds:

$$D_{ji}(\alpha_1''\alpha_2''\alpha_3''\,|\,A_i\alpha_i) = D_i(\alpha_1''\alpha_2''\alpha_3''\,|\,A_i\alpha_i) - \\ - [E(\alpha_1''\alpha_2''\alpha_3'') - \omega(A_i\alpha_i)] \int_{A_j'} \frac{C_{ji}^{+}(A_j'\alpha_j''\,|\,A_i\alpha_i)\,F_j(\{\alpha''\}_j\,A_j')}{\omega(A_j'\alpha_j'') - \omega(A_i\alpha_i) - i\varepsilon} \tag{3.14}$$

Putting this into the expression (3.13) for T_{ij} we get the following result:

$$T_{ji}(A_j''\alpha_j''\,|\,A_i\alpha_i) = \int_{\alpha_1'\alpha_2'\alpha_3'} \frac{D_i(\alpha_1'\alpha_2'\alpha_3'\,|\,A_i\alpha_i)}{E(\alpha_1'\alpha_2'\alpha_3') - \omega(A_i\alpha_i) - i\varepsilon} \Big\{ W_j(A_j''\alpha_j''\,|\,\alpha_1'\alpha_2'\alpha_3') - \\ - \int_{Aj'} W_j(A_j''\alpha_j''\,|\,A_j'\alpha_j')\,F_j^{*}(\{\alpha'\}_j\,A_j') \Big\} \tag{3.15}$$

Since D_i is uniquely determined by (3.8) the same is true for T_{ji}. Because of the different transformations leading to equation (3.12) and the term T_{ji}

(dependent on D_i) we see that the transition amplitude C_{ji}^{+} on the energy-shell depends on all the other off-shell transition amplitudes. It may seem strange that the elastic amplitudes depend on the inelastic ones, but a closer investigation of the term T_{ji}, even under threshold, will clarify this dependence (see section 5).

§ 4. *Investigation of equation* (3.8). We already mentioned that the kernel of equation (3.8) is not compact. Due to this difficulty we cannot apply Fredholm's theory and we have to rewrite the equation in another form. This we will do using the method introduced by Faddeev[1) 2) 3) 4)] and applied by Lovelace[5) 6)], Weinberg[7)] and Rosenberg[8)].

Instead of (3.8) we consider the equation

$$D_i(\alpha_1''\alpha_2''\alpha_3''\,|\,\alpha_1\alpha_2\alpha_3) = W_i(\alpha_1''\alpha_2''\alpha_3''\,|\,\alpha_1\alpha_2\alpha_3) - \int_{\alpha_1'\alpha_2'\alpha_3'} \frac{D_i(\alpha_1'\alpha_2'\alpha_3'\,|\,\alpha_1\alpha_2\alpha_3)}{E(\alpha_1'\alpha_2'\alpha_3') - \omega(A_i\alpha_i) - i\varepsilon}\, W(\alpha_1''\alpha_2''\alpha_3''\,|\,\alpha_1'\alpha_2'\alpha_3'), \tag{4.1}$$

which defines the function $D_i(\alpha_1''\alpha_2''\alpha_3''\,|\,\alpha_1\alpha_2\alpha_3)$.

Using (2.2) it is seen immediately that the solution of (3.8) is then given by

$$D_i(\alpha_1''\alpha_2''\alpha_3''\,|\,A_i\alpha_i) = \int_{\{\alpha\}_i} D_i(\alpha_1''\alpha_2''\alpha_3''\,|\,\alpha_1\alpha_2\alpha_3)\, F_i(\{\alpha\}_i A_i)$$

In operator form equation (4.1) reduces to

$$D_i(z) = W_i - WA(z)\, D_i(z), \tag{4.2}$$

for $z \to \omega(A_i\alpha_i) + i\varepsilon$. We furthermore have used the notation

$$A(z)\,|\,\alpha_1'\alpha_2'\alpha_3'\rangle = [E(\alpha_1'\alpha_2'\alpha_3') - z]^{-1}\,|\alpha_1'\alpha_2'\alpha_3'\rangle. \tag{4.3}$$

The solution of equation (4.2) can be written in the form

$$D_i(z) = W_i - D(z)\, A(z)\, W_i, \tag{4.4}$$

if $D(z)$ satisfies the equation

$$D(z) = W - WA(z)\, D(z) \tag{4.5}$$

This is the equation in which the "dangerous δ-functions" occur in their purest form. With Faddeev we transform it as follows.

The functions $\bar{D}_l(z)$ for $l = 1, 2, 3$ are defined as the solutions of the equations

$$\bar{D}_l(z) = V_l - V_l A(z)\, \bar{D}_l(z) \qquad (l = 1, 2, 3) \tag{4.6}$$

They can be expressed in terms of the two-particle scattering amplitudes $t_l(z)$ as follows

$$\langle\alpha_1''\alpha_2''\alpha_3''\,|\bar{D}_l(z)|\,\alpha_1\alpha_2\alpha_3\rangle = \delta(\alpha_l' - \alpha_l)\,\langle\{\alpha''\}_l\, t_l(z)|\{\alpha\}_l\rangle. \tag{4.7}$$

$t_l(z)$ describes the scattering of particles k and $m (k \neq m \neq l \neq k)$, but at an energy $z = \omega(A_i \alpha_i)$ which in general is not equal to the energy $\varepsilon_{\alpha_k} + \varepsilon_{\alpha_m}$ of the state $|\{\alpha\}_l\rangle$. Faddeev has shown that $D(z)$ can be written in the form

$$D(z) = D^{(1)}(z) + D^{(2)}(z) + D^{(3)}(z),$$

so that $D^{(1)}(z)$ has the same δ-singularity as $\bar{D}_l(z)$.

Equation (4.5) was then shown to be equivalent to (see also appendix)

$$\begin{pmatrix} D^{(1)}(z) \\ D^{(2)}(z) \\ D^{(3)}(z) \end{pmatrix} = \begin{pmatrix} \bar{D}_1(z) \\ \bar{D}_2(z) \\ \bar{D}_3(z) \end{pmatrix} - U(z)\, A(z) \begin{pmatrix} D^{(1)}(z) \\ D^{(2)}(z) \\ D^{(3)}(z) \end{pmatrix}, \tag{4.8}$$

with

$$U(z) = \begin{pmatrix} 0 & \bar{D}_1(z) & \bar{D}_1(z) \\ \bar{D}_2(z) & 0 & \bar{D}_2(z) \\ \bar{D}_3(z) & \bar{D}_3(z) & 0 \end{pmatrix}$$

Now we could further apply Faddeev's method[2)4)], who proved that the square of the kernel U is compact. If we assume that the potentials are separable, as e.g. in the work of Lovelace[5]). it can be shown that, under certain conditions, the kernel U itself is of the Hilbert-Schmidt type.

We observe that the integral equation (4.8) only contains the two-particle scattering amplitudes, but that the potentials still occur, viz. in equation (4.4). This equation also shows that for the determination of $D_i(z)$ we only need some integral over $D(z)$.

The function $D(z)$ contains therefore much more information than we need at this stage. It determines e.g. also the amplitudes for elastic scattering. For these, however, we will derive separate equations in the next section. We will therefore now apply Faddeev's method directly to equation (4.2). Writing $D_i(z)$ as

$$D_i(z) = D_i^{(1)}(z) + D_i^{(2)}(z) + D_i^{(3)}(z),$$

it is found that these functions can be chosen so as to satisfy the equation

$$\begin{pmatrix} D_i^{(1)}(z) \\ D_i^{(2)}(z) \\ D_i^{(3)}(z) \end{pmatrix} = \begin{pmatrix} \bar{D}_l(z) \\ \bar{D}_k(z) \\ 0 \end{pmatrix} - \\ - \begin{pmatrix} 0 & \bar{D}_l(z) & \bar{D}_l(z) \\ \bar{D}_k(z) & 0 & \bar{D}_k(z) \\ \bar{D}_i(z) & \bar{D}_i(z) & 0 \end{pmatrix} A(z) \begin{pmatrix} D_i^{(1)}(z) \\ D_i^{(2)}(z) \\ D_i^{(3)}(z) \end{pmatrix} (i \neq k \neq l \neq i) \tag{4.9}$$

where the functions $\bar{D}_m(z)$, $m = i, k, l$, are again defined by (4.6).

The proof is very much the same as that of the appendix. The singularities in $D_i(z)$ are seen to be those of $\bar{D}_l(z)$ and $\bar{D}_k(z)$ and these are indeed the

same as those of W_i in (4.4). Substitution of

$$D_i^{(3)}(z) = -D_i(z)\, A(z)(D_i^{(1)}(z) + D_i^{(2)}(z)) \tag{4.10}$$

into the other two equations of (4.9) leads to

$$\begin{pmatrix} D_i^{(1)} \\ D_i^{(2)} \end{pmatrix} = \begin{pmatrix} \bar{D}_l \\ \bar{D}_k \end{pmatrix} + \begin{pmatrix} \bar{D}_l A \bar{D}_i & \bar{D}_l A \bar{D}_i - \bar{D}_l \\ \bar{D}_k A \bar{D}_i - \bar{D}_k & \bar{D}_k A \bar{D}_i \end{pmatrix} A \begin{pmatrix} D_i^{(1)} \\ D_i^{(2)} \end{pmatrix} \tag{4.11}$$

Instead of the Faddeev equation (4.8) we now have to solve the simpler equation (4.11) for $i = 1, 2, 3$.

Another advantage over the calculation of $D_i(z)$ by way of (4.4) is that the potentials have been eliminated and that only a knowledge of the two-particle scattering amplitudes (off their energy-shell) is required.

§ 5. *Investigation of equation* (3.12). Let us assume that for $z = \omega(A_i\alpha_i) + i\varepsilon$ we have solved $D_i(z)$ from the equations of the preceding section. We can then perform the integrations in equation (3.15) and obtain $T_{ji}(A_j''\alpha_j'' | A_i\alpha_i)$.

Formally equation (3.15) can be written as

$$T_{ji}(z) = W_j(1 - P_j)\, A(z)\, D_i(z), \tag{5.1}$$

where

$$P_j = \int\limits_{A_j'\alpha_j'} |A_j'\alpha_j'\rangle \langle A_j'\alpha_j'| \tag{5.2}$$

is the projection operator on the subspace spanned by the states $|A_j'\alpha_j'\rangle$. In the same way equation (3.12) can be written as

$$\bar{C}_{ji}(z) = W_i - T_{ji}(z) - W_j A_j(z)\, P_j \bar{C}_{ji}(z) \tag{5.3}$$

with

$$A_j(z)\, |A_j'\alpha_j'\rangle = [\omega(A_j'\alpha_j') - z]^{-1}\, |A_j'\alpha_j'\rangle \tag{5.4}$$

Both (5.1) and (5.3) must be considered as multiplied by $\langle A_j''\alpha_j''|$ on the left- and $|A_i\alpha_i\rangle$ on the right hand side. Equation (5.3) can be reduced in the following way. Define the function $C_j(z)$ as the solution of the equation

$$C_j(z) = W_j - W_j A_j(z)\, P_j C_j(z) \tag{5.5}$$

which is a shorthand notation for

$$\langle A_j''\alpha_j'' \,|\, C_j(z)|\, A_j\alpha_j\rangle = \langle A_j''\alpha_j'' \,|W_j|\, A_j\alpha_j\rangle - \int\limits_{A_j'\alpha_j'} \frac{\langle A_j''\alpha_j'' \,|W_j|\, A_j'\alpha_j'\rangle \langle A_j'\alpha_j' \,|C_j(z)|\, A_j\alpha_j\rangle}{\omega(A_j'\alpha_j') - z} \tag{5.6}$$

It is than seen immediately that

$$\bar{C}_{ji}(z) = W_i - T_{ji}(z) - C_j(z)\, A_j(z)\, P_j(W_i - T_{ji}(z)) \tag{5.7}$$

is the solution of (5.3).

In order to evaluate this expression only an integration over the function $C_j(z)$ is involved. Equation (5.5) or (5.6) describes the scattering of a particle of the type j by the other two particles in a bound state. We stress that there is no exchange with one of the bound particles, nor break up of the bound state.

The only allowed processes are elastic scattering or excitation or deexcitation of the bound state. It must be noted, however, that the energy of the initial state is not $\omega(A_j\alpha_j)$, but is equal to z, which approaches $\omega(A_i\alpha_j) + i\varepsilon$. $C_j(z)$ must therefore be calculated off its energy shell.

Formula (5.7) shows the position of three types of branch points in $\bar{C}_{ji}(z)$. One given by the zero's of $A_j(z)$ and corresponds to the opening of channels for excited states. Another type comes from the branchpoints of $T_{ji}(z)$ as caused by the zero's of $A(z)$. This is connected with the opening of the breakup channel. The branchpoints of $D_i(z)$ give, through (5.1), rise to a third type of branchpoint in $\bar{C}_{ji}(z)$. These originate from the singularities in $\bar{D}_l(z)$ and $\bar{D}_k(z)$, (see (4.7)) as it should, because it might happen e.g. channel $l \neq$ j opens up before j does and this should be reflected in the scattering amplitude for $i \to j$.

Faddeev's reduction technique, as used in section 4, can also be applied to equation (5.5). For that purpose we introduce the scattering amplitudes $E_k(z)$ and $E_l(z)$ as the solutions of the equations

$$\begin{aligned} E_l(z) &= V_l - V_l A_j(z)\, P_j E_l(z) \qquad (j \neq k \neq l \neq j) \\ E_k(z) &= V_k - V_k A_j(z)\, P_j E_k(z) \end{aligned} \tag{5.8}$$

From these equations one sees that $E_l(z)$ describes how the particle j is scattered by the bound state of k and l for the case that j interacts only with particle k. In $E_k(z)$ there is only interaction between j and l. It can be shown that $C_j(z)$ can be written as

$$C_j(z) = C_j^{(1)}(z) + C_j^{(2)}(z)$$

where the two functions in the right hand side must be solved from the equation

$$\begin{pmatrix} C_j^{(1)}(z) \\ C_j^{(2)}(z) \end{pmatrix} = \begin{pmatrix} E_l(z) \\ E_k(z) \end{pmatrix} - \begin{pmatrix} 0 & E_l(z) \\ E_k(z) & 0 \end{pmatrix} A_j(z)\, P_j \begin{pmatrix} C_j^{(1)}(z) \\ C_j^{(2)}(z) \end{pmatrix} \tag{5.9}$$

Actually this reduction is not necessary since the matrix elements $\langle A_j''\alpha_j'' \,|W_j|\, A_j\alpha_j\rangle$, occurring in (5.6) do not contain "dangerous δ-functions", as was the case in equation (3.8). Although, therefore, the structure of the equations (5.8) is not different from that of equation (5.5) is there may be practical reasons why (5.8) should be preferred. Since in $E_l(z)$ j and l do not interact it could e.g. happen that under certain conditions, $E_l(z)$ is simply related to the two-particle amplitude for j–k scattering. In that

case we do not have to know the potentials in order to solve (5.9). We cannot do completely without potentials, however, because they turn up again in equation (5.7).

§ 6. *Conclusions.* In this and the previous article[9]) we have formulated the scattering by composite systems in more physical terms than has been done before. The asymptotically stationary (a.s.) states, as first introduced by Van Hove[10]), offer a natural tool for a definition of the S-matrix. Since the a.s. states form an overcomplete set, the scattering amplitudes are not unique. It was shown[9]), however, that the S-matrix was well defined and satisfied unitarity. Although in principle the calculation of scattering cross sections was now solved, we still had to give a procedure for the determination of S. For this we should solve the scattering problem and here we had to face the problem of non-uniqueness of the scattering amplitudes. This problem was solved in the present paper for the special case of three particles, any two of which may form bound states. By some simple transformations new amplitudes for elastic and inelastic scattering were introduced and it was shown that these could be determined uniquely from some integral equations. We, of course, encounter also the "dangerous δ-functions" but these are eliminated using the method of Faddeev[1]).

Our conclusions can be summed up in the following remarks:

1. Three-particle unitarity is satisfied.
2. After Faddeev's reduction, our equation for the inelastic scattering amplitude (4.11) is simpler than in ref. 6. This is caused by the use of the transformation (3.7) and also by the use of a.s. states which offer a simple apparatus for describing the observable quantities in a scattering process.
3. The equations like (5.5) for the elastic scattering amplitides are ordinary "two-particle" scattering equations in which no particle exchange occurs. For practical purposes an ansatz for T_{ji} could be made. This must be done in such a way that it incorporates other processes.
4. From the equations of section 5 we see that it would be quite natural to use composite particle models.
5. Equation (5.3) can be written as a "two-particle" Lippmann-Schwinger equation. The "potential" occurring in this equation can be obtained from the function $T_{ji}(z)$ and has all the features of an optical potential.
6. It is not necessary to assume that the two-particle potentials are separable in order to obtain simple equations. Although the occurrence of potentials in our equation (5.7) is a disadvantage, our equations have a simpler form than the ones obtained in the literature, where the potentials have been eliminated.

7. We have derived our equations only for the case of three distinct particles and under the assumption that any pair can form bound states. By trivial changes the method can be applied to all other cases.

APPENDIX

We will show that the solution of the equation

$$D(z) = W - WA(z)\, D(z) \tag{A.1}$$

with $W = \sum_{l=1}^{3} V_l$, can be written as $D(z) = \sum_{l=1}^{3} D^{(l)}(z)$ where $D^{(l)}(z)$ are solutions of

$$\begin{pmatrix} D^{(1)}(z) \\ D^{(2)}(z) \\ D^{(3)}(z) \end{pmatrix} = \begin{pmatrix} \bar{D}_1(z) \\ \bar{D}_2(z) \\ \bar{D}_3(z) \end{pmatrix} - \begin{pmatrix} 0 & \bar{D}_1(z) & \bar{D}_1(z) \\ \bar{D}_2(z) & 0 & \bar{D}_2(z) \\ \bar{D}_3(z) & \bar{D}_3(z) & 0 \end{pmatrix} A(z) \begin{pmatrix} D^{(1)}(z) \\ D^{(2)}(z) \\ D^{(3)}(z) \end{pmatrix} \tag{A.2}$$

and where $\bar{D}_l(z)$ is the solution of

$$\bar{D}_l(z) = V_l - V_l A(z)\, \bar{D}_l(z), \quad (l = 1, 2, 3) \tag{A.3}$$

From (A.3) follows

$$V_l = \bar{D}_l(z)[I - A(z)\, \bar{D}_l(z)]^{-1} \tag{A.4}$$

Defining $D^{(1)}(z)$ as the solution of the equations

$$D^{(l)}(z) = V_l - V_l A(z) \sum_{j=1}^{3} D^{(j)}(z) \tag{A.5}$$

we see by summing over l that $D(z) = \sum_{l=1}^{3} D^{(l)}(z)$ satisfies (A.1). In order to show that (A.5) is equivalent with (A.2) we replace V_l in (A.5) by the expression in (A.4) and multiply (A.5) on the left by

$$\bar{D}_l(z)[I - A(z)\, \bar{D}_l(z)][\bar{D}_l(z)]^{-1} = I - \bar{D}_l(z)\, A(z).$$

We obtain

$$[I - \bar{D}_l(z)\, A(z)]\, D^{(l)}(z) = \bar{D}_l(z) - \bar{D}_l(z)\, A(z) \sum_{j=1}^{3} D^{(j)}(z).$$

This is indeed the same equation as (A.2).

Acknowledgement: This investigation is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie" and was made possible by financial support from the "Nederlandse Organisatie Voor Zuiver Wetenschappelijk Onderzoek".

Received 27-6-66

REFERENCES

1) Faddeev, L. D., Zh. eksperim. i. teor. Fiz. **39** (1960) 1459.
(English transl.: Soviet Phys. – JETP **12** (1961) 1014.)
2) Faddeev, L. D., Dokl. Akad. Nauk SSR **138** (1961) 565.
(English transl.: Soviet Phys. – Doklady **6** (1961) 384.)
3) Faddeev, L. D., Dokl. Akad. Nauk SSR **145** (1962) 301.
(English transl.: Soviet Phys. – Doklady **7** (1963) 600.)
4) Faddeev, L. D., "Mathematical Problems of the Quantum Theory of Scattering for a Three-Particle System". (Publications of the Steklov Mathematical Institute, Leningrad, 1963, no. 69.)
(English transl.: Israel Program for Scientific Translations, Jerusalem, 1965.)
5) Lovelace, C., "Strong Interactions and High Energy Physics", edited by R. G. Moorhouse (Oliver and Boyd, London, 1964).
6) Lovelace, C., Phys. Rev. **135 B** (1964) 1225.
7) Weinberg, S., Phys. Rev. **133 B** (1964) 232.
8) Rosenberg, L., Phys. Rev. **135 B** (1964) 715.
9) Van Nieuwenhuizen, P. and Ruijgrok, T., Physica **32** (1966) 595.
10) Van Hove, L., Physica **21** (1955) 901, **22** (1956) 343.