Abelian varieties and p-divisible groups

Frans Oort

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Introduction

It is nice to give a talk at this event on the influential paper:

> [M] Yu. I. Manin – *The theory of commutative formal groups over fields of finite characteristic.* Usp. Math. **18** (1963), 3-90; Russ. Math. Surveys **18** (1963), 1-80.

As Yuri Manin explained to me:

"Formal groups" was my "habilitation" thesis:
it is the second thesis in the Russian system, earning "Doctor of science",
whereas PhD is called "Candidate of science".

As you will see soon, this thesis contained material to keep several mathematicians busy for the 40 years to follow.

In the first section I will *briefly* discuss the contents of the paper, however not using notations from 1963, but using notions the way we phrase these concepts now.

In the second section we will see *questions, suggestions and conjectures* contained in the paper.

In the last section I will discuss some material developed in 1964 – 2005, mostly stimulated by ideas and concepts of this paper. As you will see this answers almost all questions posed in this thesis. This paper has been a source of inspiration.

Although I will follow material of [M] closely, I will try to use notations which are standard now, sometimes slightly different from phrasing as used in [M].
As in [M], in this note I will use *contravariant* Dieudonné module theory.
An "equidimensional commutative formal group" as in [M] will be a called a "p-divisible formal group".
A p-divisible group called "homogeneous" in [M], see page 38 is now called "isoclinic" (all Newton slopes are equal). Something which could have been called a "formal isogeny type", as in [M], Classification Theorem on page 35, and in [M] Th. 4.1 on pp. 72/73 will be encoded by means of a Newton polygon. The Serre dual of a p-divisible group X will be denoted by

X^t. The p-divisible group denoted by Manin as $G_{m,n}$, having properties like $\dim(G_{m,n}) = m$, and $\dim((G_{m,n})^t) = n$, will give rise to the slope $m/(m+n)$ with multiplicity $m+n$.

All base fields will be of characteristic p. We write k for an algebraically closed field. We will write A and B, etc. for an abelian variety, and X and Y, etc. for a p-divisible group.

1 Contents of the paper by Manin

In Chapter I Manin discusses the theory of contravariant Dieudonné modules of finite group schemes and of formal groups over a perfect field.

In Chapter II Manin describes isogeny classes of p-divisible groups over a perfect field and over an algebraically closed field. Here is the main result: *for a pair of coprime positive integers $m, n \in \mathbb{Z}$ there exists an isosimple formal group $G_{m,n}$ of dimension m, with* $\dim((G_{m,n})^t) = n$. *Any formal p-divisible group X over k is up to isogeny a direct sum of such groups:*

$$X \quad \sim \quad \sum_i G_{m_i,n_i};$$

see [M] Classification Theorem on page 35. Note that for any abelian variety (for example defined over a field of large transcendence degree over $\mathbb{F}_p$), its p-divisible group is isogenous with a p-divisible group defined over the prime field.

Note that the results in this section show there is a bijection between the set of k-isogeny classes of p-divisible groups over k and the set of Newton polygons:

Theorem (Dieudonné and Manin), see [M], "Classification theorem " on page 35 .

$$\{X\}/\sim_k \quad \xrightarrow{\sim} \quad \{\text{Newton polygon}\}$$

In Chapter III Manin constructs a moduli space for all isomorphism classes of formal groups inside one given isogeny class. The crucial notion here is the definition of a *special p-divisible group*, see [M], page 38. In Th.3.5 on page 44 we see that *the minimal possible degree of an isogeny $X \to X_0$ from any p-divisible group X in the isogeny class of a special X_0 is universally bounded inside that isogeny class.* Once this established we see that *the moduli space of all such isogenies is given as a closed set in an appropriate Grassmannian.*

In Chapter IV Manin discusses Newton polygons coming from an abelian variety. The "formal structure of an abelian variety A", now called the Newton polygon of A, is discussed, written as $\mathcal{N}(A)$:

$$A \quad \mapsto \quad A[p^\infty] =: X \quad \mapsto \quad \mathcal{N}(X) \quad =: \quad \mathcal{N}(A).$$

For an abelian variety A over a finite field Manin proves that the Newton polygon of an abelian variety is symmetric in the sense that the isogeny factors $G_{m,n}$ and $G_{n,m}$ appear with the same multiplicity; or in other words: a slope λ appears with the same multiplicity in $\mathcal{N}(A)$ as the slope $1 - \lambda$.

Manin then formulates his celebrated
Manin Conjecture [M], Conjecture 2 on page 76:

$$\textit{a Newton polygon } \xi \textit{ comes from an abelian variety} \quad \Leftrightarrow \quad \xi \textit{ is symmetric.}$$

In [M] we find some evidence supporting this conjecture:

Theorem 4.2. *Every p-divisible group appears up to isogeny as a direct summand of an abelian variety.*
Indeed, for coprime integers $m > n > 0$ the formal group $G_{m,n}$ is contained up to isogeny in the p-divisible group of the Jacobian of the curve given by

$$Y^p - Y \quad = \quad X^{p^{m+n}-1}.$$

Example 1 on pp. 77/78:

$$\mathrm{Jac}(Y^2 = X^7 - X + 1) \quad \sim \quad G_{2,1} + G_{1,2}.$$

Example 2 on p. 78:

$$\mathrm{Jac}(Y^2 = X^9 + X^7 + X + 1) \quad \sim \quad G_{3,1} + G_{1,3}.$$

Example (c) on p.80, p=2:

$$\mathrm{Jac}(Y^2 - Y = X^5 + aX^3) \quad \sim \quad G_{1,1} + G_{1,1}.$$

2 Questions, suggestions and conjectures by Manin formulated in this paper

(2.1) A. The duality theorem. See. pp. 70, 72. 73, 74. From [M], page 70, especially displays (4.2) and (4.3): *for an isogeny φ between abelian varieties, and $G = \mathrm{Ker}(\varphi)$ it should hold that the Cartier dual G^D equals the kernel of the dual isogeny, $G^D \cong \mathrm{Ker}(\varphi^t)$.*

Manin remarks, page 70, that *"this variant of the duality of A. Weil apparently does not follow from results in the existing literature."*

Judging from remarks on page 80 one could deduce that Manin was thinking of the duality theorem for abelian varieties defined over fields.

If duality theorem holds over every field then for every abelian variety A the Newton polygon $\mathcal{N}(A)$ is symmetric.

(2.2) B. The Manin conjecture. See Conjecture 2, page 76:

$$\xi \text{ is symmetric} \quad \Leftrightarrow \quad \exists A: \ \mathcal{N}(A) = \xi.$$

On page 77 we read: *"I do not know how to construct abelian varieties with such properties in the general case; the difficulty is increased by the fact that it is clearly hopeless to try and obtain such varieties by reduction mod p an abelian variety of characteristic zero with the required properties."*

(2.3) C. Weil numbers: isogeny classes. See pp. 33, 72, 80. In II.3 on page 33 Manin focuses on the geometric Frobenius. On page 72 the eigenvalues of endomorphisms appear. On page 80 Manin stresses *"the connection between the formal structure and the characteristic polynomial of the Frobenius endomorphisms"*.

(2.4) D. Cartier theory. In [M] I.5, Comments, we read: *"A question arises naturally is the study of formal groups over rings, and not merely over fields."*

This theory was worked out by Cartier, see [1], [2]. Also see the nice survey [25].

(2.5) E. Jumps, the Grothendieck conjecture. We have seen that Manin constructed moduli spaces for isomorphism classes *within one isogeny class.* Commenting on this, Manin writes, page 45: we are not able *"to touch on the question of specializations in the moduli space. The possible 'jumps' from one component to another under specialization seem to possess a very complex character. With some definitions, specializations may even change the isogeny class ..."*.

(2.6) F. Structure and dimension of strata. On page 78 we read: *"It can be shown that the points of the parameter space for which the completion of the Jacobian variety corresponding to the curve has a given isogeny type are constructible sets. What is the structure of there sets and in particular their dimension ?"*

(2.7) G. Supersingular. On page 79 we read: *"For $g = 1$, all elliptic curves whose completion is isomorphic to $G_{1,1}$ are isogenous among themselves 'globally'. Is this also true for $g = 2$ and the case $\hat{J} \sim 2G_{1,1}$?"*

This indeed is true. Define an abelian variety A over a field $K \supset \mathbb{F}_p$ to be *supersingular* if $A \otimes_K k \sim E^g$, where E is a supersingular elliptic curve. Write σ for the Newton polygon having all slopes equal to 1/2.

$$\mathcal{N}(A) = \sigma \quad \Longleftrightarrow \quad A \text{ is supersingular.}$$

Various details were proved by: Tate, FO, Deligne, Shioda; for references see [10], 1.6.

(2.8) H. Moduli spaces for isomorphism classes. We have seen in Chapter II the construction of moduli spaces classifying isomorphisms classes inside one isogeny class.

This question was taken up again in [5], Seection 5. For one of the main ingredients, the *finiteness theorem* [M], Th. 3.5 on page 44, also see: [20], 1.6.1; we see that this theorem of Manin holds in a more general case: we need not restrict to the case of one isogeny class, uniformity holds with respect to the height.

(2.9) I. Newton polygons of curves. We have seen that the Manin conjecture was supported by examples involving constructions of algebraic curves. Describing strata of curves with a given Newton polygon, see page 78: *"A complete answer to this question presupposes fairly precise information on the moduli space of curves of a given genus."*

3 Some developments in 1964 – 2005 in this field

(3.1) A. The duality theorem. While Manin was working on his thesis, another manuscript was being prepared, see [12]. Theorem 19.1 in that reads:

Duality theorem. *Let S be a locally noetherian base scheme. Let $\varphi : A \to B$ be an isogeny of abelian schemes over S, with kernel $N = \mathrm{Ker}(\varphi)$. The exact sequence*

$$0 \quad \to \quad N \quad \longrightarrow \quad A \quad \stackrel{\varphi}{\longrightarrow} \quad B \quad \to \quad 0$$

gives rise to an exact sequence

$$0 \quad \to \quad N^D \quad \longrightarrow \quad B^t \quad \stackrel{\varphi^t}{\longrightarrow} \quad A^t \quad \to \quad 0.$$

This makes some results of [M] more general. Also it shows that some aspects of the theory are truly geometric (and it is not necessary to work over a finite field), whereas some other results are more arithmetic in nature (e.g. the Honda-Serre-Tate theory, see below), where it is essential to work over a finite field.

(3.2) B & C. The Manin conjecture, Weil numbers. Through the work by A. Weil it became clear that eigenvalues of a geometric Frobenius of a variety over a finite field are important. For curves and abelian varieties Weil proved that such a number satisfies the Riemann hypothesis.

Definition. *Let $q = p^a$. A q-Weil number π is an algebraic integer such that for every embedding $\psi : \mathbb{Q}(\pi) \to \mathbb{C}$ we have $\mid \psi(\pi) \mid = \sqrt{q}$.*

We say that π and π' are *conjugated* if there exists an isomorphism $\mathbb{Q}(\pi) \cong \mathbb{Q}(\pi')$ mapping π to π'.

Theorem (Weil). *Let A be an abelian variety over $K = \mathbb{F}_q$. Let $(F_A)^a = \pi \in \mathrm{End}(A)$. Then π is a q-Weil number.* See [11], Th. 4 on page 206.

Theorem (Honda, Serre and Tate). *Fix a finite field $K = \mathbb{F}_q$. The assignment $A \mapsto \pi = (F_A)^a$ induces a bijection from the set of K-isogeny classes of K-simple abelian varieties defined over K and the set of conjugacy classes of q-Weil numbers.*
See [24].

In the proof of this theorem it is necessary to construct "enough" abelian varieties. This is done by choosing an appropriate CM-type abelian variety in characteristic zero, and deducing from its CM-type properties of the Newton polygon of its reduction mod p, see [24], §4 and §5. In this way the Manin conjecture was proved:

See [24], page 98. For coprime integers $m > n > 0$ choose the polynomial $T^2 + p^nT + p^{m+n}$. A zero π of this polynomial is a q-Weil number with $q := p^{m+n}$. By the Honda-Serre-Tate theory this q-Weil number defines an isogeny class; properties of π show that any abelian variety A in this isogeny class has formal group $A[p^\infty] \sim G_{m,n} + G_{n,m}$. *Hence the Manin Conjecture is proved.*

Note that a characteristic zero abelian variety with "the same properties as A" was not constructed; however choosing an appropriate abelian variety in characteristic zero, and reduction mod p turned out to be a crucial step in the way of proving the Honda-Serre-Tate theory, and hence of proving the Manin conjecture.

(3.3) D. Cartier theory. In [M] I.5, Comments, we read: *"A question arises naturally is the study of formal groups over rings, and not merely over fields."*
This theory was worked out by Cartier, see [1], [2]. Also see the nice survey [25].

(3.4) E. Jumps, the Grothendieck conjecture. As Manin remarks, under deformations, or under specializations, the formal isogeny type may change. This aspect of the theory turns out to be a difficult one. The main obstacle is that, although deformation theory can be formulated quite well (in abstract terms, also in the language of "displays") it is not easy to read off from a given deformation the Newton polygon of the generic fiber. Grothendieck was very well aware of this:

(3.5) A conjecture by Grothendieck. (GC) *Suppose given a p-divisible group X_0 over a field K, with Newton polygon $\beta := \mathcal{N}(X_0)$. Suppose given a lower Newton polygon: $\gamma \succ \beta$. Conjecture: there exist an integral scheme S over K, a closed point $0 \in S(K)$, and a p-divisible group $\mathcal{X} \to S$ such that over the generic point $\eta \in S$ we have $\mathcal{N}(\mathcal{X}_\eta) = \gamma$ (?)*
See [4], page 150.

Here is a variant:
(GCpp) *Suppose given a* principally *quasi-polarized (X_0, λ_0). Suppose given symmetric Newton polygons $\xi \succ \zeta := \mathcal{N}(X_0)$. Does there exist a deformation of quasi-polarized p-divisible groups $(\mathcal{X}, \lambda) \to S$ such that $\mathcal{N}(\mathcal{X}_\eta) = \xi$?*

Remarks. It is easy to give counterexamples to an analogous conjecture for polarized p-divisible groups, or polarized abelian schemes, if the the polarization is not supposed to be *principal.*

If **(GCpp)** holds, then the analogous conjecture is true for principally polarized abelian schemes; this follows using the Serre-Tate theorem.

(GCpp) $\Rightarrow$ (MC): *If this conjecture* **(GCpp)** *holds, then the Manin conjecture follows.* Indeed, for a given symmetric Newton polygon ξ, we choose A_0 to be a principally polarized supersingular abelian variety. Clearly $\mathcal{N}(A_0) = \sigma \prec \xi$. If **(GCpp)** does hold, we can construct $\mathcal{N}(A_\eta) = \xi$, thus proving **MC**.

Note that we are free to choose the supersingular A_0; this allows us to use a weaker version of the (analogon of the) Grothendieck conjecture. This is the way the Manin conjecture is proved in [17]. Note that this proof does not use any information on abelian varieties in characteristic zero: this is a pure characteristic p proof.

The conjectures **GC** and **GCpp** have been proved, see [17] and [19], see below.

(3.6) F. Structure and dimension of strata. Manin remarks that Newton polygon strata are *constructible sets*, see page 45. In fact much more is true. We introduce a partial ordering on the set of Newton polygons. We write $\gamma \succ \beta$ if these Newton polygons have the same endpoint, and no point of γ is above β, i.e. γ is "below" β. Explanation: strata defined by γ contain strata defined by β, hence for γ is "below" β we feel as γ being larger than β. Grothendieck proved: *"Newton polygons go up under specialization"*. This can be made more precise.

For a given p-divisible group $\mathcal{X} \to S$ and given γ we define

$$\mathcal{W}_\gamma(S) := \{s \mid \mathcal{N}(\mathcal{X}_s) \prec \gamma\}.$$

Theorem (Grothendieck-Katz).

$$\mathcal{W}_\gamma(S) \quad \subset \quad S$$

is a closed subset.
See [6], Th. 2.3.1 on page 143.

We write $W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)$. In [16], [17], [5] and [19] we find:
- a proof of **(GC)**, **(GCpp)**, **(MC)**;
- a computation of dimension of Newton polygon strata in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ and of Newton polygon strata in universal deformation spaces of (quasi-polarized) p-divisible groups.

Here are some remarks on methods.

The theory of *displays*, as developed by Mumford, Norman and FO enables us to perform direct computations on deformation spaces.

Define $a(X) = \dim_K(\mathrm{Hom}(\alpha_p, X)$ for a commutative group scheme over a perfect field $K \supset \mathbb{F}_p$.

Theorem (see [17]). *Let X_0 be a p-divisible group over a field K. Let $D = \mathrm{Def}(X_0)$ be the universal characteristic p deformation space. Suppose $a(X_0) = 1$. Let $\beta := \mathcal{N}(X_0)$.* For any Newton polygon γ belonging to dimension d and height $h = d + c$ we write

$$\diamond(\gamma) := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < c, \quad y < x, \quad (x, y) \ \text{not below} \ \gamma\}.$$

For any $\gamma \succ \beta$:

$$\mathcal{W}_\gamma(D) \quad \cong \quad \mathrm{Spf}(K[[t_{(x,y)} \mid (x, y) \in \diamond(\gamma)]]).$$

In particular, these strata are nonsingular, and for $\gamma \succ \beta$ we have $\mathcal{W}^0_\gamma(D) \neq \emptyset$.
The proof uses a variant of the Cayley-Hamilton theorem, as known in linear algebra.

The analogous theorem holds for principally quasi-polarized p-divisible groups with $a(X_0) = 1$, for principally polarized abelian schemes $a(A_0) = 1$, and symmetric Newton polygons.

As there exist principally polarized supersingular abelian varieties of any dimension, we conclude from the theorem:

Corollary (the Manin conjecture). *For every prime number p, and for every symmetric Newton polygon ξ there exists an abelian variety A over a finite field in characteristic p with $\mathcal{N}(A) = \xi$.*

However, deformation theory of a p-divisible group X_0 with $a(X_0) > 1$ in most cases gives Newton polygon strata which are highly singular; direct computations seem difficult to perform. Therefore we first prove:

Lemma. *Let X_0 be a p-divisible group; there exists a deformation $\mathcal{X} \to S$ over an integral base, such that $\mathcal{N}(X_0) = \mathcal{N}(X_\eta)$ and $a(X_\eta) \leq 1$.*
I.e. we can move out of a (singular) point with $a(X_0) > 1$ to a non-singular point with $a(X_0) = 1$.

Also variants of this are proven for principally quasi-polarized p-divisible groups and for principally polarized abelian schemes.

This difficult lemma is the crucial method in the whole technique. It uses "Purity" as proven in [5]. Then it uses a variant of the methods developed by Manin in [M], Chapter II, see (2.8). A tricky computation shows that *this moduli of isomorphism classes in an isosimple isogeny class* is *irreducible*, see [5], Theorem 5.11; this shows that the $(a = 1)$-points are dense in this moduli space; hence the lemma is proved in this case. Once arrived at that point it is merely a matter of carefully bookkeeping to prove the lemma in full generality, see [19], Th. 2.10 and Th. 4.1..

The lemma reduces the general case to the case with $(a = 1)$. The Cayley-Hamilton method finishes off, and gives all information we would like to have:

Theorem. *The Grothendieck conjecture* **GC** *holds. The analogon* **GCpp** *holds for principal quasi-polarized p-divisible groups, and for principally polarized abelian schemes.*
See [17] and [19].

We see that the question of Manin for the structure and their dimension of Newton polygon strata inside $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is fully and satisfactorily answered. However, the "complete answer" which Manin asks for on page 78 seems far away, see (3.7).

(3.7) I. Newton polygons of curves. *Which Newton polygons show up on the moduli space $\mathcal{M}_g = \mathcal{M}_g \otimes \mathbb{F}_p$ of curves in characteristic p?*

This is the only question in [M] which is still unanswered. It has become clear what an answer might be, but this problem seems still wide open. There is an enormous amount of literature on this topic. Let us mention only one detail. *We expect that not all symmetric Newton polygons appear on $\mathcal{M}_g$.* Here is a candidate.

(3.8) Expectation?. *Let $g = 11$, and let ξ be the Newton polygon with slopes $5/11$ and $6/11$. We expect:*

$$\mathcal{W}^0_\xi(\mathcal{M}_g \otimes \mathbb{F}_p) \stackrel{?}{=} \emptyset.$$

Here $\mathcal{W}^0_\xi(\mathcal{M}_g \otimes \mathbb{F}_p) := \{[C] \mid \mathcal{N}(\mathrm{Jac}(C) = \xi)\}$. See [21] for details.

(3.9) Conjecture?. *Let $g', g'' \in \mathbb{Z}_{>0}$; let ξ', respectively ξ'' be a symmetric Newton polygon appearing on $\mathcal{M}_{g'} \otimes \mathbb{F}_p$, respectively on $\mathcal{M}^0_{g''} \otimes \mathbb{F}_p$; write $g = g' + g''$. Let ξ be the Newton polygon obtained by taking all slopes with their multiplicities appearing in ξ' and in ξ''. We conjecture that in this case ξ appears on $\mathcal{M}_g$.*

If this were true, then we would have as
Corollary?. *For any prime number p, and any $g \in \mathbb{Z}_{>0}$ the supersingular locus $\mathcal{W}_\sigma(\mathcal{M}_g)$ is non-empty; i.e. we conjecture that for any genus in every characteristic supersingular curves do exist.* This has been proved for many values of g and p (Van der Geer and Van der Vlugt, Scholten and Zhu, Re).

Here is the essence of (3.8) and (3.9): for slopes with "large denominators" it might be that such a Newton polygon does not show up on $\mathcal{M}_g$. However for a Newton polygon where all slopes have "small denominators" we think it does show up on $\mathcal{M}_g$.

(3.10) Local monodromy. We describe some of the ideas contained in [7]. A p-adic analogue of Grothendieck's local monodromy theorem was formulated by Crew and Tsuzuki. This concerns quasi-unipotence of the Frobenius operator for modules with a connection over the "Robba ring" (a ring with an overconvergence property for its elements). Tsuzuki proves that this quasi-unipotence would follow if a certain slope filtration would exist.

If the module would come from the Dieudonné module of a p-divisible group over a discrete valuation ring, following Grothendieck we would know that the special fiber has a Newton polygon above that of the generic fiber.

In the situation at hand there is no special fiber. But one could use the virtual presence as a guide line for investigations. In the paper [7] the author studies two different Newton polygons. One is the usual Dieudonné-Manin theory of slopes of Frobenius on the generic fiber, see Section 5. A rather difficult and virtuoso construction gives the "special slopes" (which would come from the Newton polygon of the special fiber if that would exist), see Section 4. Once these concepts are at hand, Grothendieck-comparison of special and generic Newton polygon can be proved and several slope filtrations can be constructed over various extension rings. The ascending special slope filtration finally leads to the required monodromy.

Here we see that the Dieudonné-Manin theory of slopes is the essential tool giving access to this result.

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Frans Oort
Mathematisch Instituut
P.O. Box. 80.010
NL - 3508 TA Utrecht
The Netherlands
email: oort@math.uu.nl