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THE DISPERSION EQUATION FOR PLASMA WAVES

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Synopsis

Different opinions concerning the correct form – and the very existence – of the dispersion equation for waves in plasmas have appeared in the literature. The origin of these differences is here discussed.

1. *Introduction.* The simplest kind of plasma is a gas consisting of classical, nonrelativistic particles with a constant neutralising background and no external fields. It is well-known that in such a plasma plane waves may exist with frequency ω and wave vector $\mathbf{k}$, but even for this simple model several different dispersion formulae have been given. The following results occur in the litterature.

(i) $\omega^2 = \omega_P^2 + (\kappa T/m)k^2$ [Thomson [1]), Bailey [2])]

(ii) $\omega^2 = \omega_P^2 + \frac{5}{3}(\kappa T/m)k^2$ [Gross [3])]

(iii) $\omega^2 = \omega_P^2 + 3(\kappa T/m)k^2 + \ldots$ [Bohm and Gross [4])]

(iv) no dispersion formula exists [5]) [6]) [7]).

Here ω_P denotes the plasma frequency $(4\pi e^2 n_0/m)^{\frac{1}{2}}$, m and e are the mass and the charge of the particles, n_0 their average number per unit volume. κ is Boltzmann's constant and T the equilibrium temperature, so that $\kappa T/m$ is the mean square of each velocity component.

The object of this note is to analyse the causes for these discrepancies. Our conclusion will be that (i) is incorrect, and that (ii) applies to the case where collisions are frequent enough to insure that local equilibrium is established. If, on the other hand, collisions are very rare, then strictly speaking, (iv) is correct, but (iii) is applicable in certain circumstances.

2. *The basic equation.* Boltzmann's transport equation for the particle distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is in our case

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} + \frac{e}{m}\mathbf{E}\frac{\partial f}{\partial \mathbf{v}} = \left[\frac{\partial f}{\partial t}\right]_{\text{collisions}}. \qquad (1)$$

The average electric field strength $\mathbf{E}(\mathbf{r}, t)$ is determined by

$$\operatorname{div} \mathbf{E} = 4\pi e \int \{f(\mathbf{r}, \mathbf{v}, t) - f_0(\mathbf{v})\}\, d\mathbf{v} = 4\pi e\,(n - n_0), \qquad (2)$$

where $f_0(\mathbf{v})$ is the equilibrium distribution, and $n(\mathbf{r}, t)$ the particle density.

The collision term on the right-hand side of (1) is a combination of several contributions. Firstly, there are the collisions with the heavy particles of the neutralising background, and with any uncharged particles that may be present. We shall neglect these collisions altogether. Secondly, if the plasma particles have a short-range interaction in addition to the electrostatic forces, these also contribute to the collision term. Thirdly, the total electrostatic force on each particle consists of the averaged field given by (2), plus rapid fluctuations depending on the precise position of the neighbouring particles; these local deviations from the average field **E** are also incorporated in the collision term.

3. *The limiting cases.* In principle, it is possible to write all these contributions explicitly, so that a definite expression is obtained for the right-hand side of (1). However, that would not only require a detailed knowledge of the collision mechanism, it would also give rise to an integro-differential equation, which would be hard to solve. Fortunately there are two limiting cases in which the collision term may be dealt with in a much easier way.

The first limiting case ('hydrodynamical case') is the one in which the collisions are sufficiently frequent to establish a local equilibrium, *i.e.*, in each small volume element the velocities are distributed according to Maxwell's law. Then the behaviour of the plasma can be described by macroscopic hydrodynamical equations, or, what amounts to the same, by Maxwell's transfer equations. This method was used by the Thomsons [1]) and by Bailey [2]) to derive plasma waves. For its validity the mean free path must be short compared to the wave length of the plasma wave, and the mean time of free flight must be short compared to the period of the plasma wave. We shall study this case in sections 4–7.

The other limiting case ('Vlasov case') is the one in which the collisions are so unimportant that the right-hand side of (1) may be omitted altogether *). This case was first treated by Vlasov [8]), and later by Bohm and Gross [4]) and others. It turns out that for low temperature the dispersion formula is then approximately (iii). However, there is a difficulty in the theory, which gave rise to the opinion (iv). The state of affairs is discussed in sections 8–11.

The two treatments are sometimes [9]) referred to as 'transport method' and 'Boltzmann method' respectively. These names are somewhat confusing, as both methods are based on Boltzmann's transport equation (i). Moreover – to summarise this section – *they are not alternative treatments of the same problem, but two different methods adapted to two diametrically opposite physical situations.*

4. *The transfer equations.* In this section we confine ourselves to one dimension, both for simplicity and to make a more direct comparison with

*) For an investigation on the applicability of this assumption to gas discharges see E. A. Ash and D. Gabor, Proc. roy. Soc. (A) **228** (1955) 477.

the Thomson treatment possible. One obtains from the one-dimensional analogue of (1), by integrating with respect to v, the continuity equation

$$\partial n/\partial t + \partial n\bar{v}/\partial x = 0. \tag{3a}$$

The right-hand side vanishes, as the collisions cannot alter the particle density. Similarly one finds the conservation of momentum

$$\partial n\bar{v}/\partial t + \partial n\overline{v^2}/\partial x - (e/m)\, En = 0, \tag{3b}$$

and the conservation of energy

$$\partial n\overline{v^2}/\partial t + \partial n\overline{v^3}/\partial x - (e/m)\, En\bar{v} = 0. \tag{3c}$$

These are the well-known Maxwell equations of transfer [10]). The transfer equations for higher powers of v are of no use to us, because their right-hand side no longer vanishes. Hence we are left with three equations for the unknown functions n, $\bar{v}$, $\overline{v^2}$, and $\overline{v^3}$ of x, t; E being determined by

$$\partial E/\partial x = 4\pi e(n - n_0).$$

Now suppose that the collisions are sufficiently frequent to ensure that locally a Maxwell velocity distribution is established:

$$f(x, v, t) = n(2\pi\Theta)^{-\frac{1}{2}} \exp\left[-(v - u)^2/2\Theta\right].$$

This distribution function contains three unknown parameters n, u, Θ depending on x and t. All averages of powers of v now only depend on these parameters, in particular *).

$$\bar{v} = u, \quad \overline{v^2} = u^2 + \Theta, \quad \overline{v^3} = u^3 + 3u\Theta.$$

On substituting this in the transfer equations one obtains three equations for the three unknowns $n(x, t)$, $u(x, t)$, $\Theta(x, t)$. Thus, *owing to the additional assumption of local equilibrium, the transfer equations are sufficient to describe the behaviour of the plasma.* In fact, we have only given the standard kinetic derivation of the laws of hydrodynamics, supplemented with an electrostatic term.

5. *Solution of the linearised equation.* In order to find an explicit solution in the form of a plane wave, it is of course necessary to confine oneself to the linear approximation. Hence we put $n = n_0 + n_1$, $\Theta = \Theta_0 + \Theta_1 = (\kappa T/m) + \Theta_1$ and retain only first powers of n_1, Θ_1, u. **)

$$\partial n_1/\partial t + n_0\, \partial u/\partial x = 0, \tag{5a}$$

$$n_0\, \partial u/\partial t + (\partial/\partial x)\,(n_1\Theta_0 + n_0\Theta_1) = (e/m)\, n_0 E, \tag{5b}$$

$$(\partial/\partial t)(n_1\Theta_0 + n_0\Theta_1) + 3n_0\Theta_0\, \partial u/\partial x = 0. \tag{5c}$$

*) The precise form of the velocity distribution is immaterial. We only use the fact that it is a function of $v - u \equiv \Delta v$ with $\overline{\Delta v} = 0$, $\overline{(\Delta v)^2} = \Theta$, $\overline{(\Delta v)^3} = 0$.

**) More precisely: it is sufficient to neglect second powers of n_1, the third power of u, and the products $n_1 u$ and $\Theta_1 u$.

By eliminating $\partial u/\partial x$ from (5*a*) and (5*c*) one finds one integral of the motion

$$(n_1\Theta_0 + n_0\Theta_1) - 3\Theta_0 n_1 = \text{constant}. \tag{6}$$

Inserting this into (5*b*), eliminating u by means of (5*a*) and E by means of (2), one obtains an equation for n_1 alone,

$$-\frac{\partial^2 n_1}{\partial t^2} + 3\Theta_0 \frac{\partial^2 n_1}{\partial x^2} = \frac{4\pi e^2}{m} n_0 n_1.$$

This wave equation clearly corresponds to the dispersion formula

$$\omega^2 - 3\Theta_0 k^2 = \omega_P^2$$

This is identical to the result (iii) of Bohm and Gross (disregarding the dots in (iii)!), but that is only a coincidence. The derivation of Bohm and Gross applies to the opposite case, in which collisions may be neglected; moreover, as will be shown in section 7, the numerical factor 3 is due to the one-dimensional treatment and must be replaced by 5/3 in the three-dimensional case.

6. *The dispersion formula* (i). In this section we shall show why the Thomsons found the different equation (i), although they also confined themselves to one dimension. They actually did not use (5*c*), but only (5*a*) and (5*b*). In order to get rid of the third unknown they simply put $\Theta_1 = 0$. One then finds easily by eliminating u and E as before,

$$-\frac{\partial^2 n_1}{\partial t^2} + \Theta_0 \frac{\partial^2 n_1}{\partial x^2} = \frac{4\pi e^2}{m} n_0 n_1.$$

This wave equation leads to the dispersion equation (i). Although Bailey treats a much more complicated case, it can be seen that his approximation amounts to the same.

However, from (6) follows (as the constant must be zero)

$$\Theta_1/\Theta_0 = 2(n_1/n_0), \tag{7}$$

which shows that it is inconsistent to neglect the temperature variation when computing the density variations. In fact, (7) is the differential form of the law of adiabatic compression of a one-dimensional ideal gas,

$$\Theta n^2 = \text{constant}.$$

Hence *Thomson's approximation amounts to treating the local compressions as isothermal instead of adiabatic.*

Of course, our assumption that $\overline{v^3}$ may be replaced by its equilibrium value (4) cannot be proved to be correct either, unless the total Boltzmann equation (1) is solved completely. Yet there is an essential difference between the two assumptions. As the collisions do not influence $\Theta = \overline{v^2}$, they cannot produce a tendency for Θ to return to some equilibrium value. On the other hand, $\overline{v^3}$ is not conserved, and may therefore be expected to assume its local equilibrium value, provided that the collisions are sufficiently frequent *). Hence, for a consistent approximation it is necessary to use all three transfer equations (5). That is of course equivalent with using the zeroth order approximation of the Chapman-Enskog expansion. For the higher order approximations it would be necessary to consider the collision mechanism in more detail, which will not be done here.

7. *The dispersion formula* (ii). In order to understand the origin of (ii) it is necessary to write the conservation laws in three dimensions. From (1) they are found to be (in slightly abbreviated notation, summation over repeated subscripts implied)

$$\partial_t n + \partial_\alpha(n\bar{v}_\alpha) = 0, \tag{8a}$$

$$\partial_t(n\bar{v}_\alpha) + \partial_\beta(n\overline{v_\alpha v_\beta}) - (e/m)\, E_\alpha n = 0, \tag{8b}$$

$$\partial_t(n\overline{v_\alpha^2}) + \partial_\gamma(n\overline{v_\alpha^2 v_\gamma}) - (2e/m)\, E_\alpha \overline{v_\alpha} n = 0. \tag{8c}$$

This is a set of five equations for the five unknown functions n, u_α, Θ. The other averages occurring in (8) must be expressed in terms of u_α and Θ; one finds

$$\overline{v_\alpha v_\beta} = u_\alpha u_\beta + \Theta\, \delta_{\alpha\beta},$$

$$\overline{v_\alpha^2 v_\gamma} = u^2 u_\gamma + 5u_\gamma \Theta.$$

Hence (5) becomes in linear approximation

$$\partial_t n_1 + n_0 \partial_\alpha u_\alpha = 0, \tag{9a}$$

$$n_0 \partial_t u_\alpha + \partial_\alpha(n_1\Theta_0 + n_0\Theta_1) = (e/m)\, n_0 E_\alpha, \tag{9b}$$

$$3\partial_t(n_1\Theta_0 + n_0\Theta_1) + 5n_0\Theta_0\partial_\gamma u_\gamma = 0. \tag{9c}$$

From (9*a*) and (9*b*) one now finds the integral

$$3(n_1\Theta_0 + n_0\Theta_1) - 5\Theta_0 n_1 = \text{constant} = 0, \tag{10}$$

or

$$\Theta = \text{const.}\ n^{-\frac{2}{3}}.$$

This is the law for adiabatic compression of an ideal gas; it here appears that this law holds true for the small density fuctuations of a plasma, even in the presence of electrostatic interaction.

*) One might think that there may exist plasmas in which the collisions *with the heavy particles of the background* are sufficiently frequent to keep the temperature constant. However, in that case the average velocity u would also be kept constant at zero value, so that the plasma is firmly anchored to the background, and no plasma waves can occur.

Inserting (10) into (9*b*) and using (9*a*) and (2) one finds the wave equation

$$-\partial_t^2 n_1 + \tfrac{5}{3}\,\Theta_0 \partial_\alpha^2 n_1 = (4\pi e^2 n_0/m)\, n_1,$$

which yields the dispersion formula (ii). It is not surprising that the numerical constant in front of Θ_0 depends on the number of dimensions, because it is related to the number of degrees of freedom that share the local energy increase *).

8. *The Vlasov case.* So far we have studied the hydrodynamical case. Now consider the opposite extreme, in which the collision term in (1) may be taken zero. Then (1) reduces to an integro-differential equation for $f(\mathbf{r}, \mathbf{v}, t)$ of a sufficiently simple type to solve explicitly in linear approximation. This has been done by many authors [8] [4] [5] [6] [7], but as their methods of solution differ considerably, different point of view have arisen. Actually, of course, all methods ought to lead to the same result, provided they are applied correctly.

Putting $f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t)$ and dropping second and higher powers of f_1 one may write (1) in the form

$$\frac{\partial f_1}{\partial t} = -\,\mathbf{v}\frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m}\frac{\partial f_0}{\partial \mathbf{v}}\,\mathbf{E} \equiv -\,i\Omega f_1. \tag{11}$$

Ω is a linear operator acting on f_1, since $\mathbf{E}$ stands for

$$\mathbf{E}(\mathbf{r}, t) = e\iint \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\, f_1(\mathbf{r}', \mathbf{v}', t)\, \mathrm{d}\mathbf{r}'\, \mathrm{d}\mathbf{v}'.$$

The factor i is added to make Ω hermitean. In the case of an equation of type (11) one is usually interested in the initial-value problem: to find $f_1(\mathbf{r}, \mathbf{v}, t)$ when $f_1(\mathbf{r}, \mathbf{v}, 0)$ is given. The initial-value problem for (11) has actually been solved by Landau [5] by means of a Fourier transformation with respect to $\mathbf{r}$, and a Laplace transformation with respect to t. Twiss [11] treated a more complicated case using Laplace transforms both with respect to $\mathbf{r}$ and t.

An alternative method consists in looking for stationary solutions of (11), *i.e.*, solutions of the form

$$f_1(\mathbf{r}, \mathbf{v}, t) = f_{1\omega}(\mathbf{r}, \mathbf{v})\, \mathrm{e}^{-i\omega t}.$$

This amounts to solving the eigenvalue problem

$$\omega f_{1\omega} = \Omega f_{1\omega}.$$

In order to solve the initial-value problem it is necessary to find a complete

*) This difference between one and three dimensions does not occur in the isothermal approximation; that is why Bailey's three-dimensional treatment yielded the same equation (i) that the Thomsons had found for one dimension.

set of eigenfunctions $f_{1\omega}$, so that an arbitrary $f_1(\mathbf{r}, \mathbf{v}, 0)$ can be written as a superposition of them,

$$f_1(\mathbf{r}, \mathbf{v}, 0) = \sum_\omega c(\omega)\, f_{1\omega}(\mathbf{r}, \mathbf{v}).$$

Then the solution of (11) with given initial value $f_1(\mathbf{r}, \mathbf{v}, 0)$ is

$$f_1(\mathbf{r}, \mathbf{v}, t) = \sum_\omega c(\omega)\, f_{1\omega}(\mathbf{r}, \mathbf{v})\, e^{-i\omega t}. \tag{12}$$

However, the search for stationary solutions is of more importance than merely as a device for solving the initial-value problem. For, what is observed experimentally, is not the evolution of a plasma whose initial state is known, but periodic waves [12]). (The reason for this is that the time variation of f_1 is so rapid that it can only be observed when it manifests itself in the form of a periodic motion with sufficiently long lifetime.) From this point of view one is not interested in obtaining a complete set of stationary solutions, but only those stationary solutions that satisfy certain conditions of physical observability. In the next section we show that these two points of view correspond to (iv) and (iii); *the waves that are directly observable obey the dispersion equation* (iii), *whereas for the complete set of stationary solutions, which is needed for solving the initial-value problem, no dispersion equation exists.*

9. *Stationary solutions.* In the following sections the method of stationary solutions is discussed; for actual calculations we must refer to a previous publication [6]).

Ω commutes with the displacement operator $-i\partial/\partial\mathbf{r}$. Hence it suffices to look for eigenfunctions of the form

$$f_{1\omega}(\mathbf{r}, \mathbf{v}) = g(\mathbf{v})\, e^{i\mathbf{kr}}.$$

Complex values of $\mathbf{k}$ are not needed, because the functions $e^{i\mathbf{kr}}$ with real $\mathbf{k}$ constitute already a complete set, in which an arbitrary function of $\mathbf{r}$ can be expanded. In the subspace of functions belonging to a given $\mathbf{k}$ one may replace Ω by an operator $\Omega_\mathbf{k}$ acting on $g(\mathbf{v})$, *viz.*,

$$\Omega_\mathbf{k}\, g(\mathbf{v}) = \mathbf{kv}\, g(\mathbf{v}) - \frac{4\pi e^2}{m}\frac{\mathbf{k}}{k^2}\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}\int g(\mathbf{v}')\, d\mathbf{v}'. \tag{13}$$

This reduces the problem to finding for each fixed $\mathbf{k}$ the eigenvalues and corresponding eigenfunctions $g_{\mathbf{k},\omega}(\mathbf{v})$ of $\Omega_\mathbf{k}$; that is, one has to solve the eigenvalue problem

$$\Omega_\mathbf{k}\, g_{\mathbf{k},\omega}(\mathbf{v}) = \omega g_{\mathbf{k},\omega}(\mathbf{v}). \tag{14}$$

Now one finds immediately from (14) and (13)

$$g_{\mathbf{k},\omega}(\mathbf{v}) = \frac{4\pi e^2}{m}\frac{1}{\mathbf{kv} - \omega}\frac{\mathbf{k}}{k^2}\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}\int g_{\mathbf{k},\omega}(\mathbf{v}')\, d\mathbf{v}', \tag{15}$$

where ω must clearly satisfy

$$1 = \frac{4\pi e^2}{m} \frac{\mathbf{k}}{k^2} \int \frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}} \frac{\mathrm{d}\mathbf{v}}{\mathbf{kv} - \omega}. \tag{16a}$$

An alternative form for (16*a*) obtains by partial integration:

$$1 = \frac{4\pi e^2}{m} \int \frac{f_0(\mathbf{v})}{(\mathbf{kv} - \omega)^2} \mathrm{d}\mathbf{v}, \tag{16b}$$

from which, on expanding the dominator,

$$(\omega/\omega_{\mathrm{P}})^2 = 1 + (\mathbf{k}/\omega_{\mathrm{P}})^2 \, \overline{\mathbf{v}^2} + (\mathbf{k}/\omega_{\mathrm{P}})^4 \{\overline{\mathbf{v}^4} - (\overline{\mathbf{v}^2})^2\} + \dots,$$

which is (iii).

However, since the denominator in (15) and (16) is zero for certain values of $\mathbf{v}$, these equations are meaningless, unless the numerator vanishes for the same $\mathbf{v}$. This difficulty has caused some discussion, and various physical explanations have been forwarded [4] [13] [14]. I claim that it is of purely mathematical origin, and that it can be overcome by determining the eigenvalues and eigenfunctions of $\Omega_{\mathbf{k}}$ more carefully.

10. *Cut off velocity distribution.* Suppose that $f_0(\mathbf{v}) = 0$ for $|\mathbf{v}| \geqslant v_{\max}$, so that (15) and (16) have a well-defined meaning for $|\omega| > kv_{\max}$. This hypothesis is, of course, highly artificial, and excludes all known equilibrium distributions, in particular the Maxwell distribution. It is true that these distributions fall off very rapidly, but even the smallest tail makes (16) meaningless *). It will be shown in the next section that a small tail does not materially affect the physical results, although the mathematical derivation becomes quite different. We merely adopt the hypothesis of a rigorous cut-off in this section to facilitate the discussion. Moreover, for the sake of brevity, $f_0(\mathbf{v})$ is supposed spherically symmetrical: $f_0(\mathbf{v}) = f_0(v)$.

Clearly the right-hand side of (16) is a well-defined, positive, monotonically decreasing function of ω for $\omega > kv_{\max}$. Hence it takes the value 1 just once, provided it does not stay below 1, *i.e.*, provided that

$$k^2 < \frac{16\pi^2 e^2}{m} \int_0^{v_{\max}} \frac{v^2 f_0(v)}{v_{\max}^2 - v^2} \mathrm{d}v. \tag{17}$$

Thus (16) has just one positive root, ω_{B} say, and of course a second root $-\omega_{\mathrm{B}}$. This is the result of Vlasov and of Bohm and Gross.

These two roots $\omega = \pm\,\omega_{\mathrm{B}}$ are eigenvalues of (14) with corresponding eigenfunctions given by (15). Yet they do not constitute the whole spectrum

*) Clemmow and Willson [14]) use relativity theory as a *deus ex machina* to obtain a rigorously cut off velocity distribution. Accordingly, they only find stationary plasma waves with phase velocities greater than that of light.

of the operator $\Omega_{\mathbf{k}}$, but in addition to these two isolated eigenvalues there is a continuous spectrum between $-kv_{\max}$ and $+kv_{\max}$. With each point of this continuous spectrum is associated an 'improper eigenfunction' *), and hence a possible mode of oscillation of the plasma.

Physically, this means that for a given wave vector $\mathbf{k}$ it is possible to have an oscillation with arbitrary ω, as long as the phase velocity ω/k is less than the maximum velocity $v_{\max}$ of the plasma particles. This result is perfectly reasonable. If one wants a periodic density variation with wave vector $\mathbf{k}$ to propagate with a certain phase velocity, one just has to communicate the initial density variation to one particular velocity group of particles, in such way that the disturbance is bodily carried along with the desired phase velocity.

It is clear that in general the mechanism that is responsible for the excitation of the plasma waves cannot be expected to be so subtle as to select just one velocity group among all particles. Usually the excitation will affect all particles, or at least particles with many different individual velocities. Hence the initial density variation breaks up into many different waves, each propagating with its own phase velocity, so that the variation in the total density is smeared out after a short while. This can easily be seen from the explicit expression (12) for the solution, where $\sum$ means summation over the discrete eigenvalues and integration over the continuous spectrum. It is clear that for large t the integral vanishes, whereas the two isolated terms with $e^{\mp i\omega_B t}$ do not tend to zero.

One thus arrives at the following picture for the case of a cut off velocity distribution. *In order to find the evolution of a given initial disturbance, one has to write it as a superposition of eigenfunctions; this can only be done if both proper and improper eigenfunctions are employed. However, the improper ones are only of interest for the transient effect; after a short while only a periodic oscillation survives, which is just the one found by Vlasov and Bohm and Gross.*

11. *Non-vanishing velocity distribution.* Finally, suppose that f_0 does not vanish completely beyond $v_{\max}$, but tapers off with a small tail as v tends to infinity. Physically this cannot make much difference, but the mathematics is altered materially; for the continuous spectrum of $\Omega_{\mathbf{k}}$ now extends from $-\infty$ to $+\infty$, so that there is no room for isolated eigenvalues. There are only improper eigenfunctions, which now have to be taken seriously.

It is again possible to compute explicitly the evolution of an arbitrary given initial disturbance. The result is that one can again separate the transient effect from a long-lived oscillation. This oscillation has a definite frequency ω_B, which is again connected with the wave vector $\mathbf{k}$ by the dispersion equation (16*a*), *provided one deals with the singularity by taking the*

*) 'improper' refers to the fact that they are not square integrable, which shows up by the appearance of a δ-function.

principal value. This dispersion equation can again be approximated by (iii) if ω/k is large. Hence one finds again the plasma wave of Vlasov and Bohm and Gross; the same result was also found by Landau from his solution of the initial value problem (11). Recently a different treatment was given by Berz [7]) with similar conclusions.

The difference with the case of the previous section is that the life-time of this plasma wave is no longer infinite, because the density variation is eventually smeared out by the few particles whose individual velocities coincide with the phase velocity ω_B/k. For the life-time one finds, using the same approximation that is involved in (iii),

$$\frac{n_0 k^3}{2\pi^2\,\omega_P^4 f_0(\omega_P/k)}\,.$$

If f_0 is the Maxwell-Boltzmann law, this is identical to the expression given by Landau. These plasma waves cease to have a well-defined meaning for that value of k for which this life-time becomes of the same order as ω_P^{-1}; this turns out to correspond to a wave-length of the order of the Debye-Hückel radius $(\kappa T/4\pi n_0 e^2)^{\frac{1}{2}}$.

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REFERENCES

1) Thomson, J. J. and Thomson, G. P., *Conduction of electricity through gases*, 3rd edition, vol. **2** (Cambridge University Press 1933), p. 353 ff.
2) Bailey, V. A., Phys. Rev. **78** (1950) 428.
3) Gross, E. P., Phys. Rev. **82** (1951) 232.
4) Bohm, D. and Gross, E. P., Phys. Rev. **75** (1949) 1851.
5) Landau, L., J. Phys. USSR **10** (1946) 25.
6) Van Kampen, N. G., Physica **21** (1955) 949.
7) Berz, F., Proc. Phys. Soc. **69B** (1956) 939.
8) Vlasov, A., J. Phys. USSR **9** (1945) 25.
9) Piddington, J. H., Phil. Mag. **46** (1956) 1037.
10) Maxwell, J. C., *Scientific papers* (Cambridge 1890) **2**, 26;
Jeans, J. H., *The dynamical theory of gases* (Cambridge 1925).
11) Twiss, R. Q., Phys. Rev. **88** (1952) 1392.
12) Penning, F. M., Nature **118** (1926) 301; Physica **6** (1926) 241;
Merrill, H. J., and Webb, H. W., Phys Rev. **55** (1939) 1191.
13) Ecker, G., Z. Phys. **140** (1955) 274 and 293.
14) Clemmow, P. C., and Willson, A. J., Proc. roy. Soc. **237** (1956) 117.