Monodromy and irreducibility of leaves

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Abstract

We show that non-supersingular Newton polygon strata in the principally polarized case are irreducible.

Consider the theory of *foliations* of an open Newton polygon stratum in the moduli space of abelian varieties in positive characteristic. We show that any non-supersingular leaf is *irreducible*, and that the monodromy on such a leaf is *maximal*. Note that in the final result degrees of polarizations are arbitrary.

The irreducibility of leaves, as proved here, is the *discrete part* of a proof of the Hecke orbit conjecture, which will be published in [7]. For a survey of this proof and for the terminology "discrete part" see [2].

Here is the logical dependence of results and proofs:

(1) Show irreducibility of non-supersingular NP-strata in the principally polarized case (use EO, Cayley-Hamilton, description of $\Pi_0(W_\sigma)$, Chai's result on ℓ-adic monodromy). Note: using geometry on $\mathcal{A}_{g,1}$ we translate properties of W_ξ in properties of the supersingular locus; on W_σ the description of components (using Li-Oort and Oda-Oort) translates this into an algebraic problem, which can easily be solved; we conclude $\mathcal{H}_\ell$-transitivity on $\Pi_0(W_\xi)$, hence irreducibility of W_ξ by Chai's result on ℓ-adic monodromy.

(2) Show irreducibility of central leaves in the principally polarized case (use (1), abelian varieties over finite fields, weak approximation, and Chai's result on prime-to-p monodromy).

(2-bis) Remark. Here is a variant. Using the theory of minimal p-divisible groups (Oort), we see that the central stream $\mathcal{Z}_\xi$ is an EO-stratum; using a theorem by T. Ekedahl and G. van der Geer on irreducibility of certain EO-strata, we conclude that the central stream $\mathcal{Z}_\xi$ is irreducible of every non-supersingular ξ. In (4) and (5) this version can be used instead of (2).

(3) Compute monodromy on leaves in a Hilbert modular variety (use strata as described by Goren-Oort; Honda-Tate: use a method of Ribet producing enough monodromy elements as quotients of Weil numbers).

(4) Using (2) and (3) prove maximality of monodromy on the central stream $\mathcal{Z}_\xi$.

(5) From (2) and (4) conclude (via p-adic correspondences) irreducibility of every non-supersingular leaf.

(6) From (4) conclude maximality of monodromy on every non-supersingular leaf.

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Introduction

For any symmetric Newton polygon ξ we consider the stratum $\mathcal{W}_\xi(\mathcal{A}_g)$ where

$$\mathcal{A}_g := \mathcal{A}_g \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p)\,.$$

In every component of $\mathcal{A}_g$ this stratum has many components (in general) if the polarization has a degree divisible by p or if ξ equals the supersingular Newton polygon σ.

Theorem A. *If $\xi \neq \sigma$ the stratum $W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1})$ is* geometrically irreducible.
See 2.1.

In [35] for any symmetric Newton polygon ξ we consider a foliation of the open Newton polygon stratum $\mathcal{W}^0_\xi(\mathcal{A}_g)$ where $\mathcal{A}_g := \mathcal{A}_g \otimes \mathbb{F}_p$. For a polarized abelian variety (B, μ) over a perfect field we define its quasi-polarized p-divisible group $(Y, \mu) = (B, \mu)[p^\infty]$. We define

$$\mathcal{C}_{(Y,\mu)}(\mathcal{A}_g) := \{(C, \tau) \mid \exists \Omega = \overline{\Omega} : \quad (C, \tau)[p^\infty]_\Omega \cong (Y, \mu)_\Omega\};$$

this is called the *central leaf* passing through $y = [(B, \mu)]$; notation: $C(y) = \mathcal{C}_{(Y,\mu)}(\mathcal{A}_g)$. If there is no confusion possible (with "isogeny leaves"), we will just say "leaf" instead of "central leaf".

In [35], 3.3 we see:

$C(y)$ is a closed subset of the open Newton polygon stratum $\mathcal{W}^0_\xi(\mathcal{A}_g)$,

where $\xi = \mathcal{N}(Y)$ is the Newton polygon of B.

Theorem B. *If B is not supersingular then the leaf $C(y)$ is* geometrically irreducible.
See 5.6, and see 3.1, 3.7.

On a leaf we will define the monodromy , see 1.20.

Theorem C. *If B is not supersingular then the monodromy on the central leaf $C(y)$ is* maximal.
See 5.7

We will see that the proofs of these theorems are intertwined: first we show Theorem A, see 2.1. Then we show Theorem B in the principally polarized case, see 3.1. Using this result we then show the monodromy theorem on the "central stream" 5.2; from these two results the theorems B and C in the general cases follow, see Corollary 5.6, Corollary 5.7.

Note that the proof of Theorem A is geometric in nature, while the proof of Theorem B then is more arithmetic in flavor.

Note that in Theorem A we need the polarization is principal, but in Theorem B and in Theorem C we consider central leaves in the general case (i.e. arbitrary degree of polarization).

§1. Prerequisites

(1.1) Some notation to be used below. Let p be a prime number, fixed in this article. All abelian varieties and p-divisible groups are defined over a field or a base scheme of characteristic p. We write $\mathbb{F} = \overline{\mathbb{F}_p}$ for an algebraic closure of $\mathbb{F}_p$. For an abelian variety A we write $X = A[p^\infty]$ for its p-divisible group.

We write k and Ω for algebraically closed field of characteristic p. All base fields and base schemes considered will be in characteristic p.

For a group scheme G over a field K we write $a(G)$ for the dimension of the L-vector space $\mathrm{Hom}(\alpha_{p,L}, G_L)$ where $L \supset K$ is a perfect field. Note that $\mathrm{End}(\alpha_{p,L}) = L$, hence the right-$L$-module $\mathrm{Hom}(\alpha_{p,L}, G_L)$ is a vector space over L.

For a scheme W over a field K we write $\Pi_0(W)$ for the set of geometrically irreducible components of W, i.e. we choose $K \subset k$, and we consider the set of irreducible components of W_k.

We write $\sigma = \sigma_g$ for the supersingular Newton polygon (i.e. all slopes are equal to 1/2).

We write $\mathcal{A}_{g,d,n} = \mathcal{A}_{g,d,n} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{F}_p)$ for the moduli space of polarized abelian varieties, with degree of polarization equal to d^2, and with symplectic level-n-structure; it is understood that n is a positive integer, which is assumed to be at least 3, and which is not divisible by p.

(1.2) Newton polygons with slopes between 0 and 1 will be denoted by a symbol like ζ or ξ. When we write $\zeta = \sum_i (m_i, n_i)$ we intend to say that the (lower convex) Newton polygon starting from the origin of the plane, such that the multiplicity of a slope ν in ζ is equal to $\sum_{\nu = m_i/(m_i+n_i)} (m_i + n_i)$. In the above, it is understood that $m_i, n_i \in \mathbb{Z}_{\geq 0}$ and $\gcd(m_i, n_i) = 1$ for all i; "lower convex" means that ζ is the graph of a convex piecewise linear function defined on $[0, h]$, where $h = \sum_i (m_i + n_i)$. A Newton polygon $\zeta = \sum_i (m_i, n_i)$ can be specified by its *slope sequence* $(\nu_1, \ldots, \nu_h)$, where $0 \leq \nu_1 \leq \cdots \nu_h \leq 1$, and for every $\nu \in \mathbb{Q} \cap [0, 1]$, the multiplicity of ν in the slope sequence above, defined as $\mathrm{Card}\{j \mid \mu_j = \nu,\ 1 \leq j \leq h\}$, is equal to the multiplicity of ν in ζ.

A Newton polygon ξ is said to be *symmetric* if the multiplicity of ν in ξ is equal to the multiplicity of $1-\nu$ in ξ for every slope ν that appears in ξ. We say that two Newton polygons are *disjoint* if they have no slopes in common. Every symmetric Newton polygon ξ can be written as a sum of disjoint symmetric Newton polygons, each having at most two slopes.

Every symmetric Newton polygon ξ can be written in a unique way in the following *standard form*

$$\xi = (\gamma_0 \cdot ((1, 0) + (0, 1))) + \left(\sum_{1 \leq i \leq t} \gamma_i \cdot ((m_i, n_i) + (n_i, m_i)) \right) + (\gamma_{t+1} \cdot (1, 1)).$$

where $f = \gamma_0 \in \mathbb{Z}_{\geq 0},\ \gamma_1, \ldots, \gamma_t, \in \mathbb{Z}_{>0},\ t = \gamma_{t+1} \in \mathbb{Z}_{\geq 0}$, and $m_i > n_i \geq 0$ for $1 \leq i \leq t$, and $(m_i, n_i) \neq (m_j, n_j)$ if $1 \leq i \neq j \leq t$. The coefficients $\gamma_0, \gamma_1, \ldots, \gamma_{t+1}$ are called the *multiplicities of the simple parts* of ξ. Define $g(\xi) = \gamma_0 + \sum_{1 \leq i \leq t} \gamma_i \cdot (m_i + n_i) + \gamma_{t+1}$.

(1.3) According to the Dieudonné-Manin classification of p-divisible groups over an algebraically closed field, see [23], page 35, every p-divisible group X over an algebraically closed field $k \supset \mathbb{F}_p$ is isogenous to a direct product of isoclinic p-divisible groups $G_{m,n}$, with $m, n \in \mathbb{Z}_{\geq 0}$ and $\gcd(m,n) = 1$, with $\dim(G_{m,n}) = m$; in this case $G_{m,n}$ has height $m+n$ and is isoclinic of slope $m/(m+n)$. The Newton polygon of a p-divisible group X isogenous to $\prod_i G_{m_i,n_i}$ is

$$\sum_i (m_i, n_i) =: \mathcal{N}(X)\,.$$

For an abelian variety A over a field $K \supset \mathbb{F}_p$, the Newton polygon attached to $A[p^\infty]$ is a symmetric Newton polygon $\mathcal{N}(A)$, and it can be written in standard form as above. Then we have $\dim(A) = g(\xi)$. We hope there will be no confusion caused by the formal sum expressing ξ and the summation as in the formula for g.

The results is that there is a bijection between the set of k-isogeny classes of p-divisible groups over k and the set of Newton polygons:

Theorem (Dieudonné and Manin), see [23], "Classification theorem " on page 35.

$$\{X\}/\sim_k \quad \xrightarrow{\sim} \quad \{\text{Newton polygon.}\}$$

(1.4) Let A be an abelian variety over a field K. An isogeny

$$A \quad \sim \quad \sum_{1 \leq i \leq r} \; A_i^{\beta_i},$$

is called a *primary isogeny decomposition* of A if:

- $\beta_i \in \mathbb{Z}_{>0}$;
- for every $1 \leq i \leq r$ the abelian variety A_i is simple;
- for $1 \leq i < j \leq r$ the abelian varieties A_i and A_j are non-isogenous.

The Poincaré-Weil theorem says that *every abelian variety over a field K admits a primary isogeny decomposition over K.*

(1.5) Tate: abelian varieties over finite fields. As Tate proved, see [43], an abelian variety defined over a finite field admits smCM (= sufficiently many Complex Multiplications). If an abelian variety over field $K \supset \mathbb{F}_p$ admits smCM, than over $\overline{K} = k$ this abelian variety is isogenous with an abelian variety defined over a finite field, as was proved by Grothendieck, see [28]. These results will be used without further mention.

(1.6) Hypersymmetric abelian varieties Main reference: [6].
Definition. Let B be an abelian variety over a field $K \supset \mathbb{F}_p$. We say that B is *hypersymmetric* if the natural map

$$\mathrm{End}\left(B \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\right) \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \xrightarrow{\sim} \quad \mathrm{End}\left(B[p^\infty] \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\overline{K})\right)$$

is an isomorphism. If confusion might arise we will say "K-hypersymmetric".
Using the result of Grothendieck in [28] we see that the above definition is equivalent with:

Definition. Let B be an abelian variety over a field $K \supset \mathbb{F}_p$; we say that B is *hypersymmetric* if there exist an abelian variety A defined over $\mathbb{F} := \overline{\mathbb{F}_p}$ and an isogeny

$$B \times_{\operatorname{Spec}(K)} \operatorname{Spec}(\overline{K}) \sim A \times_{\operatorname{Spec}(\overline{\mathbb{F}_p})} \operatorname{Spec}(\overline{K}),$$

such that the natural map

$$\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \xrightarrow{\sim} \quad \operatorname{End}(A[p^\infty])$$

is an isomorphism.

Remark. An abelian variety B over an algebraically closed field $k \supset \mathbb{F}_p$ is hypersymmetric if and only if

$$\operatorname{End}(B) \otimes_{\mathbb{Z}} \mathbb{Q}_p \xrightarrow{\sim} \operatorname{End}(B[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an isomorphism. In particular, if an abelian variety B is isogenous to a hypersymmetric abelian variety A, then B is hypersymmetric.

Remark. Tate proved that for an abelian variety A *defined over a finite field* $\mathbb{F}_q$, the natural homomorphism

$$\operatorname{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \quad \xrightarrow{\sim} \quad \operatorname{End}(A[p^\infty])$$

is an isomorphism, see [45], Theorem 1 on page 60. This shows that $\operatorname{End}(A)\otimes_{\mathbb{Z}}\mathbb{Z}_p$ is identified with the $\operatorname{Gal}(\mathbb{F}/\mathbb{F}_q)$-invariant endomorphisms of $\operatorname{End}\left(A[p^\infty] \times_{\operatorname{Spec}(\mathbb{F}_q)} \operatorname{Spec}(\mathbb{F})\right)$. Suppose that

$$\operatorname{End}(A) \xrightarrow{\sim} \operatorname{End}\left(A \times_{\operatorname{Spec}(\mathbb{F}_q)} \operatorname{Spec}(\mathbb{F})\right),$$

then A is hypersymmetric if and only if every element of the ring of endomorphisms of the p-divisible group $A[p^\infty] \times_{\operatorname{Spec}(\mathbb{F}_q)} \operatorname{Spec}(\mathbb{F})$ is invariant under every element of $\operatorname{Gal}(\mathbb{F}/\mathbb{F}_q)$. From this we see that there are "many" abelian varieties over a finite field which are not hypersymmetric. Also we see that for a hypersymmetric abelian variety this Galois action is "in diagonal form" for every isoclinic part of $A[p^\infty]$. This can be made precise as follows.
Let K be a finite field and let B be an abelian variety over K. Then B is hypersymmetric if and only if there exists a positive integer n such that the n-th power of the Frobenius π_B of B lies in the center of $\operatorname{End}^0\left(B[p^\infty] \times_{\operatorname{Spec}(K)} \operatorname{Spec}(\mathbb{F})\right)$*. In other words, any two eigenvalues of the action of π_B on the Dieudonné module of* $B[p^\infty] \times_{\operatorname{Spec}(K)} \operatorname{Spec}(\mathbb{F})$ *which have the same p-adic absolute value, differ by a root of unity.*
In [6] we show that for any symmetric Newton polygon there exists a hypersymmetric abelian variety.

(1.7) Newton polygon strata. Newton polygons are partially ordered; *we write $\gamma \prec \beta$ if no point of γ is below β*:

$$\gamma \prec \beta \quad \Leftrightarrow \quad \gamma \text{ is "above" } \beta.$$

If $(\nu_1, \dots, \nu_h)$ and $(\mu_1, \dots, \mu_{h'})$ are the slope sequence of γ and β respectively, $0 \leq \nu_1 \leq \cdots \leq \nu_h \leq 1$, $0 \leq \mu_1 \leq \cdots \leq \mu_{h'} \leq 1$, then $\gamma \prec \beta$ iff

$$h = h', \quad \sum_{j=1}^{m} \nu_j \geq \sum_{j=1}^{m} \mu_j \quad \text{for} \quad m = 1, \dots, h-1, \quad \text{and} \quad \sum_{j=1}^{h} \nu_j = \sum_{j=1}^{h} \mu_j\,.$$

An abelian variety A is isogenous with its dual A^t; using the duality theorem, see [29], 19.1 we conclude that $X \sim X^t$; hence $\mathcal{N}(A) =: \xi$ is symmetric iff *the slope* λ *appears in* ξ *with multiplicity* n_λ*, then* $1 - \lambda$ *also appears with that multiplicity:* $n_\lambda = 1 - n_{1-\lambda}$.

For a symmetric Newton polygon ξ we write:

$$\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) = \{[(A, \lambda)] \mid \mathcal{N}(A) \prec \xi\},$$

$$\mathcal{W}^0_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) = \{[(A, \lambda)] \mid \mathcal{N}(A) = \xi\}.$$

Grothendieck - Katz:

$$\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) \subset \mathcal{A}_g \otimes \mathbb{F}_p \textit{ is closed,}$$

see [16] page 149–150, [20] Th. 2.3.1 on page 143; hence

$$\mathcal{W}^0_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) \subset \mathcal{A}_g \otimes \mathbb{F}_p \textit{ is locally closed.}$$

These are called the **Newton polygon strata.** We write

$$W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p), \quad W^0_\xi = \mathcal{W}^0_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p).$$

(1.8) Hecke orbits. Suppose given a field K, a polarized abelian variety (A, λ) over K. We define the *Hecke orbit* of the moduli point $x := [(A, \lambda)]$ to be the set of points $y = [(B, \mu)]$ over some field L such that there exist a field Ω containing K and L,

an integer $n \in \mathbb{Z} > 0$ and an isogeny $\varphi : A \to B$ such that $\varphi^*(\mu) = n{\cdot}\lambda$.

Notation. $y \in \mathcal{H}(x)$. The set $\mathcal{H}(x)$ is called the *Hecke orbit* of x

Hecke-prime-to-p-orbits. If in the previous definition moreover the degree of φ and m are not divisible by p, we say $[(B, \mu)] = y$ is in the Hecke-prime-to-p-orbit of x.
Notation: $y \in \mathcal{H}^{(p)}(x)$.

Hecke-ℓ-orbits. Fix a prime number ℓ different from p. We say $[(B, \mu)] = y$ is in the Hecke-ℓ-orbit of x if in the previous definition moreover the degree of φ and m both are a power of ℓ.
Notation: $y \in \mathcal{H}_\ell(x)$.

Remark. (i) We have given the definition of the so-called $\mathrm{CSp}_{2g}(\mathbb{A}_f)$-Hecke orbits (resp. $\mathrm{CSp}_{2g}(\mathbb{A}_f^{(p)})$-Hecke orbit, resp. $\mathrm{CSp}_{2g}(\mathbb{Q}_\ell)$-Hecke orbit), i.e. orbits under Hecke correspondences attached to the group $\mathrm{CSp}_{2g}(\mathbb{A}_f)$ (resp. $\mathrm{CSp}_{2g}(\mathbb{A}_f^{(p)})$, resp. $\mathrm{CSp}_{2g}(\mathbb{Q}_\ell)$) of $\mathbb{A}_f$ points (resp. $\mathbb{A}_f^{(p)}$ points, resp. $\mathbb{Q}_\ell$ points) of the group of symplectic similitudes. Here $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ is the ring of finite $\mathbb{Q}$-adeles, and $\mathbb{A}_f^{(p)} = \prod'_{\ell \neq p} \mathbb{Q}_\ell$ is the ring of finite prime-to-p adeles. We write $\mathcal{H}(x) = \mathcal{H}^{\mathrm{CSp}}(x) = \mathcal{H}_{\mathrm{CSp}}(x)$ (resp. $\mathcal{H}^{(p)}_{\mathrm{CSp}}(x)$, resp. $\mathcal{H}^{\mathrm{CSp}}_\ell(x)$) if we want to make clear this is the notion we want. The above definition of CSp-Hecke orbits is compatible with the notion of Hecke correspondences in [15], VII.3.

(ii) The $\mathrm{CSp}_{2g}(\mathbb{A}_f^{(p)})$-Hecke orbit (resp. $\mathrm{CSp}_{2g}(\mathbb{Q}_\ell)$-Hecke orbit) is the image in $\mathcal{A}_{g,1}$ of the orbit of a lift $\tilde{x}$ of x to the modular $\mathrm{CSp}_{2g}(\prod_{\ell\neq p} \mathbb{Z}_\ell)$-cover (resp. $\mathrm{CSp}_{2g}(\mathbb{Z}_\ell)$-cover) of $\mathcal{A}_{g,1}$ under the natural action of the group $\mathrm{CSp}_{2g}(\mathbb{A}_f^{(p)})$ (resp. $\mathrm{CSp}_{2g}(\mathbb{Q}_\ell)$). Here the modular covers come from level structures on prime-to-p (resp. ℓ-power) torsion points of the universal abelian variety. On the other hand, the $\mathrm{CSp}_{2g}(\mathbb{A}_f)$-Hecke orbit of x is not the image in $\mathcal{A}_{g,1}$ of the $\mathrm{CSp}_{2g}(\mathbb{A}_f)$-orbit of a lift of x in some profinite cover of $\mathcal{A}_{g,1}$ on which the group $\mathrm{CSp}_{2g}(\mathbb{A}_f)$ operates.

(iii) On can also define the (slightly smaller) $\mathrm{Sp}_{2g}(\mathbb{A}_f)$-Hecke orbits (resp. $\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})$-Hecke orbits, resp. $\mathrm{Sp}_{2g}(\mathbb{Q}_\ell)$-Hecke orbits) as follows.
We write $[(B,\mu)] = y \in \mathcal{H}^{\mathrm{Sp}}(x)$ (resp. $y \in \mathcal{H}^{(p)}_{\mathrm{Sp}}(x)$, resp. $y \in \mathcal{H}^{\mathrm{Sp}}_\ell(x)$) if there exist a field Ω containing K and L, and isogenies (resp. isogenies whose kernels are killed by an integer prime to p, resp. isogenies whose kernels are killed by a power of ℓ)

$$\varphi_1 : C_\Omega \to A_\Omega, \quad \varphi_2 : C_\Omega \to B_\Omega \quad \text{such that} \quad \varphi_1^*(\mu) = \varphi_2^*(\lambda),$$

that is:

$$\exists \text{ a geometric quasi-isogeny } \varphi (= \varphi_2 \circ \varphi_1^{-1}) : A \to B$$
$$\text{which preserves the polarizations } \lambda \text{ and } \mu.$$

The $\mathcal{H}_\alpha$-orbit. If in the notation just used the kernels of φ_1 and of φ_2 are successive extensions of α_p over an algebraically closed field, we write $[(B,\mu)] = y \in \mathcal{H}_\alpha(x)$.

Hecke stable. We say a set $T \subset \mathcal{A}_g$ is $\mathcal{H}_\ell$-stable if for every $x := [(A,\lambda)] \in T$ we have $\mathcal{H}_\ell(x) \subset T$, and analogously for $\mathcal{H}^{(p)}$-stable. Note that Newton polygon strata are Hecke-stable, and EO-strata and central leaves (see below) are $\mathcal{H}^{(p)}$-stable.

(1.9) Leaves in a foliation. Main reference: [35].
In the introduction we defined the notion

$$C(y) = \mathcal{C}_{(Y,\mu)}(\mathcal{A}_g) := \{(C,\tau) \mid \exists \Omega = \overline{\Omega} : \ \ (C,\tau)[p^\infty]_\Omega \cong (Y,\mu)_\Omega\}$$

of a *central leaf*; write $y = [(B,\mu)]$, and $Y = B[p^\infty]$ with $\xi = \mathcal{N}(Y)$.

(1.10) Theorem (Oort), see [35], Th. 3.3.

$$C(y) \subset \mathcal{W}^0_\xi(\mathcal{A}_g) \quad \textit{is a closed subset.}$$

Note that a central leaf is $\mathcal{H}^{(p)}$-stable.

For (B,μ) defined over a perfect field we consider $C(y)$ as a subscheme by giving it the induced reduced scheme structure from $\mathcal{A}_g$.

By considering Hecke correspondences where all isogenies involved are geometrically successive extensions of α_p, we define the Hecke-α-orbit of a moduli point, denoted by $\mathcal{H}_\alpha(x)$. We define $I(y) \subset \mathcal{A}_g$, a maximal $\mathcal{H}_\alpha$-set, by taking the union of all irreducible components of the $\mathcal{H}_\alpha$-orbit of y containing y; this is called the isogeny leaf passing through $y = (B,\mu)$; we give this the induced reduced scheme structure.

The completion of an isogeny leaf at a point is the same as the reduction modulo p of a Rapoport-Zink space, see [41], Th. 2.16.

(1.11) Theorem (Oort) ("central leaves and isogeny leaves almost give a product structure on an irreducible component of a Newton polygon stratum"), see [35], Th. 5.3.
Work over an algebraically closed field k. Choose a symmetric Newton polygon ξ, an irreducible component W of $\mathcal{W}^0_\xi(\mathcal{A}_g)$, an irreducible component C of a central leaf, and an irreducible component I of an isogeny leaf. There exist finite surjective morphisms $T \twoheadrightarrow C$, $J \twoheadrightarrow I$ and a finite surjective morphism

$$\Phi : T \times J \quad \twoheadrightarrow \quad W$$

such that for every $u \in J$,

$$\Phi(T \times \{u\}) \text{ is a component of a central leaf,}$$

and for every $t \in T$,

$$\Phi(\{t\} \times J) \text{ is a component of an isogeny leaf.}$$

(1.12) Minimal p-divisible groups. Main reference: [37].
For a pair (m,n) of coprime non-negative integers we define $H_{m,n}$ to be a p-divisible group of dimension m, of height $m+n$, of slope $m/(m+n)$ such that over k the endomorphism ring $\mathrm{End}(H_{m,n})$ is the *maximal order* in $\mathrm{End}^0(H_{m,n}) := \mathrm{End}(H_{m,n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$; this defines this group uniquely up to isomorphism over an algebraically closed field; the group can be constructed over $\mathbb{F}_p$. For $\zeta = \sum_i \ (m_i,n_i)$ we write $H(\zeta) = \prod_i \ H_{m_i,n_i}$. This p-divisible group is defined over $\mathbb{F}_p$, and if no confusion can arise we consider $H(\zeta)$ over a field K, meaning $H(\zeta) \otimes_{\mathbb{F}_p} K$.

Definition. We say a p-divisible group X is *minimal* if there exists an isomorphism $X_\Omega \cong H(\zeta)$.

(1.13) Theorem (Oort), see [37], Th. 1.2. *Work over an algebraically closed field k. Let X be a p-divisible group, and let ζ be some Newton polygon. Suppose there exists an isomorphism $X[p] \cong H(\zeta)[p]$. Then there exists an isomorphism $X \cong H(\zeta)$.*

Suppose $[(A,\lambda)] = x$ where $A[p^\infty]$ is *minimal* and λ is a *principal* polarization. In this case the central leaf $C(x)$ is denoted by $\mathcal{Z}_\xi = C(x)$, where $\xi = \mathcal{N}(A[p^\infty])$, and it is called the *central stream* inside W^0_ξ. Note that a principal quasi-polarization on the minimal p-divisible $A[p^\infty]$ is unique, see [35], 3.7.

(1.14) Honda-Tate theory. Main reference: [44].
Definition. *Let p be a prime number, $a \in \mathbb{Z}_{>0}$; write $q = p^a$. A Weil q-number is an algebraic* integer π *such that for every embedding $\psi : \mathbb{Q}(\pi) \to \mathbb{C}$ we have*

$$|\ \psi(\pi)\ | \quad = \quad \sqrt{q}.$$

We say that π and π' are *conjugated* if there exists an isomorphism $\mathbb{Q}(\pi) \cong \mathbb{Q}(\pi')$ mapping π to π'.
Notation: $\pi \quad \sim \quad \pi'$. We write $W(q)$ for the set conjugacy classes of Weil q-numbers.

Definition. *A field L is said to be a* CM-*field if L is a finite extension of* $\mathbb{Q}$ (hence L is a number field), and *there is a subfield $L_0 \subset L$ such that $L_0/\mathbb{Q}$ is totally real* (i.e. every $\psi_0 : L_0 \to \mathbb{C}$ gives $\psi_0(L_0) \subset \mathbb{R}$) *and L/L_0 is quadratic totally imaginary,* i.e. $[L : L_0] = 2$ and for every $\psi : L \to \mathbb{C}$ we have $\psi(L) \not\subset \mathbb{R}$.

Remark. This quadratic extension L/L_0 gives an involution $\iota \in \mathrm{Aut}(L/L_0)$. For every embedding $\psi : L \to \mathbb{C}$ this involution corresponds with the restriction of complex conjugation on $\mathbb{C}$ to $\psi(L)$.

We see an equivalent definition of a Weil q-number π:

$$\beta := \pi + \frac{q}{\pi} \quad \text{is totally real,}$$

and let π is a zero of

$$T^2 - \beta{\cdot}T + q, \quad \text{with} \quad \beta < 2\sqrt{q}.$$

In this way every Weil q-numbers can be constructed.

Consider a simple abelian variety A over $\mathbb{F}_q$, where $q = p^n$. The relative Frobenius $F_{A/\mathbb{F}_q} : A \to A^{(p)}$ can be "iterated n times" giving

$$F_{A^{(p^{n-1})}} \cdots F_{A^{(p)}} = \pi_A : A \quad \longrightarrow \quad A,$$

called the *geometric Frobenius* of $A/\mathbb{F}_q$. By the Riemann hypothesis for an abelian variety over a finite field, proved by Weil, we know:

(1.15) Theorem (Weil). *Let A be a simple abelian variety over $K = \mathbb{F}_q$; consider the endomorphism $\pi_A \in \mathrm{End}(A)$. The algebraic number π_A is a Weil q-number.*
See [46], page 70; [47], page 138; [25], Theorem 4 on page 206.

(1.16) Theorem (Honda and Tate). *Fix a finite field $K = \mathbb{F}_q$. The assignment $A \mapsto \pi_A$ induces a bijection from the set of K-isogeny classes of K-simple abelian varieties defined over K and the set of conjugacy classes of Weil q-numbers.*
See [44] and [18].

Chai's theorem on ℓ and prime-to-p monodromy and irreducibility.
Main reference: [3].

(1.17) Theorem (Chai), see [3], Prop. 4.4 and Prop. 4.6. *Work over an algebraically closed field k. Suppose $n \in \mathbb{Z}_{\geq 3}$ is not divisible by p. Let $Z \subset \mathcal{A}_{g,d,n}$ be a locally closed non-singular subset not contained in the supersingular locus. Let ℓ be a prime number not dividing pn. Suppose Z is $\mathcal{H}_\ell^{\mathrm{Sp}}$-stable. Assume that $\mathcal{H}_\ell^{\mathrm{Sp}}$ acts transitively on $\Pi_0(Z)$. Then Z is* irreducible *and the ℓ-adic monodromy representation on Z equals* $\mathrm{Sp}_{2g}(\mathbb{Z}_\ell)$. *Moreover Z is $\mathcal{H}_{\mathrm{Sp}}^{(p)}$-stable.*

(1.18) Theorem (Chai), see [3], Prop. 4.5.4. *Work over an algebraically closed field k. Suppose $n \in \mathbb{Z}_{\geq 3}$ is not divisible by p. Let $Z \subset \mathcal{A}_{g,d,n}$ be a locally closed non-singular subset not contained in the supersingular locus. Let $Z \subset \mathcal{A}_{g,d,n}$ be a locally closed non-singular subset not contained in the supersingular locus. Suppose Z is $\mathcal{H}_{\mathrm{Sp}}^{(p)}$-stable. Assume that $\mathcal{H}_{\mathrm{Sp}}^{(p)}$ acts transitively on $\Pi_0(Z)$. Then Z is* irreducible *and the prime-to-p monodromy representation on Z equals*

$$\mathrm{Sp}_{2g}(\hat{\mathbb{Z}}^{(p)})(n) := \{\gamma \in \mathrm{Sp}_{2g}(\hat{\mathbb{Z}}^{(p)}) \mid \gamma \equiv 1 \pmod{n}\}.$$

(1.19) Unwinding a finite level over a leaf. Main reference: [35], Th. 1.3.
We say that a p-divisible group $X \to S$ is *geometrically fiberwise constant*, abbreviated gfc, if fibers are mutually geometrically isomorphic, see [35], 1.1 for more details. In [35], Th. 1.3 we show that *if S satisfies reasonable finiteness conditions and $X \to S$ is gfc, and $i \in \mathbb{Z}_{>0}$ then there exists a finite surjective morphism $T_i = T \twoheadrightarrow S$ such that $X[p^i] \times_S T$ is constant over T.* We need a part of the proof of this theorem which is contained in [38]. And actually we need the part of the proof which goes back to Hasse and Witt, to Dieudonné, and which was described by Zink in [51], §2, also see [38], 1.6.

(1.20) Defining p-adic monodromy.
Let $X \to S$ be a p-divisible group over an integral normal k-scheme S, where $k \supset \mathbb{F}_p$ is an algebraically closed field as before. Let s be a geometric point of S. Suppose X/S is gfc. One knows that there exists a canonical *slope filtration*

$$0 = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_m = X$$

on X with the following properties:

- X_i/X_{i-1} is an isoclinic p-divisible group for $i = 1, \ldots, m$; let μ_i be the slope of X_i/X_{i-1};
- $1 \geq \mu_1 > \mu_2 > \cdots \mu_m \geq 0$.

See [51], [38], and also [5, 3.3.1].

Consider the smooth étale $\mathbb{Z}_p$-sheaf

$$\mathcal{G} := \underline{\mathrm{Isom}}_S \left(\prod_{1\leq i\leq m} (X_{i,s}/X_{i-1,s}),\ \prod_{1\leq i\leq m} (X_i/X_{i-1}) \right) = \prod_{1\leq i\leq m} \underline{\mathrm{Isom}}_S \left(X_{i,s}/X_{i-1,s}, X_i/X_{i-1}\right) .$$

This smooth étale $\mathbb{Z}_p$-sheaf $\mathcal{G}$ "is" a right torsor for the group

$$G := \mathrm{Aut}\left(\prod_{1\le i\le m}(X_{i,s}/X_{i-1,s})\right) = \prod_{1\le i\le m}\mathrm{Aut}\,(X_{i,s}/X_{i-1,s})\ ,$$

so it corresponds to a homomorphism $\rho_X : \pi_1(S,s) \to G$ from the geometric fundamental group of S to G. This homomorphism will be called the *p-adic monodromy representation* attached to the gfc p-divisible group $X \to S$. The image of ρ_X will be called the *p-adic monodromy group* of $X \to S$.

Remark. The p-adic monodromy group can be defined in a slightly different way. Let K be the function field of S, and let let X_K be the generic fiber of $X \to S$. We have a natural action of the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $\mathrm{Isom}_{K^{\mathrm{alg}}}\left(X_s, X_K \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K^{\mathrm{alg}})\right)$. This action gives rise to a homomorphism from $\rho'_X : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to G$, which is equal to the composition of ρ_X with the natural surjection $\mathrm{Gal}(K^{\mathrm{sep}}/K) \twoheadrightarrow \pi_1(S,s)$. Notice that the above definition makes sense for every BT-divisible group Y over an extension field K of k such that there exists an algebraically closed extension field $\Omega \supseteq K$ and a BT-group Y_0 over k such that $Y_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\Omega) \cong Y \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\Omega)$. More generally, suppose that we have a BT-group Y over an extension field K of k, a BT-group Y_0 over k, and a quasi-isogeny $\phi : Y_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\Omega) \to Y \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\Omega)$, we can still define a p-adic monodromy homomorphism $\rho_Y : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to G$, where G is the locally compact p-adic group consisting of all quasi-isogenies from Y_0 to itself, using the $\mathbb{Q}_p$-sheaf of quasi-isogenies from $Y_0 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\Omega)$ to $Y \times_{\mathrm{Spec}(K)} \mathrm{Spec}(\Omega)$.

§2. Irreducibility of Newton polygon strata

In order to start the proof of irreducibility of leaves, we first show the irreducibility of Newton polygon strata in the principally polarized case. Note however that supersingular Newton polygon strata are reducible (for p large). For the case of elliptic curves in characteristic p this is classical. By Hasse, Deuring, Igusa we know:

$$\sum_{j(E)\ \text{is ss}} \frac{1}{\#(\mathrm{Aut}(E))} = \frac{p-1}{24};$$

this implies that the number of supersingular j-invariants is asymptotically $p/12$ for $p \to \infty$.

For fixed g and large p the supersingular locus has "many components", see [22], 4.9:

$$\#(\Pi_0(W_\sigma)) = H_g(p,1) \quad \text{if} \quad g \quad \text{is odd,}$$

$$\#(\Pi_0(W_\sigma)) = H_g(1,p) \quad \text{if} \quad g \quad \text{is even.}$$

Note that for g fixed, and $p \to \infty$, indeed $\#(\Pi_0(W_\sigma)) \to \infty$.

(2.1) Theorem. *Let ξ be a symmetric Newton polygon which is* not supersingular. *The stratum $W_\xi := \mathcal{W}_\xi(\mathcal{A}_{g,1})$ is* geometrically irreducible.

Details of a proof can be found in [36]. We give a *sketch* of this proof here.

Step 1. *The moduli scheme $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is geometrically irreducible.*
This was proved by Faltings, see [14], Korollar on page 364, and this was proved by Chai (in case $p > 2$) in his Harvard PhD-thesis 1984, see [15], Chap. IV Cor. 6.8; for a pure characteristic p proof see [32], Cor. 1.4.

Step 2. *Deformation to $a \leq 1$. Let (A, λ) be a* principally polarized *abelian variety. There exists a deformation to a principally polarized abelian variety B, λ with $\mathcal{N}(A) = \mathcal{N}(B)$ and $a(B) \leq 1$.*
This is a difficult theorem. For the proof we use deformation to $a \leq 1$ for simple p-divisible groups and purity, see [19]. Once this is established the general result follows by considering deformation of filtered p-divisible groups, see [34], Section 3.

Corollary. *$W^0_\xi(a \leq 1)$ is dense in W^0_ξ.*

Step 3. *The Cayley-Hamilton theorem.* In [33] we study deformations of (polarized) p-divisible groups with a-number equal to 1. As a corollary we show that the strata

$$\{W^0_\xi(a \leq 1) \mid \xi\}$$

are ordered by inclusion-in-the-boundary exactly as prescribed by the NP-graph; moreover we compute the dimension of these strata; see [33], Th. 3.4

Step 4. Corollary. *For symmetric Newton polygons $\xi_1 \prec \xi_2$ and for an irreducible component T_1 of $W^0_{\xi_1}$ there exists a irreducible component T_2 of $W^0_{\xi_2}$ such that $T_1 \subset (T_2)^{\mathrm{Zar}}$ and T_2 is unique.*
Corollary. *$W^0_\xi(a \leq 1)$ is dense in W_ξ.*
Corollary. *$\cup_{\xi' \precneqq \xi} W^0_{\xi'}$ is contained in $(W^0_\xi)^{\mathrm{Zar}}$.*

Step 5. Using the facts

- every non-supersingular Hecke-ℓ orbit is non-finite, see [1], Prop. 1 on page 448,
- EO-strata are Hecke-ℓ stable,
- EO-strata are quasi-affine, see [32], Th. 1.2,

we show:
for every symmetric non-supersingular Newton polygon ξ_2 there exists $\xi_1 \precneqq \xi_2$ such that $W^0_{\xi_1} \cap (W_{\xi_2})^{\mathrm{Zar}} \neq \emptyset$.

Step 6. For $\xi_1 \prec \xi_2$ define $i^{\xi_1}_{\xi_2} : \Pi_0(W_{\xi_1}) \to \Pi_0(W_{\xi_2})$ by:

$$i^{\xi_1}_{\xi_2}(T_1) = T_2 \quad \Longleftrightarrow \quad T_1 \subset (T_2)^{\mathrm{Zar}}.$$

Using steps 3, 4 and 5 we conclude:
Corollary. *This map is well-defined.*
Corollary. *The map*

$$i^\sigma_\xi : \Pi_0(W_\sigma) \twoheadrightarrow \Pi_0(W_\xi)$$

is surjective for every ξ. Note that this map is $\mathcal{H}^{\mathrm{Sp}}_\ell$-equivariant.
Corollary. *The map $i^{\xi_1}_{\xi_2}$ is surjective.*

Step 7. Notation. For $g \in \mathbb{Z}_{>1}$ and $j \in \mathbb{Z}_{\geq 0}$ we write $\Lambda_{g,j}$ for the set of isomorphism classes of polarizations μ on the superspecial abelian variety $A = E^g \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(k)$ such that $\mathrm{Ker}(\mu) = A[F^j]$; here E is a supersingular elliptic curve defined over $\mathbb{F}_p$. Note that $\Lambda_{g,j} \xrightarrow{\sim} \Lambda_{g,j+2}$ under $\mu \mapsto F^t{\cdot}\mu{\cdot}F$.

Fact. *Characterization of components of W_σ. There is a canonical bijective map*

$$\Pi_0(W_\sigma) \quad \xrightarrow{\sim} \quad \Lambda_{g,g-1}.$$

See [22], 3.6 and 4.2; this uses [27], 2.2 and 3.1.

Step 8. Transitivity. *The action of $\mathcal{H}_\ell^{\mathrm{Sp}}$ on $\Pi_0(W_\sigma)$ is transitive.*
By the previous step the problem is translated into a question of transitivity of the set of isomorphism classes of certain polarizations on a superspecial abelian variety. Use [11], pp. 158/159 to describe the set of isomorphism classes of such polarizations. Use the strong approximation theorem, see [39], Theorem 7.12 on page 427 in order to conclude that this $\mathcal{H}_\ell^{\mathrm{Sp}}$ action is transitive.

Step 9. *End of the proof.* We have seen that the map $i_\xi^\sigma : \Pi_0(W_\sigma) \to \Pi_0(W_\xi)$ is surjective and $\mathcal{H}_\ell^{\mathrm{Sp}}$ equivariant for every ξ; by the previous step this implies that the action of $\mathcal{H}_\ell^{\mathrm{Sp}}$ on $\Pi_0(W_\xi)$ is transitive. By 1.17 this implies that W_ξ is geometrically irreducible for $\xi \neq \sigma$. This finishes a proof of Theorem 2.1. □

Remark. The same proof shows that $\mathcal{W}_\xi(\mathcal{A}_{g,1,n})$ is geometrically irreducible for every $n \neq p$.

§3. I: Irreducibility of leaves in the principally polarized case

(3.1) Theorem. *Let (A, λ) be a* principally polarized *abelian variety which is not supersingular over an algebraically closed field k . The central leaf $C(x)$ passing through $[(A, \lambda)] =: x \in \mathcal{A}_{g,1} \otimes k$ is irreducible.*

Note that this is a special case of Corollary 5.6. In the proof of that corollary however, the result proven here is used.

(3.2) The group G. Suppose we have a principally polarized abelian variety (A, λ) over a field K. We write $* : \mathrm{End}(A) \to \mathrm{End}(A)$ for the Rosati involution given by λ. We define a linear algebraic group $G = G^{(A)} = G^{(A,\lambda)}$ over $\mathbb{Q}$ by

$$G(R) = \mathrm{U}\left(\mathrm{End}(A) \otimes R, *\right) := \{x \mid x{\cdot} *(x) = 1 = *(x){\cdot}x\}$$

for every commutative $\mathbb{Q}$-algebra R. Remark that if $K \subset k$ is an inclusion of fields, and (A, λ) over K such that $\mathrm{End}(A) = \mathrm{End}(A \otimes_K k)$ then $G^{(A)} = G^{(A \otimes_K k)}$; this follows because the definition of G just uses $(\mathrm{End}(A), *)$.

Lemma. *Let K be a finite field, (A,λ) a principally polarized abelian variety defined over K. Assume that "all endomorphisms of A are defined over K", i.e. for any field extension $K \subset k$ we have* $\mathrm{End}(A) = \mathrm{End}(A \otimes_K k)$*. The group $G = G^{(A)}$ is connected.*

Proof. It suffices to check the assertion over $\mathbb{Q}_p$. By the Tate p-conjecture, see [45], Theorem 1 on page 60, we have:

Theorem (Tate) *Let K be a finite field. Let A and B be abelian varieties over K. The natural map*

$$\mathrm{Hom}(A,B) \otimes \mathbb{Z}_p \quad \xrightarrow{\sim} \quad \mathrm{Hom}(A[p^\infty], B[p^\infty])$$

is an isomorphism.

We show that the corresponding statement is true for the group $G \otimes_{\mathbb{Q}} \mathbb{Q}_p$ derived from the semi-simple $\mathbb{Q}_p$-algebra $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}_p$. By the Poincaré-Weil theorem we can write

$$A \quad \sim \quad \sum_{1 \le j \le r} \; C_j^{\beta_j},$$

with C_j simple and $\beta_j \in \mathbb{Z}_{>0}$, and for $j < s$ there is no isogeny between C_j and C_s; we see that moreover the abelian varieties C_j are absolutely simple. Then we know that for every j either $C = C_j$ is a supersingular elliptic curve and $\mathrm{End}^0(C)$ is the definite quaternion algebra $\mathbb{Q}_{p,\infty}$ central over $\mathbb{Q}$, or $\mathrm{End}^0(C)$ is a division algebra over a CM-field $\mathbb{Q} \subset L_0 \subset L$.

If $C = E^\beta$, where E is a supersingular elliptic curve with all endomorphisms defined over the base field, we see that is associated with $(M_\beta(\mathbb{Q}_{p,\infty}), *)$ the $\beta \times \beta$ matrix algebra over the quaternion division algebra over $\mathbb{Q}_{p,\infty}$, where the involution $*$ is transposing a matrix and applying the canonical involution of $Q_{p\infty}$ to all the entries. Hence $G^{(C)} \otimes \mathbb{Q}_p$ is an inner form of $\mathrm{Sp}_{2\beta}$. We conclude that $G^{(C)}$ is connected in this case.

Suppose $C = C_j$ is not supersingular. Let $0 \le \mu < 1/2$ be a slope appearing in $\mathcal{N}(C)$. Consider the slope filtration of $Z = C^\beta[p^\infty]$, and consider the isoclinic subfactors Z_μ and $Z_{1-\mu}$ in Z. Let G_μ be the algebraic group corresponding with these isoclinic parts, i.e. using $(\mathrm{End}^0(Z_\mu \times Z_{1-\mu}), *))$, where the involution $*$ induced by the polarization λ is an isomorphism switching the factors $\mathrm{End}(Z_\mu)$, and $\mathrm{End}(Z_{1-\mu})$. Note that $\mathrm{End}(Z_\mu \times Z_{1-\mu})$ is the μ-part and $(1-\mu)$-part of $\mathrm{End}(C^\beta) \otimes_{\mathbb{Z}} \mathbb{Q}_p$. Let v respectively w be the valuations of $L = \mathbb{Q}(\pi_C)$ corresponding with these slopes. Then $(\mathrm{End}(C^\beta) \otimes \mathbb{Q}_p)_{\mu,1-\mu} = D_v \times D_w$, where D_v respectively D_w is central over L_v, respectively L_w, say, both of degree d^2. We see that in this case G_μ is an outer form of $\Pi_{L_{0,v}/\mathbb{Q}_p}(\mathrm{GL}_d \otimes \mathbb{Q}_p)$, the $L_{0,v}/\mathbb{Q}_p$-Weil restriction of the $d \times d$ matrix group over $\mathbb{Q}_p$. This group is connected. This finishes the proof of the lemma. □

(3.3) Remark. If A is moreover hypersymmetric, G is a form of a matrix algebra.

(3.4) A torsor. Let (A,λ) and (B,μ) be polarized abelian varieties over a field K (eventually we will assume all endomorphisms are defined over K, the polarizations are principal, and the polarized abelian varieties are in the same Hecke orbit).

For a commutative $\mathbb{Q}$-algebra R we define a symplectic R-isogeny from (A,λ) to (B,μ) as an element of $\mathrm{Hom}(A,B) \otimes_{\mathbb{Z}} R$ which has an inverse in $\mathrm{Hom}(B,A) \otimes_{\mathbb{Z}} R$ and which respects the polarizations. Define $\mathcal{T} = \mathcal{T}^{(A,B)} = \mathcal{T}^{((A,\lambda),(B,\mu)))}$ by requiring that $\mathcal{T}(R)$ is the set of symplectic R-isogenies from (A,λ) to (B,μ), for every commutative $\mathbb{Q}$-algebra R. (We should not confuse the notions of an R-isogeny and of an isogeny defined over R.) The linear group $G = G^{(A)}$ operates naturally on $\mathcal{T}$ on the right by composing arrows in the following pattern $A \to A \to B$. In case $[(B,\mu)] \in \mathcal{H}_{\rm Sp}([(A,\lambda)])$ this gives $\mathcal{T}^{(A,B)}$ a structure of a right $G^{(A)}$-torsor.

(3.5) We start a proof of 3.1. We write $W = W^0_\xi = \mathcal{W}^0_\xi(\mathcal{A}_{g,1})$. Note that $\xi \neq \sigma$, hence by 2.1 we know that W is geometrically irreducible.

Choose a point $z = [(C,\tau)] \in W^0(\mathbb{F})$; this means that we consider a principally polarized abelian variety over a finite field K with all endomorphisms defined over K, with $\mathcal{N}(A) = \xi$, and we consider its moduli point over $\mathbb{F} = \overline{\mathbb{F}_p}$. Let $G = G^{(C,\tau)}$ be the linear group constructed above.

Next choose an algebraically closed field k, and consider $z \otimes k \in W(k)$. From now on in the proof we work over k, and all notation will imply we work over this field (e.g. we write W instead of $W \otimes k$).

We consider a point $[(A,\lambda)] =: x \in W(k) \subset \mathcal{A}_{g,1}(k)$ and the central leaf $C(x) \subset \mathcal{A}_{g,1} \otimes k$ passing through x. We are going to show that $C(x)$ is irreducible.

As explained in 1.11 by the almost product structure we obtain finite surjective morphisms $T \twoheadrightarrow C'$ over a component of some central leaf $J \twoheadrightarrow I'$ over a component of some isogeny leaf and a finite surjective morphism

$$\Phi : T \times J \quad \twoheadrightarrow \quad W.$$

Let C_x be the irreducible component of $C(x)$ containing x. Consider $(t', j') = y' \mapsto \Phi(y') = y := [(C,\tau)] \in W$, and write

$$\Phi(t' \times J) \cap C(x) = \{z_1, \cdots, z_u\}.$$

Let $D_j \subset C(x)$ be the irreducible component containing z_j. Note that $\Phi(t' \times J)$ does intersect a given component of $C(x)$, see [35], 5.7. However it may intersect a given component of $C(x)$ more than once. Then $\cup D_j = C(x)$. In other words, the set $\{z_j \mid 1 \leq j \leq u\}$ contains a system of representatives of $\Pi_0(C(x))$. Note that an isogeny leaf is in a $\mathcal{H}^{\rm Sp}$-orbit.

Claim. For $1 \leq i \neq j \leq u$ **we have** $z_i \in \mathcal{H}^{(p)}_{\rm Sp}(z_j)$**.**

Proof. Let (A_i,λ_i), (A_j,λ_j) correspond to z_i, z_j respectively. As $z_j \in \mathcal{H}^{\rm Sp}(z_i)$ we see that the $\mathcal{T} = \mathcal{T}^{(A_i,A_j)}$ is non-empty. Hence it is a torsor, as explained above, under $G = G^{(C,\tau)}$.

We have that

$$(A_i,\lambda_i), \quad (A_j,\lambda_j) \in \Phi(t' \times J).$$

As $\Phi(t' \times J)$ is a component of an isogeny leaf, we conclude that there exists a quasi-isogeny

$$(\psi' : (A_i,\lambda_i) \dashrightarrow (A_j,\lambda_j)) \in \mathcal{T}(\mathbb{Q}).$$

As G is connected, $G \to G(\mathbb{Q}_p)$ satisfies weak approximation, see [39], 7.3, Theorem 7.7 on page 415, and we conclude that there exists

$$\psi \in \mathcal{T}(\mathbb{Q}) \quad \text{such that} \quad \psi[p^\infty] : (A_i,\lambda_i)[p^\infty] \quad \xrightarrow{\sim} \quad (A_j,\lambda_j)[p^\infty]$$

is an isomorphism. Hence $z_i \in \mathcal{H}^{(p)}_{\mathrm{Sp}}(z_j)$, and the claim has been proved. □

From the claim we see that $\mathcal{H}^{(p)}_{\mathrm{Sp}}$ operates transitively on $\Pi_0(C(x))$. By 1.18 we conclude that $C(x)$ is irreducible, i.e. $C(x) = C_x$. This proves Theorem 3.1. □

(3.6) Remark. The argument in the proof, plus the Hasse principle for simply connected groups over global fields shows: *if $y_1 = [(B_1, \mu_1)]$ and $y_2 = [(B_2, \mu_2)]$ correspond to two polarized abelian varieties over $\mathbb{F}$ in the same leaf in $\mathcal{A}_{g,1}$ such that the abelian varieties B_1 and B_2 are isogenous, then $y_1 \in \mathcal{H}^{(p)}(y_2)$. One considers the torsor $\mathcal{T}$ as above which is trivial over $\mathbb{Q}_\ell$ for all $\ell \neq p$, and it has $\mathbb{Q}_p$-points because y_1 and y_2 lie on the same leaf. That $\mathcal{T}$ has $\mathbb{R}$-points follows from the fact that any two positive involution on a semi-simple algebra E over $\mathbb{R}$ are conjugate by an element of $E^\times$.*

(3.7) Theorem. *Let (A, λ) be a* principally polarized *abelian variety which is* not supersingular. *Suppose $A[p^\infty]$ is a* minimal *p-divisible group. The central stream $\mathcal{Z}_\xi$ passing through $[(A, \lambda)] =: x \in \mathcal{A}_{g,1}$ is geometrically irreducible.*

Note that 3.7 is a special case of 3.1, and a special case of Corollary 5.6. We state this special case explicitly because there is a proof which is completely different from the proof of 3.1 given above.

PROOF. **Step 1.** *For every Newton polygon ξ the central stream $\mathcal{Z}_\xi \subset \mathcal{A}_{g,1}$ is an* EO-stratum:

$$\mathcal{Z}_\xi := C(x) = S_\varphi, \quad \text{where} \quad \varphi := ((A, \lambda)[p] \mod \cong).$$

PROOF. The central stream is defined by the fact that it is the central leaf where the p-divisible groups are minimal. For a minimal p-divisible group X over k, we know that $X[p]$ determines the isomorphism class of X, see [37], 1.2. Moreover a principal quasi-polarization on a minimal p-divisible group is unique, see [35], 3.7. This proves that the usual inclusion $C(x) \subset S_\varphi$ in this case is an equality. □

Step 2. Lemma. Use notation as in [32]. *Let $\varphi = \{\varphi(1), \cdots, \varphi(g)\}$ be an elementary sequence, and let $S_\varphi \subset \mathcal{A}_{g,1}$ be the corresponding* EO*-stratum. Write $g = 2r$, respectively $g = 2r - 1$, i.e. $r = \lceil g/2 \rceil$.*
Claim.

$$\varphi(r) = 0 \iff S_\varphi \subset W_\sigma .$$

PROOF. Let $N_1 \subset \cdots \subset N_{2g} = N = X[p]$ be a final filtration. Suppose $\varphi(r) = 0$; we see that N_{g+r}/N_r is annihilated by F and by V. Hence X/N_r is superspecial. Hence X is supersingular. We conclude $S_\varphi \subset W_\sigma$.

Define $u(\xi)$ to be the elementary sequence of the minimal p-divisible group with Newton polygon equal to ξ. Suppose ξ is "almost supersingular", i.e. either $\xi = (r, r-1) + (1,1) + (r-1, 1)$ or $\xi = (r, r-1) + (r-1, r)$. Direct computation shows that

$$u(\xi) := \mathrm{ES}(H(\xi)[p]) = \{0, \cdots, \varphi(r-1) = 0, \varphi(r) = 1, \cdots, 1\}.$$

In this case $S_{u(\xi)} \not\subset W_\sigma$. For every φ with $\varphi(r) \neq 0$ we have $\varphi \succ \mathrm{ES}(H(\xi)[p])$. Using [32], Th. 1.3 we conclude

$$\varphi(r) \neq 0 \quad \Longrightarrow \quad S_{u(\xi)} \subset S_\varphi; \quad \text{hence} \quad S_\varphi \not\subset W_\sigma.$$

□

Step 3. *End of the proof of* 3.7. In [13], Th. 11.5, we see that certain EO-strata are geometrically irreducible. By Step 2 we see that these are exactly the EO-strata not contained in the supersingular locus. Using moreover Step 1 we conclude that $\mathcal{Z}_\xi = S_{u(\xi)}$ is geometrically irreducible for every $\xi \neq \sigma$ and $u(\xi) := \mathrm{ES}(H(\xi)[p])$. This proves Theorem 3.7. □

§4. Monodromy on leaves in a Hilbert modular variety

(4.1) Let $E = F_1 \times \cdots \times F_r$, where $F_1, \ldots, F_r$ are totally real number fields. Let $\mathcal{O}_E = \mathcal{O}_{F_1} \times \cdots \mathcal{O}_{F_r}$ be the product of the ring of integers of $F_1, \ldots, F_r$. Let $k \supset \mathbb{F}_p$ be an algebraically closed filed as before. Let $n \geq 3$ be an integer such that $(n, p) = 1$. The Hilbert modular variety $\mathcal{M}_{E,n}$ over k is a smooth scheme over k of dimension $[E : \mathbb{Q}]$ such that for every k-scheme S the set of S-valued points of $\mathcal{M}_{E,n}$ is the set of isomorphism class of 5-tuples $(A \to S, \iota, \mathcal{L}, \mathcal{L}^+, \lambda, \eta)$, where

- $A \to S$ is an abelian scheme of relative dimension $[E : \mathbb{Q}]$;
- $\iota : \mathcal{O}_E \to \mathrm{End}_S(A)$ is an injective ring homomorphism which sends $1 \in \mathcal{O}_E$ to Id_A;
- $\mathcal{L}$ is a locally free $\mathcal{O}_E$-module of rank one and $\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a free E-module of rank one;
- $\mathcal{L}^+$ is a notion of positivity for $\mathcal{L}^+$, i.e. a disjoint union of connected components of $\mathcal{L} \otimes_{\mathbb{Q}} \mathbb{R}$ such that $\mathcal{L} \otimes_{\mathbb{Q}} \mathbb{R}$ is the disjoint union of $\mathcal{L}^+$ and $-\mathcal{L}^+$;
- $\lambda : \mathcal{L} \to \mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A, A^t)$ is an $\mathcal{O}_E$-linear homomorphism which sends totally positive elements in $\mathcal{L}$ to $\mathcal{O}_E$-linear polarizations of the $\mathcal{O}_E$-linear abelian scheme $A \to S$, and the homomorphism $A \otimes_{\mathcal{O}_E} \mathcal{L} \xrightarrow{\sim} A^t$ induced by λ is an isomorphism of abelian schemes.
- η is an $\mathcal{O}_E$-linear level-n structure for $A \to S$.

Remark. (i) More information about Hilbert modular varieties can be found in [10], [9], [49]. We have followed [10] in the definition of Hilbert Modular varieties.

(ii) The modular variety $\mathcal{M}_{E,n}$ is not smooth over k if any one of the totally real fields F_i is ramified above p; if so then $\mathcal{M}_{E,n}$ has moderate singularities—it is a local complete intersection.

(iii) There is a natural isomorphism $\mathcal{M}_{E,n} \xrightarrow{\sim} \mathcal{M}_{F_1,n} \times \cdots \times \mathcal{M}_{F_r,n}$

(4.2) Let $y = [(A, \iota_A, \mathcal{L}_A, \mathcal{L}_A^+, \lambda_A, \eta_A)] \in \mathcal{M}_{E,n}(k)$ be a closed point of $\mathcal{M}_{E,n}$. Denote by $C_{\mathcal{M}_{E,n}}(y)$ the smooth locally closed subscheme of $\mathcal{M}_{E,n}$ such that $C_{\mathcal{M}_{E,n}}(y)$ consists of all points $z = [(B, \iota_B, \mathcal{L}_B, \mathcal{L}_B^+, \lambda_B, \eta_A)] \in \mathcal{M}_{E,n}(k)$ such that the $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$-linear polarized BT-group $(A[p^\infty], \iota_A[p^\infty], \mathcal{L}_A \otimes_{\mathbb{Z}} \mathbb{Z}_p, \lambda_A[p^\infty])$ is isomorphic to $(B[p^\infty], \iota_B[p^\infty], \mathcal{L}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p, \lambda_B[p^\infty])$.

Remark. (i) The decomposition of $\mathcal{M}_{E,n}$ into a product $\mathcal{M}_{F_1,n} \times \cdots \times \mathcal{M}_{F_r,n}$ induces a product decomposition for every leaf in $\mathcal{M}_{E,n}$ into a product of leaves in $\mathcal{M}_{F_i,n}$, $i = 1, \ldots, r$.
(ii) It follows easily from [17] and [3] that if p is unramified in a totally real number field F, then every non-supersingular leaf in $\mathcal{M}_{F,n}$ is irreducible.

(4.3) The goal of this section is to show that Ribet's method in [40] and [10] can be use to show that the p-adic monodromy group of any leaf in a Hilbert modular variety $\mathcal{M}_F$ is maximal.

We will restrict to the case when p is unramified in the totally real number field F and the $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p$-linear BT-group in question is maximal, because we need only this case for later application. The general case will appear in [8].

We will first assume that 1/2 is not a slope of the leaf being considered, and present Ribet's method under this simplifying assumption; see 4.4. Then we will indicate how to remove this hypothesis with the help of some lemmas in group theory.

(4.4) Theorem. *Let F be a totally real number field such that p is unramified in F. Let $n \geq 3$ be an integer prime to p. Let $\mathcal{M}_{F,n}$ the Hilbert modular variety of level-n attached to F. Let $y = [(A, \iota_A, \mathcal{L}_A, \mathcal{L}_A^+, \lambda_A, \eta_A)]$ be a point of $\mathcal{M}_{F,n}(\mathbb{F})$ such that $A[\wp^\infty]$ splits into a product of two $\mathcal{O}_\wp$-linear BT-group for every prime ideal $\wp$ in $\mathcal{O}_F$ which lies above p. Then the p-adic monodromy representation*

$$\rho = \rho_{C_{\mathcal{M}_{F,n}(y)}} : \pi_1(C_{\mathcal{M}_{E,n}(y)}, y) \to \mathrm{Aut}_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p}(A[p^\infty], \lambda_A[p^\infty])$$

is surjective.

Remark. (i) The assumption on $A[\wp^\infty]$ implies that 1/2 is not a slope of A.
(ii) We have a split short exact sequence

$$0 \to S_\wp \to A[\wp^\infty] \to Q_\wp \to 0$$

of $\mathcal{O}_\wp$-linear BT-groups for every prime ideal $\wp$ of $\mathcal{O}_F$ which divides p, such that the following properties hold.

- $\mathrm{ht}(\mathrm{S}_\wp) = \mathrm{ht}(\mathrm{S}_\wp) = [\mathrm{F}_\wp : \mathbb{Q}_\mathrm{p}]$;
- we have $\mathrm{Aut}_{\mathcal{O}_{F_\wp}}(S_\wp) = \mathcal{O}_\wp^\times$ and $\mathrm{Aut}_{\mathcal{O}_{F_\wp}}(Q_\wp) = \mathcal{O}_\wp^\times$;
- both $S_\wp$ and $Q_\wp$ are isoclinic, and $\nu_\wp :=$ slope of $Q_\wp <$ slope of $S_\wp = 1 - \nu_\wp$;
- $S_\wp$ and $Q_\wp$ are dual to each other by the quasi-polarization induced by any $\mathcal{O}_\wp$-generator of $\mathcal{L} \otimes_{\mathcal{O}_F} \mathcal{O}_{F_\wp}$.

- the natural projections

$$\mathrm{Aut}_{\mathcal{O}_{F_\wp}\otimes_{\mathbb{Z}}\mathbb{Z}_p}(A[\wp^\infty],\lambda_A[\wp^\infty]) \to \mathrm{Aut}_{\mathcal{O}_{F_\wp}}(Q_\wp) = \mathcal{O}_\wp^\times$$

  and

$$\mathrm{Aut}_{\mathcal{O}_{F_\wp}\otimes_{\mathbb{Z}}\mathbb{Z}_p}(A[\wp^\infty],\lambda_A[\wp^\infty]) \to \mathrm{Aut}_{\mathcal{O}_{F_\wp}}(S_\wp) = \mathcal{O}_\wp^\times$$

  are isomorphisms.

Proof. Given any element $a \in (\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ and any positive integer m, we will show that there exists an element in the image of ρ which is congruent to a modulo p^m. The theorem will follow, because $\pi_1(C_{\mathcal{M}_{F,n}(y)}, y)$ is compact and ρ is continuous.

Choose a positive integer n_1 such that $(n_1, np) = 1$ and every element of the strict ideal class group of $\mathcal{O}_F$ represented by a prime ideals dividing n_1. Let $n_2 = n \cdot n_1$.

Recall that the Hilbert modular variety $\mathcal{M}_{F,n}$ is defined over $\mathbb{F}_p$; write $\mathcal{M}_{F,n,\mathbb{F}_p}$ for this modular variety over $\mathbb{F}_p$. Let $q = p^r$ be a power of p such that

(i) $y \in \mathcal{M}_{F,n}(\mathbb{F}_q)$, consequently $C_{\mathcal{M}_{F,n}}(y)$ is also defined over $\mathbb{F}_q$;

(ii) $q \equiv 1 \pmod{n_2^2}$;

(iii) $\wp^r$ is a principal ideal in $\mathcal{O}_F$ for each prime ideal $\wp$ in $\mathcal{O}_F$ dividing p.

We write $C(y)_{\mathbb{F}_q}$ for this smooth locally closed subscheme of $\mathcal{M}_{E,n,\mathbb{F}_p}$. We have a short exact sequence

$$1 \to \pi_1\left(C_{\mathcal{M}_{F,n}}(y), y\right) \to \pi_1\left(C(y)_{\mathbb{F}_p} \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(\mathbb{F}_q), y\right) \to \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_q) \to 1$$

of fundamental groups. We have an arithmetic monodromy representation

$$\rho_{\mathrm{arith}} : \pi_1\left(C(y)_{\mathbb{F}_p} \times_{\mathrm{Spec}(\mathbb{F}_p)} \mathrm{Spec}(\mathbb{F}_q), y\right) \to \mathrm{Aut}_{\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}_p}(A[p^\infty],\lambda_A[p^\infty])$$

extending the (geometric) monodromy representation ρ.

For each prime ideal $\wp$ of $\mathcal{O}_F$ dividing p, denote by $\nu_\wp$ the slope of $A[\wp^\infty]$ which is $< 1/2$, so that the slopes of $A[\wp^\infty]$ are $\nu_\wp$ and $1-\nu_\wp$, each with multiplicity $[F_\wp : \mathbb{Q}_p]$. Let $\wp_1, \ldots, \wp_s$ be the primes ideals of $\mathcal{O}_F$ dividing p. Choose a positive integer $N_0 > 0$ such that $\mu_i := N_0 \cdot \nu_{\wp_i} \in \mathbb{Z}$ for $i = 1, \ldots, s$. For each element $\bar{a} \in (\mathcal{O}_F \otimes_{\mathbb{Z}} (\mathbb{Z}/p^m\mathbb{Z}))^\times$, choose an element $u_{\bar{a}} \in \mathcal{O}_F$ such that

- $u_{\bar{a}} \equiv a \pmod{p^m}$
- $u_{\bar{a}} \equiv 2 \pmod{n_2^2}$

Claim 1. There exists a positive integer $N_1 > 0$ such that for every integer $N \geq N_1$, there exists an element $\xi_N \in \mathcal{O}_F$ satisfying the following properties

(i) $\xi_N \mathcal{O}_F = \prod_{i=1}^s \wp_i^{r\mu_i N}$

(ii) $u_{\bar{a}}^2 \xi_N^2 - 4q^{N_0 N}$ is totally negative for every $\bar{a} \in (\mathcal{O}_F \otimes_{\mathbb{Z}} (\mathbb{Z}/p^m\mathbb{Z}))^\times$

Proof of Claim 1. Pick a generator z_i of $\wp_i^r$ for $i = 1, \ldots, s$. Let $z = \prod_{i=1}^s z_i^{\mu_i}$. Let $q_i = \mathrm{Card}(\mathcal{O}_F/\wp_i)$. By the product formula, we have

$$\prod_{\iota: F\hookrightarrow\mathbb{R}} |\iota(z)| = \prod_{i=1}^s q_i^{r\mu_i} =: p^M ,$$

where $M \in \mathbb{N}$ is defined by the last equality above. On the other hand, we know that $\prod_{i=1}^s q_i = p^g$, where $g := [F : \mathbb{Q}]$, which results from the product formula applied to the element $p \in F$. Then $M < gN_0$ because $\mu_i/N_0 = \nu_i < 1/2$ for $i = 1, \ldots, s$.

The desired element ξ_N is necessarily of the form $\xi_N = z^N \cdot \zeta$ with $\zeta \in \mathcal{O}_F^\times$. We must show that there exists a unit $\zeta \in \mathcal{O}_F^\times$ such that

$$|u_{\bar a}|_\iota \cdot |z|_\iota^N \cdot |\zeta|_\iota < 4p^{rN_0N} \qquad \forall \iota : F \hookrightarrow \mathbb{R}\ \forall \bar a \in (\mathcal{O}_F/p^m\mathcal{O}_F)^\times .$$

The condition for the unit $\zeta \in \mathcal{O}_F^\times$ can be rewritten as

$$\log|\zeta|_\iota < A_\iota + N\left(\log(p^{rN_0} - \log|z|_\iota\right) \qquad \forall \iota : F \hookrightarrow \mathbb{R},$$

where $A_\iota := \log 4 + \mathrm{Min}\{-\log|u_{\bar a} \mid \bar a \in (\mathcal{O}_F/p^m\mathcal{O}_F)^\times\}$. Identify $\mathcal{O}_F^\times/\{\pm 1$ as a lattice in the hyperplane V_F in $F_\infty = \prod_{\iota:F\hookrightarrow\mathbb{R}} \mathbb{R}_\iota$ where the sum of the coordinates is equal to zero. The the above system of inequalities describes a family of symplexes Δ_N in V_F cut out by hyperplanes parallel to the coordinate hyperplanes of F_∞, whose (linear) size grows with a constant rate N. The key here is that $\sum_{\iota:F\hookrightarrow\mathbb{R}} \left(\log p^{rN_0} - \log|z|_\iota\right) > 0$. So for N sufficiently large, there exists lattice points in the interior of the symplex Δ_N. Claim 1 is proved.

From Claim 1, we see that for every element $a \in \mathcal{O}_F^\times$, every $m > 0$, and every $N_2 \geq N_1$, there exist an element $b \in \mathcal{O}_F$ such that $b \equiv 2 \pmod{n_2^2}$, $b \equiv aq^{N_2\mu_i} \pmod{q^{m+N_2\mu_i}}$ in $\mathcal{O}_{\wp_i}$ for $i = 1, \ldots, s$, and $b^2 - 4q^{N_0N_2}$ is totally negative; also there exists an element $c \in \mathcal{O}_F$ such that $c \equiv 2 \pmod{n_2^2}$, $c \equiv q^{N_2\mu_i} \pmod{q^{m+N_1\mu_i}}$ in $\mathcal{O}_{\wp_i}$, for $i = 1, \ldots, s$, and $c^2 - 4q^{N_0N_2}$ is totally negative. Increasing N_2 if necessary, we may and do assume that $m + N_2\mu_i < N_2N_0$. Let $N = N_2N_0$. Consider the quadratic polynomials $f_1(T) = T^2 - bT + q^N$ and $f_2(T) = T^2 - cT + q^N$, where $N := N_0N_1$.

Claim 2. There exists points $y_1, y_2 \in C_{\mathcal{M}_{F,n}}(\mathbb{F}_{q^N})$ such that $\mathrm{Fr}_{y_i} := \mathrm{Fr}_{A_{y_i},q^N}$ is a root of $f_i(T)$ for $i = 1, 2$.

Proof of Claim 2. By Thm. 1.16, there exists abelian varieties A_1, A_2 over $\mathbb{F}_{q^N}$ such that $\pi_i := \mathrm{Fr}_{A_i}$ is a root of $f_1(T)$. By [43], $\mathrm{End}(A_i) \otimes_{\mathbb{Z}} \mathbb{Q} = F(\pi_i)$ is a totally imaginary quadratic extension K_i of F which is split over every prime ideal $\wp$ of $\mathcal{O}_F$ above p, $i = 1, 2$. Saturating A_i, we may and do assume that $End(A_i) = \mathcal{O}_{K_i}$ for $i = 1, 2$. Since $\mathcal{O}_{K_i}$ splits over every prime of F above p, the $\mathcal{O}_F$-linear BT-group $A_i[p^\infty]$ splits for $i = 1, 2$.

Let $\mathcal{L}_i$ be the invertible $\mathcal{O}_F$-module of $\mathcal{O}_F$-linear symmetric homomorphisms from A_i to A_i^t, $i = 1, 2$. The invertible $\mathcal{O}_F$-module $\mathcal{L}_i$ has a natural notion of positivity $\mathcal{L}_i^+$ such that elements of $\mathcal{L}_i^+ \cap \mathcal{L}_i$ correspond to $\mathcal{O}_F$-linear polarizations on A_i. The assumption on the congruence modulo n_2^2 of b, c and q implies that $(\pi_i - 1)/n_2$ is integral for $i = 1, 2$, Hence the finite étale group schemrs $A_i[n_2]$ over $\mathbb{F}_{q^N}$. So we can find $\mathbb{F}_{q^N}$-rational level-n structure on the $\mathcal{O}_F$-linear abelian variety A_i. Moreover, we may modify A_i by a suitable $\mathcal{O}_F$-linear isogeny whose kernel is killed by n_1 so that the polarization sheaf $(\mathcal{L}_i, \mathcal{L}_i^+)$ is isomorphic to $(\mathcal{L}, \mathcal{L}^+)$. So we obtain $\mathbb{F}_q$ points y_1, y_2 on $C_{\mathcal{M}_{F,n}}$ with the required properties. The claim is proved.

The $\rho\left(\mathrm{Fr}_{y_1}\cdot\mathrm{Fr}_{y_2}^{-1}\right)$ is an element of the image of the geometric monodromy representation ρ which is congruent to a modulo q^m by construction. We have shown that the p-adic monodromy group of $C_{\mathcal{M}_{F,n}}$ contains an element which is congruent to a modulo p^m. ∎

Definition. Let F be a totally real number field such that p is unramified in F. A point $y=[(A,\iota_A,\mathcal{L}_A,\mathcal{L}_A^+,\lambda_A,\eta_A)]\in\mathcal{M}_{F,n}(k)$ is *minimal* if

- $A[\wp^\infty]$ splits into a product of two $\mathcal{O}_\wp$-linear BT-groups of height $[F_\wp:\mathbb{Q}_p]$ for every place $\wp$ of F above p such that $A[\wp^\infty]$ is not supersingular, and
- $\mathrm{End}_{\mathcal{O}_\wp} A[\wp^\infty]$ is the maximal order of the quaternion division algebra over $\mathcal{O}_\wp$ for every place $\wp$ of F above F such that $A[\wp^\infty]$ is supersingular.

An equivalent definition is that the BT-group $A[p^\infty]$ is minimal.

Remark. If z_1,z_2 are two minimal points in $\mathcal{M}_{E,n}(k)$ for some algebraically closed field $k\supset\mathbb{F}_p$ such that the slopes of $A_{z_1}[\wp^\infty]$, $A_{z_2}[\wp^\infty]$ are equal for every place $\wp$ of F above p. Then the $\mathcal{O}_F\otimes\mathbb{Z}_p$-linear BT-groups $A_{z_1}[\wp^\infty]$ and $A_{z_2}[\wp^\infty]$ are isomorphic. This fact is well-known and left as an exercise.

Remark. If $1/2$ is a slope of A_y, then the target group $\mathrm{Aut}_{\mathcal{O}_{F_\wp}}(A_y[p^\infty],\lambda_y[p^\infty])$ of the (geometric and arithmetic) monodromy representations is not commutative: If the $A_y[\wp^\infty]$ is isoclinic of slope $1/2$, then the group $\mathrm{Aut}_{\mathcal{O}_{F_\wp}}(A_y[\wp^\infty],\lambda_y[\wp^\infty])$ is the subgroup of units of the maximal order of a quaternion division algebra over $F_\wp$ with reduced norm 1.
(ii) We will see below that the above method of the proof of Thm. 4.4 can be applied to the more general situation.

(4.5) Theorem. *Let F be a totally real number field such that p is unramified in F. Let $n\geq 3$ be an integer prime to p. Let $\mathcal{M}_{F,n}$ the Hilbert modular variety of level-n attached to F. Let $y=[(A,\iota_A,\mathcal{L}_A,\mathcal{L}_A^+,\lambda_A,\eta_A)]$ be a point of $\mathcal{M}_{F,n}(\mathbb{F})$ which is minimal. Then the p-adic monodromy representation*

$$\rho=\rho_{C_{\mathcal{M}_{F,n}(y)}}:\pi_1(C_{\mathcal{M}_{E,n}(y)},y)\to\mathrm{Aut}_{\mathcal{O}_F\otimes_{\mathbb{Z}}\mathbb{Z}_p}(A[p^\infty],\lambda_A[p^\infty])$$

is surjective.

(4.6) Lemma. *Let K be a finite extension field of $\mathbb{Q}_p$, and let D be a quaternion division algebra over K. Let $\mathcal{O}_D$ be the maximal order of D, and let $\mathcal{O}_{D,1}^\times$ be the subgroup of $\mathcal{O}_D^\times$ consisting of all elements of $\mathcal{O}_D$ with reduced norm 1. Let L be a quadratic extension of K, and let $h_1:L\to D$ and $h_2:L\to D$ be two F-linear embeddings of L into D. Then there exists an element $u\in\mathcal{O}_{D,1}^\times$ such that $u\cdot h_1(L)\cdot u^{-1}=h_2(L)$.*

Proof. We show first that there exists an element $v\in\mathcal{O}_D^\times$ such that $v\cdot h_1(L)\cdot v^{-1}=h_2(L)$. By Noether-Skolem, $h_1(L)$ and $h_2(L)$ are conjugate under $D^\times$. Also, there exists a generator π_D of the maximal ideal $\mathfrak{m}_D$ of $\mathcal{O}_D$ such that $\pi_D\cdot h_1(L)\cdot\pi_D^{-1}=h_1(L)$. The last statement is obvious if L is ramified over K. See [48], Chap. I, Thm. 7 on p. 16 for the case when L is unramified over K. Since $D^\times=\pi_D^{\mathbb{Z}}\times\mathcal{O}_D^\times$, there exists an element $v\in\mathcal{O}_D^\times$ such that $v\cdot h_1(L)\cdot v^{-1}=h_2(L)$.

Suppose that L is unramified over K. Since $\mathrm{Nm}_{L/K}(\mathcal{O}_L^\times) = \mathcal{O}_K^\times$, we can modify v by a suitable element of $h_1(\mathcal{O}_K^\times)$ to get a element $u \in \mathcal{O}_D^\times$ which has the required properties.

Suppose that L is ramified over K. Then there exists an element of $w \in \mathcal{O}_D^\times$ such that conjugation by w induces the nontrivial element of $\mathrm{Gal}(L/K)$, and $\mathrm{Nrd}_{D/K}(w)$ is the nontrivial element of $K^\times/\mathrm{Nm}_{L/K}(L^\times)$. See [48], Chap. IX, §4 and Chap. XII, §2. Adjusting v by the product of a power of w with an element of $L^\times$, we obtain an element u with the required properties. ∎

We also recall the following elementary fact in group theory.

(4.7) Lemma. *Let H be a subgroup of a finite group G such that H intersects every conjugacy class of G non-trivially. Then H is equal to G.*

Proof of Thm. 4.5. Assume that $1/2$ is a slope of A. Then the target group of the p-adic monodromy representation is not commutative. Some modification of the definition/construction of the p-adic monodromy group in 1.20 will be useful. Let

$$\wp_1, \ldots, \wp_a, \cdots \wp_{a+1}, \wp_{a+b}$$

be the prime of F above p, $A[\wp_i^\infty]$ is not supersingular for $i = 1, \ldots, a$, and $A[\wp_i^\infty]$ is supersingular for $i = a+1, \ldots, a+b$. Let $(Y_i, \iota_{Y_i}, \lambda_{Y_i})$ be an minimal $\mathcal{O}_{\wp_i}$-linear BT-group over $\mathbb{F}_{p^2}$ of slope $1/2$ and height $2[F_{\wp_i} : \mathbb{Q}_p]$ such that $\mathrm{Fr}_{Y_i} = p^2$. Denote by $(\mathcal{A}, \iota_{\mathcal{A}}, \lambda_{\mathcal{A}}))$ the universal $\mathcal{O}_F$-linear abelian scheme over the leaf $\mathcal{C}_F = \mathcal{C}_{\mathcal{M}_{F,n}}(y)$. Consider the smooth $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)$-sheaf

$$\mathcal{F} := \prod_{i=1}^{a} \underline{\mathrm{Isom}}_{C_F}\left((A, \iota, \lambda)[\wp_i^\infty], (\mathcal{A}, \iota_{\mathcal{A}}, \lambda_{\mathcal{A}})[\wp_i^\infty]\right) \times \prod_{i=a+1}^{a+b} \underline{\mathrm{Isom}}_{C_F}\left((A, \iota, \lambda)[\wp_1^\infty], (\mathcal{A}, \iota_{\mathcal{A}}, \lambda_{\mathcal{A}})[\wp_1^\infty]\right)$$

where the base field is a suitable finite field $\mathbb{F}_q \supseteq \mathbb{F}_{p^2}$. We will use the above torsor when considering the arithmetic and geometric p-adic monodromy of $C_{\mathcal{M}_{F,n}}(y)$. The advantage of $\mathcal{F}$ is that the action of Frobenii of Y_i over finite overfields are contained in the center of $\mathrm{End}(Y_i)$. Consider a closed point z of $C_{\mathcal{M}_{F,n}}(y)$. Let q_z be the cardinality of the residue field $\kappa(z)$ of z. Denote by $\mathrm{Fr}_z|_{A_z[p^\infty]}$ the Frobenius conjugacy class of the action of Fr_z on $A_z[p^\infty]$. Denote by $\mathrm{Fr}_{q_z}|_{Y_i}$ the the Frobenius conjugacy class of the action of Fr_{q_z} on Y_i, $i = a+1, \ldots, a+b$. Since the $\wp_i$-factor of the action of the Frobenius attached to the closed point z of on the torsor $\mathcal{F}$ is the difference "$(\mathrm{Fr}_z|_{A_z[p^\infty]}) \cdot (\mathrm{Fr}_{q_z}|_{Y_i})^{-1}$" between $\mathrm{Fr}_z|_{A_z[p^\infty]}$ and $\mathrm{Fr}_{q_z}|_{Y_i}$, the property that Fr_{Y_i} lies in the center of $\mathrm{End}\, Y_i$ make is easier to pinpoint the conjugacy class of the image of Frobenius attached to the closed point z in the target group of the p-adic monodromy representation.

The method of the proof of Thm. 4.4, together with Lemma 4.6, now shows that for all $m > 0$, $\rho(\pi_1(C(y), y)$ modulo p^m intersects non-trivially with every conjugacy class of the target group $\mathrm{Aut}_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p}(A[p^\infty], \lambda_A[p^\infty])$ of the p-adic monodromy representation modulo p^m. Apply Lemma 4.7, we see that $\rho(\pi_1(C(y), y)$ contains $\mathrm{Aut}_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p}(A[p^\infty], \lambda_A[p^\infty])$ modulo p^m for every $m > 0$. Therefore $\rho(\pi_1(C(y), y) = \mathrm{Aut}_{\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p}(A[p^\infty], \lambda_A[p^\infty])$. ∎

Remark. There is an alternative way to take care of the factors $\wp_{a+1}, \ldots, \wp_{a+b}$ where the target group of the p-adic monodromy is not commutative: Instead of Lemmas 4.6 and 4.7, use the following properties about the group of units $\mathcal{O}_D^\times$ of the maximal order of a quaternion division algebra D over a finite extension field K of $\mathbb{Q}_p$. Consider the decreasing filtration

$$\mathcal{O}_D^\times \supset 1 + \mathfrak{m}_D \supset 1 + \mathfrak{m}_D^2 \supset \cdots$$

of $\mathcal{O}_D^\times$ by the group $\Gamma_n = 1 + \mathfrak{m}_D^n$ of principal units of level n, $n = 1, 2, 3, \ldots$, where $\mathfrak{m}_D$ is the maximal ideal of $\mathcal{O}_D$. Each Γ_n is a normal subgroup of $\mathcal{O}_D^\times$. We have $\mathcal{O}_D^\times/\Gamma_1 \cong \mathbb{F}_{q^2}^\times$, while Γ_n/Γ_{n+1} is non-canonically isomorphic to $\mathbb{F}_{q^2}$ as a one-dimensional vector space over $\mathbb{F}_{q^2}$. Here q is the cardinality of the residue field $D/\mathfrak{m}_D$. The commutativity of the successive quotients makes it easy to use Ribet's method. Another property of the group $\mathcal{O}_D^\times$ is also handy: If H is a closed subgroup of $\mathcal{O}_D^\times$ such that $H \cdot \Gamma_2 = \mathcal{O}_D^\times$, then $H = \mathcal{O}_D^\times$. So for each "quaternion factor", one only has to show that the p-adic monodromy maps surjectively to the group $\mathcal{O}_D^\times/\Gamma_2$, which is the semi-direct product of $\mathbb{F}_{q^2}^\times$ by $\mathbb{F}_{q^2}$.

§5. Monodromy

We first present a proof as in [4, 7.4] of a known fact that the p-adic monodromy group of the ordinary locus of $\mathcal{A}_{g,n}$ is maximal.

Let $\mathcal{A}_{g,n}^{\rm or}$ be the ordinary locus of a Siegel modular variety $\mathcal{A}_{g,n}$ over k, where $g \geq 1$, $n \geq 3$, $(n, p) = 1$, and the base field $k \supseteq \mathbb{F}_p$ is algebraically closed. Let $A \to \mathcal{A}_{g,n}^{\rm or}$ be the universal abelian scheme over $\mathcal{A}_{g,n}^{\rm or}$. Let $A[p^\infty]_{\rm et} \to \mathcal{A}_{g,n}^{\rm or}$ be the maximal étale quotient of $A[p^\infty] \to \mathcal{A}_{g,n}^{\rm or}$; it is an étale p-divisible group group of height g. Let E_0 be an ordinary elliptic curve defined over $\overline{\mathbb{F}_p}$, and let $x_0 = (A_0, \lambda_0)$, where A_0 is the product of g copies of E_0, and λ_0 is the product principal polarization on A_0. Let $\mathrm{T}_p = \mathrm{T}_p(A_0[p^\infty])_{\rm et})$ be the p-adic Tate module of the étale p-divisible group $A_0[p^\infty]_{\rm et}$; it is naturally isomorphic to the direct sum of g copies of $\mathrm{T}_p(E_0[p^\infty]_{\rm et}) \cong \mathbb{Z}_p$, so $\mathrm{GL}(\mathrm{T}_p)$ is naturally isomorphic to $\mathrm{GL}_g(\mathbb{Z}_p)$. Let $\rho : \pi_1(\mathcal{A}_{g,n}^{\rm or}, x_0) \to \mathrm{GL}(\mathrm{T}_p)$ be the p-adic monodromy representation of $A[p^\infty] \to \mathcal{A}_{g,n}^{\rm or}$.

(5.1) Proposition. *Notation as above. The image of the p-adic monodromy representation*

$$\rho : \pi_1(\mathcal{A}_{g,n}^{\rm or}, x_0)$$

of the ordinary locus of $\mathcal{A}_{g,n}$ is equal to $\mathrm{GL}(\mathrm{T}_p) \cong \mathrm{GL}_g(\mathbb{Z}_p)$.

Proof. Let $\mathcal{B}$ be the product of g copies of $\mathcal{A}_{1,n}$, diagonally embedded in $\mathcal{A}_{g,n}$. Let E_0 be an ordinary elliptic curve defined over a $\overline{\mathbb{F}_p}$, and let $x_0 = (A_0, \lambda_0)$, where A_0 is the product of g copies of E_0, and λ_0 is the product principal polarization on A_0. Let $\mathcal{O} = \mathrm{End}(E_0)$. Then $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$, corresponding to the natural splitting of $E_0[p^\infty]$ into the product of its toric part $E_0[p^\infty]_{\rm tor}$ and its étale part $E_0[p^\infty]_{\rm et}$. So we have an isomorphism $\mathrm{End}(A_0) \cong \mathrm{M}_g(\mathcal{O})$, and a splitting $\mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathrm{M}_g(\mathcal{O}) \times \mathrm{M}_g(\mathcal{O})$ corresponding to the splitting of $A_0[p^\infty]$ into the product its toric and étale parts. Denote by $\mathrm{pr} : (\mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times \to \mathrm{GL}(\mathrm{T}_p) \cong \mathrm{GL}_g(\mathbb{Z}_p)$ the projection corresponding to the action of $\mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ on the étale factor $A_0[p^\infty]_{\rm et}$ of $A_0[p^\infty]$. The Rosati involution $*$ on $\mathrm{End}(A_0)$ interchanges the two factors of $\mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. It follows that $\mathrm{U}(\mathcal{O}_{(p)} \otimes_{\mathbb{Z}} \mathbb{Z}_p, *)$ is isomorphic to $\mathrm{GL}(\mathrm{T}_p)$ under the projection map pr, therefore the image of $\mathrm{U}(\mathcal{O}_{(p)}, *)$ in $\mathrm{GL}(\mathrm{T}_p)$ is dense in $\mathrm{GL}(\mathrm{T}_p)$. Here $\mathcal{O}_{(p)} = \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, and $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$ is the localization of $\mathbb{Z}$ at the prime ideal $(p) = p\mathbb{Z}$.

By a classical theorem of Igusa, the p-adic monodromy group of the restriction to $\mathcal{B}$, i.e. $\rho\left(\mathrm{Im}\left(\pi_1(\mathcal{B}, x_0) \to \pi_1(\mathcal{A}_{g,n}, x_0)\right)\right)$, is naturally identified with the product of g copies of $\mathbb{Z}_p^\times$ diagonally embedded in $\mathrm{GL}(\mathrm{T}_p) \cong \mathrm{GL}_g(\mathbb{Z}_p)$. Denote by D this subgroup of $\mathrm{GL}(\mathrm{T}_p)$. See [21] Thm. 4.3 on p. 149 for an exposition of Igusa's theorem.

Let $R_{(p)} = \mathrm{End}(A_0) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \cong \mathrm{M}_g(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Every element $u \in R_{(p)}$ such that $u^*u = uu^* = 1$ gives rise to a prime-to-p isogeny from A_0 to itself respecting the polarization λ_0. Such an element $u \in R_{(p)}$ gives rise to

- a prime-to-p Hecke correspondence h on $\mathcal{A}_{g,n}$ having x_0 as a fixed point, and
- an irreducible component $\mathcal{B}'$ of the image of $\mathcal{B}$ under h such that $\mathcal{B}' \ni x_0$.

By the functoriality of the fundamental group, the image of the fundamental group $\pi_1(\mathcal{B}', x_0)$ of $\mathcal{B}'$ in $\pi_1(\mathcal{A}_{g,n}^{\mathrm{or}}, x_0)$ is mapped under the p-adic monodromy representation ρ to the conjugation of D by the element $\mathrm{pr}(h) \in \mathrm{GL}(\mathrm{T}_p)$. In particular, $\rho(\pi_1(\mathcal{A}_{g,n}^{\mathrm{or}}, x_0))$ is a closed subgroup of $\mathrm{GL}(\mathrm{T}_p)$ which contains all conjugates of D by elements in the image of $\mathrm{pr} : \mathrm{U}(E_{(p)}, *) \to \mathrm{GL}(\mathrm{T}_p)$.

Recall that the image of $\mathrm{U}(E_{(p)}, *)$ in $\mathrm{GL}(\mathrm{T}_p)$ is a dense subgroup. So $\rho(\pi_1(\mathcal{A}_{g,n}, x_0))$ is a closed normal subgroup of $\mathrm{GL}(\mathrm{T}_p) \cong \mathrm{GL}_g(\mathbb{Z}_p)$ which contains the subgroup D of all diagonal elements. An exercise in group theory shows that the only such closed normal subgroup is $\mathrm{GL}_g(\mathbb{Z}_p)$ itself.

Remark. (i) There are at least two published proofs of Prop. 5.1 in the literature, in [12] and [15, chap. V §7] respectively.

(ii) The proof above is group-theoretic in nature. The idea is to show that the subgroup of the p-adic monodromy group generated by the p-adic monodromy group of (subvarieties of) Shimura subvarieties is equal to the target of the p-adic monodromy group. One uses a hypersymmetric point x_0 as a base point, contained in many Shimura subvarieties. These Shimura varieties are obtained by hitting a Hilbert modular subvariety passing through x_0 with elements of the stabilizer subgroup of x_0 in the prime-to-p Hecke correspondences. Since one knows that the p-adic monodromy of the Hilbert modular variety, as a subgroup of automorphisms of the $A_{x_0}[p^\infty]_{\mathrm{et}}$ group theory (and the functoriality of the fundamental group) implies that the p-adic monodromy in question is "as large as possible".

(5.2) Theorem. *Let* $[(A, \lambda), b] = x \in \mathcal{A}_{g,1,n}$ *be a* principally polarized, minimal, non-supersingular *abelian variety. Write* $\xi = \mathcal{N}(A)$. *Consider the central stream* $C = C(x) = \mathcal{Z}_\xi \subset \mathcal{A}_{g,1,n}$. *Consider the p-adic monodromy*

$$\rho_C : \pi_1(C, x) \longrightarrow \mathrm{Aut}((A, \lambda)[p^\infty]).$$

The image of this monodromy map is maximal, i.e.

$$\mathrm{Im}(\rho_C) = \mathrm{Aut}((A, \lambda)[p^\infty]).$$

Thm. 5.2 will be proved by the method illustrated in the proof of Prop. 5.1. We first record some lemmas in group theory.

(5.3) Lemma. *Let D be a central division algebra over $\mathbb{Q}_p$ with $\dim_{\mathbb{Q}_p}(D) = d^2$. Let $\mathcal{O}_D$ be the maximal order of D, and let $W(\mathbb{F}_{p^d}) \subset \mathcal{O}_D$ be an embedding of $W(\mathbb{F}_{p^d})$ in $\mathcal{O}_D$. Then the smallest closed normal subgroup of $\mathcal{O}_D^\times$ containing $W(\mathbb{F}_{p^d})$ is equal to $\mathcal{O}_D^\times$*

Proof. There exists a generator $\pi = \pi_D$ of the maximal ideal $\mathfrak{m}_D$ of $\mathcal{O}_D$ such that $\pi \cdot W(\mathbb{F}_{p^d})\pi^{-1} = W(\mathbb{F}_{p^d})$ and $\sigma := \mathrm{Ad}(\pi)|_{W(\mathbb{F}_{p^d})}$ is induced by a generator of $\mathrm{Gal}(\mathbb{F}_{p^d}/\mathbb{F})$. See [48], Chap. I, §4, Thm. 7, p. 16. Notice that σ induces an automorphism of the group μ of Teichmüller representatives of $\mathbb{F}_{p^d}^\times$ in $W(\mathbb{F}_{p^d})^\times$.

Let H be the topological closure of the subgroup of $\mathcal{O}_D^\times$ generated by the $\mathcal{O}_D^\times$ conjugates of $W(\mathbb{F}_{p^d})$. Clearly H surjects to $\mathbb{F}_{p^d}^\times$ under the natural projection $\mathcal{O}_D^\times \to \mathbb{F}_{p^d}^\times$. We have

$$(1+\pi^i a)\zeta(1+\pi^i a)^{-1}\zeta^{-1} \equiv 1 + \pi_i \left(a(\zeta - \zeta^{\sigma^{-1}})\right) \pmod{\mathfrak{m}_D^{2i}} \qquad \forall a \in W(\mathbb{F}_{p^d}),\ \forall \zeta \in \mu\,.$$

This implies that $H \cap (1+\mathfrak{m}_D^i)$ surjects to $1+\mathfrak{m}_D^i/1+\mathfrak{m}_D^{i+1}$ for all $i > 0$ such that $i \not\equiv 0 \pmod{d}$. On the other hand, write $p = \pi^d \cdot u_0$ with $u_0 \in \mathcal{O}_D^\times$. We have

$$1 + p^m a \equiv 1 + \pi^{md} u_0^m a \pmod{\mathfrak{m}_D^{md+1}}, \quad \forall a \in W(\mathbb{F}_{p^d})\,.$$

The last formula shows that $H \cap (1+\mathfrak{m}_D^i)$ surjects to $1+\mathfrak{m}_D^i/1+\mathfrak{m}_D^{i+1}$ if $i \equiv 0 \pmod{d}$. We conclude that $H = \mathcal{O}_D^\times$ by successive approximation. □

(5.4) Lemma. *Notation as in Lemma 5.3. Let $n \geq 2$ be an integer. Let $\Delta \cong \mathcal{O}_D^\times \times \cdots \times \mathcal{O}_D^\times$ be the subgroup of diagonal matrices in $\mathrm{GL}_n(\mathcal{O}_D)$. Then $\mathrm{GL}_n(\mathcal{O}_D)$ is the only closed normal subgroup of $\mathrm{GL}_n(\mathcal{O}_D)$ which contains Δ.*

Proof. It suffices to check the case when $n = 2$. Let N be a normal closed normal subgroup of $\mathrm{GL}_n(\mathcal{O}_D)$ which contains Δ. The formula

$$\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & (ad^{-1}-1)u \\ 0 & 1 \end{pmatrix}$$

shows that N contains all upper triangular nilpotent 2×2 matrices with entries in $\mathcal{O}_D$. Similarly, N contains all lower triangular nilpotent 2×2 matrices with entries in $\mathcal{O}_D$. These nilpotent matrices, together with the subgroup Δ of diagonal matrices, generate $\mathrm{GL}_2(\mathcal{O}_D)$. Hence $N = \mathrm{GL}_2(\mathcal{O}_D)$. □

(5.5) Proof of Thm. 5.2. We will show that the method of Prop. 5.1, together with Thm. 4.4 and Lemmas 5.3 and 5.4, implies Thm. 5.2 if $\frac{1}{2}$ is not a slope of the abelian variety A. The proof of the general case is similar, using Thm. 4.5 instead of 4.4.

We may and do assume that the abelian variety A corresponding to the modular point x is defined over $\overline{\mathbb{F}_p}$ and hypersymmetric. Moreover, we may and do assume that A is a product of abelian varieties $A_1, \ldots, A_s$ such that each A_i has exactly two slopes, and the slopes of A_i and A_j are disjoint if $i \neq j$. Then the target group of the p-adic monodromy representation ρ_C decomposes into a product

$$\mathrm{Aut}\,((A_1, \lambda_1)[p^\infty]) \times \cdots \times \mathrm{Aut}\,((A_s, \lambda_s)[p^\infty])\ .$$

So we may and do assume that $s = 1$, i.e. A has only two slopes, is defined over $\overline{\mathbb{F}_p}$ and is hypersymmetric; $A[p^\infty]$ is minimal.

Under the above assumptions, $A[p^\infty]$ is the product of two isoclinic minimal BT-group over $\overline{\mathbb{F}_p}$, denoted S and Q, such that $\mu :=$ slope of $Q <$ slope of S $= 1 - \mu$. Moreover, S and Q are dual to each other. The two projections $\mathrm{pr}_S : \mathrm{Aut}((A, \lambda)[p^\infty] \to \mathrm{Aut}(S)$ and $\mathrm{pr}_Q : \mathrm{Aut}((A, \lambda)[p^\infty] \to \mathrm{Aut}(Q)$ are both automorphisms; we will use pr_Q. The endomorphism group $\mathrm{End}(Q)$ is (non-canonically) isomorphic to $\mathrm{M}_n(\mathcal{O}_D)$, where D is a central division algebra over $\mathbb{Q}_p$ with Brauer invariant μ, $nd = \dim(A) =: g$, $d^2 = \dim_{\mathbb{Q}_p}(D)$, and d is the denominator of the rational number μ. So we may identify the image G of the p-adic monodromy representation with $\mathrm{GL}_n(\mathcal{O}_D)$. The unitary group attached to the semisimple algebra with involution $(\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}, *_\lambda)$ "is" the stabilizer subgroup at x in the set of all prime-to-p Hecke correspondences.

Choose a totally real number field F contained in $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $[F : \mathbb{Q}] = g := \dim(A)$, p is unramified in F, and $\mathcal{O}_F \otimes \mathbb{Z}_p$ is isomorphic to a product of n copies of the $W(\mathbb{F}_{p^d})$. Then we obtain a Hilbert modular variety $\mathcal{M}_{F,n}$, a finite etale morphism, $f : \mathcal{M}_{F,n} \to \mathcal{A}_{g,1,n}$ and a point $y \in \mathcal{M}_{F,n}$ which is mapped to x under f. By functoriality of the fundamental group, we know that the image H of the p-adic monodromy representation ρ contains the product of n copies of $W(\mathbb{F}_{p^d})^\times$. The argument of Prop. 5.1 shows that H is a normal subgroup of G. By Lemma 5.3 and Lemma 5.4, $H = G$. ∎

The following Corollaries are formal consequences of Thm. 5.2.

(5.6) Corollary. (= Theorem B) *Let $[(B, \mu), b] = y \in \mathcal{A}_{g,d,n}(k)$ with B non-supersingular. The leaf $C(y) \subset \mathcal{A}_{g,d,n}$ is geometrically irreducible.*

(5.7) Corollary. (= Theorem C) *Let $[(B, \mu), b] = y \in \mathcal{A}_{g,d,n}(k)$, with B non-supersingular. Consider the leaf $C = C(y) \subset \mathcal{A}_{g,d,n}$. The image of the p-adic monodromy*

$$\rho_C : \pi_1(C, y) \longrightarrow \mathrm{Aut}((B, \mu)[p^\infty])$$

is maximal, i.e.

$$\mathrm{Im}(\rho_C) = \mathrm{Aut}((B, \mu)[p^\infty]).$$

Proof of Cor. 5.6. Let C be a leaf in $\mathcal{A}_{g,1,n}$ with the same Newton polygon as $C(y)$. For every positive integer $n > 0$, let $T_n \to C$ be the finite surjective morphism obtained from C by trivializing the universal BT_n-group $A[p^n] \to C$ over C. By [35], there exists a surjective dominant morphism $f_n : T_n \to C(y)$ if n is sufficiently large. Since T_{irred} is irreducible by Thm. 5.2, we conclude that $C(y)$ is irreducible. ∎

Proof of Cor. 5.7. Let $x = [(A, \lambda)] \in \mathcal{A}_{g,1,n}(k)$ be a k- point of $\mathcal{A}_{g,1,n}$ such that there exists an isogeny $\alpha : B \to A$ satisfying $\alpha^*(\lambda) = \mu$. We may and do assume that x is hypersymmetric and defined over $\overline{\mathbb{F}_p}$ This assumption simplifies the description of the action of Γ on $\widetilde{T}$ below.

For every integer $n \geq 1$, denote by $\mathcal{T}_n$ the perfection of the finite étal cover of $C(x)$ corresponding to the quotient $G := \mathrm{Aut}(A[p^\infty], \lambda[p^\infty]) \twoheadrightarrow \mathrm{Aut}(A[p^n], \lambda[p^n])$ of the p-adic monodromy group of $C(x)$. Notice that $\mathcal{T}_n$ is a projective limit of irreducible smooth k-schemes with the same underlying topological space, hence $\mathcal{T}_n$ is an integral scheme. In particular, $\mathcal{T}_1$ has the same underlying topological space as $C(x)$. Also, the topological space underlying $\mathcal{T}_n$ coincide with that of the scheme T_n in the proof of Cor. 5.6. Let $\widetilde{\mathcal{T}} = (\mathcal{T}_n)_{n \in \mathbb{Z}_{>0}}$ be the projective system formed by the $\mathcal{T}_n$'s, with the natural maps $\mathcal{T}_n \to \mathcal{T}_m$ as transition maps for $m|n$. The profinite group G operates on the projective system $\widetilde{\mathcal{T}}$, making G the Galois group for the profinite étal covering $\widetilde{\mathcal{T}}/\mathcal{T}_1$. The projective limit $\varprojlim \widetilde{\mathcal{T}}$, being a projective limit of connected topological spaces, is connected.

Let $\Gamma = \mathrm{U}\left(\mathrm{End}(A[p^\infty) \otimes_{\mathbb{Z}} \mathbb{Q}, *_\lambda\right) = \left\{\gamma \in (\mathrm{End}(A[p^\infty] \otimes_{\mathbb{Z}} \mathbb{Q})^\times \mid x \cdot x^* = x^* x = \mathrm{Id}\right\}$; Γ is a locally compact group which contains G as a compact open subgroup. Fix a point $\tilde{x}$ of $\mathcal{T}$ lying above x. Let $\widetilde{A} \to \widetilde{\mathcal{T}}$ be the universal abelian scheme over $\widetilde{\mathcal{T}}$. There is a natural extension of the action of G on $\widetilde{\mathcal{T}}$ to Γ, characterized by the following property: For every $\gamma \in \mathrm{End}(A) \otimes_{\mathbb{Q}} \cap \Gamma$, there exists a $\mathbb{Q}$-isogeny

$$(\tilde{\gamma}, \gamma) : (\widetilde{A} \to \widetilde{T}) \to (\tilde{A} \to \widetilde{\mathcal{T}}),$$

or equivalently, a $\mathbb{Q}$-isogeny $\tilde{\gamma}$ from $A \to \widetilde{\mathcal{T}}$ to $\gamma^*(A \to \widetilde{\mathcal{T}})$), such that the fiber of $\tilde{\gamma}$ over $\tilde{x}$ is equal to γ.

Consider the compact group $H := \mathrm{Aut}\left((B, \mu)[p^\infty]\right)$. We identify H as a compact open subgroup of Γ via the isogeny $\alpha : B \to A$. It is easy to see that there exists an abelian scheme $\widetilde{B} \to \widetilde{\mathcal{T}}$ such that α extends to an isogeny $\tilde{\alpha} : \widetilde{B} \to \widetilde{A}$ over $\widetilde{T}$. Moreover, the action of H on $\widetilde{T}$ lifts to quasi-isogenies on $\widetilde{B}$. So $\widetilde{B} \to \mathcal{T}$ descends to an abelian scheme $\mathcal{B} \to \mathcal{T}/H$, giving rise to a dominant morphism $f_H : \mathcal{T}/H \to C(y)$. The scheme $\mathcal{T}/H$ is connected and (geometrically) irreducible, because $\varprojlim \widetilde{\mathcal{T}}$ is connected and each $\mathcal{T}_n$ is smooth. So the image of the p-adic monodromy ρ_C contains H. ∎

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