Anabelian number theory and geometry
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Informal notes. Not for publication.

In June 1983 and in January 1984 Grothendieck stated a surprising conjecture, see [22], [23]. Gradually we start getting convinced that this daring idea, the "*anabelian conjecture*", is the right approach: recently several cases of this startling conjecture have been proved to be true. It might very well be that this gives a key method in number theory and in algebraic geometry. Hence we thought that we should get better acquainted with this circle of ideas. Number theorists and algebraic geometers in The Netherlands organize a Workshop in which the conjecture in cases of number fields and of algebraic curves are discussed.

In 1981 Grothendieck wrote a long manuscript "*La longue marche à travers la téorie de Galois*". Part of this has been edited, see [24]. In Section 5 of that manuscript we also find a formulation of the "anabelian conjecture", and in the sequel of that manuscript we find ideas by Grothendieck about *geometric aspects of arithmetic questions.*

In the program we will study the *anabelian conjecture* and we discuss proofs as can be found in the literature in the following cases:

(I) Anabelian results for number fields (the Neukirch-Uchida theorem); this is an analog of the Grothendieck anabelian conjecture for number fields, originally proved completely independent from the geometric ideas;
literature can be found in:
J. Neukirch, A. Schmidt & K. Wingberg – *Cohomology of Number fields*, [45].
See Chapter XII of this book for the theorems we are going to discuss. This theorem is sometimes called the Neukirch - Ikeda - Iwasawa - Uchida theorem.

(II) The Grothendieck conjecture for affine curves (Nakamura-Tamagawa);
H. Nakamura – *Galois rigidity of the étale fundamental groups of punctured projective lines*, [40]; also see [43];
A. Tamagawa – *The Grothendieck conjecture for affine curves*, [63].

(III) The Grothendieck conjecture for curves over sub-p-adic fields ;
S. Mochizuki – *The local pro-p anabelian geometry of curves*, [36].

Surveys of known cases can be found in [45], [44], [11], [42], [43], [37]. In Section 9 we discuss formulations of the anabelian Grothendieck Conjecture.

Basic remarks. 1) Let $[K:\mathbb{Q}] < \infty$, i.e. K is a number field. Its absolute Galois group $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ is *anabelian*, in the sense of (12.1); see [45], 12.1.6.
2) A complete algebraic curve X over a field K is *hyperbolic* if and only if $\pi_1(X \otimes_K K^s)$ is *anabelian*; see (7.3) [An algebraic curve X over a field K of characteristic zero is hyperbolic if and only if $\pi_1(X \otimes_K K^s)$ is anabelian.]
3) The Tate conjecture holds over global fields (Faltings, 1983), but the analog of the Tate conjecture over local fields does not hold, see (14.4); for an abelian variety A the fundamental group $\pi_1(A)$ is commutative. The anabelian Grothendieck conjecture holds for hyperbolic curves over local fields (Mochizuki, 1999).

These notes are just for internal use. We do not claim any originality. Some speakers have given their comments. I thank them for advice and remarks; however correctness of the contents of these notes is completely my responsibility, FO.

1 Some easy facts in group theory.

(1.1) Let G be a group, $N \subset G$ a normal subgroup, write $\Gamma := G/N$ for the factor group, and consider the "exact sequence"

$$1 \to N \longrightarrow G \xrightarrow{p} \Gamma \to 1. \qquad (\xi)$$

For any group N we define

$$\mathrm{Out}(N) := \mathrm{Aut}(N)/\mathrm{Inn}(N);$$

note that the group $\mathrm{Inn}(N)$ of inner automorphisms of N indeed is a normal subgroup of $\mathrm{Aut}(N)$.

The exact sequence defines a homomorphism

$$R : \Gamma \longrightarrow \mathrm{Out}(N) \quad \text{by:} \quad R(\sigma')(x) := \sigma x \sigma^{-1},$$

for $\sigma' \in \Gamma$, $\sigma \in G$ with $p(\sigma) = \sigma'$ and $x \in N$. Note that indeed $\sigma x \sigma^{-1} \in N$, because N is normal in G, and the definition of $R(\sigma') \in \mathrm{Out}(N)$ does not depend on the choice of $\sigma \in G$.

If moreover N is an abelian group, the exact sequence defines a homomorphism

$$R : \Gamma \longrightarrow \mathrm{Aut}(N) \quad \text{by:} \quad R(\sigma')(x) := \sigma x \sigma^{-1},$$

for $\sigma' \in \Gamma$, $\sigma \in G$ with $p(\sigma) = \sigma'$ and $x \in N$.

(1.2) Here is one of the fundamental examples. Let X be a scheme (or a variety, if you wish) over a field K. The we obtain an exact sequence

$$1 \to N = \pi_1(X \otimes_K K^s) \longrightarrow G = \pi_1(X) \xrightarrow{p} \Gamma_K = \mathrm{Gal}(K^s)/K) \to 1$$

as will see. Hence we obtain

$$R: \quad \Gamma_K \quad \longrightarrow \quad \mathrm{Out}(N).$$

This representation will play a fundamental role.

In particular, let $X := \mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$. We know that

$$\pi_1^{\mathrm{top}}(X(\mathbb{C})) \cong \mathbb{Z} * \mathbb{Z},$$

the free group on two generators. It follows that

$$\pi_1(X \otimes_{\mathbb{Q}} \mathbb{C}) \cong (\mathbb{Z} * \mathbb{Z})^\wedge.$$

The exact sequence

$$1 \to N = \pi_1(X \otimes_{\mathbb{Q}} \mathbb{C}) \longrightarrow G = \pi_1(X) \xrightarrow{p} \Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1$$

defines

$$R: \quad \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \quad \longrightarrow \quad \mathrm{Out}((\mathbb{Z} * \mathbb{Z})^\wedge).$$

Injectivity of this homomorphism was proved by Belyi, see [5]. Can we use this to derive information, via this *geometric approach* , on the *arithmetic object* $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$?

(1.3) Suppose given two exact sequences $G_i/N_i = \Gamma_i \quad (\xi_i)$ for $i = 1, 2$ as above. We say that these two sequences are isomorphic if there is an isomorphism $G_1 \to G_2$ mapping N_1 isomorphically onto N_2 (and hence inducing an isomorphism $\Gamma_1 \cong \Gamma_2$).

(1.4) Suppose given an exact sequence $G/N = \Gamma$ (ξ) as above, in particular N is normal in G. We have seen that this exact sequence defines the representation $R : \Gamma \to \mathrm{Out}(N)$. Conversely:

Lemma. *Suppose Γ and N are given, and suppose that the center of N is trivial: $\mathcal{Z}(N) = \{1\}$. From Γ, N and R we can reconstruct G and $G/N = \Gamma$. If G and N are topological groups, and R is continuous, the $G/N = \Gamma$ thus constructed can be equipped with the structure of an exact sequence of topological groups.* [See [11], page 134.]
The essential step: As N has trivial center, $N \hookrightarrow \mathrm{Inn}(N)$ is injective, hence the square

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}(N) \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{R} & \mathrm{Aut}(N)/\mathrm{Inn}(N) \end{array}$$

is cartesian; this shows we can reconstruct the left hand upper corner from the rest of the data. $\square$

2 Galois theory

We will use basic concepts of Galois theory. We refer to [6], V.10 and V Appendix II; [32], Chap. VI; [60]. Also see Section 5 below.

For infinite Galois theory, see [6], V. App. II. We will use the concept of a *pro-finite group*: the projective limit (inverse limit) of a system of finite groups, with the coarsest topology making all projections on the discrete finite quotients continuous.

(2.1) Warning. For a group G we define its profinite completion $G^\wedge$ as the projective limit of all finite quotients of G; there is an induced homomorphism $G \to G^\wedge$; clearly this need not be injective, e. g. for $G = \mathbb{Q}^+$, we have $G \to G^\wedge = 0$.

We will encounter this situation in algebraic geometry over $\mathbb{C}$; for example for a complete algebraic curve X over $\mathbb{C}$ we do have

$$\pi_1^{\mathrm{top}}(X(\mathbb{C})) \quad \hookrightarrow \quad (\pi_1^{\mathrm{top}}(X(\mathbb{C})))^\wedge \cong \pi_1^{\mathrm{alg}}(X) = \pi_1(X);$$

however there are algebraic varieties over $\mathbb{C}$ for which the map $\pi_1^{\mathrm{top}}(X(\mathbb{C})) \to (\pi_1^{\mathrm{top}}(X(\mathbb{C})))^\wedge = \pi_1(X)$ is not injective (in this case the group is "not residually finite"). An example was given by Toledo in 1990; for a discussion, see [4], pp. 136 - 139.

3 Number theory

We will use basic concepts about number fields, about Galois theory for number fields (decomposition group, inertia group), and some facts about Galois cohomology; we will use [45] as basic reference.

4 Algebraic curves

(4.1) We consider *non-singular, absolutely irreducible curves*: let K be a field, and let $k \supset K$ be an algebraic closure; we say that X is an *algebraic curve* over K if it is a variety (or a scheme if you like) of dimension one over K, which is regular (non-singular), such that $X \otimes_K k$ is irreducible (and reduced).

The field of rational functions on X is denoted by $K(X)$; this is an extension of K, such that the extension $K(X) \supset K$ is regular of transcendence degree one, i.e. K is algebraically closed in $K(X)$, and there exist an element $t \in K(X)$ transcendental over K such that $K(X) \supset K(t)$ is finite separable; such a field is called a "function field in one variable over K".

An algebraic curve X over a field K can be "compactified", i.e. there exists an algebraic curve X^c over K, and an embedding $X \hookrightarrow X^c$ such that X^c is complete (proper over K). Once X/K is given, $X \subset X^c/K$ with these properties is unique (note that we assume that X^c is regular and absolutely irreducible). In some situations we will write U for an algebraidc curve, and $X = U^c$ for its "compactification".

(4.2) To an algebraic curve X/K we attach the integers g, r and χ:

$g = g(X)$ is the genus of X, i.e.

$$g = \dim_K H^1(X^c, \mathcal{O}_{X^c});$$

for a complete non-singular curve the notions of "geometric genus" and "arithmetic genus" coincide;

$r = r(X)$ is the number of geometric points needed to "compactify" X:

$$r = \#((X^c - X)(k));$$

$\chi = \chi(X)$ is the "Euler number" of X:

$$\chi = -2g + 2 - r.$$

Definition. *An algebraic curve* X/K *is called* hyperbolic *if* $-\chi(X) > 0$.

Note that this is the case iff one of the following conditions hold:

- either $g(X) = 0$ and $r(X) \geq 3$; in this case $X_k^c \cong \mathbb{P}_k^1$, and the number of "deleted geometric points" is at least three;
- or $g(X) = 1$ and $r(X) \geq 1$; in this case $X^c \otimes_K k$ is an elliptic curve, after a point $0 \in X^c \otimes_K k$ has been chosen, and the number of "deleted geometric points" is at least one;
- or $g(X) = 2$, and $r(X) \geq 0$.

(4.3) Remark / Exercise. Let X/K be an algebraic curve, and $k = \overline{K}$. Then:

$$X \text{ is a hyperbolic curve} \quad \Longleftrightarrow \quad \mathrm{Aut}(X_k) \text{ is a finite group.}$$

(4.4) Remark / Exercise. Let X/K and Y/K be algebraic curves over a field K. Let $f : Y \to X$ be a quasi-finite morphism. Then

$$X \text{ is a hyperbolic curve} \quad \Longrightarrow \quad Y \text{ is a hyperbolic curve.}$$

If moreover f is finite and étale and K is of characteristic zero, then

$$X \text{ is a hyperbolic curve} \quad \Longleftrightarrow \quad Y \text{ is a hyperbolic curve.}$$

Note that there exist counterexamples to "$\Leftarrow$" over fields of positive characteristic.

(4.5) For statements and proofs of the Hurwitz-Zeuthen formula linking invariants in a separable finite cover $f : Y \to X$, see [26], IV.2, see [38], Chap. 7 , see [19], Chap. 2, in particular pp. 216-219.

(4.6) Remark. Let K be a field and let X be an algebraic curve over K. Suppose that X is not complete, i.e. $X \subsetneq X^c$. Then X is an affine curve. In fact, if $X \subsetneq X^c$ there is a divisor D on X which is effective, non-zero, with support on X^c X. An effective, non-zero divisor on a curve is ample. Hence there exists a positive integer n such that nD is very ample; the embedding $\varphi_{nD} : X^c \hookrightarrow \mathbb{P}^N$ maps X onto a closed curve in $\mathbb{A}^N$; hence X is affine.

(4.7) Suppose our ground field is $K = \mathbb{C}$, the field of complex numbers. An algebraic curve $X/\mathbb{C}$ gives rise to a complex manifold $X(\mathbb{C})$; this is a real, orientable surface, hence $X(\mathbb{C})$ is a Riemann surface. Conversely for every (open) Riemann surface S there is an algebraic curve $X(\mathbb{C})$ such that $X(\mathbb{C}) \cong S$. This correspondence $X \leftrightarrow X(\mathbb{C}) = S$ is one-to-one on isomorphism classes. An algebraic curve $X/\mathbb{C}$ is complete (proper over $\mathbb{C}$) iff the associated Riemann surface $X(\mathbb{C})$ is compact. An algebraic curve $X/\mathbb{C}$ is of genus g iff the associated compact Riemann surface $X^c(\mathbb{C}) = S^c$ is of topological genus g, i.e. $\mathrm{rk}_{\mathbb{Z}}(\mathrm{H}_1(S^c, \mathbb{Z})) = 2g$.

5 Fundamental groups: Galois theory and topology

Classically there are these two basic notions: Galois extensions in algebra, and coverings in topology, which seemed to be unrelated, but which turn out to be two disguises of one notion: *the algebraic fundamental group*, as introduced by Grothendieck in 1960. This was a breakthrough in number theory and algebraic geometry. It enables us to use geometric methods in arithmetic situations on the one hand, and it also provides arithmetic techniques in algebraic geometry.

(5.1) Galois theory. Let K be a field, and let $f \in K[T]$ be a polynomial in one variable over K. We write Ω_K^f for the splitting field of f over k; this can be constructed as $\Omega_K^f \cong K(a \in k \mid f(a) = 0)$, where $k \supset K$ is an algebraic closure.

Suppose that all zeros of f are simple, i.e. $\{a \in k \mid f(a) = 0\}$ is supposed to have the same cardinality as the degree of f. The field extension $L = \Omega_K^f \supset K$ is called a *finite Galois extension.* The automorphism group $\mathrm{Aut}(L/K)$ is called the Galois group of this extension, $\mathrm{Gal}(L/K) := \mathrm{Aut}(L/K)$. A union of finite Galois extensions is called a Galois extension.

A separable closure $K^s \supset K$ is a union of finite Galois extensions. The group $\mathrm{Gal}(K^s/K) := \mathrm{Aut}(K^s/K)$ is called the *absolute Galois group* of K; this is a pro-finite group. This will be denoted by

$$\Gamma_K = \mathrm{Gal}(K^s/K) := \mathrm{Aut}(K^s/K).$$

Galois theory describes the connection between (closed) subgroups of the Galois group $\mathrm{Gal}(L/K)$ of a Galois extension and intermediate fields between K and L.

(5.2) The topological fundamental group. We use topological spaces with "reasonable properties"; without further mention we suppose that a topological space S is *connected, locally pathwise connected and locally simply connected.* Note that if $V/\mathbb{C}$ is an (irreducible) variety over the complex numbers then the complex space $V(\mathbb{C})$ satisfies the properties just mentioned, see [25], II.2.4 (Local structure of analytic varieties).

A continuous map $f : T \to S$ between topological spaces is called a (topological) *covering* if every point in S has a neighborhood $s \in U \subset S$ such that its inverse image $f^{-1}(U)$ is disjoint union of open sets in T, each of which is mapped homeomorphically by f onto U.

Let S be a topological space, and let $s_0 \in S$ be a chosen (base) point. We define $\pi_1^{\mathrm{top}}(S, s_0)$ to be the *topological fundamental group* of (S, s_0); this is the set of homotopy classes of loops in (S, s_0) with composition as group law. Note (we supposed S to be connected) that two choices $s_0, s_1 \in S$ yield isomorphic groups $\pi_1^{\mathrm{top}}(S, s_0) \cong \pi_1^{\mathrm{top}}(S, s_1)$.

(5.3) Note that if $V/\mathbb{C}$ is a regular (irreducible) variety over the complex numbers then the complex space $V(\mathbb{C})$ satisfies the properties just mentioned:

$$(V \text{ is irreducible}) \Rightarrow (V(\mathbb{C}) \text{ is connected}),$$

for example, see [57], Vol. 2, page 126, Th. 1, and:

$$(V \text{ is regular}) \Rightarrow (V(\mathbb{C}) \text{ is locally euclidean}),$$

hence in this case $V(\mathbb{C})$ is locally pathwise connected and locally simply connected.

Some references: [16], [59], [61], [62].

(5.4) Two warnings. In what follows, either on topological fundamental groups, or on algebraic fundamental groups we should mention and write the notion of the base points, but often we will assume and then ignore the choice and the notation of the base point.

Let $f : V \to W$ be a covering in the sense of algebraic geometry; this is a finite morphism between algebraic varieties; suppose these are defined over $\mathbb{C}$ and consider the induced map $V(\mathbb{C}) \to W(\mathbb{C})$. This need not be a topological covering: in case the morphism f is *ramified* the related topological map is not a topological covering. Sometimes a warning is given by saying something like "consider a (ramified) cover" in order to distinguish the habits of algebraic geometers on the one hand, and the usage in topology on the other hand.

Let S be a topological space (with properties mentioned above !). There exits a "universal covering" $S^\sim \to S$; this is a covering, $S^\sim$ is connected and simply connected; these properties characterize this covering up to $S-$homeomorphism. Moreover $\pi := \pi_1^{\rm top}(S, s_0)$ acts on $S^\sim \to S$ and the orbit space of $S^\sim$ under the fundamental group π is S:

$$S^\sim \to \pi\backslash S^\sim = S.$$

(5.5) Remark / Exercise. Let X/k be an algebraic curve over an algebraically closed field k of characteristic zero. Then:

$$X \text{ is a hyperbolic curve} \quad \Longleftrightarrow \quad \pi_1(X) \text{ is a non-commutative group.}$$

[For a more precise formulation, see (7.3).] [Note that there exist counterexamples to "$\Leftarrow$" over fields of positive characteristic.] More precisely:

(5.6) Let $X/\mathbb{C}$ be an algebraic curve, and $S := X(\mathbb{C})$ the related Riemann surface.

- Suppose $X \cong \mathbb{P}^1_{\mathbb{C}}$, the case $g = 0 = r$; then $\pi = \{1\}$ is trivial, and $S^\sim = S = X(\mathbb{C})$, the Riemann sphere.
- Suppose $X \cong \mathbb{P}^1_{\mathbb{C}} - \{\infty\}$, the case $g = 0, \quad r = 1$; then $\pi = \{1\}$ is trivial, and $S^\sim = S = X(\mathbb{C}) \approx \mathbb{C}$.
- Consider the case $g = 0$ and $r = 2$ or the case $g = 1$ and $r = 0$; in these cases $\pi_1^{\rm top}(S)$ is commutative (free of rank one, respectively two), and $S^\sim = X(\mathbb{C}) \approx \mathbb{C}$.
- In case $X/\mathbb{C}$ is hyperbolic, the fundamental group $\pi_1^{\rm top}(X(\mathbb{C}))$ is non-abelian, and the corresponding universal covering space is homeomorphic with the upper half plane:

$$X(\mathbb{C})^\sim \approx \mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}.$$

(5.7) Suppose X is an algebraic curve over $\mathbb{C}$. Let $S = X(\mathbb{C})$ be the related Riemann surface. As a real manifold, S is orientable, and classification of real surfaces shows that S is completely described by g and r: a Riemann surface of genus g with r punctures, see [18]; see [16], Chapter 17. In this case the fundamental group

$$\pi_1^{\rm top}(S) \cong \Gamma_{g,r},$$

where $\Gamma_{g,r}$ is the group defined by: it is generated by

$$\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_r$$

satisfying

$$(\alpha_1{\cdot}\beta_1{\cdot}\alpha_1^{-1}{\cdot}\beta_1^{-1}){\cdot}\quad\cdots\quad{\cdot}(\alpha_g{\cdot}\beta_g{\cdot}\alpha_g^{-1}{\cdot}\beta_g^{-1}){\cdot}\gamma_1{\cdot}\quad\cdots\quad{\cdot}\gamma_r.$$

Note: If $r > 0$, i.e. if $X(\mathbb{C})$ is not compact, then $\Gamma_{g,r} \cong F_{2g+r-1}$, the free group on $2g + r - 1$ generators.

6 The algebraic fundamental group

Basic reference: SGA1. We assume the reader is familiar with the definition and some basic properties of the algebraic fundamental group of a scheme with a base point; sometimes we will say the "étale fundamental group", or "algebraic fundamental group", or we just say "the fundamental group".

There are some properties which need special attention:

(6.1) The fundamental exact sequence, see SGA1, Th. IX.6.1. Let K be a field, let $k \supset K$ be an algebraic closure, let $X \to \mathrm{Spec}(K)$ be a scheme over K, which is algebraic and geometrically connected over K (e.g. this is the case if we consider a variety V/K). Let $\overline{a}$ be a base point for $\overline{X} = X \otimes_K k$, i.e. a morphism $\overline{a} : \mathrm{Spec}(\Omega) \to \overline{X}$ where Ω is an algebraically closed field; let $a : \mathrm{Spec}(\Omega) \to X$ and $b : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K)$ the induced base points (obtained by composition of morphisms). Hence we obtain morphisms of pointed schemes:

$$(\overline{X}, \overline{a}) \longrightarrow (X, a) \longrightarrow (\mathrm{Spec}(K), b);$$

these induce an *exact sequence*

$$1 \to \pi_1(\overline{X}, \overline{a}) \longrightarrow \pi_1(X, a) \longrightarrow \pi_1(\mathrm{Spec}(K), b) \cong \mathrm{Gal}(K^s/K) \to 1.$$

(6.2) Comparison theorems for étale morphisms, see SGA1 XII.5. Let X be a $\mathbb{C}$-scheme, locally of finite type, and let $X^{\mathrm{an}} = X(\mathbb{C})$ be the corresponding analytic space [in our case, we will often consider an algebraic curve over $\mathbb{C}$, and $X^{\mathrm{an}} = X(\mathbb{C}) = S$ will be a Riemann surface.] EGA1, Th. XII.5.1 says that *the category of finite étale morphisms above X and the same above X^{an} are equivalent.*

From this we conclude, see EGA1 Cor. XII.5.2 that for a connected, locally of finite type $\mathbb{C}$-scheme $X/\mathbb{C}$ we have the following comparison:

$$\pi_1^{\mathrm{top}}(X(\mathbb{C})) \quad \longrightarrow \quad \left(\pi_1^{\mathrm{top}}(X)\right)^{\wedge} \quad \xrightarrow{\sim} \quad \pi_1(X),$$

i.e. *the pro-finite completion of the topological fundamental group of $X(\mathbb{C})$ is equal to the fundamental group of X.*

(6.3) Let $f : X \to Y$ be a morphism between algebraic curves over a field K. This extends uniquely to a morphism $f^c : X^c \to Y^c$. Note that if f is étale, the morphism f^c may ramify above $Y^c - Y$.

In higher dimensions analogous considerations can be used, using the Gauert-Remmert theorem, which however is a non-trivial machinery, see EGA1, XII.5.4.

(6.4) Here is a corollary of this comparison theorem, using the extension theorem of morphisms. Let k be an algebraically closed field of characteristic zero. Let X/k be an algebraic curve of genus g with $r(X) = r$ (the number of "punctures, the number of geometric points needed to "compactify" X). Then

$$\pi_1(X) \quad \cong \quad (\Gamma_{g,r})^\wedge ;$$

for the notation $\Gamma_{g,r}$, see (5.7).

(6.5) Remark/Example. Note that we use topological considerations in order to describe the structure of an algebraic-geometric object like the algebraic fundamental group of a curve. As far as I know there is proof of the previous fact directly. For example, we now know that

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} - \{0, 1, \infty\}) \cong (\mathbb{Z} * \mathbb{Z})^\wedge$$

but we do not know a proof of this without topological considerations (and, as Gabber told me, one can prove that this isomorphism suffices to determine the structure of the fundamental group of any algebraic curve along "algebraic" lines).

Open Problem. *Can we determine the fundamental group of any algebraic curve along algebraic lines?*

(6.6) Geometric class field theory. We will give a topological and an algebraic formulation.
Suppose X is a complete curve over $\mathbb{C}$; let $J = \mathrm{Jac}(X)$ be its Jacobian. Then we know that we have isomorphisms

$$\left(\pi_1^{\mathrm{top}}(X(\mathbb{C}))\right)_{\mathrm{ab}} \quad \xrightarrow{\sim} \quad \pi_1(J(\mathbb{C})) \quad \cong \quad \mathrm{H}_1(X(\mathbb{C}), \mathbb{Z}) \quad \cong \quad \mathbb{Z}^{2g}.$$

We see that *any finite, étale, abelian Galois cover can be obtained from pulling back an isogeny over* J; here we regard J as the Albanese variety of X; see Section 7 for a further discussion. This follows because the topological considerations prove the statement for analytic varieties, and GAGA principles (the Chow theorem) then finishes the argument.

Suppose X is a complete curve over an algebraically closed field, and let $J = \mathrm{Jac}(X)$ be its Jacobian. Here again, as in the previous paragraph, *any finite, étale, abelian Galois cover can be obtained from pulling back an isogeny over* J; this time we can invoke directly [52], Cor. on page 128.

Note that certain finite, étale, abelian Galois covers of affine curves can be described in an analogous way, see [52], Prop. 9 on page 126.

(6.7) Notation. Let G be a group, and let p be a prime number. We write $G^{(p)}$ for the pro-finite group obtained as projective limit of all finite factor groups of G with order prime to p.
Let G be a group, and let ℓ be a prime number. We write $G(\ell)$ for the "*ell-primary component*": the pro-finite group obtained as projective limit of all finite factor groups of G with order a power of ℓ. [This notation is not standard. E.g. in [63] this group is written as G^ℓ. Such a notation might cause confusion, I think.]

Note that the kernel of $G \to G^{(p)}$ is a characteristic subgroup of G; the same for the kernel of $G \to G(\ell)$.

(6.8) The p-rank. Let A be an abelian variety over a field K of characteristic p; let $k \supset K$ be an algebraic closure. We write $f = f(A)$, and we say f is the p-rank of A if

$$A[p](k) \quad \cong \quad (\mathbb{Z}/p)^f.$$

Note that $0 \leq f \leq \dim(A) =: g$, and indeed all these values for f can be achieved by considering all abelian varieties of dimension g. For an algebraic curve X we say its p-rank equals f if its Jacobian $J = \mathrm{Jac}(X)$ has p-rank equal to f. An abelian variety is called *ordinary* if $f(A) = \dim(A)$. A curve is called *ordinary* if its Jacobian is ordinary.

(6.9) Fundamental groups in positive characteristic. In EGA1, X.7, in EGA1, XII, in [25] we find descriptions of various properties of fundamental groups of schemes in positive characteristic. We will give some references which focus on the case of algebraic curves.

(6.10) Theorem (Grothendieck, EGA 1, X.3.10). *Let k be an algebraically closed field of characteristic p, and let X/k be a complete algebraic curve over k of genus g. Then*

$$\pi_1(X)^{(p)} \quad \cong \quad (\Gamma_{g,0})^{(p)}.$$

For a prime number $\ell \neq p$ we have

$$\pi_1(X)(\ell) \quad \cong \quad (\Gamma_{g,0})\,(\ell).$$

This is proved by considering a lift of X to characteristic zero, use the specialization theorem EGA1, X.2.1, and ensure that the property of "being prime to p" reduces Galois covers of degree prime to p to separable covers, etc.

Furthermore, for X a complete curve over k, note that $\pi_1(X)$ is a quotient of $(\Gamma_{g,0})^\wedge$, where the map on the p-primary component is not an isomorphism if $\mathrm{char}(k) = p > 0$ and $g > 0$.

We like to describe analogous results for Galois covers which have a degree divisible by p, and especially also these results for open curves.

(6.11) Note that Galois coverings of degree divisible by p in general are not so easy to describe. However class field theory for algebraic curves, see [52], VI.12 gives a results for abelian Galois covers.
Theorem. *Let k be an algebraically closed field of characteristic p, and let X/k be a complete algebraic curve over k. Let $J = \mathrm{Jac}(X)$ be its Jacobi variety. Then we have an isomorphism*

$$(\pi_1(X))_{\mathrm{ab}}\,(p) \quad \cong \quad \varprojlim J[p^i](k);$$

here the index "ab" denotes the abelianization of that group.

(6.12) A covering $X \to Y$ of algebraic curves in characteristic p is called *tame* if all ramification indices are not divisible by p. If U is a curve and $X = U^c$ its compactification, we write $\pi_1^{\mathrm{tame}}(U)$ by taking limits over all étale Galois coverings of U which can be extended to tame covers of X. Note that a Galois cover of U with Galois group of order prime to p extends to a Galois cover which is tame.

Consider a discrete valuation ring R, with $L = Q(R) \subset R \to K$, here the field of fractions is L and residue class field is K; consider schemes $\mathcal{U} \subset \mathcal{X} \to \mathrm{Spec}(R)$, with generic fiber $(\mathcal{U} \subset \mathcal{X}) \otimes L = (U \subset X)$ and special fiber $(\mathcal{U} \subset \mathcal{X}) \otimes K = (U_0 \subset X_0)$. In order to formulate a theorem on tame covers, and to be used in considering the paper [63], we say that this gives a *good reduction for the pair* $(U, X)/L$ if:

X is a complete curve over L,

and $U \subsetneq X$ is a dense open inside X (and hence U is an affine curve over L),

and $\mathcal{X} \to \mathrm{Spec}(R)$ is smooth and proper (and hence X_0/K is a proper curve),

and $(\mathcal{X} - \mathcal{U})$ is étale over $\mathrm{Spec}(R)$.

If this is the case, we obtain a surjective homomorphism:

$$\pi_1(U) \quad \to \quad \pi_1^{\mathrm{tame}}(U_0);$$

this is an isomorphism on the prime-to-p pro-finite quotient.
See EGA1, pp. 392 - 394.

7 Anabelian fundamental groups.

We have seen that class field theory for curves uses the Jacobian. We should me more precise.

(7.1) Alb and Pic. For a regular, complete variety V over a field K, and the choice of $P \in V(K)$ we can define the Albanese variety of V:

$$\tau : V \quad \longrightarrow \quad \mathrm{Alb}(V) = A_V;$$

this is the universal solution of mapping V into an abelian variety; see [31], II.3

For a complete variety V over a field K one can define its Picard variety (the connected component of a "universal solution to the problem of parameterizing divisor classes on V"), see [31], II.3, VI.1 and VI.4 for the case V is complete and regular (much more general theorems about the existence of Picard schemes exist; we will not go into those details here). In case V is regular, we obtain an abelian variety $\mathrm{Pic}_V = P_V$, and a divisor class (the Poincaré bundle) on the product $V \times P_V$ describing the parameterization of divisor classes.

For a regular, complete variety V over a field K, with $V(K) \neq \emptyset$ both notions are defined, and in fact the abelian variety A_V is isomorphic with the dual abelian variety of P_V:

$$A_V \quad \cong \quad (P_V)^t,$$

see [31], Th. 1 on page 148. As $(P_V)^{tt} \cong P_V$ we also obtain $(A_V)^t \cong P_V$ (double duality on abelian varieties is the identity: Cartier and Nishi; for references, see [48], Section 20).

Suppose A_V admits a principal polarization; in that case $A_V \cong (A_V)^t \cong P_V$; in this case, sometimes we "identify" the Albanese and the Picard variety; however we should realize that A_V comes with $\tau : V \to A_V$, and P_V comes with the Poincaré class. For a complete, nonsingular algebraic curve X these abelian varieties have a principal polarization, "the canonical polarization", we write $A_X = \mathrm{Jac}(X) = P_X$, and we call this "the Jacobian variety"; however, in applications, please realize well which aspect of the Jacobian is used. Here is an example, which is not essential, but it gives a motivation for the sequel:

(7.2) Cyclic covers of a curve. We suppose that Y is a *complete*, regular curve over a field K. We choose a prime number ℓ different from the characteristic of K, and we suppose that all ℓ-roots of unity are in K. We suppose that $Y(K) \neq \emptyset$. Then: *There is a bijective correspondence*

$$(\alpha \in P_Y(K)) \quad \leftrightarrow \quad (a \in \mathrm{Gal}(T/Y)),$$

where:
- *the order of $\alpha \in P_Y(K)$ is exactly ℓ, and*
- *$T \to Y$ is an étale Galois cover, with $\mathrm{Gal}(T/Y) =< a >$ cyclic of order ℓ.*

Here is a proof. The element α generates a "cyclic" subgroup scheme

$$\underline{\mathbb{Z}/\ell} \cong < \alpha > \subset P = P_Y, \quad \alpha \in < \alpha > (K),$$

and we obtain an exact sequence

$$0 \to < \alpha > \longrightarrow P \longrightarrow P/ < \alpha > =: B \to 0. \qquad \text{(pic)}$$

By the duality theorem, see [48], Th. 19.1 we obtain an exact sequence

$$0 \to < \alpha >^D \longrightarrow B^t \longrightarrow P^t \to 0. \qquad \text{(alb)}$$

Using $P^t \cong A_Y$ and $\tau : Y \to A = A_Y$ we obtain by pulling back:

$$\begin{array}{ccl} T & \hookrightarrow & B^t \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & A = A_Y = P^t. \end{array}$$

Moreover $\alpha^{\perp} =: a \in \mathrm{Gal}(T/Y)$ generates this cyclic group; we see that the choice of α indeed gives the desired covering plus a generator for the Galois group. Conversely, the data $a \in \mathrm{Gal}(T/Y)$, of a cyclic Galois cover with a generator in the cyclic covering group, by class field theory for curves, see [52], page 128, Cor., the covering gives rise to the exact sequence (alb), and by duality we recover (pic), and $\alpha = a^{\perp} \in < a >^D (K)$.

Note that we have used that for the étale commutative group scheme $\underline{\mathbb{Z}/\ell} \cong < \alpha >$ we can consider its Cartier dual $< \alpha >^D$, and we have $< \alpha >^D (K) = (< \alpha > (\overline{K}))^D$, where in this case the upper D denotes duality of abelian groups, in our case, duality of vector spaces over $\mathbb{F}_\ell$, and $\alpha \in < \alpha > (K)$ defines $\alpha^{\perp} \in (< \alpha > (K))^D$. We have seen that from α we construct a and conversely. □

(7.3) Lemma. *Let k be an algebraically closed field, and let X be a* hyperbolic *curve over k. The fundamental group $\pi_1(X)$ is anabelian.*
For the case that X is complete, see [11], Lemma 1 on page 133.

The rest of this section is devoted to a sketch of a proof of this lemma. We suppose $\Sigma \in \mathcal{Z}(\pi_1(X))$. For a Galois étale cover $Y \to X$ we define $\Sigma_{|Y} =: \sigma_Y = \sigma \in \mathrm{Aut}(Y/X)$. We are going to show that $\sigma = \mathrm{id} \in \mathrm{Aut}(Y/X)$. If this holds for all σ_Y, then $\Sigma = \{\sigma_Y\} = \mathrm{id}$.

We choose one Galois étale cover $Y \to X$. Note that "X is hyperbolic" implies that "Y is hyperbolic".

Consider $Y \subset Y^*$, where $Y^* \leftarrow Y^c$ is obtained from the completion Y^c by identifying all points of $Y^c - Y$ to one point on Y^* via mutually normal crossings of the branches; i.e. consider the "module" m in the sense of [52] III.1, and let $Y^* = (Y^c)_{\mathsf{m}}$. We write $A = J_{\mathsf{m}}(Y) = \mathrm{Jac}(Y^*)$. In case Y is complete, hence $Y = Y^c$, and of genus g, this is an abelian variety of dimension g. In case $\#((Y^c - Y)(k)) = r > 0$ this is a semi-abelian variety of dimension $g + r - 1$, an extension of $\mathrm{Jac}(Y^c)$, an abelian variety of dimension $g = \mathrm{genus}(Y^c)$, by $(\mathbb{G}_m)^{r-1}$. We have an immersion $Y^* \supset Y \hookrightarrow A$.

We choose a prime number ℓ such that: $\ell > 2$ and ℓ different from the characteristic of k. We use the Picard aspect of $Y^* \mapsto A = \mathrm{Jac}(Y^*)$, and hence obtain $\sigma^* : A \to A$.

We are going to prove that $(\sigma^*_{|A[\ell]} : \quad A[\ell] \to A[\ell]) = \mathrm{id}$.

Let $Z \to Y$ be the "maximal abelian ℓ-cover", i.e. this cover is the smallest one dominating all étale, cyclic Galois covers of Y of degree ℓ; such a cover may ramify "above Y^c". In this case we have, by class field theory, a covering

$$\begin{array}{ccc} Z & \hookrightarrow & A \\ \downarrow & & \downarrow \times\ell \\ Y & \hookrightarrow & A. \end{array}$$

Write $\Pi := A[\ell](k)$; note that $\Pi \cong (\mathbb{Z}/\ell)^{2g}$, respectively $\Pi \cong (\mathbb{Z}/\ell)^{2g+r-1}$. Note that $\mathrm{Gal}(Z \to Y) = \Pi$. We see that $Z \to Y \to X$ is a Galois cover. Hence we can choose a surjective homomorphism $\pi_1(X) \to \mathrm{Gal}(Z/X)$; we write $\Psi = \Sigma_{|Z} \in \mathrm{Gal}(Z/X)$ for the image of $\Sigma \in \pi_1(X)$ in $\mathrm{Gal}(Z/X)$.

Suppose that $T_a \to Y$ is a cyclic Galois cover with given generator $a \in \mathrm{Gal}(T_a/Y)$. Let

$$(b \in \mathrm{Gal}(T_b/Y)) \quad = \quad \sigma^*(a \in \mathrm{Gal}(T_a/Y))$$

be the pull back. Then there exists ψ fitting into a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\Psi} & Z \\ \downarrow & & \downarrow \\ T_b & \xrightarrow{\psi} & T_a. \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\sigma} & Y \\ \downarrow & & \downarrow \\ X & = & X; \end{array}$$

writing $I_a = \mathrm{Gal}(Z/T_a)$ and $I_b = \mathrm{Gal}(Z/T_b)$ we obtain:

$$\Psi \cdot (I_a \to \Pi \to \Pi/I_a =< a >) \cdot \Psi^{-1} \quad = \quad (I_b \to \Pi \to \Pi/I_b =< b >).$$

This proves that for

$$(\Pi \to \Pi/I_a =< a >)^D = (< \alpha > \subset \Pi^D)$$

we have:

$$\sigma^*(\Pi \to \Pi/I_a =< a >) = \sigma_*((\Pi \to \Pi/I_a =< a >)^D).$$

Note that $\Sigma \in \mathcal{Z}(\pi_1(X))$, hence $\Psi \in \mathcal{Z}(\Pi)$. We conclude that

$$(\sigma^* : A[\ell] \to A[\ell]) = \mathrm{id}.$$

By the semi-stable version of Serre's lemma, see [1], 3.5 (for the oroginal version of Serre's Lemma, see [53]), we conclude that $\sigma^* : A \to A$ is the identity. From this we conclude that $(\sigma : Y \to Y) = $id. (7.3)□

Remark. In case Y is complete, every $\mathrm{Gal}(T_a/Y) =< a >$ as above corresponds with $\alpha \in P[\ell](k)$, and we see that in this case $\sigma^* : P \to P$ is the identity. In case the Jacobian A is not an abelian variety, we do not use duality as in (7.2); therefore we took in this case the detour via Π^D.

8 Properties of good reduction

(8.1) The Tate-ℓ-group. Let K be a field, and let A be an abelian variety over K. Choose a prime number ℓ, *different from the characteristic of* K. We define

$$\mathrm{T}_\ell(A) \quad := \quad \lim_{\leftarrow} A[\ell^i](K^s);$$

this is a group which is isomorphic with $(\mathbb{Z}_\ell)^{2g}$, where $g = \dim A$, equipped with a continuous Galois action

$$\mathrm{Gal}(K^s/K) \to \mathrm{Aut}(\mathrm{T}_\ell(A)) \cong \mathrm{GL}(2g, \mathbb{Z}_\ell).$$

We might as well have written $\mathrm{T}_\ell(A) = \lim_{\leftarrow} A[\ell^i]$, intending the pro-finite group scheme. Note that giving a finite *étale* group scheme N over K is the same as giving the group $N(K^s)$ plus the continuous Galois action on this discrete group.

N.B. In case the characteristic of K is $p > 0$, we do not use the notation $T_p(A)$. However some authors use this notation, creating a confusion between the (very different) concepts: $\lim_{\leftarrow} A[p^i](K^s)$, sometimes called "the physical Tate-p-module" on the one hand and $\lim_{\leftarrow} A[p^i]$ which is a pro-finite group scheme on the other hand; the last one has a much richer structure than the first one; in general, in characteristic p we cannot recover the last one from the first one.

(8.2) The Weil pairing. This notion was for the first time introduced in [66], page 150. In the literature two different notions are called "the Weil pairing". Here is an explanation. For an abelian scheme A, we denote by A^t its dual abelian scheme. For every $n \in \mathbb{Z}_{>0}$ there is a natural isomorphism $A[n]^D \cong A^t[n]$, as follows from the duality theorem, see [48], Th. 19.1. For an abelian variety A over a field of characteristic different from ℓ this implies the existence of a bilinear pairing

$$<,>: \quad \mathrm{T}_\ell(A) \times \mathrm{T}_\ell(A^t) \quad \longrightarrow \quad \mathrm{T}_\ell(\mathbb{G}_m),$$

which is a perfect pairing of Γ_K-modules. Suppose moreover that

$$\lambda : A \quad \xrightarrow{\sim} \quad A^t$$

is a principal polarization. Then we obtain

$$< -, -\cdot\lambda >=:<>_\lambda : \quad \mathrm{T}_\ell(A) \times \mathrm{T}_\ell(A) \quad \longrightarrow \quad \mathrm{T}_\ell(\mathbb{G}_m);$$

this is called the Weil paring of the polarized abelian variety (A, λ). You might consult SGAD, or [49] for further properties.

(8.3) Good and stable reduction of abelian varieties. References: [55]; see SGAD. Let R be a discrete valuation ring, with field of fractions $Q(R) = K$ and residue class field $R \to R/m = k$. Let A be an abelian variety over K we say that A has stable reduction (or stable reduction at R, or stable reduction at v, where v is the discrete valuation given on K by R) if there exists a flat group scheme $\mathcal{A} \to \mathrm{Spec}(R)$ such that $\mathcal{A} \otimes_R K \cong A$, and the connected component of 0 of $A_0 := \mathcal{A} \otimes_R k$ is an extension of an abelian variety by a torus L, i.e. A_0^0/L is an abelian variety, and $L \otimes \overline{k} \cong (\mathbb{G}_m)^s$ for some non-negative integer s.

We say A has good reduction at R if the model $\mathcal{A} \to \mathrm{Spec}(R)$ can be chosen such that A_0 is an abelian variety.

Properties of good reduction and of stable reduction can be read off from the Galois representation ρ restricted to an inertia group $I(\overline{v})$ of v on $\mathrm{T}_\ell(A)$:

the criterion of Néron-Ogg-Shafarevich, see [55], Th. 1 on page 473, says that A has good reduction at v iff $\rho(I(\overline{v})) = \{1\}$;

A has stable reduction at v iff all eigenvalues of elements in $\rho(I(\overline{v}))$ are equal to 1.

In these cases, we have chosen en extension of v to a place $\overline{v}$ of K^s, and by $I(\overline{v})$ we intend the inertia group with respect to this choice; different choices of this extension give conjugate subgroups.

Using the fact that eigenvalues of monodromy are roots of unity, see [55], page 515, and using properties of the Néron minimal model, one can show the stable reduction theorem for abelian varieties, see SGAD, Th. IX.3.6 on page 351: *Let A be an abelian variety over a filed K, let v be a discrete valuation on K; there exists a finite extension $[L : K] < \infty$, and a discrete valuation w of L dividing v such that $A \otimes_K L$ has stable reduction at w.*

(8.4) Stable reduction of algebraic curves. Basic reference: [8]. Let R be a discrete valuation ring, with field of fractions $Q(R) = K$ and residue class field $R \to R/m = k$. Let X be a complete algebraic curve over K (hence we suppose that X is proper, smooth and geometrically irreducible of dimension one over K), of genus g at least 2. Following Deligne and Mumford, we say that X has *stable reduction at v* if there exists a model $\mathcal{X} \to \mathrm{Spec}(R)$, i.e. $\mathcal{X} \otimes_R K \cong X$, which is flat and proper over R, and such that all geometric fibers of $\mathcal{X}/R$ are connected, reduced curves with at most ordinary double points as singularities, such that any rational components contains at least 3 singular points. In [8], page 88, we find:

$$(\mathrm{Jac}(X) \text{ has stable reduction at } v) \Rightarrow (X \text{ has stable reduction at } v).$$

(8.5) Good reduction of algebraic curves. Let X be a complete algebraic curve over K. We say that X has *good reduction at v*, if there exists a flat, smooth proper model $\mathcal{X} \to \mathrm{Spec}(R)$. If such a model does not exist we say that X has bad reduction at v.

Warning. Let X be a complete algebraic curve over K, and let $J = \mathrm{Jac}(X)$ be its Jacobian. Note that

$$(\mathrm{Jac}(X) \text{ has good reduction at } v) \Leftarrow (X \text{ has good reduction at } v);$$

this is clear. However:

$$(\mathrm{Jac}(X) \text{ has good reduction at } v) \not\Rightarrow (X \text{ has good reduction at } v).$$

Here is an example. Let R be a discrete valuation ring, and let $\mathcal{X} \to \mathrm{Spec}(R)$ be a flat, proper morphism, relative of dimension one, such that the generic fiber is an algebraic curve of genus $g \geq 2$, and the special fiber is the union of a curve of genus 1 and a curve of genus $g-1$, attached to each other via a normal crossing (such examples are easy to construct). One can prove that $X = \mathcal{X} \otimes_R K$ has it bad reduction at v; however, we see that $J = \mathrm{Jac}(X)$ has good reduction at v.

Basic references: [46], [47], [3]. Takayuki Oda proved a remarkable criterion for *good reduction of algebraic curves*. As before, let X be a complete regular curve over a field K, and let R be a discrete valuation ring of K. Then we have the usual exact sequence

$$1 \to N = \pi_1(X \otimes K^s) \longrightarrow G = \pi_1(X) \longrightarrow \Gamma = \mathrm{Gal}(K^s/K) \to 1.$$

We choose a prime number ℓ different from the residue characteristic of R. We denote by $\pi_1(X \otimes K^s)(\ell)$ the ℓ-profinite completion. Thus we obtain a representation

$$\rho: \quad I(v) \quad \longrightarrow \quad \mathrm{Out}(\pi_1(X \otimes K^s)(\ell).$$

Theorem (Takayuki Oda) *The curve X has good reduction at v iff $\rho(I(v)) = \{1\}$.*

Note that

$$(\pi_1(X \otimes K^s)(\ell))_{\mathrm{ab}} = \mathrm{T}_\ell(J).$$

The central idea in the proof by Oda starts with: the fact that ρ has trivial image, the image $\rho_{ab}(I(v)) \subset \mathrm{Aut}(\mathrm{T}_\ell(J))$, hence J has good reduction, hence the reduction X_0 of X is a tree of regular curves; studying the representation in the lower central series in $\pi_1(X \otimes K^s)$ shows that if this representation ρ already has trivial image in $\mathrm{Out}(\Pi/\Pi_3)$, modding out the the third step in the lower central series in $\Pi := \pi_1(X \otimes K^s)$, then the reduction is good: then X_0 is irreducible (and hence X_0 is irreducible and regular).

Homology of abelian varieties versus the fundamental group of an algebraic curve. [For an abelian variety A/K we know that $\pi_1(A \otimes K^s)$ is an abelian group.] We see that the subtle difference between good reduction of X and good reduction of J is reflected in the difference between the representation in $\mathrm{Out}(\pi_1(X \otimes K^s)(\ell)$ and the related abelian representation in $\mathrm{Aut}(\mathrm{T}_\ell(J))$. This seems to be a very fundamental fact. We will encounter this again in the difference between:

the Tate conjecture (which holds over global fields, but not over local fields),

and

the Grothendieck conjecture (which, as Mochizuki shows, holds for hyperbolic curves over local fields).

And, yes, we see in the proof by Moichizuki that the non-abelian representations play an essential role. I think that these subtle differences explain that indeed the Grothendieck conjecture could be called the "anabelian Tate conjecture" if we take into account the anabelian aspects of $\pi(X)$ (for X hyperbolic) versus the fact that $\pi_1(X)(\ell)_{\mathrm{ab}} = \mathrm{T}_\ell(J)$ is abelian.

9 The anabelian conjectures by Grothendieck

(9.1) Grothendieck writes about his “anabelian conjecture” in “*Esquisse d'un programme*”, 1984, see [22], in his letter to Faltings, 1983, see [23], and in his manuscript “La longue marche ...”, 1981, see [24]. However it is not completely clear how to phrase the conjecture precisely for two reasons:

- Which varieties should be baptized as “anabelian varieties”?
- Which form of the conjecture should be taken, the Isom-formulation, or the Hom-formulation?

Grothendieck expects there is a certain class of algebraic varieties, defined over a global field (or, fields of finite type over their prime field), such that these up to isomorphism can be recovered form their (algebraic fundamental group), the Isom-formulation; in his case Grothendieck mentions the representation of the absolute Galois group on the geometric fundamental group; see page 6 of his letter ([51], page 54).

In these cases one certainly expects the fundamental group to be very non-commutative, see page 14 of “Esquisse ..”, [51], page 17, made precise in (12.1). However we now know, e.g. see [29], that we have to be careful in imposing only this condition, and Grothendieck was very well aware of that. For *number fields*, and for *algebraic curves* it turns out that the anabelian condition indeed suffices for formulating the correct conjecture. In more general situations, perhaps one should like to impose a condition which is something like “rigidity” which could result from the condition that a variety is “very hyperbolic”, see [42]. However we do not have a satisfactory definition of “anabelian varieties” in the sense that the Grothendieck conjectures holds for this class.

As yet we do not see a definite version of the anabelian conjectures for higher dimensions: it is not clear which varieties should be called “anabelian” in the sense of Grothendieck: either $\pi_1^{\mathrm{top}} V(\mathbb{C})$ being anabelian, or $\pi_1(V \otimes \overline{K}$ being anabelian does not seem the right condition.

In the Workshop, and in these notes, *we will restrict ourselves to the case of number fields and to the case of algebraic curves over finite, over global or over sub-p-adic fields.* The case of number fields is completely covered by the Neukirch-Uchida theorem. Below we formulate the anabelian Grothendieck conjectures only for algebraic curves:

(9.2) Following Grothendieck, e.g. in [51], page 54, and in reading various papers of Nakamura, Tamagawa and Mochizuki we will formulate the conjectures as follows, e.g. see [44], pp. 33-35: Let X, Y be curves over a field K; the exact sequence

$$1 \to N = \pi_1(X \otimes \overline{K}) \longrightarrow G = \pi_1(X) \xrightarrow{p} \Gamma = \Gamma_K = \mathrm{Gal}_1(\overline{K}/K) \to 1. \qquad (\xi_X)$$

defines

$$R : \Gamma_K \longrightarrow \mathrm{Out}(\pi_1(X \otimes \overline{K})),$$

and for anabelian N we can recover G from R, see (1.4).

(GC1)$_?$, the fundamental conjecture, the Isom-Conjecture. *Suppose X and Y are hyperbolic curves over fields K_1, K_2; then*

$$(\xi_X) \cong (\xi_Y) \quad \Rightarrow \quad X/K_1 \cong Y/K_2.$$

In this form the conjecture can reasonably be formulated over finite fields, over fields of finite type over the prime field, in particular over number fields, or over local fields (finite extensions of the field of p-adic numbers).

Here is another formulation of the conjecture: *Let K_i, $i = 1, 2$ be fields* (satisfying certain properties) *and let X_i/K_i be hyperbolic curves. Then*

$$\mathrm{Isom}(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}(\pi_1(X_1), \pi_1(X_2))/\mathrm{Inn}(\pi_1(X_2)).$$

(GC2)$_?$, the Hom-Conjecture. *Let X/K and Y/K be hyperbolic algebraic curves over field K which is finitely generated over $\mathbb{Q}$; the natural map*

$$\mathrm{Hom}_K^{\mathrm{dom}}(X, Y) \quad \xrightarrow{\sim} \quad \mathrm{Hom}^{\mathrm{open}}_{\mathrm{Gal}(\overline{K}/K)}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}(\pi_1(Y \otimes \overline{K}))$$

is bijective; here the superscript "dom" stand for the subset of "dominant morphisms" (in the case of curves this is equivalent by saying "non-constant"); we will write

$$\mathrm{Hom}^{\mathrm{open}}_{\mathrm{Gal}(\overline{K}/K)}(\pi_1(X), \pi_1(Y))/\mathrm{Inn}(\pi_1(Y \otimes \overline{K})) = \mathrm{OutHom}^{\mathrm{open}}_{\Gamma_K}(\pi_1(X), \pi_1(Y)).$$

(GC3)$_?$, the Section Conjecture. *Let K be a field* (satisfying certain conditions) *and let X/K be a hyperbolic algebraic curve. Every section homomorphism*

$$\alpha : \Gamma_K := \mathrm{Gal}(\overline{K}/K) \quad \longrightarrow \quad \pi_1(X)$$

of the projection homomorphism $\pi_1(X) \to \Gamma_K$ comes from an element in $X^c(K)$ (a K-rational point of the "compactification" of X).

See [22], pp. 4 - 8; [23]; [24], §5. For more details and references, see [44], Section 1.2; [42]; [43]; see [45], XII.3: *Anabelian conjectures*; F. Pop in [51], pp. 113 - 126; [37].

In the Workshop we shall say that we study "the Grothendieck conjecture" or just say GC, where we intend to say in (I) and in (II) the anabelian Grothendieck conjecture in the form of the Isom-conjecture (GC1); in (III) we study the Hom version (GC2) over sub-p-adic fields.

10 p-adic Hodge theory

In the third part of the Workshop part of the result of [36] will be discussed. A survey can be found in [11]. Also see [7].

On 23-XI B. Moonen will present some prerequisites about p-adic Hodge theory. Especially:

(a) We assume that people in the audience are familiar with the notion of a Hodge-Tate representation. See for instance [54] or [65]; this notion will also be discussed by B. Moonen at the ICS of 23 November. The following theorem was proven by Faltings:

Theorem. *Let K be a p-adic local field, X a smooth proper K-variety. Then $H^n(X_{\overline{K}}, \mathbf{Q}_p)$ with its natural action of* $\mathrm{Gal}(\overline{K}/K)$ *is a Hodge-Tate module of Hodge-Tate weights in* $[0, n]$.

Though not strictly necessary for the workshop it is good to know that this result is only the beginning of a much more powerful p-adic Hodge theory. For an overview see [14], [15], [30]. Note: almost all conjectures that are stated in these papers are now theorems. Hopefully Colmez's Bourbaki lecture of November 2001 will give an overview of the present state of the art.

(b) One of the main tools in Mochizuki's work [36] is the use of "anabelian Hodge-Tate structures", i.e., Hodge-Tate structures on the pro-p fundamental group. To make this precise one uses (a truncated version of) the unipotent algebraic envelope of $\pi_1^{(p)}$; see [7], §9. Some familiarity with this notion of a unipotent algebraic envelope (also referred to as the Malçev completion of $\pi_1^{(p)}$) is useful. Note: more refined results in "anabelian" Hodge-Tate theory have been obtained by Shiho [58]; cf. the notes at the end of §3 in [36].

Relevant literature: [10], [13], [14], [15], [30], [54], [58], [65].

11 Checklist prerequisites

In the Workshop we suppose certain notions in number theory and algebraic geometry to be known. Here is a list of topics, definitions and methods we hope/expect/suppose the participants are at least familiar with:

- Basic notions on Galois theory and on number theory.
- Elementary properties about algebraic curves, see Section 4. In the Workshop we will mainly consider algebraic curves over finite fields, over number fields, and over local fields. Equivalence of (complete algebraic curves) $\longleftrightarrow$ (function fields in one variable); equivalence of (algebraic curves over $\mathbb{C}$) $\longleftrightarrow$ (Riemann surfaces).
- Notions on topological fundamental groups, and description of these for Riemann surfaces, see Section 5.
- Basic facts about the (algebraic) fundamental group Section 6: see the Intercity Seminar talk by H. W. Lenstra on 23-XI. Especially we need the description of the (arithmetic)

fundamental group as an extension over the absolute Galois group of the base field with kernel the fundamental group of the geometric fiber.

- We need some facts about the description of covers of algebraic curves via class field theory, see (6.6) and Section 7.
- We need some facts about counting points over finite fields, the Weil-bound etc. (not described in these notes).
- We need properties of good reduction, and of stable reduction for algebraic curves; see Section 8.
- Notions on p-adic Hodge theory will be used in (III); see the Intercity Seminar talk by B. Moonen on 23-XI, and see Section 10.
- Theory mentioned in Section 14 will not be used in the Workshop.

12 Some terminology

We collect some of the definitions used, and we try to homogenize notation.

(12.1) A topological group G is called "anabelian" if for every closed, finite index subgroup $H \subset G$ has trivial center: $\mathrm{Cent}(H) = \{1\}$. [A group G is called "anabelian" if for every finite index subgroup $H \subset G$ has trivial center.]
Examples:
For every number field $[K : \mathbb{Q}] < \infty$ the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ is anabelian, [45], 12.1.6.
For every hyperbolic curve X over $\mathbb{C}$ the topological fundamental group $\pi_1^{\mathrm{top}}(X)$ is anabelian. For every hyperbolic curve X over an algebraicaly closed field $\pi_1(X)$ is an anabelian groups, see (7.3).

(12.2) For a field K we write K^s for a separable closure, and $\overline{K}$ for an algebraic closure of K. We write $\Gamma_K := \mathrm{Gal}(K^s/K)$ for the "absolute Galois group". Sometimes we write $k = \overline{K}$; in mixed characteristic situations we sometimes use k for the residue class field (not necessarily algebraicaly closed).

(12.3) We write $\pi_1^{\mathrm{top}}(S)$ for the topological fundamental group of a topological space S. We write $\pi_1^{\mathrm{alg}}(X) = \pi_1(X)$ for the étale fundamental group of a scheme (or of a variety) X. In this notation, for a field K we have $\Gamma_K = \mathrm{Gal}(K^s/K) = \pi_1(\mathrm{Spec}(K))$.

(12.4) Sometimes we use the terminology of varieties, sometimes we use schemes. A scheme X over a field $X \to \mathrm{Spec}(K)$ is called a "variety" defined over K if X is separated, algebraic over K and geometrically reduced and geometrically irreducible, see [39], especially II.5; also see HAG, II.2.6 for the case that $K = k$ is algebraically closed. We wrote "variety": there is an equivalence of categories, but the set of points on a variety V and the corresponding scheme X are different in the positive dimensional case (i.e. as soon as V does not consist of just one point); going from varieties to schemes and backwards, please be careful what is meant by "a point on V" etc.

13 Theorems to be discussed at the Workshop

From the existing proved cases of the GC we will discuss a small part. [Reminder: algebraic curves over a field considered are supposed to be regular and absolutely irreducible.]

(I) The analog of the Grothendieck conjecture for number fields.

Theorem (Uchida, Neukirch). *Suppose* $[K_1 : \mathbb{Q}] < \infty$ *and* $[K_2 : \mathbb{Q}] < \infty$ (i.e. these are number fields); *the natural map*

$$\mathrm{Isom}(K_1, K_2) \quad \xrightarrow{\sim} \quad \mathrm{OutIsom}(\Gamma_{K_1}, \Gamma_{K_2})$$

is an isomorphism.
See [45], XII.2 for several versions of this theorem.

(II) The Grothendieck conjecture for hyperbolic curves over finite and over number fields.

Theorem (Nakamura). *Let* K_i *be a number fields,* $i = 1, 2$, *and let* X_1/K_1, *and* X_2/K_2 *be hyperbolic curves of genus zero. There are exact sequences*

$$1 \to \pi_1(X_i \otimes \overline{K_i} \longrightarrow \pi_1(X_i) \xrightarrow{p} \Gamma_{K_i} = \mathrm{Gal}(\overline{K_i}/K_1) \to 1. \qquad (\xi_i).$$

Suppose that the exact sequences are isomorphic:

$$(\xi_1) \quad \cong \quad (\xi_2) \quad \textit{then} \quad X_1/K_1 \cong X_2/K_2.$$

See [43].

Theorem (Tamagawa). *Let* K_i *be finite fields,* $i = 1, 2$, *and let* X_i *be affine hyperbolic curves over* K_i. *In this case*

$$\mathrm{Isom}(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}(\pi_1^{\mathrm{tame}}(X_1), \pi_1^{\mathrm{tame}}(X_2))/\mathrm{Inn}(\pi_1^{\mathrm{tame}}(X_2))$$

is bijective.

Let K_i *be number fields,* $i = 1, 2$, *and let* X_i *be affine hyperbolic curves over* K_i. *In this case*

$$\mathrm{Isom}(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}(\pi_1(X_1), \pi_1(X_2))/\mathrm{Inn}(\pi_1(X_2))$$

is bijective.
See [63], 0.5, 0.4.

(III) The Grothendieck conjecture for hyperbolic curves over sub-p-adic fields.

Definition(Mochizuki). *A field* K *is called a* sub-p-adic field *if there exist:*

– *a prime number* p

– *a finitely generated field extension* L *of* $\mathbb{Q}_p$ *such that* K *is isomorphic to a subfield of* L *.*

Clearly, every finite, or finitely generated extension of $\mathbb{Q}_p$ is sub-p-adic. Note that also number fields are sub-p-adic! Note that even certain infinite algebraic extensions of $\mathbb{Q}$ are sub-p-adic; see [36], 15.4.

Notation: $\Delta_X = (\pi_1(X \otimes \overline{K}))^{(p)}$.

Theorem (Mochizuki). *Let K be a sub-p-adic field. Let X, and Y be algebraic curves over K, such that Y is hyperbolic. Then we have an isomorphism*

$$\mathrm{Hom}_K^{\mathrm{dom}}(X,Y) \quad \xrightarrow{\sim} \quad \mathrm{OutHom}_{\Gamma_K}^{\mathrm{open}}(\Delta_X, \Delta_Y).$$

See [36].

Remark. Note that, in contrast with the results in (I) and (II), this proves a "Hom"-version of the Grothendieck conjecture. Further note that in the theorem we only use the maximal pro-p-quotient of the fundamental group. The Hom-conjecture $(\mathrm{GC2})_?$ for this situation, as stated in Section 9, is an almost immediate corollary of this theorem; also it reproves the above result of Tamagawa for affine curves over a number field.

14 An analogy: the Shafarevich conjecture, the Tate conjecture

(14.1) This section will not be of importance in the topics (I) $\sim$ (III) to be discussed in the Workshop. However, we like to indicate an analogy, and a fundamental difference between two conjectures. The Grothendieck conjecture sometimes is called "the anabelian Tate conjecture", see [43], 1.3.2; also see [37], Section 1; we will see we have to be careful to stretch this analogy too far; especially see what Mochizuki writes about this on page 324 of [36].

(14.2) In Stockholm, 1962 Shafarevich proposed a conjecture, see [56]. It generalized a classical theorem in number theorem, a finiteness theorem by Hermite to "higher dimensions".

Theorem (Hermite), see [27], pag. 595: *Let* $[K:\mathbb{Q}] < \infty$, i.e. K is a number field; *suppose given $n \in \mathbb{Z}_{>0}$ and a finite set of places S of K. The set of field extensions $K \subset L$ up to K-isomorphism, of degree n, unramified outside S is* finite.
This idea was generalized:

Conjecture (Shafarevich, 1964) / **Theorem** (Faltings, 1983), see [9]: *Let* $[K:\mathbb{Q}] < \infty$, *let $g \in \mathbb{Z}_{\geq 2}$ and let S be a finite set of places of K. The set of K-isomorphism classes of complete algebraic curves over K of genus g having good reduction outside S is* finite.

(14.3) It turned out that this conjecture was related to a conjecture by Tate, proved in [64] over finite fields, later over function fields in positive characteristic by Mori and Zarhin, and finally in 1983 by Faltings over number fields, and later in general, see [12], Ch. VI.

Conjecture (Tate, 1966) / **Theorem** (Faltings, 1983), see [9]: *Let K be a field of finite type over its prime field; let ℓ be a prime number different from the characteristic of K; let X and Y be abelian varieties over K. The natural map*

$$\mathrm{Hom}(X,Y) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \quad \longrightarrow \quad \mathrm{Hom}_{\Gamma_K}(\mathrm{T}_\ell(X), \mathrm{T}_\ell(Y))$$

is an isomorphism. [Here the Tate-ℓ-groups, $\mathrm{T}_\ell(X) = \lim_{\leftarrow} X[\ell^i](K^s)$, are to be understood as Galois modules under $\Gamma_K = \mathrm{Gal}(K^s/K)$, and homomorphisms considered are supposed to be Galois-invariant.] Note that for an abelian variety we have $\mathrm{T}_\ell(X) \cong_{\Gamma_K} \pi_1(X_k)(\ell)$.

In a sense, history seems to repeat itself. The Uchida-Neukirch theorem, studied in our part (I) in the workshop is a theorem about number fields. The fact that the absolute Galois group of a number field is very non-abelian, is "anabelian" see (12.1), seems to imply a certain "rigidity", and the theorem that number fields with isomorphic absolute Galois groups are isomorphic comes out. The Grothendieck "anabelian conjectures" seem to generalize this to "higher dimensions".

We have seen that the conjectures by Shafarevich and by Tate turned out to be crucial ingredients in proofs in arithmetic geometry, such as the implication by Parshin: (Shafarevich Conjecture) $\Rightarrow$ (Mordell Conjecture).

It might very well be that the anabelian conjecture by Grothendieck is going to play an analogous role in the near future. However we like to make one remark.

(14.4) Example (Lubin and Tate, [33], 3.5) *There exist: a local field K* (a finite extension of $\mathbb{Q}_p$) *and an elliptic curve E of K such that:*

- $\mathrm{End}(E \otimes \overline{K}) = \mathbb{Z}$, i.e. *"E has no CM", and*
- $\mathrm{End}(T_p(E)) \supsetneq \mathbb{Z}_p$*, and we can say "$T_p(E)$ does have CM".*

This shows that the condition "K is of finite type over its prime field" is essential in the sense that the analogon of the Tate conjecture does not hold in general for abelian varieties over local fields.

Grothendieck in his letter to Faltings, see [51], page 53, clearly states that the base field should be a a field of finite type over the prime field. Many people thought that this was an essential condition, as in the case of the Tate conjecture. *However in* [36] *we see that the analog of the Grothendieck anabelian conjecture holds for hyperbolic curves over local fields*; apparently the fact that the geometric fundamental group is is very non-commutative makes it possible to relax the relevant condition on the base field. The subtle differences between a curve and its Jacobian again show up: we can retrieve a curve X from its canonically polarized Jacobian (J, λ) (Torelli's theorem), but representations in $\pi_1(J)$ on the one hand and in $\pi_1(X)$ on the other hand are quite different!

(14.5) Exercise. Let K be a number field, and let X be a complete curve over K of genus $g \geq 2$. *Show that the set of complete algebraic curves Y over K of genus g, up to K-isomorphism such that*

$$(\pi_1(X))_{\mathrm{ab}} \cong_{\Gamma_K} (\pi_1(Y))_{\mathrm{ab}}$$

is finite.

15 Some more results, not discussed at the Workshop

Results mentioned in the previous sections, to be discussed in the Workshop, only form a part of what has been proved concerning the Grothendieck Conjecture.

Complete hyperbolic curves. Note that the GC has been proved for complete curves over number fields. This has been done by Mochizuki in his paper [34], where results from [63] and lifting properties were used; one of the basic results of [34] also follows from [36].

It could very well be that indeed the GC is correct for complete hyperbolic curves over finite fields, but this result has not yet been fully claimed, not yet written up (private communication from S. Mochizuki and Y. Tamagawa).

The birational Grothendieck conjecture. This has been proved, see [50], and literature cited in that paper.

The higher dimensional GC ? It is not so clear how to formulate the anabelian Grothendieck conjecture for higher dimensional varieties. In [29] we find a discussion on this topic; in particular see their "Test for anabelianity". Then we see in that paper that the Siegel moduli space $\mathcal{A}_{g,n} \otimes \mathbb{C}$ for principally polarized abelian varieties with level-n-structure over the complex numbers has an anabelian topological fundamental group $\Gamma_g(n)$; moreover it is "hyperbolic" in the sense of Kobayashi; it is a $K(\pi, 1)$ space; hence we could dream that this variety falls under the Grothendieck anabelian programme; then Ihara and Nakamura show that this should not be the case. Here we see an example of a complex variety which has an anabelian topological fundamental group, while the algebraic fundamental group has a large center. All this makes us cautious to formulate a form of the GC in higher dimensions.

(15.1) Grothendieck has a proposal for studying $\Gamma_{\mathbb{Q}}$, the absolute Galois group of $\mathbb{Q}$, i.e. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This is explained in [24]. A survey can be found in [51], pp. 183 - 203. It is not clear whether the interesting question whether $\Gamma_{\mathbb{Q}} \hookrightarrow \mathrm{GT}^{\wedge}$ is surjective.

(15.2) Tamagawa proved the "weak weak Grothendieck conjecture", using results by Pop-Saidi and Raynaud: *Consider $m := \overline{\mathbb{F}_p}$; the map from the set of hyperbolic curves over m to the set of topological groups given by $(X/m) \mapsto \pi_1(X)$ has finite fibers* [To appear].

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