

Interaction between Two Independent Recurrent Time Series

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Two mutually independent recurrent processes, each consisting of a time series of events, are considered. The durations of the intervals between the events in each series are independent of each other and identically distributed with a probability density function $\varphi(t)$ and $\psi(t)$. Every event of the $\psi(t)$ process annihilates the next event of the $\varphi(t)$ process. The probability density function of the intervals of the thus transformed $\varphi(t)$ process is derived, and possible fields of application are mentioned. The $\varphi(t)$ or the $\psi(t)$ process, being Poisson, are treated as special cases.

I. INTRODUCTION

In a previous paper (Ten Hoopen and Reuver, 1965) the following problem was posed. Consider two mutually independent processes, each consisting of a time series of events. The durations of the intervals between the events in each series are independent of each other and identically distributed with a probability density function $\varphi(t)$ and $\psi(t)$, respectively. Every event of the $\psi(t)$ process annihilates the next event of the $\varphi(t)$ process with the annotation that if there are two (or more) $\psi(t)$ events without a $\varphi(t)$ event only one subsequent $\varphi(t)$ event is deleted. The probability density function $p(t)$ of the intervals of the thus transformed $\varphi(t)$ process was derived for either the $\varphi(t)$ or the $\psi(t)$ process, being Poisson.

This problem was raised in pursuance of a possible interpretation of unitary discharge patterns of single neurons in the central nervous system, first suggested by Bishop *et al.* (1964) for geniculate neuron activity and worked out in some detail for other neurons by Ten Hoopen (1966). However, with the just mentioned mode of interaction in mind it is conceivable that there are applications to other classes of problems. In general, one can think of a stream of events in time, defined completely by the time interval distribution thereof, $\varphi(t)$, and which are to arrive

at a certain point or goal, but being distorted by other events with an interval distribution $\psi(t)$. This distortion may stand for an intentional obstruction or may be caused by a natural and neutral source. As such it recalls a comparison with hit- and antihit-problems in game theory and the military sciences, prey and predator systems in ecology, pathogenic germs and antibodies in epidemiology, distorted communication channels, etc.

For example, in terms of an attacker φ and a defender ψ with strategies characterized by a $\varphi(t)$ and a $\psi(t)$ process one may ask, given $\psi(t)$, what $\varphi(t)$ optimizes $p(t)$ according to certain criteria with the constraining condition that $\int_0^\infty t\varphi(t) dt$ equals a constant. A possible criterion might be that $\int_0^\infty tp(t) dt$ is minimal, which is for φ a direct measure of success for coming through a barrier caused by ψ . Equally, given $\varphi(t)$ and $\int_0^\infty t\psi(t) dt$, what $\psi(t)$ maximizes $\int_0^\infty tp(t) dt$ which is now a measure for success of the defense's part. Most of these problems can be solved by approximating the function being sought by orthogonal functions, and optimizing the generalized Fourier coefficients by the method of Lagrange multipliers, similar to the direct methods of the calculus of variations.

In this communication we present an expression for $p(t)$ with the only restriction that not both the $\varphi(t)$ and the $\psi(t)$ processes are periodic events with a rational ratio of the periods. However, this particular case is trivial. As special cases we shall insert in the general formulas for the $\varphi(t)$ or the $\psi(t)$ process a Poisson process, and arrive at expressions for $p(t)$ that were previously obtained separately along two quite different lines of reasoning. The solution for $p(t)$ can be expressed in a compact form by introducing the Laplace transform method, the Laplace transform of a function being designated by the corresponding capitals with argument s , e.g., $P(s)$.

II. GENERAL CASE

Let an undeleted $\varphi(t)$ event have occurred at $t = 0$. If $p_k(t) dt$ denotes the probability that the next undeleted $\varphi(t)$ event after $t = 0$ occurs in $(t, t + dt)$, while k $\psi(t)$ events have occurred in between, then $p(t) = \sum_{k=0}^{\infty} p_k(t)$.

Let $\chi_k(t, \tau) dt d\tau$ denote the probability that the k th $\psi(t)$ event after the last undeleted $\varphi(t)$ event at $t = 0$ occurs in $(t, t + dt)$, and that the last $\varphi(t)$ event before that k th $\psi(t)$ event and after $t = 0$ occurs in $(\tau, \tau + d\tau)$ divided by the probability that the interval between two

$\varphi(t)$ events is larger than $t - \tau$. Then $\chi_1(t, \tau) = 0$. Let $\chi_k(t) dt$ denote the probability that the k th $\psi(t)$ event after $t = 0$ is in $(t, t + dt)$.

To compute $\chi_k(t, \tau)$ for $k \geq 2$ we note that $\chi_k(t, \tau) dt d\tau$ equals the sum of:

(A) the integral of the product of the probabilities that

(a) the $(k - 1)$ th $\psi(t)$ event after the last undeleted $\varphi(t)$ event at $t = 0$ occurs in $(v, v + dv)$ while the last $\varphi(t)$ event before that $(k - 1)$ th $\psi(t)$ event occurs in $(u, u + du)$ with $0 < u < v < \tau$,

(b) the first $\varphi(t)$ event after that $(k - 1)$ th $\psi(t)$ event occurs in $(\tau, \tau + d\tau)$, and

(c) the k th $\psi(t)$ event occurs in $(t, t + dt)$.

(B) the integral of the product of the probabilities that

(a) the $(k - 1)$ th $\psi(t)$ event after $t = 0$ occurs in $(v, v + dv)$ with $0 < v < \tau$,

(b) the first $\varphi(t)$ event after $t = 0$ occurs in $(\tau, \tau + d\tau)$, and

(c) the k th $\psi(t)$ events occurs in $(t, t + dt)$.

(C) the integral of the product of the probabilities that

(a) the $(k - 1)$ th $\psi(t)$ event after the last undeleted $\varphi(t)$ event at $t = 0$ occurs in $(v, v + dv)$ with $\tau < v < t$ while the last $\varphi(t)$ event occurs in $(\tau, \tau + d\tau)$, and

(b) the k th $\psi(t)$ event occurs in $(t, t + dt)$.

It follows that

$$\begin{aligned} \chi_k(t, \tau) = & \iint_{\tau > v > u > 0} \psi(\tau - v) \varphi(\tau - u) \chi_{k-1}(v, u) du dv \\ & + \int_0^\tau \psi(t - v) \varphi(\tau) \chi_{k-1}(v) dv + \int_\tau^t \psi(t - v) \chi_{k-1}(v, \tau) dv. \end{aligned}$$

To compute $\chi_1(t)$ we argue as follows. For a $\varphi(t)$ event and a $\psi(t)$ event to be respectively an undeleted $\varphi(t)$ event and the first $\psi(t)$ event after that undeleted $\varphi(t)$ event, it is necessary and sufficient that the preceding $\psi(t)$ event occur before that $\varphi(t)$ event while in the interval between that preceding $\psi(t)$ event and that $\varphi(t)$ event one or more $\varphi(t)$ events have occurred.

It follows that

$$\chi_1(t) = \gamma \int_t^\infty \psi(v) \int_0^{v-t} \varphi(w) dw dv$$

where γ is a constant given by the condition $\int_0^\infty \chi_1(t) dt = 1$. To compute $\chi_k(t)$ for $k \geq 2$ we note that $\chi_k(t)$ is the convolution of $\chi_{k-1}(t)$ and $\psi(t)$. Or,

$$\chi_k(t) = \int_0^t \chi_{k-1}(v) \psi(t-v) dv.$$

To compute $p_0(t)$ we note that $p_0(t) dt$ equals the product of the probabilities that (a) the first $\varphi(t)$ event occurs in $(t, t+dt)$ and (b) the first $\psi(t)$ event does not occur in $(0, t)$.

To compute $p_k(t)$ for $k \geq 1$ we note that $p_k(t) dt$ equals the sum of:

(A) the integral of the product of the probabilities that

(a) the k th $\psi(t)$ event after the last undeleted $\varphi(t)$ event at $t=0$ occurs in $(v, v+dv)$ while the last $\varphi(t)$ event before that k th $\psi(t)$ event occurs in $(u, u+du)$ with $0 < u < v < w$,

(b) the first $\varphi(t)$ event after that k th $\psi(t)$ event occurs in $(w, w+dw)$ with $v < w < t$,

(c) the second $\varphi(t)$ event occurs in $(t, t+dt)$, and

(d) the $(k-1)$ th $\psi(t)$ event does not occur in (v, t) .

(B) the integral of the product of the probabilities that

(a) the k th $\psi(t)$ event after $t=0$ occurs in $(v, v+dv)$ with $0 < v < w$,

(b) the first $\varphi(t)$ event after $t=0$ occurs in $(w, w+dw)$ with $v < w < t$,

(c) the second $\varphi(t)$ event occurs in $(t, t+dt)$, and

(d) the $(k+1)$ th $\psi(t)$ event does not occur in (v, t) .

Therefore

$$p_0(t) = \varphi(t) \int_t^\infty \chi_1(v) dv$$

and

$$\begin{aligned} p_k(t) = & \iiint_{t > w > v > u > 0} \omega(t-v) \varphi(t-w) \varphi(w-u) \chi_k(v, u) du dv dw \\ & + \iint_{t > w > v > 0} \omega(t-v) \varphi(t-w) \varphi(w) \chi_k(v) dv dw \end{aligned}$$

where

$$\omega(t-v) = \int_{t-v}^\infty \psi(w) dw.$$

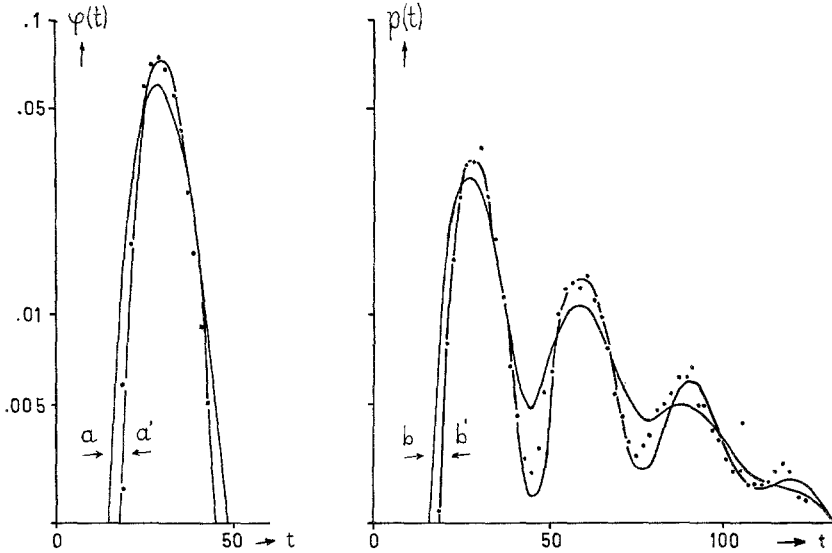


FIG. 1. For explanation see text. Abscissas in msec; ordinates in msec⁻¹

III. POISSON PROCESS DELETING

$$\begin{aligned}
 p(t) - p_0(t) - p_1(t) &= \sum_{k=2}^{\infty} p_k(t) \\
 &= \iiint_{t>w>v>u>0} \omega(t-v)\varphi(t-w)\varphi(w-u) \\
 &\quad \cdot \sum_{k=2}^{\infty} \chi_k(v, u) du dv dw \\
 &\quad + \iint_{t>w>v>0} \omega(t-v)\varphi(t-w)\varphi(w) \sum_{k=2}^{\infty} \chi_k(v) dv dw.
 \end{aligned}$$

With $\psi(t) = \mu e^{-\mu t}$ one has $\omega(t) = e^{-\mu t}$, $\chi_k(t) = \mu^k t^{k-1} e^{-\mu t} / (k-1)!$ for $k \geq 1$, and $\sum_{k=2}^{\infty} \chi_k(t) = \mu(1 - e^{-\mu t})$. The second term of the right side of this expression equals

$$e^{-\mu t} \iint_{t>w>v>0} e^{\mu v} \varphi(t-w)\varphi(w) \mu(1 - e^{-\mu v}) dv dw.$$

As to the first term we note that $\sum_{k=2}^{\infty} \chi_k(t, \tau)$ is independent of t and a function of τ only, denoted by $\chi(\tau)$ hereafter. This follows from

$$\frac{\partial}{\partial t} \sum_{k=2}^{\infty} \chi_k(t, \tau) = 0$$

and from the definition of $\chi_k(t, \tau)$.¹

From the recurrence relation for $\chi_k(t, \tau)$ it can be derived that

$$\begin{aligned} \chi(\tau) &= \int_0^\tau \chi(u) \varphi(\tau - u) du \\ &\quad - e^{-\mu\tau} \int_0^\tau e^{\mu v} \chi(v) \varphi(t - v) dv + \mu \varphi(\tau) (1 - e^{-\mu\tau}). \end{aligned}$$

With $p_0(t) = \varphi(t)e^{-\mu t}$ and

$$p_1(t) = \mu e^{-\mu t} \iint_{t > w > v > 0} \varphi(t - w) \varphi(w) dv dw$$

one has

$$\begin{aligned} p(t) &= \varphi(t)e^{-\mu t} + e^{-\mu t} \int_0^t (e^{\mu w} - 1) \varphi(w) \varphi(t - w) dw \\ &\quad + \frac{e^{-\mu t}}{\mu} \iint_{t > w > u > 0} (e^{\mu w} - e^{\mu u}) \varphi(t - w) \varphi(w - u) \chi(u) du dw. \end{aligned}$$

From

$$X(s) = X(s)\Phi(s) - X(s)\Phi(s + \mu) + \mu\Phi(s) - \mu\Phi(s + \mu)$$

one has

$$P(s) = \Phi(s + \mu) / \{1 - \Phi(s) + \Phi(s + \mu)\}.$$

IV. POISSON PROCESS DELETED

With $\varphi(t) = \mu e^{-\mu t}$ one has

$$p_0(t) = \mu e^{-\mu t} \int_t^\infty \chi_1(v) dv$$

and for $k \geq 1$

¹ $\sum_{k=2}^{\infty} \chi_k(t, \tau) dt d\tau$ is the probability that a $\psi(t)$ event occurs in $(t, t + dt)$, while the last $\varphi(t)$ event before t occurs in $(\tau, \tau + d\tau)$ and no undeleted $\varphi(t)$ event occurred after $t = 0$, divided by the probability that no $\varphi(t)$ event occurs in (τ, t) . Since the $\psi(t)$ process is a Poisson process the probability that a $\psi(t)$ event occurs in $(t, t + dt)$ is independent of t and thus $\sum_{k=2}^{\infty} \chi_k(t, \tau)$ is only dependent on τ .

$$\begin{aligned}
p_k(t) &= \mu^2 e^{-\mu t} \iiint_{t>w>v>u>0} \omega(t-v) e^{\mu u} \chi_k(v, u) du dv dw \\
&\quad + \mu^2 e^{-\mu t} \iint_{t>w>v>0} \omega(t-v) \chi_k(v) dv dw \\
&= \mu^2 t e^{-\mu t} \int_0^t \omega(t-v) \left\{ \int_0^v e^{\mu u} \chi_k(v, u) du + \chi_k(v) \right\} dv \\
&\quad - \mu^2 e^{-\mu t} \int_0^t w \omega(t-w) \left\{ \int_0^w e^{\mu u} \chi_k(w, u) du + \chi_k(w) \right\} dw.
\end{aligned}$$

For $k \geq 2$

$$\begin{aligned}
\nu_k(t) &\equiv \int_0^t e^{\mu \tau} \chi_k(t, \tau) d\tau + \chi_k(t) \\
&= \mu \iiint_{t>\tau>v>u>0} \psi(t-v) e^{\mu u} \chi_{k-1}(v, u) du dv d\tau \\
&\quad + \mu \iint_{t>\tau>v>0} \psi(t-v) \chi_{k-1}(v) dv d\tau \\
&\quad + \iint_{t>v>\tau>0} e^{\mu \tau} (t-v) \chi_{k-1}(v, \tau) d\tau dv + \int_0^t \psi(t-v) \chi_{k-1}(v) dv \\
&= \mu \left[\tau \int_0^\tau \psi(t-v) \int_0^v e^{\mu u} \chi_{k-1}(v, u) du dv \right]_0^t \\
&\quad - \mu \int_0^t \tau \psi(t-\tau) \int_0^\tau e^{\mu u} \chi_k(\tau, u) du d\tau \\
&\quad + \mu \left[\tau \int_0^\tau \psi(t-v) \chi_{k-1}(v) dv \right]_0^t \\
&\quad - \mu \int_0^t \tau \psi(t-\tau) \chi_{k-1}(\tau) d\tau \\
&\quad + \int_0^t \psi(t-v) \int_0^v e^{\mu \tau} \chi_{k-1}(v, \tau) d\tau dv \\
&\quad + \int_0^v \psi(t-v) \chi_{k-1}(v) dv.
\end{aligned}$$

For $k \geq 2$ one finds after elaboration

$$N_k(s) = \{\Psi(s) - \mu \dot{\Psi}(s)\} N_{k-1}(s) \quad \text{with} \quad \dot{\Psi}(s) = \frac{d}{ds} \Psi(s).$$

From $\chi_1(t, \tau) = 0$ one has $N_1(s) = X_1(s)$. As

$$\chi_1(t) = \mu \left[\int_t^\infty \psi(v) dv - e^{\mu t} \int_t^\infty e^{-\mu v} \psi(v) dv \right] / \gamma$$

one has

$$N_1(s) = \mu \{ [1 - \Psi(s)]/s - [\Psi(\mu) - \Psi(s)]/(s - \mu) \} / \gamma$$

with

$$\gamma = -1 + \Psi(\mu) + \mu \int_0^\infty t\psi(t) dt,$$

$$\Omega(s) = \{1 - \Psi(s)\}/s,$$

$$P_0(s) = \mu \{1 - N_1(s + \mu)\}/(s + \mu),$$

$$P_k(s) = -\mu^2 \{ \Omega(s + \mu) \dot{N}_k(s + \mu) + \dot{\Omega}(s + \mu) N_k(s + \mu) \} \\ + \mu^2 \Omega(s + \mu) \dot{N}_k(s + \mu).$$

It follows that

$$P(s) = \mu/(s + \mu) - \mu N_1(s + \mu)/(s + \mu) \\ - \{ \mu^2 \dot{\Omega}(s + \mu) N_1(s + \mu) \} / \{ 1 - \Psi(s + \mu) + \mu \dot{\Psi}(s + \mu) \} \\ = \mu/(s + \mu) + \mu^2 / \{ \gamma(s + \mu)^3 \} \\ \times \{ 1 - \Psi(s + \mu) \} \{ -s + s\Psi(\mu) - \mu\Psi(s + \tau) + \mu\Psi(\mu) \} / \\ \{ 1 - \Psi(s + \mu) + \mu \dot{\Psi}(s + \mu) \}.$$

RECEIVED: October 14, 1965

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APPENDIX

POISSON PROCESS DELETING—AN EXAMPLE

In this section we shall analyze simulation results published by Bishop *et al.* (1964) and carried out electronically. These authors have deleted a series of pulses interspaced according to a gamma distribution by

means of Poisson distributed pulses. In our nomenclature this situation amounts to the special case

$$\psi(t) = \mu \exp(-\mu t) \quad \text{and} \quad \varphi(t) = \lambda(\lambda t)^{n-1} \exp(-\lambda t)/(n-1)!$$

The undeleted process, with $n = 20$, consisted of 51,523 intervals in 1530 seconds and the deleted process of 23,084 intervals in the same time span; the mean interval durations therefore equal 29.7 msec and 66.3 msec. The two interval histograms are given as points in Fig. 1, and were computed from Fig. 7 of Bishop *et al.* (1964). Elsewhere (Ten Hoopen and Reuver, 1965) we have found, by evaluating the mean interval durations, which are respectively equal to n/λ and $(n/\lambda)(1 + \mu/\lambda)^n$, that $\mu = 0.027 \text{ msec}^{-1}$ and $\lambda = 0.67 \text{ msec}^{-1}$. The corresponding interval probability density functions are redrawn in Fig. 1 as curves *a* and *b*. The agreement is fairly good except that interval histogram of Bishop *et al.* for the gamma process is slightly narrower than ours, and consequently the maxima and minima for the deleted process are more pronounced. At the former occasion we have suggested the discrepancy to be due to errors in the Poisson process, from which the gamma distribution was obtained by frequency division. Although only of academic interest, at least in this particular example, we shall take the opportunity to examine quantitatively another source of error, to mention the frequency division procedure. Suppose that in the course of counting, utilized in this method of generating a gamma distribution, due to finite reset times of the electronic equipment a dead time of duration δ occurs after the arrival of a Poisson pulse such that during these periods other incoming Poisson pulses are neglected and not counted. Then, $\varphi(t)$ being the n -fold convolution of an exponential distribution with dead time δ

$$\begin{aligned} \varphi(t) &= 0 \text{ for } 0 < t \leq n\delta \\ &= \lambda' \{\lambda'(t - n\delta)\}^{n-1} \exp\{-\lambda'(t - n\delta)\}/(n-1)! \quad \text{for } t > n\delta. \end{aligned}$$

The mean interval duration of the undeleted series equals $n\delta + n/\lambda' = 29.7 \text{ msec}$. For $n = 20$, $n\delta = 5.9 \text{ msec}$, and thus $\lambda' = 0.84 \text{ msec}^{-1}$, an excellent fit is obtained (cf. curve *a'*). From

$$\begin{aligned} \Phi(s) &= (\lambda')^n (s + \lambda')^{-n} \exp(-n\delta s) \quad \text{and} \\ P(s) &= \Phi(s + \mu') \{1 + \Phi(s + \mu') - \Phi(s)\}^{-1} \end{aligned}$$

one finds after differentiation of $P(s)$ with respect to s for the mean

interval duration of the deleted series

$$(n/\lambda' + n\delta)(1 + \mu'/\lambda')^n \exp(n\delta\mu').$$

Setting this expression equal to 66.3 msec yields a value for μ' that differs from μ by less than 1%. The probability density function of the deleted process for the thus computed parameter values proves to give a notably better agreement with the experimental results (cf. curve b').