A THEORETICAL ELUCIDATION OF THE NOTION "VENTRICULAR GRADIENT"

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IT MAY be supposed as generally known that the spatial ventricular gradient is the time integral of the heart vector $\vec{H}$:

$$\vec{G} = \int \vec{H}\, dt \tag{1}$$

It is the vectorial sum of the infinitesimal small products of the heart vector $\vec{H}$ and the infinitesimal small time interval dt. It can be extended over the period of depolarization (QRS), over the period of repolarization (T) or over the total heart period:

$$\vec{G}_{QRS\text{-}T} = \vec{G}_{QRS} + \vec{G}_T \tag{2}$$

The sum has to be taken as a vectorial one.

The vectorial ventricular gradient owes its significance to the allegation that its value depends only on the state of the heart muscle and is independent of the origin of the excitation. So it should be a means to discriminate between a failure of the heart muscle and of the Purkinje system. The clinical significance of the gradient must remain undiscussed here.

In view of equation (1) the name "gradient" is paradoxical. While this word denotes in physics a differential quotient with respect to position or a coordinate it appears here as indicating an integral with respect to time. It is the purpose of this paper to show that, in a schematic case, $\vec{G}$ defined according to (1) has, indeed, a relation to a gradient in the physical meaning.

We will consider first the schematic case of a narrow homogeneous muscle strip as depicted in Fig. 1. An analogous case was treated some years ago by Wilson[3] and by Cabrera.[1] The present one, however, is somewhat less specialized. It is supposed that the boundary between depolarized and repolarized muscle tissue is a plane perpendicular to the strip. So the "heart vector" has the direction of the strip and is supposed to have a constant magnitude, the same for the depolarization and the repolarization wave.

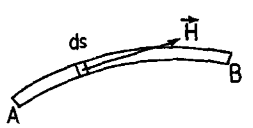

Fig. 1.—Excitation of a muscle strip AB, beginning at A. ds = element of muscle strip. $\vec{H}$ = heart vector.

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If the depolarization (*QRS*) is started at *A* (Fig. 1), the gradient of depolarization can be calculated according to equation (1):

$$\vec{G}_{QRS} = \int \vec{H}\, dt \qquad (2a)$$

We assume the velocity of propagation of the depolarization to be constant (c). Then the length of an infinitesimally small element ds of the strip equals the product of velocity c and time dt:

$$ds = c\, dt, \text{ or } dt = ds/c. \qquad (3)$$

This substituted in (2) gives:

$$\vec{G}_{QRS} = \int_A^B \vec{H}\frac{ds}{c} = \frac{1}{c}\int_A^B \vec{H}\, ds.$$

As $\vec{H}$ and $\vec{ds}$ (considered as a vector) have the same direction, we can put the arrow over $\vec{ds}$ as well as over $\vec{H}$:

$$\vec{G}_{QRS} = \frac{1}{c}\int_A^B \vec{H}\, ds.$$

H, the magnitude of the "heart vector" is supposed to be constant, so:

$$\vec{G}_{QRS} = \frac{H}{c}\int_A^B \vec{ds}. \qquad (4)$$

The vectorial sum of all elements $\vec{ds}$ is the vector $\vec{AB}$, independent of the shape of the arbitrarily curved muscle strip, and therefore:

$$\vec{G}_{QRS} = \frac{H}{c}\int_A^B \vec{ds} = \frac{H}{c}\vec{AB} \qquad (5)$$

Fig. 2.—Excitation of a muscle strip, beginning at *C* and proceeding to *A* and *B*.

If the starting point of the excitation is not at the end of the strip but in arbitrary point *C* (Fig. 2) we can consider both parts *CA* and *CB* separately. According to equation (4) we have then:

$$\vec{G}_{QRS} = \frac{H}{c}\int_C^A \vec{ds} + \frac{H}{c}\int_C^B \vec{ds} = \frac{H}{c}(\vec{CA} + \vec{CB}).$$

It can be easily seen that this vector sum generally depends on the position of the starting point *C* on the muscle strip *AB*.

After the depolarization (= excitation) there follows repolarization. If the time τ taken by the muscle tissue to repolarize is constant all over the strip, it is easily seen that the repolarization process follows the depolarization with the same velocity c. The only difference is that now the direction of the vector $\vec{H}$ is reversed. So:

$$\left(\int \vec{H}\, dt\right)_{\text{rep.}} = -\left(\int \vec{H}\, dt\right)_{\text{dep.}} \quad \text{or}$$

$$\vec{G}_T = -\vec{G}_{QRS}, \quad \text{and}$$

$$\vec{G}_{QRS\text{-}T} = 0$$

We have, therefore, to accept some heterogeneity as to the time of repolarization τ, in order to be able to explain a finite value of $\overrightarrow{G}_{QRS\text{-}T}$. We will, therefore, suppose henceforth that the time of repolarization τ is a function of the position on the muscle strip.

In order to realize the consequence of this supposition we return to the simple case of Fig. 1, where the depolarization starts at the beginning A of the muscle strip. When τ does not depend greatly on the position, i.e., on s, the distance of a point from A measured along the strip, the repolarization follows the same course from A to B. But the velocity is not c; it can be calculated in the following way.

If s is again the length of the muscle strip from A to an arbitrary point P on it, the depolarization, starting at A, takes a time s/c to reach P. If $\tau(s)$ is the time taken for the repolarization, depending on the position of P and so on the length s, the repolarization takes place at the time $s/c + \tau(s)$, after the starting of the depolarization at A. So:

$$t_{\text{rep.}} = s/c + \tau(s).$$

By differentiating this equation with respect to s, we obtain:

$$\frac{(dt)_{\text{rep.}}}{ds} = \frac{1}{c} + \tau'(s),$$

when $\tau'(s) = \dfrac{d\tau(s)}{ds}$ is the derivative of the function $\tau(s)$.

Now $ds/(dt)_{\text{rep.}}$ is the velocity $c_{\text{rep.}}$ of the repolarization wave T, so:

$$\frac{1}{c_{\text{rep.}}} = \frac{1}{c} + \tau'(s) \tag{6}$$

The contribution of the T wave to the spatial ventricular gradient is expressed by the general equation (1), but since $\overrightarrow{H}$ and $\overrightarrow{ds}$ have now an opposite direction, we get:

$$\overrightarrow{G_T} = -\int_A^B \overrightarrow{H}\,dt = -\int_A^B \overrightarrow{H}\,\frac{ds}{c_{\text{rep.}}}$$

$1/c_{\text{rep.}}$ can be replaced by its value according to (6), and $\overrightarrow{H}\,ds$ can be written as $H\,\overrightarrow{ds}$ just as in the QRS case. Then we obtain:

$$\overrightarrow{G_T} = -H\int_A^B \overrightarrow{ds}\left\{\frac{1}{c} + \tau'(s)\right\} = -H\int_A^B \frac{\overrightarrow{ds}}{c} - H\int_A^B \tau'(s)\,\overrightarrow{ds}.$$

The first integral can be evaluated as in the preceding case:

$$-H\int_A^B \frac{\overrightarrow{ds}}{c} = -\frac{H}{c}\int_A^B \overrightarrow{ds} = -\frac{H}{c}\,\overrightarrow{AB}.$$

According to (2) the total ventricular gradient is:

$$\overrightarrow{G}_{QRS\text{-}T} = \overrightarrow{G}_{QRS} + \overrightarrow{G_T} = \frac{H}{c}\,\overrightarrow{AB} - \frac{H}{c}\,\overrightarrow{AB} - H\int_A^B \tau'(s)\,\overrightarrow{ds} = -H\int_A^B \tau'(s)\,\overrightarrow{ds}. \tag{7}$$

From this equation we can only derive a distinct result, if we suppose that $\tau'(s)$ is constant along the muscle strip AB. Then (7) reduces to:

$$\overrightarrow{G}_{QRS\text{-}T} = -H\tau'(s) \int_A^B \overrightarrow{ds} = -H\tau'(s)\ \overrightarrow{AB}. \qquad (8)$$

In order to understand the next step, it should be borne in mind that $\tau'(s)$ is positive if the time τ increases from A to B. This next step is that we return to the situation of Fig. 2, where the excitation starts at an arbitrary point C of the muscle strip. From C the excitation proceeds along the muscle strip to A and to B. Both processes give their contribution to $\overrightarrow{G}$ according to (8). For the propagation from C to B we can apply (8) directly and have:

$$(\overrightarrow{G}_{QRS\text{-}T})_{CB} = -H\tau'(s)\ \overrightarrow{CB} \qquad (9a)$$

But for the propagation from C to A the direction of the propagation has a sign opposite the direction in which $\tau'(s)$ is taken positive. We must, therefore, give $\tau'(s)$ in this integral a negative sign, but according to our assumption the same value as in the preceding case, so:

$$(\overrightarrow{G}_{QRS\text{-}T})_{CA} = +H\tau'(s)\ \overrightarrow{CA} \qquad (9b)$$

The total gradient is the sum of the contributions (9a) and (9b):

$$\overrightarrow{G}_{QRS\text{-}T} = H\tau'(s)\ (\overrightarrow{CA} - \overrightarrow{CB}) = H\tau'(s)\ (\overrightarrow{CA} + \overrightarrow{BC}) = H\tau'(s)\ \overrightarrow{BA}. \qquad (10)$$

From (10) the important conclusion may be drawn that the ventricular gradient, which takes all our assumptions for granted, is independent of the position of the point C, the starting point of the excitation. It is this property that gives the gradient its importance.

Two remarks may be made with respect to equation (10): (a) The gradient is proportional to the differential quotient $\tau'(s)$, which is a real gradient, i.e., a differential quotient of a property τ of the muscle with respect to a "coordinate" s. For a homogeneous muscle strip the gradient is zero; (b) We may suppose that the retardation time τ of the repolarization is greater the more the muscle is injured or strained. If the strain is greatest at B then the time τ is greatest there and $\tau'(s)$ is positive. Then $\overrightarrow{G}$ has, according to (10), the same direction as $\overrightarrow{BA}$, so the gradient is directed from the more injured or strained part B to the less injured or strained part A.

The muscle strip, dealt with above, may be curved and may even be curved in space. It need not be flat. So we have solved a spatial problem; but on the other hand it is a linear object, although curvilinear. It is possible, however, to solve the problem with some restrictions for a real spatial case, i.e., for a muscle mass extending in three dimensions and having an arbitrary shape. But in this case we need more mathematics than in the preceding one. In the schematic (Fig. 3) part of the muscle mass (shaded) is excited. As the depolarization is depicted, the shaded part is increasing.

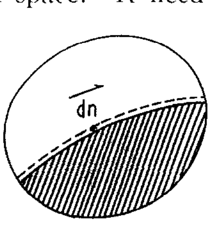

Fig. 3.—Heart muscle partly depolarized (shaded). The excitation proceeds along a normal of the boundary surface between polarized and depolarized. $\overrightarrow{dn}$ = infinitesimally small element of normal. This is a vector whose direction gives the direction of propagation of the excitation.

As in the former case, we suppose that the boundary between excited and nonexcited is sharp (Durrer and Van der Tweel[2]). This boundary surface may be described by the equation:

$$F\ (x,\ y,\ z,\ t)\ =\ 0.$$

x, y, and z are orthogonal coordinates, and t is the time. The occurrence of t in this equation means that the boundary surface depends on time, i.e., that it proceeds.

In order to make the calculation as simple as possible we think t resolved from the last equation:

$$f\ (x,\ y,\ z)\ =\ t \tag{11}$$

The way in which f is derived from F is of no importance for the following deductions. By equation (11) is expressed that, at each moment, for any value of t, the shape of the boundary surface is determined and dependent upon t. At a time $t' = t + dt$, somewhat later than t, the boundary surface has proceeded and is depicted by the dotted line, which represents a cross section of this surface and the plane of the drawing. By means of elementary analytic geometry it can be shown that the small distance dn of the two surfaces at the point $P\ (x, y, z)$ is:

$$dn = \frac{dt}{\left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right\}^{\frac{1}{2}}} \tag{12}$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are the partial differential quotients of the function f, with respect to x, y, z. The equation (12) can be used to express dt in a linear quantity dn in a way analogous to that in the case of the muscle strip.

We first calculate the depolarization part of the gradient:

$$\vec{G}_{QRS} = \int_{QRS} \vec{H}\ dt \tag{2a}$$

According to (12) dt may be substituted:

$$dt = dn\left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right\}^{\frac{1}{2}} \tag{12a}$$

The heart vector $\vec{H}$ at the moment t is a surface integral, extending over the boundary surface $t = f\ (x, y, z)$. If dS is an infinitesimal element of this surface, the contribution of dS to the heart vector $\vec{H}$ is $\vec{h}\ dS$. In this product $\vec{h}$ is a vector, directed normally to the surface $t\ (x, y, z)$ and from excited to unexcited. It is well known that the amount of $\vec{h}$, denoted by h, in various cases is not much different and of the order of magnitude of 100 mv. It is the potential jump at the boundary layer. We will suppose it to be constant, i.e., independent of place and time during the propagation of the boundary surface.

The total heart vector is the surface of $\vec{h}\ dS$, extended over the area of the surface $t = f\ (x, y, z)$:

$$\vec{H} = \int_S \vec{h}\ dS \tag{13}$$

Substitution of (12a) and (13) in (2a) gives:

$$\vec{G}_{QRS} = \int_n \int_S \vec{h} \left\{ \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 \right\}^{\frac{1}{2}} dn\, dS \tag{14}$$

The first integral sign denoted originally an integration with respect to time, but by the conversion (12a) it is now a spatial integration, and both integral signs can be replaced by an integration over the volume of the muscle. This is in accordance with the fact that $dn\, dS$ is volume element, a small cylinder with dS as base and dn as height:

$$dn\, dS = dv.$$

So $\vec{G}_{QRS}$ is a volume integral:

$$\vec{G}_{QRS} = \int_{\text{vol}} \vec{h} \left\{ \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 \right\}^{\frac{1}{2}} dv \tag{14a}$$

In order to transform this integral so that it is suited for calculation of $\vec{G}_T$, we can introduce the *gradient* of f. This is a vector, the components of which are the differential quotients $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$. It is denoted by $\overrightarrow{\nabla f\ (x, y, z)}$ or more simply by $\overrightarrow{\nabla f}$. Its value is computed from the components in the ordinary way as square root of the sum of the squares of the components:

$$|\overrightarrow{\nabla f}| = \left\{ \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 \right\}^{\frac{1}{2}}$$

Before substituting this in (14a), we may remark that the direction of $\overrightarrow{\nabla f}$ is the same as that of the normal ($\overrightarrow{dn}$, Fig. 3) on the surface $t = f\ (x, y, z)$. This follows immediately from the well-known expression of differential analytic geometry. Since $\vec{h}$ has the direction of the normal too, we can transform the integrant of (14a) in this way:

$$\vec{h} \left\{ \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 \right\}^{\frac{1}{2}} = h\, \overrightarrow{\nabla f}$$

and since h is a constant, we get:

$$\vec{G}_{QRS} = h \int_{\text{vol}} \overrightarrow{\nabla f}\ dv. \tag{15}$$

This simple formula allows us to compute the rest of the gradient, $\vec{G}_T$. To this end we suppose again that repolarization follows depolarization after a time τ. This time is in the present case a function of the place in the muscle so it is a function $\tau\ (x, y, z)$ of the coordinates. With assumptions analogous to those made in the first part, the propagation of the boundary surface, on the analogy of equation (11), can now be expressed by:

$$t = f\ (x, y, z) + \tau\ (x, y, z). \tag{16}$$

Since in repolarization (T), accepting our simplifying assumptions as in the first case, $\vec{h}$ has just the opposite direction as in depolarization, substitution of (16), i.e., $f + \tau$ for f in (15), gives:

$$\vec{G}_T = -h \int_{vol} (\vec{\nabla f} + \vec{\nabla \tau})\, dv \cdot \tag{17}$$

Addition of (15) and (17) gives the total ventricular gradient:

$$\vec{G}_{QRS\text{-}T} = -h \int_{vol} \vec{\nabla \tau}\, dv. \tag{18}$$

If $\vec{\nabla \tau}$ is constant over the whole muscle, we can write it before the integral sign and, keeping in mind that $\int_{vol} dv = V$ is the total muscle volume, the gradient amounts to:

$$\vec{G}_{QRS\text{-}T} = -h\ V\ \vec{\nabla \tau}. \tag{18a}$$

From the formulae (18) and (18a) it appears that the starting point of the depolarization has no influence on the gradient. This influence is present in both parts $\vec{G}_{QRS}$ and $\vec{G}_T$ as it is represented by the function $f\ (x, y, z)$. But the total gradient depends only on the lag time τ, in the state of the myocardium. It is interesting to remark that in the final result it is the *gradient* of this time that determines $\vec{G}$. The name gradient appears to be well chosen; the word has the same meaning as in physics.

The direction of $\vec{G}_{QRS\text{-}T}$ follows from (18a). It points from parts of the muscle with greater τ to such with smaller τ. So it is directed from the more injured or strained part to the less injured or strained part, just as in the first case, that of the narrow muscle strip.

SUMMARY

In simple cases it can be shown, theoretically, that the ventricular gradient is independent of the point of excitation. It can be expressed in the gradient of the time interval between depolarization and repolarization.

REFERENCES

1. Cabrera, E.: Bases electrophysiologiques de l'électrocardiographie, Applications cliniques, Paris, 1948, Masson & Cie.
2. Durrer, D., and Van der Tweel, L. H.: Am. Heart J. **47**:192, 1954.
3. Wilson, F. N., Macleod, A. G., Barker, P. S., and Johnston, F. D.: Am. Heart J. **10**:46, 1935.