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SPECTRAL DECOMPOSITION OF THE OPERATOR $p^2 - q^2$

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Synopsis

It is shown that the operator $p^2 - q^2$ has a continuous spectrum extending from $-\infty$ to $+\infty$. The expansion of an arbitrary function with respect to the eigenfunctions is given. The action of the operators q and p on the eigenfunctions can be written explicitly by means of symbolic formulae. Finally, an alternative method for studying the behaviour of self-accelerating wave packets is given.

1. *Introduction.* The interaction of a non-relativistic electron with the electromagnetic field gives rise to a term of the type $p^2 - q^2$ in the Hamiltonian, where p and q are two canonically conjugate variables [1] [2] [3] [4]. Such a term may be regarded as the Hamiltonian of a harmonic oscillator with a negative binding force, and hence with an imaginary frequency i (or $-i$). Classically the equations of motion of such an 'oscillator' can easily be solved. It turns out that $p(t)$ and $q(t)$ contain a time factor e^t, so that they increase exponentially with time. This corresponds to the well known self-accelerating solution of Lorentz' equation of motion for the electron [5].

In order to deal quantummechanically with this pathological oscillator, one has to find a representation in which the operator $p^2 - q^2$ is diagonal. If one represents the operators p and q in the usual way by the operators $-i(\mathrm{d}/\mathrm{d}x)$ and x acting in function space, the problem is to find a complete set of solutions of the eigenvalue problem

$$(-\,\mathrm{d}^2/\mathrm{d}x^2 - x^2)\,\varphi(x) = \lambda\varphi(x). \tag{1}$$

Some time ago I have stated [1] that this cannot be done in a proper way. This statement was criticised by Steinwedel [2], but reasserted by Arnous [6] and Enz [4]. On the other hand, according to the general theory of differential operators [7], there should be a continuous spectrum from $-\infty$ to $+\infty$, with a complete set of eigenfunctions. In view of this controversy it may be of some use to give explicit formulae for the eigenfunctions of (1) and for

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the expansion in these eigenfunctions, thus proving that Steinwedel was right.

Equation (1) is a special case of the confluent hypergeometric differential equation and is called Weber's equation [8]. Its solution in terms of Weber's function $D_\nu(z)$ is *)

$$\varphi(x) = D_{-\frac{1}{2}-\frac{1}{2}i\lambda}\,[\pm\,(1+i)\,x]. \tag{2}$$

In section 2 we write this function in a more explicit form, and list some properties. In section 3 it is shown that an arbitrary square integrable function $f(x)$ can be expanded with respect to the functions (2), when λ ranges from $-\infty$ to $+\infty$. In section 4 the explicit form of the expansion is given. Finally in section 5 the argument used by Arnous [6] to show that no complete set of eigenfunctions exists, is discussed, and the explicit form of the operators q and p in the representation of these eigenfunctions is given.

2. *Solution of equation* (1). Putting $\varphi(x) = e^{-\frac{1}{2}ix^2}\psi(x)$ one finds for $\psi(x)$ the equation

$$\psi'' - 2ix\psi' + (\lambda - i)\psi = 0. \tag{3}$$

This equation can be solved by the standard Laplace method [9]; one is thus led in a straightforward way to the solution

$$\psi\,(x) = \int_0^\infty t^{-\frac{1}{2}+\frac{1}{2}i\lambda}\, e^{-\frac{1}{4}it^2+itx}\,\mathrm{d}t. \tag{4}$$

The integration path in (4) may be turned into the half-line $(0, \infty\, e^{-i\vartheta})$ where ϑ is an arbitrary angle between 0 and $\frac{1}{2}\pi$. It can then readily be verified that (4) actually satisfies (3) for every real λ. Furthermore $\psi_\lambda(x)$ is an entire function of x; it tends to zero like $|x|^{-\frac{1}{2}}$ for $x \to \pm\,\infty$ and is therefore not square integrable along the real axis (see appendix A).

A second solution of (3) is $\psi_\lambda(-x)$. (Indeed, $\psi_\lambda(-x)$ cannot be a multiple of $\psi_\lambda(x)$, because $\psi_\lambda(x)$ tends to zero as $x \to \infty$ in the upper half of the complex x-plane, and $\psi_\lambda(-x)$ tends to zero in the lower half.) Accordingly, we have the following two solutions of (1) **)

$$\varphi_\lambda(x) = e^{-\frac{1}{2}ix^2}\psi_\lambda(x) \quad \text{and} \quad \bar\varphi_\lambda(x) = e^{-\frac{1}{2}ix^2}\,\psi_\lambda(-x) = \varphi_\lambda(-x). \tag{5}$$

These solutions only differ by a constant factor from the solutions (2). Further details are provided in appendix A.

3. *Completeness*. We shall prove the completeness of the set of functions

$$\{\varphi_\lambda(x), \bar\varphi_\lambda(x)\} \qquad (-\infty < \lambda < +\infty)$$

*) Steinwedel [2]) mentioned this solution and suggested that it should be possible to formulate the corresponding expansion theorem.

**) The bar in $\bar\varphi$ serves as a distinctive mark; complex conjugation will be indicated by *.

by showing that any function $f(x)$ that is square integrable,

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx < \infty \quad \text{or} \quad f(x) \in L^2(-\infty, +\infty),$$

can be expanded in the form

$$f(x) = \int_{-\infty}^{+\infty} \{a(\lambda)\, \varphi_\lambda(x) + b(\lambda)\, \bar{\varphi}_\lambda(x)\} \, d\lambda. \tag{6}$$

Obviously $e^{\frac{1}{4}ix^2} f(x) \equiv g(x)$ is again square integrable and should satisfy

$$g(x) = \int_{-\infty}^{+\infty} \{a(\lambda)\, \psi_\lambda(x) + b(\lambda)\, \psi_\lambda(-x)\} \, d\lambda. \tag{7}$$

Hence it suffices to prove the existence of an expansion (7). Since $g(x)$ is square integrable it has a Fourier-transform $G(t)$ such that

$$g(x) = \int_{-\infty}^{+\infty} G(t)\, e^{itx} \, dt.$$

It is convenient to decompose $g(x)$ in a "positive-frequency part" $g_+(x)$ and a "negative-frequency part" $g_-(x)$, defined by

$$g_+(x) = \int_0^{\infty} G(t)\, e^{itx} \, dt, \quad g_-(x) = \int_{-\infty}^{0} G(t)\, e^{itx} \, dt.$$

We shall first prove the expansion

$$g_+(x) = \int_{-\infty}^{\infty} a(\lambda)\, \psi_\lambda(x) \, d\lambda. \tag{8}$$

Equation (8) may also be written

$$\int_0^{\infty} G(t)\, e^{itx} \, dt = \int_{-\infty}^{+\infty} a(\lambda) \, d\lambda \int_0^{\infty} t^{-\frac{1}{2}+\frac{1}{2}i\lambda}\, e^{-\frac{1}{4}it^2+itx} \, dt, \tag{9}$$

which suggests that one put *)

$$G(t) = e^{-\frac{1}{4}it^2} \int_{-\infty}^{+\infty} a(\lambda)\, t^{-\frac{1}{2}+\frac{1}{2}i\lambda} \, dt. \qquad (0 < t < \infty) \tag{10}$$

Now since $G(t)$ is square integrable on $(0 < t < \infty)$, so is $e^{\frac{1}{4}it^2}\, G(t) = H(t)$. Hence it remains to be shown that any function $H(t) \in L^2(0, \infty)$ can be written in the form

$$H(t) = \int_{-\infty}^{+\infty} a(\lambda)\, t^{-\frac{1}{2}+\frac{1}{2}i\lambda} \, d\lambda. \qquad (0 < t < \infty) \tag{11}$$

This is the Mellin transformation, which associates with every $H(t) \in L^2(0,\infty)$ an $a(\lambda) \in L^2(-\infty, +\infty)$ and *vice versa*. The reciprocal formula is **)

$$a(\lambda) = (1/4\pi) \int_0^{\infty} H(t)\, t^{-\frac{1}{2}-\frac{1}{2}i\lambda} \, dt. \tag{12}$$

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*) The reversal of the order of integrations is justified in appendix B.

**) In order to prove this, put $t = e^\tau$, so that

$$\int_0^{\infty} |H(t)|^2 \, dt = \int_{-\infty}^{+\infty} |e^{\frac{1}{2}\tau}\, H(e^\tau)\,|^2 \, d\tau.$$

Hence $e^{\frac{1}{2}\tau} H(e^\tau) \in L^2\,(-\infty, +\infty)$, so that there is a Fourier transform $h(\omega)$,

$$e^{\frac{1}{2}\tau}\, H(e^\tau) = \int_{-\infty}^{+\infty} h(\omega)\, e^{i\omega\tau} \, d\omega. \tag{*}$$

This equation reduces to (11) on putting $\omega = \frac{1}{2}\lambda$ and $\frac{1}{2}h(\frac{1}{2}\lambda) = a(\lambda)$. The reciprocal formula to (* furnishes (12).

Thus it has been shown that the expansion (8) is possible. The result may also be stated as follows. The subspace of Hilbert space spanned by the $\psi_\lambda(x)$ $(-\infty < \lambda < +\infty)$ coincides with the space of positive-frequency functions, *i.e.*, the subspace spanned by the functions e^{itx} with $0 < t < \infty$. The transformation from the expansion coefficients $a(\lambda)$ to the function $g_+(x)$ is the product of a Mellin transformation to give $H(\lambda)$, a trivial multiplication to give $G(t)$, and finally a Fourier transformation to give $g_+(x)$.

It is now easy to show that one may similarly write

$$g_-(x) = \int_{-\infty}^{+\infty} b(\lambda)\, \psi_\lambda(-x)\, \mathrm{d}\lambda.$$

For, by changing x into $-x$, this equation reduces to an expansion of the type (8), because $g_-(-x)$ is a positive-frequency function:

$$g_-(-x) = \int_{-\infty}^{0} G(t)\, e^{-itx}\, \mathrm{d}t = \int_0^\infty G(-t)\, e^{itx}\, \mathrm{d}t.$$

Thus the possibility of (7) and therefore of (6) has been proved.

4. *Explicit form of the expansion.* The formulae in the previous section also permit to write the expansion coefficients $a(\lambda)$, $b(\lambda)$ in terms of the given function $f(x)$. With the aid of (12) one obtains

$$\begin{aligned} a(\lambda) &= (1/4\pi) \int_0^\infty t^{-\frac{1}{2}-\frac{1}{2}i\lambda}\, e^{\frac{1}{4}it^2}\, G(t)\, \mathrm{d}t = \\ &= (1/8\pi^2) \int_0^\infty t^{-\frac{1}{2}-\frac{1}{2}i\lambda}\, e^{\frac{1}{4}it^2}\, \mathrm{d}t \int_{-\infty}^{+\infty} e^{-itx+\frac{1}{2}ix^2}\, f(x)\, \mathrm{d}x \\ &= (1/8\pi^2) \int_{-\infty}^{+\infty} \varphi_\lambda{}^*(x)\, f(x)\, \mathrm{d}x. \end{aligned}$$

Similarly one finds

$$b(\lambda) = (1/8\pi^2) \int_{-\infty}^{+\infty} \bar{\varphi}_\lambda{}^*(x)\, f(x)\, \mathrm{d}x.$$

The result may be expressed symbolically by

$$\int_{-\infty}^{+\infty} \{\varphi_\lambda(x)\, \varphi_\lambda{}^*(x') + \bar{\varphi}_\lambda(x)\, \bar{\varphi}_\lambda{}^*(x')\}\, \mathrm{d}\lambda = 8\pi^2\, \delta(x-x').$$

This formula shows in a condensed way that the spectrum extends from $-\infty$ to $+\infty$, that there are two eigenfunctions for each eigenvalue λ, and that these eigenfunctions constitute a complete set. In addition one has the symbolic formulae

$$\int_{-\infty}^{+\infty} \varphi_\lambda{}^*(x)\, \varphi_{\lambda'}(x)\, \mathrm{d}x = \int_{-\infty}^{+\infty} \bar{\varphi}_\lambda{}^*(x)\, \bar{\varphi}_{\lambda'}(x)\, \mathrm{d}x = 8\pi^2\, \delta(\lambda - \lambda'),$$

$$\int_{-\infty}^{+\infty} \varphi_\lambda{}^*(x)\, \bar{\varphi}_{\lambda'}(x)\, \mathrm{d}x = \int_{-\infty}^{+\infty} \bar{\varphi}_\lambda{}^*(x)\, \varphi_{\lambda'}(x)\, \mathrm{d}x = 0.$$

They exhibit more clearly the orthogonality and normalization of the φ_λ and $\bar{\varphi}_\lambda$.

5. *The operators q and p.* The formula (4) for $\psi_\lambda(x)$ may be used to extend the definition of $\psi_\lambda(x)$ to complex values $\lambda + i\mu$ of the index. For $|\mu| < \frac{1}{2}$ the integral remains convergent as it stands. For $\mu \leqslant -\frac{1}{2}$ one first has to

turn the integration path into a direction $e^{-i\vartheta}$ in the fourth quadrant. For $\mu \geqslant +\frac{1}{2}$ one must replace the integral by a contour integral along a loop surrounding the origin. It is then easy to prove the identities

$$x\psi_\lambda = -\tfrac{1}{2}(\lambda + i)\,\psi_{\lambda+2i} + \tfrac{1}{2}\psi_{\lambda-2i}, \tag{13a}$$

$$p\psi_\lambda = -i\psi_\lambda' = \psi_{\lambda-2i}. \tag{13b}$$

These identities are merely the recursion formulae for Weber's function [8]. In terms of φ_λ they read

$$x\varphi_\lambda = -\tfrac{1}{2}(\lambda + i)\,\varphi_{\lambda+2i} + \tfrac{1}{2}\varphi_{\lambda-2i}, \tag{14a}$$

$$p\varphi_\lambda = \tfrac{1}{2}(\lambda + i)\varphi_{\lambda+2i} + \tfrac{1}{2}\varphi_{\lambda-2i}. \tag{14b}$$

Incidentally, with the aid of these formulae it is very easy to check that φ_λ actually satisfies (1).

From these equations Arnous [6]) concluded that the application of the operators q and p to φ_λ leads to new functions, $\varphi_{\lambda+2i}$ and $\varphi_{\lambda-2i}$, which cannot be expressed in terms of φ_λ and $\bar{\varphi}_\lambda$. This is correct inasmuch as $\varphi_{\lambda+2i}$ and $\varphi_{\lambda-2i}$ are not square integrable. However, the same objection could be raised against the use of plane waves for the eigenfunctions of a free particle. The correct way to deal with this difficulty is to construct 'wave packets' that are square integrable. These wave packets may still be subject to certain restrictions; in fact, it is evident that the operations x and p cannot be defined for every function in Hilbert space. It suffices that the restricted class of wave packets envisaged is everywhere dense in Hilbert space, *i.e.*, that every square integrable function can be approximated by them (in mean) *).

As an example we show that the wave packet

$$f(x) = \int_{-\infty}^{+\infty} A(\lambda)\,\varphi_{\lambda-2i}(x)\,\mathrm{d}\lambda \tag{15}$$

can be expanded in the form

$$f(x) = \int_{-\infty}^{+\infty} a(\lambda')\,\varphi_{\lambda'}(x)\,\mathrm{d}\lambda' \tag{16}$$

under suitable restrictions to be imposed on $A(\lambda)$. If $A(\lambda + i\mu)$ is a holomorphic function in the strip $0 < \mu < 2$ and continuous in $0 \leqslant \mu \leqslant 2$, then the integration path in (15) may be shifted upwards so that (15) becomes

$$f(x) = \int_{-\infty}^{+\infty} A(\lambda + 2i)\,\varphi_\lambda(x)\,\mathrm{d}\lambda, \tag{17}$$

which is the desired expansion (16). However, two questions remain. First we have to establish the restriction to be imposed on $A(\lambda)$ in order that while shifting the integration path the infinitely large values of λ may be neglected.

*) An example of such a class of wave packets is provided by the cut off functions $a_\Lambda(\lambda)$ employed in appendix B. They are dense in that half of the Hilbert space that is spanned by the φ_λ.

Secondly we have to show that the $A(\lambda)$ restricted in this way constitute an everywhere dense set.

In order to anwer the first question we need the estimate

$$\varphi_{\lambda+i\mu}(x) = O(|\lambda|) \text{ for } \lambda \to \pm\infty, \quad -2 \leqslant \mu \leqslant 0.$$

which is derived in appendix C. It can then be shown, using a device due to Paley and Wiener [10]), that the shifting of the integration path is allowed, provided that $\lambda A(\lambda)$ is uniformly square integrable in the strip, *i.e.*,

$$\int_{-\infty}^{+\infty} |(\lambda + i\mu)^2 A(\lambda + i\mu)|^2 \, \mathrm{d}\lambda \leqslant M, \; (-2 \leqslant \mu \leqslant 0) \tag{18}$$

the constant M not depending on μ.

In order to answer the second question we employ the Fourier transformation of $A(\lambda)$,

$$A(\lambda) = \int_{-\infty}^{+\infty} \mathfrak{A}(\omega)\, e^{i\omega\lambda} \, \mathrm{d}\omega.$$

One then has

$$\int_{-\infty}^{+\infty} |(\lambda + i\mu)^2 A(\lambda + i\mu)|^2 \, \mathrm{d}\lambda = 2\pi \int_{-\infty}^{+\infty} |e^{-\mu\omega}\, \mathfrak{A}'(\omega)|^2 \, \mathrm{d}\omega.$$

Thus our question reduces to the question whether an arbitrary square integrable function $\mathfrak{A}_0(\omega)$ can be approximated in mean by an $\mathfrak{A}(\omega)$ such that $e^{-2\omega}\, \mathfrak{A}'(\omega)$ is square integrable. Now $\mathfrak{A}_0$ can certainly be approximated by a function $\mathfrak{A}_1$ which is zero for large $|\omega|$. This $\mathfrak{A}_1$ is summable in the Lebesgue sense, and can therefore be approximated by a piecewise constant function $\mathfrak{A}_2$. Finally by rounding off the jumps in $\mathfrak{A}_2$ one may construct an $\mathfrak{A}$ whose derivative exists and is zero for large $|\omega|$, and such that the norm of $\mathfrak{A} - \mathfrak{A}_0$ is as small as desired. Consequently, after restricting the wave packets envisaged in (15) to those for which (18) holds and therefore also (17), one is still left with an everywhere dense set in Hilbert space. In this sense one may write for the expansion coefficient $a(\lambda)$ in (16)

$$a(\lambda) = A(\lambda + 2i).$$

Generalising this result to the equations (14) one may put

$$\begin{aligned} xf(x) &= \int_{-\infty}^{+\infty} A(\lambda)\{-\tfrac{1}{2}(\lambda + i)\, \varphi_{\lambda+2i} + \tfrac{1}{2}\varphi_{\lambda-2i}\} \, \mathrm{d}\lambda \\ &= \int_{-\infty}^{+\infty} \{-\tfrac{1}{2}(\lambda - i)\, A(\lambda - 2i) + \tfrac{1}{2}A(\lambda + 2i)\}\, \varphi_\lambda(x) \, \mathrm{d}\lambda; \\ pf(x) &= \int_{-\infty}^{+\infty} \{\tfrac{1}{2}(\lambda - i)\, A(\lambda - 2i) + \tfrac{1}{2}A(\lambda + 2i)\}\, \varphi_\lambda(x) \, \mathrm{d}\lambda. \end{aligned}$$

Here $A(\lambda)$ is subject to the additional restriction that $A(\lambda + i\mu)$ must also be holomorphic and satisfy (18) in the strip $-2 \leqslant \mu \leqslant 2$. To eliminate the use of wave packets one might introduce the following symbolic notation

$$x\varphi_\lambda(x) = \int_{-\infty}^{+\infty} \{-\tfrac{1}{2}(\lambda' - i)\, \delta(\lambda' - \lambda - 2i) + \tfrac{1}{2}\delta\,(\lambda' - \lambda + 2i)\}\varphi_{\lambda'}(x) \, \mathrm{d}\lambda';$$

$$p\varphi_\lambda(x) = \int_{-\infty}^{+\infty} \{\tfrac{1}{2}(\lambda' - i)\, \delta(\lambda' - \lambda - 2i) + \tfrac{1}{2}\delta(\lambda' - \lambda + 2i)\}\, \varphi_{\lambda'}(x) \, \mathrm{d}\lambda'.$$

6. *Propagation of a wave packet.* If the wave packet is given at time $t = 0$, its shape at time t can be found by applying the 'propagator' or 'evolution matrix'

$$U_t(x, x') = \frac{1}{8\pi^2} \int_{-\infty}^{+\infty} \{\varphi_\lambda(x)\, \varphi_\lambda^*(x') + \bar{\varphi}_\lambda(x)\, \bar{\varphi}_\lambda^*(x')\}\, e^{-\frac{1}{2}i\lambda t}\, d\lambda.$$

It is not easy, however, to reach general conclusions concerning the evolution of the wave packet in this way. Recently, Wildermuth and Baumann [11]) studied the evolution of a wave packet by using the WKB approximation for the function $\varphi_\lambda(x)$, and showed that it behaved in a way similar to the classical self-accelerating solution. In this section we want to show that the more important features of the behaviour of a wave packet can be found directly, without explicit use of the stationary solutions $\varphi_\lambda(x)$.

The essential point is that like for the ordinary harmonic oscillator the equations of motion for p and q (in Heisenberg representation) are linear:

$$\dot{p} = q, \quad \dot{q} = p.$$

Consequently, the expectation values $\langle p \rangle$ and $\langle q \rangle$ satisfy the same equations

$$(d/dt)\, \langle p \rangle = \langle \dot{p} \rangle = \langle q \rangle, \quad (d/dt)\, \langle q \rangle = \langle p \rangle.$$

They are identical with the classical equations and can be solved to give

$$\langle p \rangle_t = \langle p \rangle_0 \cosh t + \langle q \rangle_0 \sinh t,$$
$$\langle q \rangle_t = \langle p \rangle_0 \sinh t + \langle q \rangle_0 \cosh t.$$

These results are rigorously valid for every choice of the initial wave packet; they show that both the average position and average momentum increase exponentially like e^t for $t \to +\infty$.

Similarly one finds for the second-order momenta

$$(d/dt)\, \langle p^2 + q^2 \rangle = 2\langle pq + qp \rangle,$$
$$(d/dt)\, \langle pq + qp \rangle = 2\langle p^2 + q^2 \rangle.$$

These equations can be solved in the same way. By means of the identity

$$\langle p^2 - q^2 \rangle = \langle \lambda \rangle = 2\langle E \rangle$$

($\langle E \rangle$ is the average energy of the wave packet) one then obtains for the mean square fluctuation of q

$$\langle q^2 \rangle_t - \langle q \rangle_t^2 = \tfrac{1}{4}\{\langle (p + q)^2 \rangle_0 - (\langle p + q \rangle_0)^2\}\, e^{2t} + \text{finite terms.}$$

This shows that the wave packet spreads out at the same rate at which its centre of gravity runs away.

For special initial wave packets it may happen that $\langle p \rangle_t$ and $\langle q \rangle_t$ remain finite as $t \to +\infty$, namely if $\langle p + q \rangle_0 = 0$. If, moreover, the fluctuation of $p + q$ vanishes in the initial wave packet, the mean square fluctuation of q will also remain finite. These wave packets correspond to those special solutions of the classical equations which, owing to a suitable choice of the initial values of p and q, do not increase exponentially.

In order that $\langle p\rangle_t$ and $\langle q\rangle_t$ remain finite as $t \to -\infty$, one must have $\langle p - q\rangle_0 = 0$. Hence, for those wave packets for which $\langle p\rangle_0 = \langle q\rangle_0 = 0$, the centre of gravity does not run away exponentially for $t \to \pm\infty$, but remains fixed at the origin. In order that the mean square fluctuation of q remains finite as $t \to -\infty$, the fluctuation of $p - q$ must vanish at $t = 0$. There are no wave packets for which the fluctuations of both $p + q$ and $p - q$ vanish, because these operators do not commute. Hence every wave packet spreads out at least in one time direction.

APPENDIX A. *Properties of* $\psi_\lambda(x)$. Comparing (4) with the integral expression for Weber's function one finds

$$\varphi_\lambda(\pm x) = 2^{\frac14+\frac14 i\lambda}\, e^{-\frac18\pi i+\frac18\pi\lambda}\, \Gamma(\tfrac12 + \tfrac12 i\lambda)\, D_{-\frac12-\frac12 i\lambda}[\mp(1 + i)\, x].$$

Hence all properties of φ_λ and $\bar\varphi_\lambda$ can be found from those of the well known Weber function [8]. However, we shall here derive the asymptotic expansions more directly by applying the method of stationary phase [12] to the integral (4).

First let x tend to $-\infty$. For negative x there is no t for which the exponent of e is stationary. Hence there remains only one critical point, the end-point $t = 0$ of the integration interval. The contribution of this critical point can be computed by putting $|x|\, t = \tau$,

$$\psi_\lambda(x) = |x|^{-\frac12-\frac12 i\lambda} \int_0^\infty \tau^{-\frac12+\frac12 i\lambda}\, e^{-i\tau} \exp(-i\tau^2/4x^2)\, \mathrm{d}\tau,$$

and expanding $\exp(-i\tau^2/4x^2)$:

$$\psi_\lambda(x) \simeq |x|^{-\frac12-\frac12 i\lambda}\{\Gamma(\tfrac12 + \tfrac12 i\lambda)\, e^{-\frac12\pi i(\frac12+\frac12 i\lambda)} + O(|x|^{-2})\}.$$

Now let x tend to $+\infty$. Again the origin gives a contribution, which can be computed in the same way to be

$$x^{-\frac12-\frac12 i\lambda}\, \{\Gamma(\tfrac12 + \tfrac12 i\lambda)\, e^{\frac12\pi i(\frac12+\frac12 i\lambda)} + O(x^{-2})\}. \tag{19}$$

In addition there is now a contribution from the neighbourhood of the stationary point $t_s = 2x$, viz.,

$$t_s^{-\frac12+\frac12 i\lambda}\, e^{-\frac14 it_s^2+it_s x}\, \{\textstyle\int_{-\infty}^{+\infty} e^{-\frac14 i(t-t_s)^2}\, \mathrm{d}t + O(t_s^{-1})\} =$$
$$= (2x)^{-\frac12+\frac12 i\lambda}\, e^{ix^2}\, \{\sqrt{4\pi}\, e^{-\frac14\pi i} + O(x^{-1})\}. \tag{20}$$

Collecting results one has for $x \to +\infty$

$$\bar\varphi_\lambda(x) \simeq e^{-\frac14\pi i+\frac14\pi\lambda}\, \Gamma(\tfrac12 + \tfrac12 i\lambda)\, x^{-\frac12-\frac12 i\lambda}\, e^{-\frac12 ix^2};$$
$$\varphi_\lambda(x) \simeq e^{\frac14\pi i-\frac14\pi\lambda}\, \Gamma(\tfrac12 + \tfrac12 i\lambda)\, x^{-\frac12-\frac12 i\lambda}\, e^{-\frac12 ix^2} +$$
$$+ e^{-\frac14\pi i}\, 2^{\frac12 + \frac12 i\lambda} \sqrt{\pi}\, x^{-\frac12+\frac12 i\lambda}\, e^{\frac12 ix^2}.$$

This shows that $|x^{\frac12}\, \bar\varphi_\lambda(x)|$ tends to a finite limit so that $\bar\varphi_\lambda$ cannot be square integrable. Nor can φ_λ be square integrable, because $\varphi_\lambda(x) = \bar{\varphi}_\lambda(-x)$.

Incidentally, a second pair of solutions of (1) is obtained by taking the complex conjugates of (5)

$$\varphi_\lambda{}^*(x) = e^{\frac{1}{4}ix^2}\,\psi_\lambda{}^*(x), \quad \bar{\varphi}_\lambda{}^*(x) = \varphi_\lambda{}^*(-x)$$

By comparing the asymptotic expansions one finds

$$\varphi_\lambda{}^* = \pi^{-\frac{1}{2}}\, 2^{-\frac{1}{4}-\frac{1}{4}i\lambda}\, \Gamma(\tfrac{1}{2} - \tfrac{1}{2}i\lambda)\,\{e^{-\frac{1}{4}\pi\lambda}\,\varphi_\lambda + ie^{\frac{1}{4}\pi\lambda}\,\bar{\varphi}_\lambda\}.$$

APPENDIX B. *Justification of equation* (10). The work in section 3 is rigorous but for the interchange of integrations between (9) and (10). In order to justify this, we start from a given square integrable function $f(x)$ and define $g(x)$, $G(t)$, and $H(t)$ as in section 3. Finally $a(\lambda)$ may be determined from (11), so that (10) is an identity. In order to prove (9) and hence (8) it remains to show the validity of

$$\int_0^{[\infty]} e^{-\frac{1}{4}it^2+itx}\,\mathrm{d}t \int_{[-\infty]}^{[+\infty]} a(\lambda) t^{-\frac{1}{2}+\frac{1}{2}i\lambda}\,\mathrm{d}\lambda = \int_{[-\infty]}^{[+\infty]} a(\lambda)\,\mathrm{d}\lambda \int_0^\infty t^{-\frac{1}{2}+\frac{1}{2}i\lambda}\, e^{-\frac{1}{4}it^2+itx}\,\mathrm{d}t. \quad (21)$$

The limits indicated by $[\pm\infty]$ are limits in mean.

In order to render the infinite range of λ harmless, we define a cut off function $a_\Lambda(\lambda)$ by putting

$$a_\Lambda(\lambda) = a(\lambda) \text{ for } |\lambda| < \Lambda,$$
$$a_\Lambda(\lambda) = 0 \text{ for } |\lambda| > \Lambda.$$

Furthermore, we put

$$G_\Lambda(t) = e^{-\frac{1}{4}it^2}\,H_\Lambda(t) = e^{-\frac{1}{4}it^2}\int_{-\infty}^{+\infty} a_\Lambda(\lambda) t^{-\frac{1}{2}+\frac{1}{2}i\lambda}\,\mathrm{d}\lambda. \quad (22)$$

Clearly when Λ tends to infinity, $a_\Lambda(\lambda)$ tends to $a(\lambda)$ in mean:

$$\lim \int_{-\infty}^{+\infty} |a(\lambda) - a_\Lambda(\lambda)|^2\,\mathrm{d}\lambda = 0 \text{ as } \Lambda \to \infty.$$

Because the transformation (22) preserves the norm, one also has

$$\lim \int_{-\infty}^{+\infty} |G(t) - G_\Lambda(t)|^2\,\mathrm{d}t = 0 \text{ as } \Lambda \to \infty.$$

Consequently the Fourier transform of $G_\Lambda(t)$ – that is the left-hand side of (21) – tends to the Fourier transform $g_+(x)$ of $G(t)$. Hence we only have to prove that the integrations in (21) may be interchanged for the cut off wave packet $a_\Lambda(\lambda)$ instead of for $a(\lambda)$.

The interchange of integrations in (21) is certainly allowed if not only the λ-interval, but also the t-interval is cut off by some large constant T (ref. [13]), p. 390).

$$\int_0^T e^{-\frac{1}{4}it^2+itx}\,\mathrm{d}t \int_{-\Lambda}^{+\Lambda} a_\Lambda(\lambda)\, t^{-\frac{1}{2}+\frac{1}{2}i\lambda}\,\mathrm{d}\lambda = \int_{-\Lambda}^{\Lambda} a_\Lambda(\lambda)\,\mathrm{d}\lambda \int_0^T t^{-\frac{1}{2}+\frac{1}{2}i\lambda}\, e^{-\frac{1}{4}it^2+itx}\,\mathrm{d}t. \quad (23)$$

Hence it has to be shown that for $T \to \infty$ the right-hand side of (23) tends in mean to the right-hand side of (21) (with $a_\Lambda(\lambda)$ instead of $a(\lambda)$), or that

$$\int_{-\infty}^{+\infty} \left| \int_{-\Lambda}^{\Lambda} a_\Lambda(\lambda)\,\mathrm{d}\lambda \int_T^\infty t^{-\frac{1}{2}+\frac{1}{2}i\lambda}\, e^{-\frac{1}{4}it^2+itx}\,\mathrm{d}t\right|^2 \mathrm{d}x \quad (24)$$

tends to zero for $T \to \infty$.

First decompose the integration on x into two integrals, I and II, covering the intervals $(-\infty, X)$ and $(X, +\infty)$ respectively. For the first integral the inequality of Schwarz yields

$$\mathrm{I} \leqslant \int_{-\Lambda}^{\Lambda} |a_\Lambda(\lambda)|^2 \,\mathrm{d}\lambda \int_{-\infty}^{X} \mathrm{d}x \int_{-\Lambda}^{\Lambda} \left| \int_T^\infty t^{-\frac12+\frac12 i\lambda}\, e^{-\frac14 it^2+itx}\,\mathrm{d}t\right|^2 \mathrm{d}\lambda. \tag{25}$$

Here the t-integral becomes by partial integration (supposing $T > 2X$)

$$\int_T^\infty \frac{t^{-\frac12+\frac12 i\lambda}}{-\frac12 it+ix}\,\mathrm{d}e^{-\frac14 it^2+itx} = \frac{T^{-\frac12+\frac12 i\lambda}}{\frac12 iT-ix}\,e^{-\frac14 iT^2+iTx} + \\ + \int_T^\infty e^{-\frac14 it^2+itx}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{t^{-\frac12+\frac12 i\lambda}}{\frac12 it-ix}\right)\mathrm{d}t. \tag{26}$$

An upper bound for the absolute value is

$$2\,\frac{T^{-\frac12}}{\frac12 T-x} + \tfrac12|\lambda| \int_T^\infty \frac{t^{-3/2}}{\frac12 t-x}\,\mathrm{d}t$$

$$\leqslant 2\,\frac{T^{-\frac12}}{\frac12 T-x} + \tfrac12|\lambda|\,\frac{1}{x}\left\{\frac{1}{\sqrt{2x}}\log\frac{\sqrt{T}+\sqrt{2x}}{\sqrt{T}-\sqrt{2x}} - \frac{2}{\sqrt{T}}\right\}.$$

This is square integrable in $(-\infty, X)$ and vanishes uniformly for $T \to \infty$ with fixed X.

The integral II must again be decomposed into two parts, because the t-integral is asymptotically equal to (20), and hence not square integrable. The first part takes out the dangerous term, so that the second part is again square integrable. To this end we put for the t-integral in (24)

$$(2x)^{-\frac12+\frac12 i\lambda}\int_T^\infty e^{-\frac14 it^2+itx}\,\mathrm{d}t + \int_T^\infty \{t^{-\frac12+\frac12 i\lambda} - (2x)^{-\frac12+\frac12 i\lambda}\}\, e^{-\frac14 it^2+itx}\,\mathrm{d}t. \tag{27}$$

Inserting the first term in II one has

$$\int_{-\Lambda}^{\Lambda} a_\Lambda(\lambda)(2x)^{-\frac12+\frac12 i\lambda}\,\mathrm{d}\lambda \,.\, \int_T^\infty e^{-\frac14 it^2+itx}\,\mathrm{d}t.$$

The first factor is a square integrable function of x, whereas the second factor is bounded for all x and T. Hence the square integral of this term between X and $+\infty$ tends to zero as X goes to $+\infty$, uniformly with respect to T.

The second term of (27) becomes by partial integration similar to (26)

$$\frac{T^{-\frac12+\frac12 i\lambda} - (2x)^{-\frac12+\frac12 i\lambda}}{\frac12 iT-ix} + \int_T^\infty e^{-\frac14 it^2+itx}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{t^{-\frac12+\frac12 i\lambda} - (2x)^{-\frac12+\frac12 i\lambda}}{\frac12 it-ix}\right)\mathrm{d}t. \tag{28}$$

The first term can be shown in an elementary fashion to be in absolute value less than

$$\frac{|-\frac12+\frac12 i\lambda|}{-\frac12}\,\frac{T^{-\frac12}-(2x)^{-\frac12}}{\frac12 T-x} \leqslant \frac{2|1+i\Lambda|}{\sqrt{T}(\sqrt{T}+\sqrt{2x})\sqrt{2x}}.$$

Hence its square can be integrated from X to $+\infty$ and the result vanishes as T goes to $+\infty$. The second term in (28) is in absolute value less than

$$2\int_T^\infty \left| \frac{\mathrm{d}}{\mathrm{d}t} \frac{t^{-\frac{1}{2}+\frac{1}{2}i\lambda} - (2x)^{-\frac{1}{2}+\frac{1}{2}i\lambda}}{t-2x} \right| \mathrm{d}t$$

$$\leqslant 2|1 - i\lambda| \,.\, |1 - \tfrac{1}{3}i\lambda| \,.\, \int_T^\infty \frac{\mathrm{d}}{\mathrm{d}t} \frac{t^{-\frac{1}{2}} - (2x)^{-\frac{1}{2}}}{t-2x} \,\mathrm{d}t$$

$$\leqslant \frac{2|1 + i\Lambda| \,.\, |1 + \frac{1}{3}i\Lambda|}{\sqrt{T}(\sqrt{T} + \sqrt{2x})\,\sqrt{2x}}$$

This is again square integrable and vanishes for $T \to \infty$.

Finally it must be remarked that, on inserting for the t-integral in II the three terms into which we have decomposed it according to (27) and (28), there appear not only the squares of these terms – which were shown to vanish when T tends to infinity – but in addition the cross products. These, however, cannot be larger than the squared terms, owing to Schwarz' inequality, and must therefore also vanish in the limit. This completes the proof of the expansion derived in section 3.

APPENDIX C. *Behaviour of $\psi_{\lambda+i\mu}$ for large λ.* In section 5 an estimate was needed of $\psi_{\lambda+i\mu}(x)$ for $\lambda \to \pm\infty$, while x and μ are kept constant. Let the interval of integration in (4) be decomposed into $(0, a)$ and (a, ∞). The first part yields

$$\left|\int_0^a t^{-\frac{1}{2}+\frac{1}{2}i\lambda-\frac{1}{2}\mu}\, e^{-\frac{1}{2}it^2+itx}\,\mathrm{d}t\right| \leqslant \frac{a^{\frac{1}{2}-\frac{1}{2}\mu}}{\frac{1}{2}-\frac{1}{2}\mu}.$$

The second part becomes by partial integration

$$\left|2i\int_a^\infty t^{-\frac{3}{2}+\frac{1}{2}i\lambda-\frac{1}{2}\mu}\, e^{ixt}\,\mathrm{d}e^{-\frac{1}{2}it^2}\right| \leqslant 2a^{-\frac{3}{2}-\frac{1}{2}\mu}$$

$$+\, 2|x|\int_a^\infty t^{-\frac{3}{2}-\frac{1}{2}\mu}\,\mathrm{d}t + |-3 + i\lambda - \mu|\int_a^\infty t^{-\frac{5}{2}-\frac{1}{2}\mu}\,\mathrm{d}t.$$

The right-hand side is clearly of order $|\lambda|$, so that one has

$$\psi_{\lambda+i\mu}(x) = O(|\lambda|), \tag{29}$$

uniformly for $|\mu| \leqslant 1 - \delta < 1$.

A similar calculation leads to the same result for $\psi'_{\lambda+i\mu}(x)$. Hence, from (13*b*) follows that $\psi_{\lambda+i\mu}(x)$ is also of order $|\lambda|$ in the strip $-3+\delta \leqslant \mu \leqslant -1-\delta$. In addition, it can easily be shown that in the gap $-1-\delta \leqslant \mu < -1+\delta$ the increase of $\psi_{\lambda+i\mu}(x)$ is at most exponentially; according to the theorem of Phragmén and Lindelöf [13]) it can therefore be asserted that (29) holds true uniformly for $-3 + \delta \leqslant \mu \leqslant 1 - \delta$. This implies the result needed in section 5.

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RECTIFICATION

ON THE NOISE GENERATED BY DIFFUSION MECHANISMS

by K. M. VAN VLIET and A. VAN DER ZIEL

In the paper quoted (Physica, **24** (1958) 415) equation (33) should be altered as follows:

$$D(\partial\varphi/\partial y)_{\pm B} = \mp s\varphi; \quad D(\partial\varphi/\partial z)_{\pm C} = \mp s\varphi \tag{33}$$

In (35) and (36) D^2 should be replaced by D everywhere. The line below eq. (37) should read: By multiplication with $e^2E^2(\mu_n + \mu_p)^2/4A^2$ one obtains the current noise $S_i(f)$.