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THE GENERATION-RECOMBINATION THEOREM AND NOISE IN PHOTOCONDUCTORS

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Synopsis

The validity of the well-known generation-recombination (g–r) theorem is examined for the case of noise in photoconductors. A master equation for the conditional probability of the level occupancies is set up in which the generation and recombination rates are functions of the incident light intensity. A more general g–r theorem is then derived, in which the light intensity appears explicitly as a driving force. From this theorem the variances and autocorrelation functions of the density perturbations are found, and averaged with respect to the light intensity fluctuations for the case that the incident light beam displays shot noise. The g–r theorem turns out to be not rigorously valid, correction terms being required due to the fact that the photoconductor is coupled not to the incident light beam intensity but to the number of photons in the crystal volume. These correction terms are small for the case of weak absorption.

1. *Introduction.* Two statistical*) methods are generally used to calculate the fluctuations of the energy level occupancies of semiconductors. The first begins with the Langevin differential equations for the time variation of the occupancies, and the second with the master equation, a differential equation for the probability distribution of the level occupancies. The Langevin method gives in a simple way the spectral density of the electron fluctuations, in terms of the spectral density of the Langevin stochastic source functions[1] [2] [3]. One usually then assumes that the Langevin functions are purely random variables; equivalently, that the essence of the noise is shot noise in the transition rates between the various electron levels. In contrast, the analysis based on the master equation gives the variances and autocorrelation functions of the number fluctuations, and, through the Wiener-Khintchine theorem, the spectral densities[4] [5]**). An advantage

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*) We leave out the thermodynamic approaches (Van Vliet and Fassett, 1965) since we are primarily interested in photoconductors.

**) Reference 4, the review article by Van Vliet and Fassett, will hereafter be denoted by vVF.

of this method is that no "random forces" such as the Langevin functions need be considered, although it may be shown[4)6)] that with the shot noise assumption for the Langevin functions the two methods are completely equivalent.

With photoconductors the situation becomes more interesting, since generation and recombination rates are functions of the fluctuating incident light intensity. The problem is to determine the effect of the light intensity fluctuations on the instantaneous electron occupancies of the various energy levels.

It is generally approached in the following manner. In nearly all cases in which photoconductors are used in practice, the incident photons may be considered to be noninteracting particles obeying classical Boltzmann statistics, and thus the light intensity displays only shot noise. A comparison may, then, be made with a simple photo-electric detector, in which input particles (photons) are transformed into output particles (electrons in a higher energy state). Burgess[7)] has shown that if the input particles obey Poisson statistics (i.e., exhibit shot noise), the output particles do also. Therefore, the noise formulas valid for semiconductors apply for photoconductors, since the photoconductor, although in a driven state as far as time averaged level occupancies are concerned, is not in a driven state from a noise viewpoint, because the fluctuations of the driving forces are of the same nature as the fluctuations intrinsic to the photoconductor. The only difference between the semiconductor and usual photoconductor theory is that in the latter the principle of detailed balance of the transition rates[8)] may not be used to simplify the generation-recombination theorem (the main result of the master equation approach), which gives the noise variances. These arguments are commonly accepted; we know of no attempt to derive, for example, the important generation-recombination theorem specifically for photoconductors.

In the present paper we begin a study of generation-recombination noise in many-level photoconductors by setting up a master equation in which the generation and recombination rates are explicit functions of the incident light intensity. A generation-recombination ($g - r$) theorem is then derived in which the effect of the fluctuations in the light intensity is clearly shown. An average over all possible time variations of the light intensity is carried out. The final equations we only evaluate for the special case when the light intensity is purely random, i.e., that excess photon noise is absent.

Before tackling the many-variable problem we treat the simpler two-level, single-variable case with two methods. One of these methods we then use to solve the many-variable problem.

2. *Noise in two-level photoconductors.* Consider a photoconductor having two electron energy levels, for example a donor level and the conduction

band. Let electron transitions occur between these levels, as illustrated in the figure. The total number of electrons in the higher energy level is n, that in the lower level n'. Since $n + n'$ is constant we need only deal with, say, n. The number of excitations per unit time is given by g, the number of de-excitations per unit time by r. In general g and r are functions of both the number n and the intensity of the incident light of the appropriate wavelength. The light intensity fluctuates, and we want to compute the fluctuations in n, taking these light fluctuations into account.

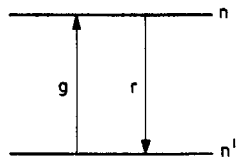

Fig. 1

Let $Z(t)$ be the number of photons in the volume of the crystal at time t, and let $P(n, t)$ be the probability that at time t the upper level is occupied by n electrons. The fluctuations in Z are not the same as the fluctuations in the incident light intensity, but we shall ignore this distinction for the moment. Then

$$P(n, t + \mathrm{d}t) = g(n - 1, Z(t))\,\mathrm{d}t\,P(n - 1, t) + r(n + 1, Z(t))\,\mathrm{d}t\,P(n + 1, t) + (1 - g(n, Z(t))\,\mathrm{d}t - r(n, Z(t))\,\mathrm{d}t)\,P(n, t). \quad (1)$$

This leads to the master equation of this problem:

$$\frac{\partial P(n, t)}{\partial t} = g(n - 1, Z(t))\,P(n - 1, t) - g(n, Z(t))\,P(n, t) + r(n + 1, Z(t))\,P(n + 1, t) - r(n, Z(t))\,P(n, t). \quad (2)$$

Here $Z(t)$ has been taken as a given function. If $Z(t)$ is a purely random function, that is, $Z(t)$ for all values of t is completely uncorrelated with the values Z took on for the previous t, then by an intuitive argument it may be shown that we need not explicitly consider $Z(t)$ in the master equation. For, if $Z(t)$ is purely random, we can immediately average the master equation over all occurring $Z(t)$. $P(n, t)$ depends upon Z for previous times, since it has a nonzero relaxation time, but $Z(t)$ is not correlated with the previous values of $P(n, t)$, nor is it correlated with $P(n, t)$ for the same t. Bars can be written over each term in the master equation to denote this averaging process, and terms such as $\overline{r(n + 1, Z)}\ \overline{P(n + 1, t)}$ result. But $\overline{r(n + 1, Z)}$ must be independent of $Z(t)$: we can instead write $r(n + 1)$. The customary master equation is therefore found, with $Z(t)$ not considered explicitly.

A rigorous approach is as follows.

An expansion of the right-hand side of the master equation shall be made. In order to determine correctly which terms of the expansion must be kept

and which ignored it is necessary to make a systematic expansion in one parameter. The parameter we take is the reciprocal of the volume Ω of the photoconductor, since fluctuations are of order $\Omega^{\frac{1}{2}}$, whereas steady state processes are of order Ω^1. This approach has been used successfully in the theory of fluctuations in nonlinear systems[9]). It remains to be seen what an expansion in powers of Ω leads to in our case.

Let $g(n, Z(t))$ be given by $g_0 Z(t)$, and $r(n, Z(t))$ by $r_0 n^2/\Omega$. The constants g_0, r_0 are independent of Ω; g and r are then of order Ω^1 as they must be*). Further put

$$Z(t) = \Omega z_0 + \Omega^{\frac{1}{2}}\zeta(t), \tag{3}$$

where z_0 is independent of Ω, and $\Omega^{\frac{1}{2}}\zeta(t)$ gives the fluctuations about Ωz_0, the time averaged value of $Z(t)$. Write also

$$n = \Omega\varphi(t) + \Omega^{\frac{1}{2}}x \tag{4}$$

to introduce x as new time-independent variable. The function $\varphi(t)$ shall be found below. When we write $Z(t)$ in this way we ignore spatial variations, taking absorption to be weak.

With our choice of g and r, equation (2) becomes

$$\frac{\partial P(n,t)}{\partial t} = g_0 Z(t)(P(n-1,t) - P(n,t)) + \\ + r_0\left\{\frac{(n+1)^2}{\Omega}P(n+1,t) - \frac{n^2}{\Omega}P(n,t)\right\}. \tag{5}$$

The Kramers-Moyal expansion of this master equation is

$$\frac{\partial P(n,t)}{\partial t} = g_0 Z(t)\left\{-\frac{\partial P}{\partial n} + \frac{1}{2}\frac{\partial^2 P}{\partial n^2} - \frac{1}{3!}\frac{\partial^3 P}{\partial n^3} + \ldots\right\} + \\ + \frac{r_0}{\Omega}\left\{\frac{\partial}{\partial n} + \frac{1}{2}\frac{\partial^2}{\partial n^2} + \frac{1}{3!}\frac{\partial^3}{\partial n^3} + \ldots\right\} n^2 P. \tag{6}$$

Write $P(n, t) = \Pi(x, t)$ so that

$$\frac{\partial \Pi}{\partial x} = \Omega^{\frac{1}{2}}\frac{\partial P}{\partial n}, \quad \frac{\partial \Pi}{\partial t} = \frac{\partial P}{\partial t} + \frac{\partial P}{\partial n}\Omega\dot{\varphi}. \tag{7}$$

The Kramers-Moyal equation for this problem becomes

$$\frac{\partial \Pi}{\partial t} - \Omega^{\frac{1}{2}}\frac{\partial \Pi}{\partial x}\dot{\varphi} = \\ g_0(\Omega z_0 + \Omega^{\frac{1}{2}}\zeta(t))\left\{-\Omega^{-\frac{1}{2}}\frac{\partial \Pi}{\partial x} + \frac{1}{2}\Omega^{-1}\frac{\partial^2 \Pi}{\partial x^2} - \frac{1}{3!}\Omega^{-\frac{3}{2}}\frac{\partial^3 \Pi}{\partial x^3} + \ldots\right\} + \\ + \frac{r_0}{\Omega}\left\{\Omega^{-\frac{1}{2}}\frac{\partial}{\partial x} + \tfrac{1}{2}\Omega^{-1}\frac{\partial^2}{\partial x^2} + \frac{1}{3!}\Omega^{-\frac{3}{2}}\frac{\partial^3}{\partial x^3} + \ldots\right\}(\Omega\varphi + \Omega^{\frac{1}{2}}x)^2\Pi. \tag{8}$$

*) Taking r, for example, r must vary as Ω^1, since each of the n electrons can recombine with n/Ω holes in its immediate neighbourhood (cf. ref. 10).

The terms of order $\Omega^{\frac{1}{2}}$ give an equation for $\varphi(t)$,

$$\dot{\varphi} = g_0 z_0 - r_0 \varphi^2. \tag{9}$$

From (3) and (4) and the expressions for g and r we see that this is the macroscopic equation for the expected value of n. After a first transient*) has died out φ takes on the value $\varphi(\infty) = \sqrt{g_0 z_0 / r_0}$; the average generation rate is equal to the average recombination rate.

Gathering the terms of order Ω^0,

$$\frac{\partial \Pi}{\partial t} = \frac{\partial}{\partial x}(2 r_0 \varphi x - g_0 \zeta(t))\, \Pi + \frac{\partial^2}{\partial x^2} \frac{g_0 z_0 + r_0 \varphi^2}{2} \Pi \tag{10}$$

a Fokker-Planck equation for the fluctuations x. One is usually interested in the behaviour for large t, and we therefore put $\varphi = \varphi(\infty)$.

The first Fokker-Planck moment, the jumps in x per unit time averaged over the probability that these jumps occur, is given by $2 r_0 \varphi(t)\, x - g_0 \varphi(t)$, which tends to $\tau x - g_0 \zeta(t)$, where $\tau = (4 g_0 z_0 r_0)^{-\frac{1}{2}}$. The second moment, $(g_0 z_0 + r_0 \varphi^2)/2$, the mean of the square of the jumps per unit time, tends to the constant value $g_0 z_0$.

Using angular brackets to denote an x-average of a function $f(x)$: $\langle f(x) \rangle_t = \sum_x f(x)\, \Pi(x, t)$, we find from the Fokker-Planck equation,

$$\frac{\mathrm{d}\langle x \rangle_t}{\mathrm{d}t} = -\frac{\langle x \rangle_t}{\tau} + g_0 \zeta(t) \tag{11}$$

$$\frac{\mathrm{d}\langle x^2 \rangle_t}{\mathrm{d}t} = -\frac{2\langle x^2 \rangle_t}{\tau} + 2 g_0 \zeta(t) \langle x \rangle_t + 2 g_0 z_0. \tag{12}$$

The same results may be obtained through the use of the characteristic function for $\Pi(x, t)$.

It may be concluded that if $\zeta(t)$ were put equal to zero, $\langle x \rangle_t$ would tend to zero and $\langle x^2 \rangle_t$ would approach the value $g_0 z_0 \tau$. The latter result is equivalent to the generation-recombination theorem for two-level semiconductors[11]). In the present problem, however, we must obtain $\langle x \rangle_t$ and $\langle x^2 \rangle_t$ for a given $\zeta(t)$ from (11) and (12), and then average over all possible fluctuations $\zeta(t)$.

In the general many-level problem, considered in the next section, equations are found of which (11) and (12) are special cases. These more general differential equations shall be treated in detail.

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*) The complete solution is

$$\varphi(t) = \sqrt{\frac{g_0 z_0}{r_0}} \frac{\varphi(0) + \sqrt{\frac{g_0 z_0}{r_0}} \tanh(\sqrt{g_0 z_0 r_0}\, t)}{\sqrt{\frac{g_0 z_0}{r_0}} + \varphi(0) \tanh(\sqrt{g_0 z_0 r_0}\, t)}$$

The transient dies out after a time of the order of $1/\sqrt{g_0 z_0 r_0}$.

Before going on to the many-level problem we add some further comments. Generally g and r may vary with both $Z(t)$ and n. We write

$$g(n, Z) = g_0\Omega + g_1 n + g_2 Z + g_3 \frac{n^2}{\Omega} + g_4 \frac{nZ}{\Omega} \tag{13}$$

$$r(n, Z) = r_0\Omega + r_1 n + r_2 Z + r_3 \frac{n^2}{\Omega} + r_4 \frac{nZ}{\Omega}. \tag{14}$$

The r_i's and g_i's are independent of Ω. We find a Kramers-Moyal equation again, of which the terms of order $\Omega^{\frac{1}{2}}$ give the macroscopic equation:

$$\dot{\varphi} = g_0 - r_0 + (g_2 - r_2)\, z_0 + (g_1 - r_1 + (g_4 - r_4)\, z_0)\, \varphi + (g_3 - r_3)\, \varphi^2. \tag{15}$$

The Fokker-Planck equation results from the terms of order Ω^0.

$$\frac{\partial \Pi}{\partial t} = \frac{\partial}{\partial x}\{x(r_1 - g_1 + (r_4 - g_4)\, z_0 + 2(r_3 - g_3)\, \varphi) + \zeta(t)((r_4 - g_4)\, \varphi + \\ + (r_2 - g_2))\}\, \Pi + \frac{1}{2} \frac{\partial^2}{\partial x^2}\{r_0 + g_0 + (r_2 + g_2)\, z_0 + (r_1 + g_1 + \\ + (r_4 + g_4)\, z_0)\, \varphi + (r_3 + g_3)\, \varphi^2\}\, \Pi. \tag{16}$$

Again $\varphi(t)$ tends to a constant value for large t and the Fokker-Planck equation is essentially the same as equation (10). Equations (11) and (12) are therefore of a quite general nature.

With the foregoing analysis differential equations linking $\langle x\rangle_t$, $\langle x^2\rangle_t$ and $\zeta(t)$ were found with a series expansion of the probability distribution $P(n, t)$ or $\Pi(x, t)$. An equivalent method, bearing a close resemblance to the two level analysis of vVF, is to multiply the master equation (2) by n and n^2 in turn and to sum over n to find the exact equations

$$\frac{\mathrm{d}\langle n\rangle_t}{\mathrm{d}t} = \langle g(n, Z(t))\rangle_t - \langle r(n, Z(t))\rangle_t \tag{17}$$

$$\frac{\mathrm{d}\langle n^2\rangle}{\mathrm{d}t} = \langle (2n + 1)\, g(n, Z(t))\rangle_t - \langle (2n - 1)\, r(n, Z(t))\rangle_t. \tag{18}$$

Now carry out an expansion in Ω^{-1}, using equations (3) and (4) and again putting $g = g_0 Z(t)$, $r = r_0 n^2/\Omega$. From (17),

$$\Omega\dot{\varphi} + \Omega^{\frac{1}{2}} \frac{\mathrm{d}\langle x\rangle_t}{\mathrm{d}t} = \Omega(g_0 z_0 - r_0\varphi^2) + \Omega^{\frac{1}{2}}(g_0\zeta(t) - 2r_0\langle x\rangle_t) - \Omega^0 r_0\langle x^2\rangle_t. \tag{19}$$

The interpretation is clear. The lowest order terms give of course the macroscopic equation (9) again for the difference between the generation and recombination rates, and the second order terms give the time variation

of $\langle x\rangle_t$ ((11)). Similarly, from (18), making use of (19),

$$\Omega \frac{\mathrm{d}\langle x^2\rangle_t}{\mathrm{d}t} = \Omega^{\frac{1}{2}}(2g_0 z_0 - 2r_0\varphi^2 - 2\dot{\varphi})\langle x\rangle_t + \\ + \Omega(g_0 z_0 + r_0\varphi^2 + 2g_0\zeta(t)\langle x\rangle_t - 4r_0\varphi\langle x\rangle_t) + \\ + \Omega^{\frac{1}{2}}(g_0\zeta(t) + 2r_0\varphi\langle x\rangle_t - 2r_0\langle x^3\rangle_t) + \Omega^0 r_0\langle x^2\rangle_t \qquad (20)$$

giving (9) and (12) again.

This alternative method will now be used to solve the general many-level problem.

3. *The many-level photoconductor.* We turn to the general s-dimensional case of a system of $s + 1$ electron levels. Electron transitions between these levels take place; these transitions may occur spontaneously or they may be stimulated by light incident on the system*). Our treatment will parallel the theory of multivariate Markov processes as reviewed in vVF.

Let the occupancies n_i of s levels be given by the (column) vector $\boldsymbol{a}$, where $a_i = n_i$. Let $P(\boldsymbol{a}'', t''\,|\,\boldsymbol{a}', t')$ be the conditional probability that the vector $\boldsymbol{a}$ takes on the value $\boldsymbol{a}''$ at time t'' assuming its value was $\boldsymbol{a}'$ at time t'. Then P satisfies the Smoluchowski equation,

$$P(\boldsymbol{a}, t\,|\,\boldsymbol{a}', t') = \sum_{\boldsymbol{a}''} P(\boldsymbol{a}, t\,|\,\boldsymbol{a}'', t'')\, P(\boldsymbol{a}'', t''\,|\,\boldsymbol{a}', t'). \qquad (21)$$

From this equation we find that the time variation of P is given by the following master equation:

$$\frac{\partial P(\boldsymbol{a}, t\,|\,\boldsymbol{a}, 0)}{\partial t} = \lim_{\Delta t\to 0} \frac{1}{\Delta t} \sum_{\boldsymbol{a}''} [P(\boldsymbol{a}'', t\,|\,\boldsymbol{a}', o)\, P(\boldsymbol{a}, t + \Delta t\,|\,\boldsymbol{a}'', t) - \\ - P(\boldsymbol{a}, t\,|\,\boldsymbol{a}', o)\, P(\boldsymbol{a}'', t + \Delta t\,|\,\boldsymbol{a}, t)]. \qquad (22)$$

In the derivation of this equation, which is similar to the derivation of the master equation given by vVF, it is important to bear in mind that P is not stationary, for P is a function of the incident light intensity.

The incident light beam we shall take to be represented by a given matrix $Z(t)$, where the element $Z_{ij}(t)$ is the number of photons in the volume of the photoconductor at time t of that wavelength that can stimulate a transition from level i to level j. Therefore Z is symmetric; put $Z_{ii}(t) = 0$. The indices i, j run from 1 to $s + 1$.

Let $Q_t(\boldsymbol{a}; \boldsymbol{a}'')$ be the transition probability per unit time from $\boldsymbol{a}''$ to $\boldsymbol{a}$: $Q_t(\boldsymbol{a}; \boldsymbol{a}'')\,\Delta t = P(\boldsymbol{a}, t + \Delta t\,|\,\boldsymbol{a}'', t)$ in the limit of small Δt. We put $Q_t(\boldsymbol{a}; \boldsymbol{a}) = 0$. $Q_t(\boldsymbol{a}; \boldsymbol{a}'')$ is not only a function of the occupancies $\boldsymbol{a}$ and $\boldsymbol{a}''$ but also of $Z(t)$

*) Photons emitted by the system itself shall be ignored, as the subsequent interactions of such photons with the system is a second order process.

and hence of t. The master equation takes the form

$$\frac{\partial P(\boldsymbol{a}, t \mid \boldsymbol{a}', o)}{\partial t} = \sum_{\boldsymbol{a}''} [P(\boldsymbol{a}'', t \mid \boldsymbol{a}', o)\, Q_t(\boldsymbol{a}; \boldsymbol{a}'') - P(\boldsymbol{a}, t \mid \boldsymbol{a}', o)\, Q_t(\boldsymbol{a}''; \boldsymbol{a})]. \tag{23}$$

The conditional expected value of $\boldsymbol{a}$ at time t, $\langle \boldsymbol{a} \rangle_{t;\boldsymbol{a}'}$, is given by

$$\langle \boldsymbol{a} \rangle_{t;\boldsymbol{a}'} = \sum_{\boldsymbol{a}} \boldsymbol{a} P(\boldsymbol{a}, t \mid \boldsymbol{a}', o). \tag{24}$$

The matrix $\langle \boldsymbol{a}\boldsymbol{a}^T \rangle_{t;\boldsymbol{a}'}$ of the (Z-dependent) variances and crossvariances is defined in similar fashion. The transpose of $\boldsymbol{a}$ is here denoted by $\boldsymbol{a}^T$. Multiply equation (23) by $\boldsymbol{a}$ and sum over $\boldsymbol{a}$ to obtain

$$\frac{\mathrm{d}\langle \boldsymbol{a} \rangle_{t;\boldsymbol{a}'}}{\mathrm{d}t} = \sum_{\boldsymbol{a}''} \sum_{\boldsymbol{a}} (\boldsymbol{a} - \boldsymbol{a}'')\, P(\boldsymbol{a}'', t \mid \boldsymbol{a}', o)\, Q_t(\boldsymbol{a}; \boldsymbol{a}''). \tag{25}$$

Define

$$\boldsymbol{A}_t(\boldsymbol{a}'') = \sum_{\boldsymbol{a}} (\boldsymbol{a} - \boldsymbol{a}'')\, Q_t(\boldsymbol{a}; \boldsymbol{a}''), \tag{26}$$

the first Fokker-Planck moments; then (25) becomes

$$\frac{\mathrm{d}\langle \boldsymbol{a} \rangle_{t;\boldsymbol{a}'}}{\mathrm{d}t} = \langle \boldsymbol{A}_t(\boldsymbol{a}) \rangle_{t;\boldsymbol{a}'}. \tag{27}$$

Define the second Fokker-Planck moments as

$$\mathsf{B}_t(\boldsymbol{a}'') = \sum_{\boldsymbol{a}} (\boldsymbol{a} - \boldsymbol{a}'')(\boldsymbol{a} - \boldsymbol{a}'')^T\, Q_t(\boldsymbol{a}; \boldsymbol{a}''). \tag{28}$$

Multiply (23) by $\boldsymbol{a}\boldsymbol{a}^T$ and sum over $\boldsymbol{a}$ to find

$$\frac{\mathrm{d}\langle \boldsymbol{a}\boldsymbol{a}^T \rangle_{t;\boldsymbol{a}'}}{\mathrm{d}t} = \langle \mathsf{B}_t(\boldsymbol{a}) \rangle_{t;\boldsymbol{a}'} + \langle \boldsymbol{a}, \boldsymbol{A}_t^T(\boldsymbol{a}) \rangle_{t;\boldsymbol{a}'} + \langle \boldsymbol{A}_t(\boldsymbol{a})\, \boldsymbol{a}^T \rangle_{t;\boldsymbol{a}'}. \tag{29}$$

Up till now the equations derived bear a close resemblance to the corresponding equations in the treatment of vVF. We have in effect only added the subscript t to point out the Z- and hence time-dependence of Q, $\boldsymbol{A}$, B, and the expectation values. Due to this time dependence the left-hand side of (29) does not tend to zero for large t, and hence the transition from equation (29) to the generalized generation-recombination theorem cannot be made. An average over all possible forms of Z must be carried out.

Proceeding, we shall make use of an expansion in the reciprocal of the crystal volume Ω, as was done in the one-dimensional case. To this end, separate the light intensity matrix $Z(t)$ into a time averaged part and the fluctuations away from this average:

$$Z(t) = \Omega Z_0 + \Omega^{\frac{1}{2}} X(t). \tag{30}$$

Introduce the new time-independent vector $\boldsymbol{x}$ instead of $\boldsymbol{a}$,

$$\boldsymbol{a} = \Omega \boldsymbol{\varphi}(t) + \Omega^{\frac{1}{2}} \boldsymbol{x}, \tag{31}$$

where $\boldsymbol{\varphi}(t)$ will be found.

Put

$$\Pi(\boldsymbol{x}, t \mid \boldsymbol{x}', o) = P(\boldsymbol{a}, t \mid \boldsymbol{a}', o). \tag{32}$$

Making an expansion about ΩZ_0 and $\Omega\boldsymbol{\varphi}$, we write

$$A_i(\boldsymbol{a}) \equiv A(\boldsymbol{a}, Z(t)) = A(\Omega\boldsymbol{\varphi}(t), \Omega Z_0) - \Omega^{\frac{1}{2}}\mathsf{M}\boldsymbol{x} + \Omega^{\frac{1}{2}}\boldsymbol{\delta}(t) + \ldots, \tag{33}$$

where

$$M_{ij} = -\left[\frac{\partial A_i(\boldsymbol{a}, Z)}{\partial a_j}\right]_{a=\Omega\varphi,\, Z=\Omega Z_0} \tag{34}$$

and

$$\delta_i(t) = \sum_{j \neq k} X_{jk}(t)\left[\frac{\partial A_i(\boldsymbol{a}, Z)}{\partial Z_{jk}}\right]_{a=\Omega\varphi,\, Z=\Omega Z_0}. \tag{35}$$

Some comments are in order. $\boldsymbol{\varphi}$ is a function of time, and hence M is also. The definition of the relaxation matrix M corresponds with that of vVF (see their footnote, p. 279). We chose to define $\boldsymbol{\delta}$ as in (35) to keep the equation (39) below in the conventional form, also working in the fluctuations $X(t)$ to reduce (33) to a simpler form. The vector $\boldsymbol{\delta}$ may be interpreted as a product of appropriate quantum efficiencies multiplied by the light intensity fluctuations (compare equations (11), (27) and (33)).

The Ω-dependence of $A(\Omega\boldsymbol{\varphi}(t), \Omega Z_0)$ is to be given explicitly. The correct power of Ω must be obtained from the physical meaning of A in our problem, which is that of fluctuations of electron occupancies in an $s+1$-level system. $Q_t(\boldsymbol{a}; \boldsymbol{a}'')$ is zero unless exactly two of the occupancies n_i change by addition or subtraction of one unit. As a result in equation (26) the effective order of the differences $(\boldsymbol{a} - \boldsymbol{a}'')$ is Ω^0. By considering a two or three level problem (see the preceding section) it may be shown that $Q_t(\boldsymbol{a}; \boldsymbol{a}'')$ is of order Ω^1. Consequently, we may write

$$A(\Omega\boldsymbol{\varphi}, \Omega Z_0) = \Omega A_0, \tag{36}$$

giving the Ω-dependence explicitly. (Again, A_0 is a function of time). It also follows that M and $\boldsymbol{\delta}(t)$ are of order Ω^0.

Equation (27) may now be written:

$$\Omega\dot{\boldsymbol{\varphi}} + \Omega^{\frac{1}{2}}\frac{\mathrm{d}\langle \boldsymbol{x}\rangle_{t;\boldsymbol{x}'}}{\mathrm{d}t} = \Omega A_0 - \Omega^{\frac{1}{2}}\mathsf{M}\langle \boldsymbol{x}\rangle_{t;\boldsymbol{x}'} + \Omega^{\frac{1}{2}}\boldsymbol{\delta}(t) + \ldots, \tag{37}$$

in which all terms of order Ω^1 and $\Omega^{\frac{1}{2}}$ are given.

The terms of order Ω^1 give

$$\dot{\boldsymbol{\varphi}} = A_0, \tag{38}$$

those of order $\Omega^{\frac{1}{2}}$

$$\frac{\mathrm{d}\langle \boldsymbol{x}\rangle_{t;\boldsymbol{x}'}}{\mathrm{d}t} = -\mathsf{M}\langle \boldsymbol{x}\rangle_{t;\boldsymbol{x}'} + \boldsymbol{\delta}(t). \tag{39}$$

Like (9), (38) is the macroscopic equation. For from the definition of A,

we may write

$$A_i(\boldsymbol{a}, Z(t)) = \sum_{j=1}^{s} [P_{ji}(\boldsymbol{a}, Z(t)) - P_{ij}(\boldsymbol{a}, Z(t))], \tag{40}$$

in which $P_{ij}\,\mathrm{d}t$ is the probability that a transition from level i to level j takes place in the interval $(t, t + \mathrm{d}t)$. During the initial transient ($\dot{\boldsymbol{\varphi}} \to 0$) non-zero net electron currents flow between the levels but after some time $\boldsymbol{A}_0 \cong 0$. Ignoring the perturbations $\boldsymbol{x}$ and $\boldsymbol{\delta}$, the net electron current to each level is zero and a steady state is reached. In the present analysis we are only interested in the fluctuations about such a steady state, and therefore take $\boldsymbol{A}_0$ to equal zero and $\boldsymbol{M}$ to be time independent: the transients have passed.

Equation (39) is like the usual Langevin equations with the essential difference that it is not an equation for the perturbation $\boldsymbol{x}$, but for the expected value of $\boldsymbol{x}$ at time t.

The equation for the second order moments may be dealt with in similar fashion. From (28), $\boldsymbol{B}$ may be approximated by $\boldsymbol{B}(\Omega\boldsymbol{\varphi}(t), \Omega Z_0) \equiv \Omega \boldsymbol{B}_0$ (see (36)). Now substitute $\Omega \boldsymbol{B}_0$ for $\langle \boldsymbol{B} \rangle_{t;a'}$ in equation (29); using (37), we find (omitting explicit mention of the initial conditions)

$$\Omega^{\frac{1}{2}}[\langle \boldsymbol{x} \rangle_t \dot{\boldsymbol{\varphi}}^T + \dot{\boldsymbol{\varphi}} \langle \boldsymbol{x}^T \rangle_t] + \Omega^1 \frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t =$$

$$\Omega^{\frac{1}{2}}[\langle \boldsymbol{x} \rangle_t \boldsymbol{A}_0^T + \boldsymbol{A}_0 \langle \boldsymbol{x}^T \rangle_t] + \Omega^1[\boldsymbol{B}_0 - \langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t \boldsymbol{M}^T - \boldsymbol{M} \langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t +$$

$$+ \langle \boldsymbol{x} \rangle_t \boldsymbol{\delta}^T(t) + \boldsymbol{\delta}(t) \langle \boldsymbol{x}^T \rangle_t]. \tag{41}$$

This equation is exact to terms of order Ω^1. Gathering terms of order $\Omega^{\frac{1}{2}}$ and rearranging,

$$\langle \boldsymbol{x} \rangle_t (\dot{\boldsymbol{\varphi}}^T - \boldsymbol{A}_0^T) + [\langle \boldsymbol{x} \rangle_t (\boldsymbol{\varphi}^T - \boldsymbol{A}_0^T)]^T = 0. \tag{42}$$

This is equivalent to (38), since $\langle \boldsymbol{x} \rangle_t$ and $\dot{\boldsymbol{\varphi}} - \boldsymbol{A}_0$ are effectively uncorrelated. The Ω^1-terms give

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t = -\langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t \boldsymbol{M}^T - \boldsymbol{M} \langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t + \boldsymbol{B}_0 + \langle \boldsymbol{x} \rangle_t \boldsymbol{\delta}^T(t) + \boldsymbol{\delta}(t) \langle \boldsymbol{x}^T \rangle_t. \tag{43}$$

This result may be compared with the generation-recombination theorem. In the present case we are not yet finished, since an equilibrium state does not exist:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t \neq 0$$

since $\langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t$ must follow the light intensity fluctuations $\boldsymbol{\delta}(t)$. Our course shall now be to solve the system of differential equations (43) and average $\langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t$ over all possible $\boldsymbol{\delta}(t)$. Before proceeding along these lines (43) shall

be cast into a simpler form. Transform $\boldsymbol{x}$ with the matrix c [4) 12)],

$$\boldsymbol{y} = \mathsf{c}\boldsymbol{x} \tag{44}$$

where c diagonalizes M*)

$$\mathsf{M}' = \mathsf{c}\mathsf{M}\mathsf{c}^{-1} \tag{45}$$

so that M' has the eigenvalues λ_i along the diagonal.

Multiply (39) on the left by c; then

$$\frac{\mathrm{d}\langle \boldsymbol{y}\rangle_t}{\mathrm{d}t} = -\mathsf{M}'\langle \boldsymbol{y}\rangle_t + \xi(t) \tag{46}$$

where $\xi(t) = \mathsf{c}\boldsymbol{\delta}(t)$. Multiply (43) by c and c^T on the left and right respectively;

$$\frac{\mathrm{d}\langle \boldsymbol{y}\boldsymbol{y}^T\rangle_t}{\mathrm{d}t} = -\langle \boldsymbol{y}\boldsymbol{y}^T\rangle_t \mathsf{M}' - \mathsf{M}'\langle \boldsymbol{y}\boldsymbol{y}^T\rangle_t + \mathsf{B}_0' + \langle \boldsymbol{y}\rangle_t \xi(t)^T + \xi(t)\langle \boldsymbol{y}^T\rangle_t, \tag{47}$$

here $\mathsf{B}_0' \equiv \mathsf{c}\mathsf{B}_0\mathsf{c}^T$.

The expectation values are obtained with $T(\boldsymbol{y}', t\,|\,\boldsymbol{y}_0, t_0) = \Pi(\boldsymbol{x}', t\,|\,\boldsymbol{x}_0, t_0)$. Since M' is diagonal,

$$\frac{\mathrm{d}\langle y_i\rangle_t}{\mathrm{d}t} = -\lambda_i\langle y_i\rangle_t + \zeta_i(t)\ , \qquad i = 1, 2, \ldots, s \tag{48}$$

$$\frac{\mathrm{d}\langle y_i y_j\rangle_t}{\mathrm{d}t} =$$

$$-(\lambda_i + \lambda_j)\langle y_i y_j\rangle_t + B'_{0_{ij}} + \langle y_i\rangle_t\zeta_j(t) + \langle y_j\rangle_t\zeta_i(t). \qquad i, j = 1, 2, \ldots, s \tag{49}$$

Equation (48) has already been encountered in the two level problem; the differential equation for the second order moments in the two level case is equivalent to equation (12) with $i = j$:

$$\frac{\mathrm{d}\langle y_i^2\rangle_t}{\mathrm{d}t} = -2\lambda_i\langle y_i^2\rangle + B'_{0_{ii}} + 2\langle y_i\rangle_t\zeta_i(t). \tag{50}$$

The present many variable problem is much like that of the single variable problem; the only complication is formed by the cross terms $\langle y_i\rangle_t\zeta_j(t)$ and $\langle y_j\rangle_t\zeta_i(t)$ in (49). As by assumption $\dot{\varphi} = 0$ and M is time independent, the solution of (48) is**):

$$\langle y_i\rangle_t = \langle y_i\rangle_0 \mathrm{e}^{-\lambda_i t} + \mathrm{e}^{-\lambda_i t}\int_0^t \mathrm{e}^{\lambda_i u}\zeta_i(u)\,\mathrm{d}u, \tag{51}$$

---

*) M is in general not symmetric (it is symmetric only for the case of thermal equilibrium), but in most cases of practical interest all eigenvalues are different and M may be diagonalized with c as in (45).

**) See also ref. 7 for a one-dimensional treatment, with $\zeta(t) = 0$.

and, using this, of (49):

$$\langle y_i y_j\rangle_t = \langle y_i y_j\rangle_0 \, e^{-(\lambda_i+\lambda_j)t} + \frac{B'_{0ij}}{\lambda_i+\lambda_j}\,(1 - e^{-(\lambda_i+\lambda_j)t}) +$$

$$+ e^{-(\lambda_i+\lambda_j)t}[\langle y_i\rangle_0 \int_0^t e^{\lambda_j u}\,\zeta_j(u)\,du + \int_0^t e^{\lambda_j u}\int_0^u e^{\lambda_i v}\,\zeta_i(v)\,\zeta_j(u)\,dv\,du] +$$

$$+ e^{-(\lambda_j+\lambda_i)t}[\langle y_j\rangle_0 \int_0^t e^{\lambda_i u}\,\zeta_i(u)\,du + \int_0^t e^{\lambda_i u}\int_0^u e^{\lambda_i v}\,\zeta_j(v)\,\zeta_i(u)\,dv\,du]. \qquad (52)$$

Note the symmetry of the last two expressions. For t large compared with all relaxation times $1/\lambda_i$,

$$\langle y_i\rangle_t = \int_0^t e^{-\lambda_i v}\,\zeta_i(t-v)\,dv, \qquad (53)$$

$$\langle y_i y_j\rangle_t = \frac{B'_{0ij}}{\lambda_i+\lambda_j} + \int_0^t\int_0^t e^{-\lambda_i u-\lambda_j v}\,\zeta_i(t-u)\,\zeta_j(t-v)\,dv\,du. \qquad (54)$$

In writing (54), use has been made of the fact that the sum of the two double integrals of (52) is symmetric in i and j to simplify the integration bounds. For large t the mean square noise is

$$\langle (y_i - \langle y_i\rangle_t)(y_j - \langle y_j\rangle_t)\rangle_t = \langle y_i y_j\rangle_t - \langle y_i\rangle_t\langle y_j\rangle_t \to \frac{B'_{0ij}}{\lambda_i+\lambda_j}, \qquad (55)$$

for any $\xi(t)$ given in the interval $(0, t)$. Now average (51) and (52) over all possible $\xi(t)$:

$$\overline{\langle y_i\rangle_t} = \overline{\langle y_i\rangle_0}\, e^{-\lambda_i t}$$

which tends to zero as t becomes large. Similarly,

$$\overline{\langle y_i y_j\rangle_t} \to \frac{B'_{0ij}}{\lambda_i+\lambda_j} + \int_0^t\int_0^t e^{-\lambda_i u-\lambda_j v}\,\overline{\zeta_i(t-u)\,\zeta_j(t-v)}\,dv\,du. \qquad (56)$$

Here we see that the (cross) autocorrelation function of the fluctuations of the incident photon beam plays a role in the (cross) variances of the electron fluctuations.

We not only want to know the variances, but also the autocorrelation functions $\overline{\langle y_i(t)\,y_j(t+\varepsilon)\rangle}$ for all i, j. Specifically, let $T(\mathbf{y}_1, t_1; \mathbf{y}_2, t_2|\mathbf{y}_0, t_0)$ be the probability that $\mathbf{y}$ takes on the value $\mathbf{y}_1$ at time t_1, and $\mathbf{y}_2$ at time t_2, given that its value at time t_0 was $\mathbf{y}_0$. This is for one particular $\xi(t)$. Then write

$$\langle y_i(t)\,y_j(t+\varepsilon)\rangle = \lim_{t_0\to-\infty} \sum_{\mathbf{y}_1}\sum_{\mathbf{y}_2} \mathbf{y}_{1i}\mathbf{y}_{2j}\,T(\mathbf{y}_1, t; \mathbf{y}_2, t+\varepsilon|\mathbf{y}_0, t_0). \qquad (57)$$

The i-th component of the vector $\boldsymbol{y}_1$ has been written as y_{1_i}. Also, although $\boldsymbol{y}$ is not a function of time, the left-hand side is written in this form to keep the notation simple.

We assume $\boldsymbol{y}$ to be Markovian: for $t_1 < t_2$,

$$T(\boldsymbol{y}_1, t_1; \boldsymbol{y}_2, t_2 | \boldsymbol{y}_0, t_0) = T(\boldsymbol{y}_1, t_1 | \boldsymbol{y}_0, t_0)\, T(\boldsymbol{y}_2, t_2 | \boldsymbol{y}_1, t_1). \tag{58}$$

Substituting this into (57),

$$\langle y_i(t)\, y_j(t+\varepsilon)\rangle \lim_{t_0 \to -\infty} \sum_{\boldsymbol{y}_1} \langle y_j(t+\varepsilon)\rangle_{\boldsymbol{y}(t)=\boldsymbol{y}_1} y_{1_i} T(\boldsymbol{y}_1, t | \boldsymbol{y}_0, t_0) \tag{59}$$

where

$$\langle y_j(t+\varepsilon)\rangle_{\boldsymbol{y}(t)=\boldsymbol{y}_1} = \sum_{\boldsymbol{y}_2} y_{2_j} T(\boldsymbol{y}_2, t+\varepsilon | \boldsymbol{y}_1, t) \tag{60}$$

is the expected value of y_j with initial conditions $\boldsymbol{y}(t) = \boldsymbol{y}_1$. This value is obtained from (51):

$$\langle y_j(t+\varepsilon)\rangle_{\boldsymbol{y}(t)=\boldsymbol{y}_1} = y_{1_j}\, \mathrm{e}^{-\lambda_j \varepsilon} + \int_0^{\varepsilon} \mathrm{e}^{\lambda_j (v-\varepsilon)}\, \zeta_j(v+t)\, \mathrm{d}v. \tag{61}$$

Consequently,

$$\begin{aligned}\langle y_i(t)\, y_j(t+\varepsilon)\rangle = \lim_{t_0 \to -\infty} [\mathrm{e}^{-\lambda_j \varepsilon} \sum_{\boldsymbol{y}_1} y_{1_i} y_{1_j}\, T(\boldsymbol{y}_1, t | \boldsymbol{y}_0, t_0) + \\ + \int_0^{\varepsilon} \mathrm{e}^{\lambda_j (v-\varepsilon)}\, \zeta_j(v+t)\, \mathrm{d}v \sum_{\boldsymbol{y}_1} y_{1_i} T(\boldsymbol{y}_1, t | \boldsymbol{y}_0, t_0)].\end{aligned} \tag{62}$$

The first expression may be obtained from (54), the second from (51) again. After some simplification,

$$\begin{aligned}\langle y_i(t)\, y_j(t+\varepsilon)\rangle = \mathrm{e}^{-\lambda_j \varepsilon}\Big[\frac{B'_{0_{ij}}}{\lambda_i + \lambda_j} + \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-\lambda_i u - \lambda_j v}\, \zeta_i(t-u)\, \zeta_j(t-v)\, \mathrm{d}u\, \mathrm{d}v + \\ + \int_0^{\varepsilon}\int_0^{\infty} \mathrm{e}^{-\lambda_i u + \lambda_j v}\, \zeta_i(t-u)\, \zeta_j(t+v)\, \mathrm{d}u\, \mathrm{d}v\Big].\end{aligned} \tag{63}$$

It now becomes necessary to average over all forms of $Z(t)$ or $\xi(t)$.

As in (56), we have to deal with such expressions as $\overline{\zeta_i(t_1)\, \zeta_j(t_2)}$.

The elements $\zeta_i(t_1)\, \zeta_j(t_2)$ of the matrix $\xi(t_1)\, \xi^T(t_2)$ are related to the elements of $\boldsymbol{\delta}(t_1)\, \boldsymbol{\delta}^T(t_2)$ since

$$\xi(t_1)\, \xi^T(t_2) = c\boldsymbol{\delta}(t_1)\, \boldsymbol{\delta}^T(t_2)\, c^T. \tag{64}$$

From the definition of $\boldsymbol{\delta}(t)$,

$$\delta_p(t_1)\, \delta_q(t_2) = \sum_{k \neq l} \sum_{m \neq n} \frac{\partial A_p}{\partial Z_{kl}} \frac{\partial A_q}{\partial Z_{mn}}\, X_{kl}(t_1)\, X_{mn}(t_2). \tag{65}$$

Here the terms $\partial Ap/\partial Z_{kl}$ and $\partial A_q/\partial Z_{mn}$ are evaluated at $\boldsymbol{a} = \Omega\boldsymbol{\varphi}(t)$ and $Z(t) = \Omega Z_0$ for $t = t_1$ and $t = t_2$ respectively. As we shall only consider times

so large that the first transients have died out, φ is constant in time and the partial derivatives are time independent.

We have then

$$\overline{\zeta_i(t_1)\,\zeta_j(t_2)} \sum_{p,q} c_{ip} \sum_{k\neq l} \sum_{m\neq n} \frac{\partial A_p}{\partial Z_{kl}} \frac{\partial A_q}{\partial Z_{mn}} \overline{X_{kl}(t_1)\,X_{mn}(t_2)}\, c_{jq}. \tag{66}$$

It shall now be assumed that the light intensity fluctuations at different wavelengths are completely uncorrelated; specifically, using Kronecker delta's,

$$\overline{X_{kl}\ (t_1)X_{mn}(t_2)} = (\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm})\,\chi_{kl}(t_1 - t_2) \tag{67}$$

where all functions $\chi_{kl}(t)$ are sharply peaked with respect to all the photoconductor relaxation times $1/\lambda_j$.

With equation (67) we take the simplest possible solution to the difficult problem of time correlations between photon numbers of the same and different wavelengths. Our assumption implies that the incident light intensities at the various wavelengths are independent Markov variables, and that this Markov character is not altered by photon absorption and emission events taking place in the photoconductor. This assumption is not valid for times shorter than the time a photon needs to go through the crystal, but these times are usually much smaller*) than the photoconductor relaxation times.

As a consequence of (67), (66) simplifies to

$$\overline{\zeta_i(t_1)\,\zeta_j(t_2)} = 2 \sum_{p,q} c_{ip} \sum_{k\neq l} \frac{\partial A_p}{\partial Z_{kl}} \frac{\partial A_q}{\partial Z_{kl}} \chi_{kl}(t_1 - t_2)\, c_{jq} \tag{68}$$

From (56) and (63), using (68),

$$\overline{\langle y_i(t)\, y_j(t+\varepsilon)\rangle} = \overline{\langle y_i(t)\, y_j(t)\rangle}\, \mathrm{e}^{-\varepsilon\lambda_j} \tag{69}$$

and

$$\overline{\langle y_i(t)\, y_j(t)\rangle} = \frac{B'_{0ij}}{\lambda_i + \lambda_j} + 2\sum_{p,q} c_{ip} \sum_{k\neq l} \frac{\partial A_p}{\partial Z_{kl}} \frac{\partial A_q}{\partial Z_{kl}} \frac{\hat{\chi}_{kl}}{\lambda_i + \lambda_j}\, c_{jq}, \tag{70}$$

where

$$\hat{\chi}_{kl} = \int_{-\infty}^{\infty} \chi_{kl}(t)\,\mathrm{d}t = 2\int_0^{\infty} \chi_{kl}\,(t)\,\mathrm{d}t. \tag{71}$$

We add that upon Z-averaging the ratio of the third term to the second term of the right-hand side of (63) is of the order of the ratio of the widths of the χ-functions to the photoconductor relaxation times. By writing (69) the contribution of the third term is therefore ignored.

These equations show that the irradiation fluctuations do not fundamental-

*) For a crystal dimension of 1 mm and a refractive index of 3, this transit time is 10^{-11} s.

ly alter the autocorrelation function of the carrier fluctuations, but increase the mean square fluctuations. In order to evaluate the magnitude of this increase, we turn to the two level single variable problem of the previous section. In equation (70), then, $i = j = p = q = 1$; $k, l = 1, 2$; $c_{11} = 1$. For y_i write x again, and write τ for $1/\lambda_i$. A comparison of (43) and (12) shows that $\partial A_1/\partial Z_{21}$ becomes $\frac{1}{2}g_0$ (see also (39) and (11)), and $B_{0_{11}}$ becomes $2g_0z_0$. $\hat{\chi}_{21}$ equals $\hat{\chi}$, the time integrated correlation function of $\zeta(t)$, defined by (3). As a result,

$$\overline{\langle x^2\rangle_t} = g_0z_0\tau\left(1 + \frac{\hat{\chi}g_0}{2z_0}\right). \tag{72}$$

The factor $g_0\hat{\chi}/2z_0$ remains to be evaluated.

Introduce $\gamma(\varepsilon)$ as the autocorrelation coefficient of $\zeta(t)$:

$$\gamma(\varepsilon) = \frac{\overline{\zeta(t)\,\zeta(t+\varepsilon)}}{\overline{\zeta(t)^2}}. \tag{73}$$

Then for weak absorption $\gamma(\varepsilon)$ equals $1 - \varepsilon/\tau_t$ if $\varepsilon < \tau_t$, τ_t the time taken by the photons to travel through the crystal. For $\varepsilon > \tau_t$, $\gamma(\varepsilon) = 0$. With (73) and (71) we find that $\hat{\chi} = \tau_t\overline{\zeta(t)^2}$. Therefore

$$\overline{\langle x^2\rangle_t} = g_0z_0\tau\left(1 + \frac{\tau_tg_0}{2}\,\frac{\overline{\Delta Z^2}}{\bar{Z}}\right). \tag{74}$$

writing ΔZ for $Z(t) - \Omega z_0$.

The product τ_tg_0 is equal to the probability p that a photon while traversing the crystal is absorbed. Equivalently, τ_t is the relaxation time of $Z(t)$ if the photon beam were suddenly cut off, and no absorption could occur. If all photons are absorbed by the photoconductor, $Z(t)$ will have $1/g_0$ as relaxation time. Note that p varies linearly with the thickness of the crystal, and may well approach unity.

The ratio $\overline{\Delta Z^2}/\bar{Z}$ will in general vary with p, but for small p will be insensitive to the magnitude of p. Consider the case in which no reflection occurs at the crystal surfaces, and no absorption in the bulk. Then Carson's theorem[1]) may be applied as follows. $Z(t)$ consists of a superposition of elementary events $q(t)$, where

$$q(t) = \begin{cases} 1 & t_0 \leq t \leq t_0 + \tau_t \\ 0 & \text{elsewhere} \end{cases}$$

Each such an event is the response to the entry of a photon at the crystal surface, and these primary events occur at the average rate $\bar{Z}/\tau_t$. Carson's theorem then gives $\overline{\Delta Z^2}/\bar{Z} = 1$. If g_0, p differ from zero each photon may either be absorbed in the crystal or leave the crystal at the far surface. In this case the calculation of $\overline{\Delta Z^2}/\bar{Z}$ is similar to the calculation of carrier

density fluctuations in semiconductors, which are due to generation-recombination events in the bulk and carrier injection and extraction at the contacts[8]). By analogy it may be said that $\overline{\Delta Z^2}/\bar{Z} = 1$ for $p \ll 1$, and hence

$$\overline{\langle x^2 \rangle_t} = g_0 z_0 \tau \left(1 + \frac{p}{2}\right). \qquad (p \ll 1) \tag{75}$$

We need not discuss the value of $\overline{\Delta Z^2}/\bar{Z}$ when p approaches unity, because in this case the photon density (and possibly the carrier density) will be spatially dependent. The calculation of the noise must then consider transport processes, i.e. the probability distribution functions become spatially dependent, and, besides time correlations, spatial correlations of the density fluctuations must be taken into account (vVF). Such effects are ignored in the present analysis, which is therefore only valid for weak absorption. This restriction has also been pointed out in connection with the definition of Z and $\bar{Z}$.

The results of the present analysis may be summarized as follows. If the incident photon beam is taken to be completely free of fluctuations, the $g - r$ theorem

$$\overline{\langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t} \boldsymbol{M}^T + \boldsymbol{M} \overline{\langle \boldsymbol{x}\boldsymbol{x}^T \rangle_t} = \boldsymbol{B}_0$$

is valid, since it follows directly from (43). If the incident photon beam displays shot noise, this theorem is closely obeyed; the departures are given by equation (70). These departures are due to the fact that the electrons of the photoconductor are coupled not to the incident photon beam intensity, which fluctuates in a purely random way, but to the numbers of photons in the crystal volume, which, due to transit time effects, do not fluctuate purely randomly (compare ref. 13).

In both these cases it is necessary to assume that the photoconductor only weakly absorbs all the incident radiation.

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