Multiplicities in the Plancherel decomposition for a semisimple symmetric space

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Introduction. Let G/H be a semisimple symmetric space, where G is a connected semisimple Lie group provided with an involution σ, and $H = G^\sigma$ is the subgroup of fixed points for σ. Assume moreover that G is linear (for the purpose of the introduction, the assumptions on G and H are stronger than necessary). Then G has a σ-stable maximal compact subgroup K; the associated Cartan involution θ commutes with σ. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decompositions of the Lie algebra $\mathfrak{g}$ induced by σ and θ, then $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{k}$ is the Lie algebra of K.

The fundamental problem in harmonic analysis on the symmetric space G/H is to obtain an explicit direct integral ('Plancherel') decomposition

$$\mathcal{L} \simeq \int_{\hat{G}}^{\oplus} m_\pi \pi \, d\mu(\pi) \tag{1}$$

of the regular representation $\mathcal{L}$ of G on $L^2(G/H)$ into irreducible unitary representations. The number m_π, which is known to be finite (cf. [**Ba 87**]), is called the multiplicity of π in the decomposition.

The group G itself is a symmetric space for the left times right action of $G \times G$. In this case ('the group case'), an explicit decomposition of the form (1) has been determined by Harish-Chandra ([**H-C 75**, **76a**, **76b**]). In particular, the multiplicities are one (or zero).

The decomposition (1) has also been determined in all cases where G/H has rank one. Recall that the rank of G/H is the dimension of any Cartan subspace (i.e. a maximal abelian subspace of $\mathfrak{q}$ consististing of semisimple elements). The spaces G/H of rank one have been treated separately by several authors (see e.g. [**Fa 79**], [**vDP 86**], [**Mo 86**] and references given in these papers). In all these cases it turns out that $L^2(G/H)$ decomposes into two series: a discrete and a continuous series. For the discrete series all multiplicities are one, but for the continuous series higher multiplicities do occur (cf. [**vD 86**]).

In general, $L^2(G/H)$ is expected to decompose into several series (in analogy with the group case), one for each H-conjugacy class of Cartan subspaces of $\mathfrak{q}$, the most extreme of these being respectively the 'most discrete series' (in the group case called the fundamental series), corresponding to the conjugacy class of Cartan subspaces with maximal $\mathfrak{k}$ part, and the 'most continuous series', corresponding to the conjugacy class of Cartan subspaces with maximal $\mathfrak{p}$ part.

It is known from work of Flensted-Jensen ([**F-J 80**]) and Oshima and Matsuki ([**OM 84**]) that the decomposition of $L^2(G/H)$ has a discrete part (the irreducible subrepresentations of which then constitute the 'most discrete series'), if and only if $\mathfrak{q}$ has a purely compact Cartan subspace. The discrete series has been extensively studied and is by now quite well understood (cf. also [**BS 87, 89**], [**Ma 88**], [**Vo 88**]). In particular (except perhaps for a few exceptional spaces), all multiplicities are one (cf. [**Bi 90**]).

In [**BS** $*$] we study the part $L^2_{\mathrm{mc}}(G/H)$ of $L^2(G/H)$ which corresponds to the most continuous series, and determine its Plancherel decomposition. In the present paper we give a survey of some of the results of [**BS** $*$], and compare them with Harish-Chandra's work for the group case. We also discuss some new results on the multiplicities m_π in this most continuous part of the Plancherel decomposition for G/H. These results are illustrated by examples.

Notation. In the following G will be a real reductive Lie group of Harish-Chandra's class (cf. [**H-C 75**]), σ an involution of G, and H an open subgroup of G^σ. Then G/H is a reductive symmetric space. Let θ be a Cartan involution commuting with σ, with corresponding maximal compact subgroup K, and let $\mathfrak{h}$, $\mathfrak{q}$, $\mathfrak{k}$, $\mathfrak{p}$ be as in the introduction.

Fix a maximal abelian subspace $\mathfrak{a}_{\mathrm{q}}$ of $\mathfrak{p} \cap \mathfrak{q}$ and denote by Σ the root system of $\mathfrak{a}_{\mathrm{q}}$ in $\mathfrak{g}$. Choose a positive system Σ^+ for Σ, let $\mathfrak{n}$ (resp. $\bar{\mathfrak{n}}$) be the sum of the corresponding positive (negative) root spaces and let $N = \exp \mathfrak{n}$ ($\bar{N} = \exp \bar{\mathfrak{n}}$). Let $\mathfrak{a}_{\mathrm{q}}^+$ and $\mathfrak{a}_{\mathrm{q}}^{*+}$ denote the positive open Weyl chambers in $\mathfrak{a}_{\mathrm{q}}$ and $\mathfrak{a}_{\mathrm{q}}^*$, respectively. Let M_1 denote the centralizer of $\mathfrak{a}_{\mathrm{q}}$ in G, and $\mathfrak{m}_1$ its Lie algebra. Then $P = M_1 N$ is a *σ-minimal parabolic subgroup* of G (i.e. it is minimal among the $\sigma\theta$-stable parabolic subgroups). Let $P = MAN$ be its Langlands decomposition. Then $\mathfrak{a} := \mathrm{Lie}(A) = \mathrm{center}(\mathfrak{m}_1) \cap \mathfrak{p}$. Put $\mathfrak{a}_{\mathrm{h}} = \mathfrak{a} \cap \mathfrak{h}$, then $\mathfrak{a} = \mathfrak{a}_{\mathrm{h}} \oplus \mathfrak{a}_{\mathrm{q}}$. Via this decomposition we view the complexified dual $\mathfrak{a}_{\mathrm{qc}}^*$ as a subspace of $\mathfrak{a}_{\mathbf{c}}^*$. Put $\rho = \rho_P = \frac{1}{2}\mathrm{tr}\ \mathrm{ad}(\cdot)|_{\mathfrak{n}} \in \mathfrak{a}^*$. Then $\sigma\theta\rho = \rho$, hence $\rho \in \mathfrak{a}_{\mathrm{q}}^*$.

The Weyl group W of Σ is naturally isomorphic to $N_K(\mathfrak{a}_{\mathrm{q}})/Z_K(\mathfrak{a}_{\mathrm{q}})$, the normalizer modulo the centralizer of $\mathfrak{a}_{\mathrm{q}}$ in K. Let $W_{K\cap H}$ be the canonical image of $N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$ in W, then the open double $P \times H$ cosets in G are parametrized by the quotient $W/W_{K\cap H}$, in the obvious way.

Principal series for G/H. Let $\widehat{M}_{\mathrm{fu}}$ be the set of equivalence classes of irreducible finite dimensional unitary representations $(\xi, \mathcal{H}_\xi)$ of M (they are supported on the compact factors of M). We consider the series of representations $\pi_{\xi,\lambda} = \pi_{P,\xi,\lambda}$ ($\xi \in \widehat{M}_{\mathrm{fu}}, \lambda \in \mathfrak{a}_{\mathrm{qc}}^*$) induced from the representation $\xi \otimes e^\lambda \otimes 1$ of $P = MAN$. Here we use left induction; thus the space $C^\infty(\xi\colon\lambda)$ of smooth vectors for $\pi_{\xi,\lambda}$ is the space of smooth functions $f\colon G \to \mathcal{H}_\xi$ satisfying the transformation rule

$$f(manx) = a^{\lambda+\rho}\xi(m)f(x) \qquad (m \in M, a \in A, n \in N, x \in G),$$

and G acts from the right.

If X is a C^∞ manifold, we let $C^{-\infty}(X)$ denote the space of generalized functions on X, i.e. the topological linear dual of the space of C_c^∞ densities on X. Then we have a natural embedding $C^\infty(X) \hookrightarrow C^{-\infty}(X)$.

This being said, let $C^{-\infty}(\xi\colon\lambda)$ denote the set of generalized functions $f\colon G \to \mathcal{H}_\xi$ satisfying the above transformation rule; it is the space of generalized vectors for $\pi_{\xi,\lambda}$. It will also be useful to work with the *compact picture* of these representations; this is obtained by taking restrictions to K of the above functions f. We denote the corresponding function spaces, consisting of smooth, resp. generalized, functions from K to $\mathcal{H}_\xi$ satisfying the transformation rule

$$f(mx) = \xi(m)f(x), \qquad (m \in M, x \in K),$$

by $C^\infty(K\colon\xi)$ and $C^{-\infty}(K\colon\xi)$. Similarly the space of L^2 functions from K to $\mathcal{H}_\xi$ satisfying this rule is denoted $L^2(K\colon\xi)$. In the compact picture this Hilbert space is the representation space for $\pi_{\xi,\lambda}$.

The representations $\pi_{\xi,\lambda}$ are irreducible for generic λ, and they are unitary for $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$. For generic λ, λ' we have that $\pi_{\xi,\lambda}$ and $\pi_{\xi',\lambda'}$ are equivalent if and only if $\xi' = w\xi$ and $\lambda' = w\lambda$ for some $w \in W$ (with respect to the natural action of W on $\widehat{M}_{\mathrm{fu}}$).

Let $\widehat{M}_H$ denote the set of $\xi \in \widehat{M}_{\mathrm{fu}}$ for which there exists a $w \in W$ such that $M_{w\xi}^{M\cap H} \neq 0$. Then the series $\pi_{\xi,\lambda}$ ($\xi \in \widehat{M}_H, \lambda \in \mathfrak{a}_{\mathrm{qc}}^*$) is called the (minimal) principal series for G/H. The representation $\mathcal{L}|L^2_{\mathrm{mc}}(G/H)$ will be a direct integral over the unitary principal series (i.e. the subseries with λ imaginary).

Example 1. Let $H = K$, then G/H is a *Riemannian* symmetric space. The explicit decomposition of $L^2(G/K)$ is well known by the work of Harish-Chandra and Helgason ([**H-C 58**, **66**], and [**He 70**], Thm I.2.6). The only principal series representations contributing are the $\pi_{\xi,\lambda}$ with trivial M-representation ξ. Let $B = M\backslash K$ be provided with the normalized invariant measure db. Then for a function f (for example compactly supported and smooth) on G/K, the Fourier transform is the function on $\mathfrak{a}_{\mathrm{qc}}^* = \mathfrak{a}_{\mathrm{c}}^*$ with values in $L^2(K\colon 1) = L^2(B)$ given by $\hat{f}(\lambda) = \pi_{1,\lambda}(f)1_\lambda$, where 1_λ is the unique K-fixed vector satisfying $1_\lambda(e) = 1$. Thus

$$\hat{f}(\lambda, Mk) = \int_G f(g)e^{(-\lambda-\rho)H(g^{-1}k^{-1})}\,dg,$$

where $H\colon G \to \mathfrak{a}$ is the usual Iwasawa projection according to $G = KAN$. The map $f \mapsto \hat{f}$ extends to an isometry of $L^2(G/K)$ into $L^2(i\mathfrak{a}^* \times B)$ with respect to an explicitly known measure $d\mu$ on $i\mathfrak{a}^*$ ($|\mathbf{c}(\lambda)|^{-2}$ times suitably normalized Lebesque measure, where $\mathbf{c}$ is Harish-Chandra's c-function), and the decomposition (1) of $L^2(G/K)$ is then given by

$$\mathcal{L} \simeq \int_{i\mathfrak{a}^*+}^{\oplus} \pi_{1,\lambda}\,d\mu(\lambda). \tag{2}$$

In particular, the representations occur with multiplicity one.

***H*-fixed distribution vectors.** The purpose of this section is to construct analogs for general reductive symmetric spaces G/H of the K-fixed vectors 1_λ in the example above. The analogs are obtained only as generalized functions; they will be H-fixed elements of $C^{-\infty}(\xi\colon\lambda)$.

We fix, once and for all, a set $\mathcal{W}$ of representatives in $N_K(\mathfrak{a}_{\mathrm{q}})$ for the quotient $W/W_{K\cap H}$. Recall that $w \mapsto PwH$ is a bijective map from $\mathcal{W}$ onto the collection of open double $P \times H$-cosets in G.

Clearly every element φ of $C^{-\infty}(\xi\colon\lambda)^H$ restricts to a smooth function on the open subset $\Omega = \bigcup_{w\in\mathcal{W}} PwH$ of G, hence it makes sense to evaluate φ in each $w \in \mathcal{W}$. It is easily seen that $\varphi(w) \in \mathcal{H}_\xi^{w(M\cap H)w^{-1}}$.

For each $\xi \in \widehat{M}_{\mathrm{fu}}$ let $V(\xi)$ denote the formal sum

$$V(\xi) = \coprod_{w\in\mathcal{W}} \mathcal{H}_\xi^{w(M\cap H)w^{-1}}, \tag{3}$$

provided with the direct sum inner product. Then we have an evaluation map

$$\mathrm{ev}\colon C^{-\infty}(\xi\colon\lambda)^H \to V(\xi),$$

defined by $\mathrm{ev}_w(\varphi) = \varphi(w),\ w \in \mathcal{W}$.

Notice that $V(\xi) \neq 0$ if and only if $\xi \in \widehat{M}_H$. Notice also that by definition the summands of (3) are mutually orthogonal in $V(\xi)$, even though this may not be the case in $\mathcal{H}_\xi$ (for example when ξ is the trivial representation). For $\eta \in V(\xi),\ w \in \mathcal{W}$ we denote by η_w the w-component of η, viewed as an element of $\mathcal{H}_\xi$.

Using Matsuki's description of $P\backslash G/H$ ([**Ma 79**]) in combination with Bruhat theory it can be shown that ev is injective for generic $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ (cf. [**Ba 88**], Cor. 5.3).

We now define a linear map $j(\xi\colon\lambda) = j(P\colon\xi\colon\lambda)$ from the finite dimensional space $V(\xi)$ into $C^{-\infty}(\xi\colon\lambda)^H$ by

$$j(\xi\colon\lambda)(\eta)(x) = \begin{cases} a^{\lambda+\rho}\xi(m)\eta_w & \text{for } x\in\Omega,\ x = manwh \\ & (m\in M, a\in A, n\in N, w\in\mathcal{W}, h\in H); \\ 0 & \text{for } x\notin\Omega. \end{cases}$$

for $\eta \in V(\xi)$, $\xi \in \widehat{M}_{\mathrm{fu}}$ and $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ with $\mathrm{Re}\langle\lambda+\rho,\Sigma^+\rangle > 0$ (it can be seen that this condition on λ implies continuity of the function $j(\xi\colon\lambda)(\eta)$ on G, hence $j(\xi\colon\lambda)(\eta) \in C^{-\infty}(\xi\colon\lambda)$). Then clearly $\mathrm{ev}\circ j(\xi\colon\lambda) = I_{V(\xi)}$, and hence $j(\xi\colon\lambda)$ is injective. Moreover, by the above-mentioned injectivity of ev, $j(\xi\colon\lambda)$ is surjective for generic λ.

THEOREM 1. ([**Ba 88**]) *The map $\lambda \mapsto j(\xi\colon\lambda) \in \mathrm{Hom}(V(\xi), C^{-\infty}(K\colon\xi))$ extends meromorphically to $\mathfrak{a}_{\mathrm{qc}}^*$, and $j(\xi\colon\lambda)$ is a bijection from $V(\xi)$ onto $C^{-\infty}(\xi\colon\lambda)^H$ for generic $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$.*

REMARK: The meromorphic extension of $j(\xi\colon\lambda)$ has also been established by [**Os 79**] and by [**'Ol 87**], independently. Taking matrix coefficients of $j(P\colon\lambda)$ with K-finite vectors, one obtains Eisenstein integrals depending meromorphically on the parameter λ, cf. [**Ba 91**]. For non-minimal $\sigma\theta$-stable parabolic subgroups meromorphic families of Eisenstein integrals are obtained in [**BD 91**].

Normalization of $j(\xi,\lambda)$. For the Plancherel decomposition of $L^2(G/H)$ we are interested in the *imaginary* values of λ. Notice however that for $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$ the map $j(\xi\colon\lambda)$ is obtained by meromorphic continuation. In particular, singularities may occur at the imaginary points. This unpleasantness can be overcome by a suitable renormalization.

Let $A(\bar{P}\colon P\colon \xi\colon \lambda)\colon C^{-\infty}(P\colon \xi\colon \lambda) \to C^{-\infty}(\bar{P}\colon \xi\colon \lambda)$ be the standard intertwining operator (cf. [**KS 80**] for its definition on smooth functions, and [**Ba 88**] for its extension to generalized functions), and define $j^1(\xi\colon \lambda) = j^1(P\colon \xi\colon \lambda) \in \mathrm{Hom}(V(\xi), C^{-\infty}(P\colon \xi\colon \lambda)^H)$ by

$$j^1(P\colon \xi\colon \lambda) = A(\bar{P}\colon P\colon \xi\colon \lambda)^{-1} j(\bar{P}\colon \xi\colon \lambda).$$

The following theorem is valid under a technical condition on the pair (G, H) that will be explained at a later stage. It is fulfilled in the Riemannian and in the group case, and also in case G has abelian Cartan subgroups and H is the full fixed point group for σ. From now on we assume this condition, denoted (F), to be fulfilled.

THEOREM 2. *The meromorphic function* $\lambda \mapsto j^1(P\colon \xi\colon \lambda)$ *has no singularities on* $i\mathfrak{a}_{\mathrm{q}}^*$.

In Example 1 one has $\xi = 1$, $V(1) = \mathbf{C}$, and $j(1, \lambda) = 1_\lambda$. Hence $j^1(\xi\colon \lambda) = \mathbf{c}(\lambda)^{-1} 1_\lambda$. It is known from [**H-C 58**], p. 580, or from the formula of [**GK 62**], that c is nowhere zero on $i\mathfrak{a}^*$. Finally, notice that if the above normalization is inserted in the definition of the Fourier transform $\hat{f}$, it has the effect that the Plancherel measure $d\mu$ in (2) becomes just Lebesgue measure.

Example 2: the group case. Let ${}^{\backprime}G$ be a group of Harish-Chandra's class, let $G = {}^{\backprime}G \times {}^{\backprime}G$, and define $\sigma\colon G \to G$ by $\sigma(x, y) = (y, x)$, and $H\colon = G^\sigma = \mathrm{diagonal}({}^{\backprime}G \times {}^{\backprime}G)$. Then as a homogeneous space for the left times right action of ${}^{\backprime}G \times {}^{\backprime}G$, we have ${}^{\backprime}G \simeq G/H$.

Let ${}^{\backprime}\theta$ be a Cartan involution for ${}^{\backprime}G$, let ${}^{\backprime}\mathfrak{g} = {}^{\backprime}\mathfrak{k} \oplus {}^{\backprime}\mathfrak{p}$ be the associated Cartan decomposition, and let ${}^{\backprime}\mathfrak{a} \subset {}^{\backprime}\mathfrak{p}$ be a maximal abelian subspace. Then $\theta\colon = {}^{\backprime}\theta \times {}^{\backprime}\theta$ is a Cartan involution for G commuting with σ, and $\mathfrak{a}_{\mathrm{q}} = \{(X, -X)\colon X \in {}^{\backprime}\mathfrak{a}\}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$.

We denote the root system of ${}^{\backprime}\mathfrak{a}$ in ${}^{\backprime}\mathfrak{g}$ by ${}^{\backprime}\Sigma$, and fix a system ${}^{\backprime}\Sigma^+$ of positive roots. Then every element $\alpha \in \Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathrm{q}})$ is of the form $\alpha(X, -X) = \tilde{\alpha}(X)$, with $\tilde{\alpha} \in {}^{\backprime}\Sigma$. Moreover, the map $\alpha \mapsto \tilde{\alpha}, \Sigma \to {}^{\backprime}\Sigma$ is a bijection. The inverse image Σ^+ of ${}^{\backprime}\Sigma^+$ under this map is a positive system for Σ. If $\alpha \in \Sigma$, then the associated root space is given by $\mathfrak{g}^\alpha = {}^{\backprime}\mathfrak{g}^{\tilde{\alpha}} \times {}^{\backprime}\mathfrak{g}^{-\tilde{\alpha}}$. Let ${}^{\backprime}\mathfrak{n}$ be the sum of the positive root spaces in ${}^{\backprime}\mathfrak{g}$, and put ${}^{\backprime}\bar{\mathfrak{n}} = {}^{\backprime}\theta\, {}^{\backprime}\mathfrak{n}$. Then $\mathfrak{n} = {}^{\backprime}\mathfrak{n} \times {}^{\backprime}\bar{\mathfrak{n}}$. Let ${}^{\backprime}P$ be the minimal parabolic subgroup of ${}^{\backprime}G$ with Langlands decomposition ${}^{\backprime}M\, {}^{\backprime}A\, {}^{\backprime}N$, and let ${}^{\backprime}\bar{P} = {}^{\backprime}\theta\, {}^{\backprime}P$ be the opposite parabolic subgroup. Then $P = {}^{\backprime}P \times {}^{\backprime}\bar{P}$ is a minimal $\sigma\theta$-stable parabolic subgroup of G. Moreover, its Langlands decomposition is $P = MAN$, where $M = {}^{\backprime}M \times {}^{\backprime}M$, and $A = {}^{\backprime}A \times {}^{\backprime}A$.

Let ${}^{\backprime}M^*$ denote the normalizer of ${}^{\backprime}\mathfrak{a}$ in K. Then $N_K(\mathfrak{a}_{\mathrm{q}}) = \mathrm{diagonal}({}^{\backprime}M^* \times {}^{\backprime}M^*) = N_{K\cap H}(\mathfrak{a}_{\mathrm{q}})$, and we see that W and $W_{K\cap H}$ are equal and isomorphic to ${}^{\backprime}W$, the Weyl group of ${}^{\backprime}\Sigma$. In particular $\#(W/W_{K\cap H}) = 1$, and we may take $\mathcal{W} = \{e\}$.

The induction data for the principal series can now be described as follows. $\widehat{M_H}$ equals the set of (equivalence classes of) representations $\xi = {}^{\backprime}\xi \otimes {}^{\backprime}\xi^\vee$, where ${}^{\backprime}\xi \in {}^{\backprime}\widehat{M}$. Moreover, $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbf{c}}^*$ corresponds to an element of $\mathfrak{a}_{\mathbf{c}}^*$ of the form $(X, Y) \mapsto {}^{\backprime}\lambda(X) - {}^{\backprime}\lambda(Y)$, with ${}^{\backprime}\lambda \in {}^{\backprime}\mathfrak{a}_{\mathbf{c}}^*$. On the K-finite level we have a natural isomorphism

$$\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1) \simeq \mathrm{Ind}_{{}^{\backprime}P}^{{}^{\backprime}G}({}^{\backprime}\xi \otimes {}^{\backprime}\lambda \otimes 1) \otimes \mathrm{Ind}_{{}^{\backprime}\bar{P}}^{{}^{\backprime}G}({}^{\backprime}\xi^\vee \otimes -{}^{\backprime}\lambda \otimes 1), \tag{4}$$

where each representation is induced from the left. Let $\langle\cdot,\cdot\rangle_\xi$ denote the bilinear pairing of $\mathcal{H}_\xi^*$ with $\mathcal{H}_\xi$. Then we recall that the map $(f,g)\mapsto\langle f,g\rangle=\int_K\langle f(k),g(k)\rangle_\xi\,dk$ establishes a non-degenerate equivariant bilinear pairing of Harish-Chandra modules

$$\mathrm{Ind}_{{}^\backprime P}^{{}^\backprime G}({}^\backprime\xi^\vee\otimes-{}^\backprime\lambda\otimes 1)\times\mathrm{Ind}_{{}^\backprime P}^{{}^\backprime G}({}^\backprime\xi\otimes{}^\backprime\lambda\otimes 1)\to\mathbf{C}.$$

The pairing naturally induces a topological linear isomorphism from $C^{-\infty}({}^\backprime P\colon{}^\backprime\xi^\vee\colon-{}^\backprime\lambda)$ onto the topological linear dual of $C^\infty({}^\backprime P\colon{}^\backprime\xi\colon{}^\backprime\lambda)$, which we will use to identify these spaces. For the intertwining operators we have:

$$\langle A({}^\backprime P\colon{}^\backprime\bar{P}\colon{}^\backprime\xi\colon{}^\backprime\lambda)\varphi,\psi\rangle=\langle\varphi,A({}^\backprime\bar{P}\colon{}^\backprime P\colon{}^\backprime\xi^\vee\colon-{}^\backprime\lambda)\psi\rangle,\tag{5}$$

for $\varphi\in C^\infty({}^\backprime\bar{P}\colon{}^\backprime\xi\colon{}^\backprime\lambda)$, $\psi\in C^\infty({}^\backprime P\colon{}^\backprime\xi^\vee\colon-{}^\backprime\lambda)$. Indeed this follows from [**KS 80**], Prop. 7.1, if one uses the anti-linear identification of $\mathcal{H}_\xi$ with its linear dual.

On the level of generalized functions the isomorphism (4) gives rise to an equivariant topological linear isomorphism

$$\Phi=\Phi_P\colon\quad C^{-\infty}(P\colon\xi\colon\lambda)\xrightarrow{\simeq}\mathrm{Hom}(C^\infty({}^\backprime\bar{P}\colon{}^\backprime\xi\colon{}^\backprime\lambda),C^{-\infty}({}^\backprime P\colon{}^\backprime\xi\colon{}^\backprime\lambda)),$$

where Hom denotes the space of continuous linear operators. Indeed, if $u\in C^{-\infty}(P\colon\xi\colon\lambda)$ and $\varphi\in C^\infty({}^\backprime\bar{P}\colon{}^\backprime\xi\colon{}^\backprime\lambda)$, then $\Phi(u)\varphi$ is the element of $C^{-\infty}({}^\backprime P\colon{}^\backprime\xi\colon{}^\backprime\lambda)$ defined by

$$\langle\Phi(u)\varphi,\psi\rangle=\langle u,\psi\otimes\varphi\rangle,$$

for $\psi\in C^\infty({}^\backprime P\colon{}^\backprime\xi^\vee\colon-{}^\backprime\lambda)$. In other words, $\Phi(u)$ may be interpreted as the operator with generalized integral kernel u. The equivariance of Φ is displayed by the formula:

$$\Phi(\pi_{\xi,\lambda}(x,y)u)=\pi_{{}^\backprime\xi,{}^\backprime\lambda}(x)\circ\Phi(u)\circ\pi_{{}^\backprime\xi,{}^\backprime\lambda}(y)^{-1}.$$

From this we see that Φ establishes an isomorphism

$$C^{-\infty}(P\colon\xi\colon\lambda)^H\simeq\mathrm{Hom}_{{}^\backprime G}(C^\infty({}^\backprime\bar{P}\colon{}^\backprime\xi\colon{}^\backprime\lambda),C^\infty({}^\backprime P\colon{}^\backprime\xi\colon{}^\backprime\lambda)),$$

where we have used that intertwining operators map C^∞ vectors to C^∞ vectors.

We have natural isomorphisms $V(\xi)\simeq(\mathcal{H}_{{}^\backprime\xi}\otimes\mathcal{H}_{{}^\backprime\xi}^*)^{M\cap H}\simeq\mathrm{End}_{{}^\backprime M}(\mathcal{H}_{{}^\backprime\xi})$. In particular $V(\xi)$ is one dimensional. Let I_ξ denote the element of $V(\xi)$ corresponding to the identity map of $\mathcal{H}_{{}^\backprime\xi}$. We normalize the Haar measure $d\bar{n}$ of ${}^\backprime\bar{N}$ so that

$$\int_{{}^\backprime\bar{N}}e^{-2{}^\backprime\rho\,{}^\backprime H(\bar{n})}\,d\bar{n}=1.$$

Here ${}^\backprime\rho$ is half the sum of the positive roots, and ${}^\backprime H$ is the Iwasawa map ${}^\backprime G\to{}^\backprime\mathfrak{a}$, determined by ${}^\backprime G={}^\backprime K\,{}^\backprime A\,{}^\backprime N$. Moreover, we normalize Haar measure dn of ${}^\backprime N$ by $dn={}^\backprime\theta^*(d\bar{n})$. This normalization induces a normalization for the standard intertwining operators, cf. also formula (6) below.

LEMMA 1. *For $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ we have:*

$$j(P{:}\, \xi{:}\, \lambda)(I_\xi) = \Phi^{-1}(A({}^{\backprime}P{:}\, {}^{\backprime}\bar{P}{:}\, {}^{\backprime}\xi{:}\, {}^{\backprime}\lambda)).$$

PROOF: Let $\vartheta(\lambda) = \mathrm{ev} \circ \Phi^{-1}(A({}^{\backprime}P{:}\, {}^{\backprime}\bar{P}{:}\, {}^{\backprime}\xi{:}\, {}^{\backprime}\lambda))$. Then $\vartheta(\lambda) \in V(\xi)$ depends meromorphically on λ, by [**Ba 88**], Lemma 4.13. Hence by Thm. 1 it suffices to show that $\vartheta(\lambda) = I_\xi$ for λ in a non-empty open subset of $\mathfrak{a}_{\mathrm{qc}}^*$. For Re ${}^{\backprime}\lambda$ strictly anti-dominant, the intertwining operator $A({}^{\backprime}\lambda) = A({}^{\backprime}P{:}\, {}^{\backprime}\bar{P}{:}\, {}^{\backprime}\xi{:}\, {}^{\backprime}\lambda)$ is given by an absolutely convergent integral:

$$A({}^{\backprime}\lambda)\varphi(x) = \int_{{}^{\backprime}N} \varphi(nx)dn \quad (x \in {}^{\backprime}G), \tag{6}$$

for $\varphi \in C^\infty({}^{\backprime}\bar{P}{:}\, {}^{\backprime}\xi{:}\, {}^{\backprime}\lambda)$. Let $M_{{}^{\backprime}\lambda}$ be the measurable map ${}^{\backprime}G \to \mathrm{End}(\mathcal{H}_{{}^{\backprime}\xi})$ defined by

$$M_{{}^{\backprime}\lambda}(nma\bar{n}) = a^{{}^{\backprime}\lambda + {}^{\backprime}\rho}\xi(m),$$

for $(n, m, a, \bar{n}) \in {}^{\backprime}N \times {}^{\backprime}M \times {}^{\backprime}A \times {}^{\backprime}\bar{N}$, and by zero outside ${}^{\backprime}P\,{}^{\backprime}\bar{N}$. Then by the usual transformation of variables applied to (6), we find that

$$A({}^{\backprime}\lambda)\varphi(x) = \int_{{}^{\backprime}K} M_{{}^{\backprime}\lambda}(k^{-1})\varphi(kx)\, dk = \int_{{}^{\backprime}K} M_{{}^{\backprime}\lambda}(xk^{-1})\varphi(k)\, dk,$$

where dk denotes the normalized Haar measure on ${}^{\backprime}K$. It follows from this that in a neighborhood of (e, e) we have $\Phi^{-1}(A({}^{\backprime}\lambda))(x, y) = M_{{}^{\backprime}\lambda}(xy^{-1})$. Hence $\vartheta(\lambda) = M_{{}^{\backprime}\lambda}(e) = I_\xi$. ∎

LEMMA 2. *For $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$ we have:*

$$j^1(P{:}\, \xi{:}\, \lambda)(I_\xi) = \Phi^{-1}(A({}^{\backprime}\bar{P}{:}\, {}^{\backprime}P{:}\, {}^{\backprime}\xi{:}\, {}^{\backprime}\lambda)^{-1}).$$

PROOF: Via the isomorphisms (4) for P and for $\bar{P}$, the intertwining operator $A(\bar{P}{:}\, P{:}\, \xi{:}\, \lambda)$ corresponds to $A({}^{\backprime}\bar{P}{:}\, {}^{\backprime}P{:}\, {}^{\backprime}\xi{:}\, {}^{\backprime}\lambda) \otimes A({}^{\backprime}P{:}\, {}^{\backprime}\bar{P}{:}\, {}^{\backprime}\xi^\vee{:}\, -{}^{\backprime}\lambda)$. Hence if $u \in C^{-\infty}(\bar{P}{:}\, \xi{:}\, \lambda)$, then taking (5) into account we obtain:

$$\Phi_P(A(\bar{P}{:}\, P{:}\, \xi{:}\, \lambda)^{-1}u) = A({}^{\backprime}\bar{P}{:}\, {}^{\backprime}P{:}\, {}^{\backprime}\xi{:}\, {}^{\backprime}\lambda)^{-1} \circ \Phi_{\bar{P}}(u) \circ A({}^{\backprime}\bar{P}{:}\, {}^{\backprime}P{:}\, {}^{\backprime}\xi{:}\, {}^{\backprime}\lambda)^{-1}.$$

Substituting $u = j(\bar{P}{:}\, \xi{:}\, \lambda)(I_\xi)$, and applying Lemma 1 with $\bar{P}$ instead of P, we obtain

$$\Phi_P(j^1(P{:}\, \xi{:}\, \lambda)(I_\xi)) = A({}^{\backprime}\bar{P}{:}\, {}^{\backprime}P{:}\, {}^{\backprime}\xi{:}\, {}^{\backprime}\lambda)^{-1}. \quad ∎$$

The Fourier transform. Let f be a smooth compactly supported function on G/H. We define the *Fourier transform* $\hat{f}$ of f by

$$\hat{f}(\xi\colon\lambda) = \pi_{\xi,\lambda}(f) j^1(\xi\colon\lambda) = \int_{G/H} f(xH)\pi_{\xi,\lambda}(x) j^1(\xi\colon\lambda)\, dxH \in \mathrm{Hom}(V(\xi), C^\infty(\xi\colon\lambda))$$

for $\xi \in \widehat{M}_H$ and $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$. Notice that the map $f \mapsto \hat{f}$ is G-equivariant in the sense that $(\mathcal{L}(x)f)\hat{}\,(\xi\colon\lambda) = \pi_{\xi,\lambda}(x)\hat{f}(\xi\colon\lambda)$. In the compact picture we have that

$$\hat{f}(\xi\colon\lambda)(k) = \int_{G/H} f(xH) j^1(\xi\colon\lambda)(kx) dxH,$$

and we view $\hat{f}(\xi\colon\lambda)$ as an element of $V(\xi)^* \otimes C^\infty(K\colon\xi)$.

For each $\xi \in \widehat{M}_H$, let $V(\xi)^* \otimes L^2(K\colon\xi)$ be endowed with the tensor product Hilbert structure. Consider the algebraic direct sum

$$\mathcal{H}_{\mathrm{alg}} = \bigoplus_{\xi\in\widehat{M}_H} V(\xi)^* \otimes L^2(K\colon\xi),$$

and let $\mathcal{H}$ be its Hilbert completion with respect to the inner product

$$(\varphi, \psi) = \sum_{\xi\in\widehat{M}_H} \dim(\xi)(\varphi_\xi, \psi_\xi).$$

Let $d\lambda$ denote a choice of Lebesgue measure on $i\mathfrak{a}_{\mathrm{q}}^*$, and let $\mathfrak{L}^2$ denote the space of $d\lambda$ square integrable functions $i\mathfrak{a}_{\mathrm{q}}^* \to \mathcal{H}$. If $F \in \mathfrak{L}^2$, we write $F(\xi\colon\lambda) = F(\lambda)_\xi$. Then we have a natural unitary representation π of G on $\mathfrak{L}^2$, given by $(\pi(x)F)(\xi\colon\lambda) = \pi_{\xi,\lambda}(x)F(\xi\colon\lambda)$.

THEOREM 3. *For suitably normalized Lebesgue measure $d\lambda$, the following holds.*

(a) *If $f \in C_c^\infty(G/H)$ then $\hat{f} \in \mathcal{L}^2$, and*

$$\|\hat{f}\|_{\mathfrak{L}^2}^2 = \sum_{\xi\in\widehat{M}_H} \int_{i\mathfrak{a}_{\mathrm{q}}^*} \dim(\xi) \|\hat{f}(\xi\colon\lambda)\|^2 d\lambda \leq \|f\|_{L^2(G/H)}^2.$$

In particular, $f \mapsto \hat{f}$ extends uniquely to a G-equivariant continuous linear map $\mathfrak{F}$ from $(\mathcal{L}, L^2(G/H))$ into $(\pi, \mathfrak{L}^2)$. Moreover:

(b) *The restriction of $\mathfrak{F}$ to the orthocomplement $L^2_{\mathrm{mc}}(G/H)$ of $\ker\mathfrak{F}$ in $L^2(G/H)$ is an isometry.*

(c) *The map $\mathfrak{F}$ induces the following Plancherel decomposition:*

$$\mathcal{L}|_{L^2_{\mathrm{mc}}(G/H)} \simeq \sum_{\xi\in\widehat{M}_H} \int^{\oplus}_{i\mathfrak{a}_{\mathrm{q}}^{*+}} V(\xi)^* \otimes \pi_{\xi,\lambda}\, d\lambda. \tag{7}$$

It follows from the proof of Theorem 3 that will be given in [**BS** $*$], that the kernel of $\mathfrak{F}$ is small in a certain spectral sense. In particular, a compactly supported smooth function is uniquely determined on the most continuous part of the spectrum:

Theorem 4. *If $f \in C_c^\infty(G/H)$ and $\hat{f} = 0$ then $f = 0$.*

If G/H has *split rank one*, that is $\dim \mathfrak{a}_\mathrm{q} = 1$, then the complement of $L^2_{\mathrm{mc}}(G/H)$ in $L^2(G/H)$ decomposes discretely, but in general there will occur intermediate series as well (for an example different from the group case, see [**Bo 87**] and [**BH 90**]).

If G/H is Riemannian, i.e. H is compact, then it follows from our proof of Thm. 3 that $\ker \mathfrak{F} = 0$, and we retrieve (2) from (7).

Normalization of measures. At this point we shall specify the normalization of the Lebesgue measure $d\lambda$ in Thm. 3. We start by specifying normalizations of various other measures involved. Firstly, the definition of j^1 depends on the normalization of the standard intertwining operator $A(\bar{P}\colon P\colon \xi\colon \lambda)$, which in turn depends on the normalization of the Haar measure $d\bar{n}$ of $\bar{N}$. We assume the Haar measure $d\bar{n}$ to be normalized so that

$$\int_{\bar{N}} e^{-2\rho H(\bar{n})}\, d\bar{n} = 1,$$

where H is the Iwasawa map specified in Example 1.

Let dx denote a choice of invariant measure on G/H, and dk the normalized Haar measure of K. Let da denote a choice of Haar measure for A_q. Associated with the Cartan decomposition $G = KA_\mathrm{q}H$, there exists a Jacobian $J\colon A_\mathrm{q} \to [0, \infty[$ such that for all $f \in C_c(G/H)$ we have:

$$\int_{G/H} f(x)\, dx = (\# W_{K\cap H})^{-1} \int_K \int_{A_\mathrm{q}} f(kaH) J(a)\, da dk.$$

From the explicit formula for J in terms of root data (cf. [**Sc 84**], p. 149) it follows that $J(a) \sim C\, a^{2\rho}$ as $a \to \infty$, radially in the positive chamber in A_q. Here C is a positive constant, which we may assume to be 1 after a suitable renormalization of da. Consider the classical Fourier transform $\varphi \mapsto \hat{\varphi}, C_c(A_\mathrm{q}) \to C(i\mathfrak{a}_\mathrm{q}^*)$, defined by the formula $\varphi(\lambda) = \int_{A_\mathrm{q}} \varphi(a) a^\lambda\, da$. Then the appropriate normalization of $d\lambda$ is $(\# W)^{-1}$ times the one which allows an extension of this transform to an isometry from $L^2(A_\mathrm{q}, da)$ onto $L^2(i\mathfrak{a}_\mathrm{q}^*, d\lambda)$.

The dependence on $\mathcal{W}$. We shall now discuss the dependence of the Fourier transform on the choice of the set $\mathcal{W}$ of representatives for $W/W_{K\cap H}$. Let $\mathcal{W}' \subset N_K(\mathfrak{a}_\mathrm{q})$ be a second set of representatives, and let $V'(\xi)$, $j'(\xi\colon \lambda)$ and $\mathfrak{F}'$ be defined as before, but with $\mathcal{W}'$ instead of $\mathcal{W}$. Let $w \mapsto w'$ denote the bijection $\mathcal{W} \to \mathcal{W}'$ which induces the identity map on $W/W_{K\cap H}$. Then for every $w \in \mathcal{W}$ there exists an element $l(w) \in M \cap K$ such that $w' \in l(w) w N_{K\cap H}(\mathfrak{a}_\mathrm{q})$. Let $R_\xi\colon V(\xi) \to V'(\xi)$ be the direct sum of the maps

$$\xi(l(w))\colon\ \mathcal{H}_\xi^{w(M\cap H)w^{-1}} \to \mathcal{H}_\xi^{w'(M\cap H)w'^{-1}}.$$

Then R_ξ is an isometry, and by [**Ba 88**], Lemma 5.8, we have:

$$j'(\xi\colon \lambda) \circ R_\xi = j(\xi\colon \lambda),$$

for $\lambda \in \mathfrak{a}_{\mathrm{qc}}^*$. This implies that for $f \in C_c^\infty(G/H)$ we have $(R_\xi^t \otimes I)\, \mathfrak{F}' f(\xi\colon \lambda) = \mathfrak{F} f(\xi\colon \lambda)$. Let $T\colon \mathcal{H} \to \mathcal{H}$ be the direct sum of the maps $(R_\xi^t)^{-1} \otimes I$, and let $\mathcal{T}\colon \mathfrak{L}^2 \to \mathfrak{L}^2$ be defined by $F \mapsto T \circ F$. Then $\mathcal{T}$ is a bijective isometry intertwining π with itself. Moreover, we have:

$$\mathfrak{F}' = \mathcal{T} \circ \mathfrak{F}.$$

Example 2, continuation. Let us look at what Theorems 2 and 3 mean for the group case. Pull-back by the map ${}'G \times {}'G \to {}'G, (x, y) \mapsto xy^{-1}$ induces an isomorphism from $C_c^\infty({}'G)$ onto $C_c^\infty(G/H)$. We denote the inverse of this isomorphism by $f \mapsto {}'f$. If $u \in C^{-\infty}(P\colon \xi\colon \lambda)^H$, then for $f \in C_c^\infty(G/H)$ we have:

$$\pi_{\xi,\lambda}(f)u = \int_{{}'G} {}'f(x)\pi_{\xi,\lambda}(x, e)u \; dx.$$

Using this with $u = j^1(P\colon \xi\colon \lambda)(I_\xi)$, and applying Lemma 2, we obtain

$$\Phi(\hat{f}(\xi\colon \lambda)(I_\xi)) = \pi_{{}'\xi,{}'\lambda}({}'f) \circ A({}'P\colon {}'\bar{P}\colon {}'\xi\colon {}'\lambda)^{-1}.$$

Now Φ induces an isometry of $L^2(K\colon \xi)$ onto the space of Hilbert-Schmidt operators from $L^2({}'K\colon {}'\xi)$ into itself, provided with the Hilbert-Schmidt norm $T \mapsto \|T\|_{\mathrm{HS}} = \sqrt{\mathrm{tr}(T \circ T^*)}$. Observe that

$$\mu({}'\xi\colon {}'\lambda)\, A({}'P\colon {}'\bar{P}\colon {}'\xi\colon {}'\lambda)^* \circ A({}'P\colon {}'\bar{P}\colon {}'\xi\colon {}'\lambda) = I,$$

with ${}'\lambda \mapsto \mu({}'\xi\colon {}'\lambda)$ a non-negative meromorphic function on $i\,{}'\mathfrak{a}^*$ (here we use the term meromorphic to indicate that it is the restriction of a meromorphic function defined on a complex neighborhood). Hence

$$\|I_\xi\|^2\, \|\hat{f}(\xi\colon \lambda)\|^2 = \mu({}'\xi\colon {}'\lambda)\, \|\pi_{{}'\xi,{}'\lambda}({}'f)\|_{\mathrm{HS}}^2.$$

Since $\|I_\xi\|^2 = \dim({}'\xi)$, this implies that

$$\dim(\xi)\, \|\hat{f}(\xi\colon \lambda)\|^2 = \dim({}'\xi)\, \mu({}'\xi\colon {}'\lambda)\, \|\pi_{{}'\xi,{}'\lambda}({}'f)\|_{\mathrm{HS}}^2. \tag{8}$$

For every $f \in C_c^\infty(G/H)$, the function $\psi_f\colon \lambda \mapsto \|\pi_{{}'\xi,{}'\lambda}({}'f)\|_{\mathrm{HS}}^2$ is analytic on $i\mathfrak{a}_{\mathrm{q}}^*$. On the other hand, from the identity $\|\hat{f}(\xi\colon \lambda)\|^2 = \langle \hat{f}(\xi\colon \lambda), \hat{f}(\xi\colon -\bar{\lambda})\rangle$ for $\lambda \in i\mathfrak{a}_{\mathrm{q}}^*$, one sees that the function $\varphi_f\colon \lambda \mapsto \|\hat{f}(\xi\colon \lambda)\|^2$ extends meromorphically to $\mathfrak{a}_{\mathrm{qc}}^*$, for every $f \in C_c^\infty(G/H)$. Hence the assertion of Thm. 2 is equivalent to the assertion that φ_f is analytic on $i\mathfrak{a}_{\mathrm{q}}^*$, for every $f \in C_c^\infty(G/H)$. In the group case the latter assertion is by (8) equivalent to the assertion that $\lambda \mapsto \mu({}'\xi\colon {}'\lambda)\psi_f(\lambda)$ is analytic on $i\mathfrak{a}_{\mathrm{q}}^*$, for every $f \in C_c^\infty(G/H)$. Given $\xi \in \widehat{M_H}$ and $\lambda_0 \in i\mathfrak{a}_{\mathrm{q}}^*$, there exists a $f \in C_c^\infty(G/H)$ such that $\psi_f(\lambda_0) \neq 0$. Hence in the group case, Thm. 2 is equivalent to Harish-Chandra's result that the function $\mu({}'\xi\colon \cdot)$ is analytic on $i\,{}'\mathfrak{a}^*$ (cf. [**H-C 76b**], Sections 12 and 13, and Thm. 25.1). Notice that our μ differs by a positive factor from that of loc. cit., because of different normalizations of the measures involved.

The analyticity of $\mu({}'\xi\colon \cdot)$ thus having been established, we see from (8) that in the group case Thm. 3 gives us the most continuous part of the Plancherel decomposition as described in [**H-C 76b**], Thm. 27.3. Notice however that we do not compute the measure explicitly.

A technical condition. Extend $\mathfrak{a}_\mathfrak{q}$ to a maximal abelian subspace $\mathfrak{a}_\mathfrak{p}$ of $\mathfrak{p}$, and let $M_\mathfrak{p}$ denote its centralizer in K. Then it is well known that there exists a finite subgroup $F \subset M_\mathfrak{p} \cap G_0$ such that

$$\begin{cases} M_\mathfrak{p} \cap G_0 = M_{\mathfrak{p}0} F \\ F \text{ centralizes } M_{\mathfrak{p}0} \\ F \text{ is } \sigma\text{-invariant.} \end{cases} \tag{9}$$

Indeed, if G_1 is the analytic subgroup of G with Lie algebra $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$, then

$$F = G_1 \cap \mathrm{Ad}^{-1}(\mathrm{Ad}(K) \cap \exp i \,\mathrm{ad}\, \mathfrak{a}_\mathfrak{p}) \tag{10}$$

fulfills the above requirements. (cf. [**He 78**] p. 435, Exercise A3 for the linear case, from which it is easily deduced). The technical condition on the pair (G, H), referred to before Theorem 2, is the following:

Condition (F). *There exists a finite group F satisfying (9), and in addition:*

$$x \mapsto \sigma(x)^{-1} \text{ induces the identity map on } F \cap H \backslash F / F \cap H. \tag{11}$$

Condition (F) is trivially fulfilled in the Riemannian case. Moreover, in the group case we may define $F = {}'F \times {}'F$ with ${}'F$ defined as in (10), but for ${}'G$. Let $(x, y) \in F$. Then $\sigma(x, y)^{-1} = (y^{-1}, x^{-1}) = (x^{-1}, x^{-1})(x, y)(y^{-1}, y^{-1})$, and we see that (11) holds. Finally, assume that G has abelian Cartan subgroups, and that $H = G^\sigma$, the full fixed point group. Then the group F defined by (10) is contained in a Cartan subgroup, hence abelian, and it follows that $x\sigma(x) \in F \cap H$ for all $x \in F$. We see that (11) holds in this situation as well.

In our derivation of the Plancherel decomposition we need Conditon (F) just once, namely for the proof of Thm. 6.3 of [**Ba 88**], in the case that $\dim \mathfrak{a}_\mathfrak{q} = 1$ and $\#\mathcal{W} = 1$. In [**Ba 88**] it is assumed that all Cartan subgroups are abelian, but it was overlooked that the assumption $H = G^\sigma$ is needed in the proof of Lemma 6.16. On the other hand, by a minor modification of its proof one can show that Thm. 6.3 in loc. cit. is valid under Condition (F). Under this condition it is not required that the Cartan subgroups are abelian (this is of importance for the example discussed at the end of the present paper).

Remark: Observe that (11) implies that the space $C(F/F \cap H)$ of functions on $F/F \cap H$ decomposes multiplicity free for the left regular representation $\mathcal{L}_F$. Indeed, let $\mathcal{A}$ be the algebra of bi-$F \cap H$-invariant functions on F, provided with the convolution product. Then the involution σ induces an automorphism on $\mathcal{A}$, which at the same time is an anti–automorphism, in view of (11). Hence $\mathcal{A}$ is commutative, and this implies that $\mathcal{L}_F$ decomposes multiplicity free.

The following example shows that Condition (F) does not always hold. Notice however that nevertheless there exists a group F as in (9), such that $C(F/F \cap H)$ decomposes multiplicity free.

Example 3. Let $G = \mathrm{SL}(3, \mathbf{R})$, and consider the involution $\sigma\colon x \mapsto JxJ$, where J denotes the diagonal matrix with entries $-1, -1, 1$ respectively. Then $H = (G^\sigma)_0$ equals

$S(GL(2, \mathbf{R}) \times \mathbf{R}^+)$. The Cartan involution $\theta: x \mapsto (x^t)^{-1}$ commutes with σ, and for $\mathfrak{a}_\mathfrak{q}$ we may take the line generated by the matrix

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

which is the image of the diagonal matrix with entries $0, 1, -1$ under conjugation by the rotation c around the first coordinate axis by an angle $\pi/4$. Thus for $\mathfrak{a}_\mathfrak{p}$ we may take $c\mathfrak{b}c^{-1}$, where $\mathfrak{b}$ denotes the algebra of diagonal matrices in $\mathfrak{sl}(3, \mathbf{R})$. Let $D \subset G$ be the subgroup of diagonal matrices with entries ± 1, then $M_\mathfrak{p} = cDc^{-1}$. By finiteness we must have $F = M_\mathfrak{p}$. One readily verifies that $F \cap H = \{I\}$, so that (11) is equivalent to $\sigma f^{-1} = f$ for all $f \in F$. Let $f(\epsilon_1, \epsilon_2, \epsilon_3)$ denote the c conjugate of the diagonal matrix with entries $\epsilon_j \in \{\pm 1\}$ respectively. Then one readily verifies that $\sigma(f(-1, -1, 1))^{-1} = f(-1, 1, -1)$. Hence (11) does not hold.

The multiplicities. It follows from (7) that the unitary principal series $\pi_{\xi,\lambda}$ has the multiplicity $m_\xi = \dim V(\xi)$ in $L^2_{\mathrm{mc}}(G/H)$. Notice that

$$\dim V(\xi) = \sum_{w \in \mathcal{W}} \dim \mathcal{H}_\xi^{w(M\cap H)w^{-1}}.$$

In general $\dim \mathcal{H}_\xi^{w(M\cap H)w^{-1}}$ can be greater than one (cf. the example given in [**Ba 88**], p. 362). However, if every Cartan subgroup of G is abelian, we have

$$\dim \mathcal{H}_\xi^{w(M\cap H)w^{-1}} \leq 1 \tag{12}$$

for all $\xi \in \widehat{M}_H, w \in N_K(\mathfrak{a}_\mathfrak{q})$, cf. [**Ba 88**], Lemma 5.4. On the other hand, if G is connected, but not necessarily with abelian Cartan subgroups, it follows from [**Os 88**], Thm. 4.9 that (12) holds.

From now on we assume that G is connected and that (G, H) fulfills Condition (F). In particular we then have (12). For completeness, and for later reference, the proof of (12) is given below. It follows that $m_\xi \leq \#\mathcal{W}$ for all ξ.

LEMMA 3. *Assume that G is connected and that (G, H) satisfies Condition (F). Let $\xi \in \widehat{M}_{\mathrm{fu}}$. Then for each $w \in N_K(\mathfrak{a}_\mathfrak{q})$ we have (12).*

PROOF: Since $\mathcal{H}_\xi^{w(M\cap H)w^{-1}} \simeq \mathcal{H}_{w^{-1}\xi}^{M\cap H}$, it suffices to prove this for $w = 1$. Let M_n be the connected normal subgroup of M which is maximal subject to the condition that it contains no non-trivial compact normal subgroups.

We claim that $M = M_n M_\mathfrak{p}$. To see this, first notice that $\mathfrak{a}_M = \mathfrak{a}_\mathfrak{p} \cap \mathfrak{m}$ is maximal abelian in $\mathfrak{m} \cap \mathfrak{p}$. Let Σ_M^+ be a choice of positive roots for the root system Σ_M of $\mathfrak{a}_M$ in $\mathfrak{m}$, let $\mathfrak{n}_M$ denote the sum of the positive root spaces, and let $\bar{\mathfrak{n}}_M = \theta(\mathfrak{n}_M)$. Then $\mathfrak{m}_n = \mathrm{Lie}(M_n)$ equals the subalgebra generated by $\mathfrak{n}_M$ and $\bar{\mathfrak{n}}_M$ (which is an ideal). Since M has no split component, $\mathfrak{a}_M \subset \mathfrak{m}_n$, hence $\mathfrak{m} = \mathfrak{m}_n + \mathfrak{m}_\mathfrak{p}$ and it follows that $M_0 = (M_n M_\mathfrak{p})_0$. But $M = M_0 M_\mathfrak{p}$ and the claim follows.

If $\xi \in \widehat{M}_{\rm fu}$, then ξ is trivial on M_n, hence $\xi|M_\mathfrak{p}$ is irreducible. Conversely, if $\xi_\mathfrak{p} \in \widehat{M}_\mathfrak{p}$ has a trivial restriction to $M_n \cap M_\mathfrak{p}$ then it is the restriction of a unique $\xi \in \widehat{M}_{\rm fu}$.

Let F be as in Condition (F). Then $M_{\mathfrak{p}0} \cap F$ is central in $M_\mathfrak{p}$. Hence if $\xi \in \widehat{M}_{\rm fu}$, then its restriction $\xi|M_{\mathfrak{p}0} \cap F$ is a multiple of a character χ_ξ.

The natural multiplication map $\varphi: M_{\mathfrak{p}0} \times F \to M$ is a group homomorphism, because F centralizes $M_{\mathfrak{p}0}$. It follows that $\xi \circ \varphi$ is a finite dimensional irreducible representation of $M_{\mathfrak{p}0} \times F$. Hence it is equivalent to an exterior tensor product of the form $\xi_\mathfrak{p} \widehat{\otimes} \xi_F$, for unique $\xi_\mathfrak{p} \in \widehat{M}_{\mathfrak{p}0}, \xi_F \in \widehat{F}$. Notice also that $\xi_\mathfrak{p}|M_{\mathfrak{p}0} \cap F$ and $\xi_F|M_{\mathfrak{p}0} \cap F$ are both multiples of χ_ξ. The representation $\xi_\mathfrak{p}$ is trivial on $M_n \cap M_{\mathfrak{p}0}$ so that it is the restriction of a unique $\xi_0 \in \widehat{M}_{0{\rm fu}}$. Hence there exists an isomorphism $\mathcal{H}_\xi \simeq \mathcal{H}_{\xi_0} \otimes \mathcal{H}_{\xi_F}$ so that for $m \in M_0, f \in F$ the endomorphism $\xi(mf)$ corresponds to $\xi_0(m) \otimes \xi_F(f)$.

Now clearly one has that $\mathcal{H}_\xi^{M \cap H} \subset \mathcal{H}_{\xi_0}^{M_0 \cap H} \otimes \mathcal{H}_{\xi_F}^{F \cap H}$. Moreover, $\mathcal{H}_{\xi_0}^{M_0 \cap H} \subset \mathcal{H}_{\xi_0}^{\mathfrak{m} \cap \mathfrak{h}}$, and by standard semisimple theory, the latter space has dimension at most 1. Hence it suffices to show that $\dim \mathcal{H}_{\xi_F}^{F \cap H} \leq 1$. Now this is a straightforward consequence of the remark below Condition (F). ∎

Lemma 4. *Let $\xi \in \widehat{M}_{\rm fu}$ and assume that $\mathcal{H}_\xi^{M_0 \cap H} \neq 0$. Then for each $w \in W$ there exists a $v \in W_{K \cap H_0}$, such that $wv\xi|_{M_0}$ is equivalent to $\xi|_{M_0}$.*

Proof: Let $w \in W$. Then the above assertion with w^{-1} instead of w is equivalent to the existence of a $v \in W_{K \cap H_0}$, such that $w\xi|_{M_0} \sim v\xi|_{M_0}$. For this it suffices to show that $w\xi_0 \sim v\xi_0$ for some $v \in W_{K \cap H_0}$: here we have used the notations of the proof of Lemma 3. By an easy induction on the length of w we can reduce to the case that w is the reflection s_α in a simple root $\alpha \in \Sigma^+$. As in [**Ba 88**], p. 392 we may then reduce to the case that $\dim \mathfrak{a}_\mathrm{q} = 1$. Then W has two elements, and we may assume that $w \notin W_{K \cap H_0}$: for otherwise the assertion of the lemma would be trivially true. Hence $W_{K \cap H_0}$ is trivial and it suffices to prove that $s_\alpha \xi_0 \sim \xi_0$. Let $\mathfrak{g}_+ = \mathfrak{g}^{\sigma\theta} = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q}$. Then the root system $\Sigma(\mathfrak{g}_+, \mathfrak{a}_\mathrm{q})$ has $W_{K \cap H_0}$ as its Weyl group, hence must be trivial. It follows that $\mathfrak{a}_\mathrm{q}$ centralizes $\mathfrak{g}_+$. By maximality of $\mathfrak{a}_\mathrm{q}$, this implies $\mathfrak{p} \cap \mathfrak{q} = \mathfrak{a}_\mathrm{q}$. Hence $\mathfrak{g}_+ = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{a}_\mathrm{q}$. According to [**Ba 88**], Lemma 8.7 we have that $\Sigma(\mathfrak{g}, \mathfrak{a}_\mathrm{q}) = \{-\alpha, \alpha\}$. Moreover, the associated root spaces are contained in the -1 eigenspace of $\sigma\theta$, and it follows that $[\mathfrak{g}^{-\alpha}, \mathfrak{g}^\alpha] \subset \mathfrak{g}_+$. Let $\mathfrak{g}'$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}$. Then in fact $\mathfrak{g}'$ is an ideal, and we may select a complementary ideal $\mathfrak{g}''$.

Since $\mathfrak{a}_\mathrm{q} \subset \mathfrak{g}'$, we have that $\mathfrak{m} = \mathfrak{m}' \oplus \mathfrak{m}''$, a direct sum of ideals, where $\mathfrak{m}' = \mathfrak{m} \cap \mathfrak{g}'$, $\mathfrak{m}'' = \mathfrak{m} \cap \mathfrak{g}''$. We now observe that the centralizer of $\mathfrak{a}_\mathrm{q}$ in $\mathfrak{g}'$ equals $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$, hence is contained in $\mathfrak{g}_+$. Hence $\mathfrak{m}' \subset \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{a}_\mathrm{q}$, and we conclude that $\mathfrak{m}' \subset \mathfrak{m} \cap \mathfrak{h}$. The infinitesimal representation ξ_0 of $\mathfrak{m}$ has a $\mathfrak{m} \cap \mathfrak{h}$-, and therefore a $\mathfrak{m}'$-fixed vector. It follows that $\xi_0|\mathfrak{m}' = 0$. There exists a representative s of s_α in K which centralizes $\mathfrak{g}''$, hence $\mathfrak{m}''$. Hence $s_\alpha \xi_0 \sim s\xi_0 = \xi_0$. ∎

Corollary 1. *If $\mathcal{H}_\xi^{M_0 \cap H} = 0$, then $m_\xi = 0$.*

Proof: If $m_\xi \neq 0$ then $\mathcal{H}_\xi^{w(M \cap H)w^{-1}} \neq 0$ for some w. By Lemma 4 there exists an element $v \in W_{K\cap H}$ such that $wv\xi|_{M_0} \simeq \xi|_{M_0}$. Hence

$$\dim \mathcal{H}_\xi^{M_0\cap H} = \dim \mathcal{H}_{(wv)^{-1}\xi}^{M_0\cap H} = \dim \mathcal{H}_\xi^{wv(M_0\cap H)v^{-1}w^{-1}} = \dim \mathcal{H}_\xi^{w(M_0\cap H)w^{-1}} \neq 0. \quad \blacksquare$$

REMARK: Notice that with the present assumptions on G and H, it follows from Lemma 3 that (7) can be rewritten as follows

$$\mathcal{L}|_{L^2_{\mathrm{mc}}(G/H)} \simeq \sum_{\xi\in\widehat{M}^{M\cap H}} \int_{i\mathfrak{a}_{\mathrm{q}}^{*\sigma\theta+}}^{\oplus} \pi_{\xi,\lambda}\, d\lambda. \tag{13}$$

Here $\widehat{M}^{M\cap H}$ denotes the set of $\xi \in \widehat{M}_{\mathrm{fu}}$ having a non-zero $M\cap H$-fixed vector, and $\mathfrak{a}_{\mathrm{q}}^{*\sigma\theta+}$ denotes the positive Weyl chamber in $\mathfrak{a}_{\mathrm{q}}^*$ for $\Sigma(\mathfrak{g}^{\sigma\theta}, \mathfrak{a}_{\mathrm{q}})^+$.

The case of connected H. In this section we assume that G is connected linear and that $H = (G^\sigma)_0$. Let $F = K \cap \exp i\mathfrak{a}_{\mathfrak{p}}$. Then F satisfies the properties listed in (9). We will compute m_ξ under the assumption that F satisfies (11). Put $F_{\mathrm{q}} = K\cap H\cap \exp i\mathfrak{a}_{\mathrm{q}}$. Then $F_{\mathrm{q}} \subset F\cap H$.

LEMMA 5. $\quad M\cap H = (M^\sigma)_0\; F_{\mathrm{q}}$.

PROOF: In view of the fact that $\mathfrak{a}_{\mathrm{q}}$ is maximal abelian in $\mathfrak{g}^{\sigma\theta}\cap\mathfrak{p}$, whereas $K\cap H$ is a maximal compact subgroup of $(G^{\sigma\theta})_0$, the group F_{q} is the analog of F for the connected linear group $(G^{\sigma\theta})_0$. Hence

$$M\cap (G^{\sigma\theta})_0 = (M^\sigma \cap G^{\sigma\theta})_0 F_{\mathrm{q}} \subset (M^\sigma)_0 F_{\mathrm{q}}.$$

This implies that $M\cap H\cap K \subset (M^\sigma)_0 F_{\mathrm{q}}$, whence $M\cap H \subset (M^\sigma)_0 F_{\mathrm{q}}$. The reversed inclusion is obvious. $\blacksquare$

By linearity of G, the group F is central in M. Hence by the same arguments as in the proof of Lemma 3, $\mathcal{H}_{\xi_F}$ is one dimensional, ξ_F is a character, and $\xi|_F$ is a multiple of ξ_F.

LEMMA 6. *Let $\xi \in \widehat{M}_{\mathrm{fu}}$. If $\mathcal{H}_\xi^{M_0\cap H} = 0$, then $m_\xi = 0$. Otherwise m_ξ equals the number of $w \in \mathcal{W}$ for which the character ξ_F is trivial on $wF_{\mathrm{q}}w^{-1}$.*

PROOF: The first statement of the Lemma is asserted in Cor. 1. Thus assume that $\mathcal{H}_\xi^{M_0\cap H} \neq 0$. By Lemma 4 we have $\mathcal{H}_{w^{-1}\xi}^{M_0\cap H} \neq 0$ for all $w \in W$. Now it suffices to prove that $\mathcal{H}_\xi^{w(M\cap H)w^{-1}}$ has dimension 1 if ξ_F is trivial on $wF_{\mathrm{q}}w^{-1}$, and has dimension 0 otherwise. Since $\mathcal{H}_\xi^{w(M\cap H)w^{-1}} = \mathcal{H}_{w^{-1}\xi}^{M\cap H}$, it suffices to prove this for $w = 1$. Then with notations as in the proof of Lemma 3, we have that $\mathcal{H}_{\xi_0}^{M_0\cap H} = \mathcal{H}_{\xi_0}^{\mathfrak{m}\cap\mathfrak{h}} = \mathcal{H}_{\xi_0}^{(M^\sigma)_0}$ is one dimensional. Moreover, in view of the equality of Lemma 5, the argumentation of the proof of Lemma 3 yields that

$$\mathcal{H}_\xi^{M\cap H} \simeq \mathcal{H}_{\xi_0}^{(M^\sigma)_0} \otimes \mathcal{H}_{\xi_F}^{F_{\mathrm{q}}} \simeq \mathcal{H}_{\xi_F}^{F_{\mathrm{q}}}.$$

This implies the result. $\blacksquare$

Example 4. Let $G/H = \mathrm{SL}(n, \mathbf{R})/\mathrm{SO}_0(1, n-1)$. Here the involution is given by $\sigma x = J(x^t)^{-1}J$, where J is the diagonal matrix with entries 1, except for the upper left corner entry, which is -1. A maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ is the space $\mathfrak{a}_\mathrm{q}$ of diagonal matrices in $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{R})$. Then $\mathfrak{a}_\mathrm{q}$ is maximal abelian in $\mathfrak{g}$, and hence $\mathfrak{m} = 0$. We then have that $M = F$ is the abelian group of diagonal matrices $d(\epsilon) \in \mathrm{SL}(n, \mathbf{R})$ with entries $\epsilon_j = \pm 1 (j = 1, \dots, n)$ in the diagonal, and $M \cap H$ is the subgroup given by $\epsilon_1 = 1$ (notice that if H had been taken as the full fixed point group, $\mathrm{SO}(1, n-1)$, then we would have had $M \subset H$). The finite dimensional irreducible representations ξ of M are parametrized by the coset space $S = \mathbf{Z}_2^n/R$, where R is the subgroup generated by $(1, 1, \dots, 1)$. If $\nu \in S$, then the associated representation ξ_ν is defined by $\xi_\nu(d(\epsilon)) = \prod_{j=1}^n \epsilon_j^{\nu_j}$. Hence ξ_ν has a $M \cap H$-fixed vector if and only if $\nu = (0, ..., 0)$ (corresponding to the trivial representation) or $\nu = (1, 0, \dots, 0)$, modulo R. The Weyl group W, which is the permutation group of n entries, is generated by the reflections $s_{ij} (1 \le i < j \le n)$, and it is easily seen that $W_{K \cap H}$ is the subgroup leaving the first entry fixed. Hence the quotient $W/W_{K \cap H}$ has n elements, and as representatives we can take $s_1 = 1, s_2 = s_{12}, \dots, s_n = s_{1n}$. Let $n > 2$. Then $s_j(M \cap H)s_j^{-1}$ is the subgroup of M given by $\epsilon_j = 1$, and hence ξ_ν has a $s_j(M \cap H)s_j^{-1}$-fixed vector if and only if ν is trivial or $\nu = (0, \dots, 1, \dots, 0)$ with 1 on the j'th entry. Thus we get the following multiplicities of $\pi_{\xi,\lambda}$ in $L^2_{\mathrm{mc}}(G/H)$:

$$m_\xi = \begin{cases} n & \text{if } \xi = \xi_{(0,\dots,0)} = 1, \text{ the trivial representation} \\ 1 & \text{if } \xi = \xi_{(0,\dots,1,\dots,0)} \\ 0 & \text{otherwise.} \end{cases}$$

The case of the full fixed point group. If we assume that G is connected linear and that H is the *full* fixed point group G^σ, then the computation of m_ξ simplifies. In fact it was proved in [**'Ol 87**], Cor. 3.5, that ξ has a $M \cap H$-fixed vector if and only if it has a $w(M \cap H)w^{-1}$-fixed vector, for all $w \in W$, and hence the multiplicity of $\pi_{\xi,\nu}$ in $L^2_{\mathrm{mc}}(G/H)$ is either 0 or $\sharp\mathcal{W}$. There is a simple explanation to this result:

LEMMA 7. *Assume that G is linear and let $H = G^\sigma$. Then $w(M \cap H)w^{-1} = M \cap H$ for all $w \in W$.*

PROOF: This is an immediate consequence of the following lemma.

LEMMA 8. *Assume that G is linear. Then every $w \in W$ has a representative $y \in N_K(\mathfrak{a}_\mathrm{q})$ such that*

$$y\sigma(x)y^{-1} = \sigma(yxy^{-1}) \tag{14}$$

for all $x \in M$.

PROOF: As in the proof of Lemma 4 we may reduce to the case that $\dim \mathfrak{a}_\mathrm{q} = 1$, $W_{K \cap H_0} = \{1\}$, $\Sigma = \{-\alpha, \alpha\}$ and $w = s_\alpha$. The ideal $\mathfrak{g}'$ defined in the proof of Lemma 4 is σ-stable, hence we may select a σ-stable complementary ideal $\mathfrak{g}''$. There exists a representative $y \in N_K(\mathfrak{a}_\mathrm{q})$ which centralizes $\mathfrak{g}''$.

Being connected and linear, G has a connected complexification G_c. Let M_{1c} be the centralizer of $\mathfrak{a}_\mathrm{q}$ in G_c. Then it suffices to prove (14) for all $x \in M_{1c}$. Since M_{1c} is connected, it actually suffices to show that $\mathrm{Ad}(y) \circ \sigma = \sigma \circ \mathrm{Ad}(y)$ on $\mathfrak{m}_1$. Now $\mathfrak{m}_1 = \mathfrak{a}_\mathrm{q} \oplus \mathfrak{m}' \oplus \mathfrak{g}''$. On $\mathfrak{a}_\mathrm{q}$

we have $\mathrm{Ad}(y)\circ\sigma = -\mathrm{Ad}(y) = \mathrm{Ad}(y)\circ\sigma$. On $\mathfrak{m}'$ we have $\mathrm{Ad}(y)\circ\sigma = \mathrm{Ad}(y) = \sigma\circ\mathrm{Ad}(y)$, because $\mathfrak{m}' \subset \mathfrak{h}$. Finally, on $\mathfrak{g}''$ we have $\mathrm{Ad}(y)\circ\sigma = \sigma = \sigma\circ\mathrm{Ad}(y)$. ∎

Notice that if G is simply connected, then the full fixed point group is automatically connected.

LEMMA 9. *Let G be a connected, simply connected and semisimple real Lie group, and assume that σ is an involution of G. Then G^σ is connected.*

PROOF: Let θ be a Cartan involution commuting with σ. Then $K = G^\theta$ is σ-invariant. Since $G = K\exp\mathfrak{p}$, K is simply connected. Let K_1 be the semisimple part of K, then K_1 is compact, simply connected and σ-invariant, and $K = K_1 \times C$, with C a σ-invariant vector subgroup. It follows from [**He 78**], Thm VII.8.2, that K_1^σ is connected. Moreover, C^σ is clearly connected (it is a vector subgroup). Hence $G^\sigma = K^\sigma\exp(\mathfrak{h}\cap\mathfrak{p})$ is also connected. ∎

However, the assumption in Lemma 7 that G is linear is important, as can be seen from the following example.

Example 4, continuation. Let $n > 2$ and let $\tilde{G}$ be the universal covering of $\mathrm{SL}(n, \mathbf{R})$, with covering map π. The cover is two fold. Hence $\ker\pi$ is a σ-invariant set consisting of two elements; one of these, the identity element, is fixed under σ, and therefore the other is fixed as well, i.e. $\ker\pi \subset \tilde{G}^\sigma$. It follows from Lemma 9 that $\tilde{H} = \tilde{G}^\sigma$ is connected, hence $\pi(\tilde{H})$ equals $H = \mathrm{SO}_0(1, n-1)$.

Let $\tilde{M} = \pi^{-1}(M)$. Then $\tilde{F} := \pi^{-1}(F) = \tilde{M}$. We will first show that $\tilde{F}$ satisfies condition (11). Indeed, let $x \in \tilde{F}$. Then $\pi(x)$ is σ-fixed, hence $\sigma x = \epsilon x$, for some $\epsilon \in \ker\pi$. Using that the elements of F have order at most two, we obtain that

$$(\sigma x)^{-1} = x x^{-2}\epsilon \in x\ker\pi \subset x(F\cap\tilde{H}).$$

Therefore Theorem 3 applies to the pair $(\tilde{G}, \tilde{H})$. The multiplicities $m_{\tilde{\xi}}$ can be determined as follows. Let $\tilde{\xi} \in \widehat{\tilde{M}}_{\mathrm{fu}}$, and suppose that $V(\tilde{\xi}) \neq 0$. Then there exists a $1 \leq j \leq n$ such that $\tilde{\xi}$ possesses a non-trivial $s_j(\tilde{M}\cap\tilde{H})s_j^{-1}$-fixed vector. Therefore it has a non-trivial $\ker\pi$-fixed vector, and we see that $\tilde{\xi}$ factorizes to a representation $\xi \in \widehat{M}_{\mathrm{fu}}$, possessing a $s_j(M\cap H)s_j^{-1}$-fixed vector. In other words, $\tilde{\xi} = \xi_\nu\circ\pi$ for some $\nu \in S$. Since $\pi(\tilde{M}\cap\tilde{H}) = M\cap H$, it follows that $m_{\tilde{\xi}}$ equals the earlier determined multiplicity m_{ξ_ν}. Thus both multiplicities 1 and n occur in $L^2_{\mathrm{mc}}(\tilde{G}/\tilde{H})$, and we see that linearity of G is essential for the conclusion of Lemma 7.

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