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FINITENESS OF RADIATIVE CORRECTIONS IN ALL ORDERS TO μ-DECAY

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Abstract: The well known result that second order radiative corrections to μ-decay are finite is generalized to all orders.

1. INTRODUCTION

Recently, Cabibbo et al. [1] derived conditions on the vector and axial-vector current by postulating that β-decay for the pion in order e^2 be finite. Since the quark model does not fulfil these conditions, they reject it in favour of another model (in which e.g. all elementary constituents have integral charge). It is natural, then, to ask if the theory is finite in any order of radiative corrections. In μ-decay one can check whether all radiative corrections are finite or not, since here no complications of strong interactions are present. It is well known that the radiative corrections in second order to μ-decay are finite. They have been calculated and the result is that the weak coupling constant is increased by 0.2% (ref. [2]). It is also well known that the infrared divergences of inner radiative corrections to μ-decay cancel in each order against the infrared divergences from internal Bremsstrahlung [3]. It will be shown here that the ultraviolet divergences which arise in μ-decay cancel in each order. This means that it is possible to calculate the radiative corrections to μ-decay to arbitrary order. Or phrased differently: there is no need to renormalize the weak coupling constant since adding electrodynamics to first order weak interactions does not introduce divergences.

In sect. 2 we show that in order to prove the finiteness of μ-decay in all order radiative corrections, it is sufficient to prove that the *sum* of all proper weak vertex parts equals a *product* of a certain infinite constant (which cancels the other ultraviolet divergences that are present) and a finite function. In sect. 3 we write down a Bethe-Salpeter type integral equation for the sum of all proper weak vertex parts and then show that it is sufficient for our purposes to prove: (i) that $Z_1(\mu) - Z_1(e)$ is finite (μ and e denote muon and electron) and (ii) a relation for the homogeneous part of the integral equation. These relations are proved in sect. 4. In sect. 5 we discuss the renormalization of quantum electrodynamics and conclude that it goes on, even when two (or more) types of fermions are present that inter act only through the electromagnetic field, a result that is used in the preceding sections. Since we only need the V-A character of the weak interactions after the Fierz transformation in sect. 2, and assume the left-handedness of neutrinos, our results apply to an interaction of the V-A form for the original interaction, to which can be added a term of the form

$$[\bar{\psi}_e (1+\gamma_5)\psi_{\nu_e}][\bar{\psi}_{\nu_\mu} (1-\gamma_5)\psi_\mu] \,.$$

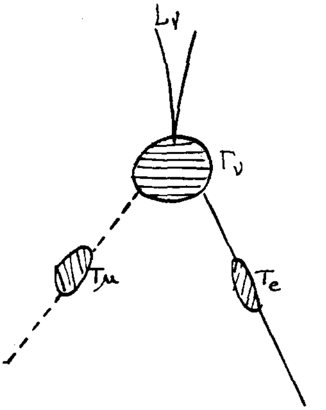
2. A CONDITION ON THE WEAK VERTEX

The sum M of all Feynman diagrams representing the radiative corrections to μ-decay can be factorized into a product of four factors:

$$M = T_e \, \Gamma^\nu \, T_\mu \, L_\nu \tag{1}$$

where T_e is the sum of all self-energy terms of the external electron line and gives rise to the wave function renormalization of the electron, T_μ does the same for the muon, Γ^ν is the sum of all proper weak vertex parts and L_ν is the neutrino current (times a factor $g/\sqrt{2}$). We postpone a discussion of Bremsstrahlung in μ-decay until the end of sect. 4. Fig. 1a illustrates this, in fig. 1b we give a term occurring in Γ^ν. Note that the diagrams corresponding to the terms in Γ^ν have no free electron or muon lines. We can forget about the neutrino lines, since we have written the V-A interaction Hamiltonian by a Fierz transformation in the form:

$$\int d_3x\mathscr{H}(x) = \frac{ig}{\sqrt{2}} \int d_3x \, [\bar{\psi}_{\nu_\mu}(x)\, \gamma_\nu (1+\gamma_5)\psi_{\nu_e}(x)][\bar{\psi}_e(x)\, \gamma_\nu (1+\gamma_5)\psi_\mu(x)] \,. \tag{2}$$



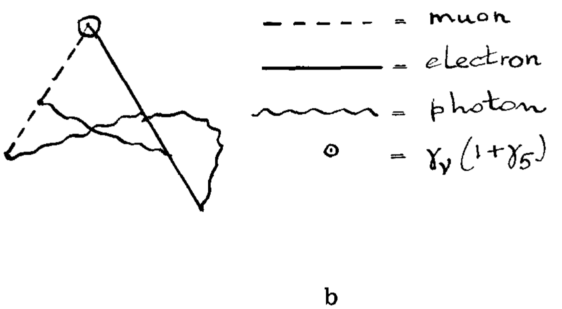


Fig. 1.

As in quantum electrodynamics $T_\mu = \sqrt{Z_2(\mu)}\, \psi_\mu(0)$ and $T_e = \sqrt{Z_2(e)}\, \bar{\psi}_e(0)$ (see sect. 5). If we can show that

$$\Gamma^\nu = \left[\frac{1}{Z_2(e) Z_2(\mu)}\right]^{\frac{1}{2}} \overline{\Gamma^\nu} \tag{3}$$

with finite $\overline{\Gamma^\nu}$, then it follows upon inserting eq. (3) in eq. (1) that the radiative corrections to μ-decay are finite.

3. AN INTEGRAL EQUATION FOR THE WEAK VERTEX

The sum of all proper, unrenormalized, weak vertex parts Γ^ν can be written graphically in the integral equation shown in fig. 2, where S'_e and S'_μ are the unrenormalized propagators.

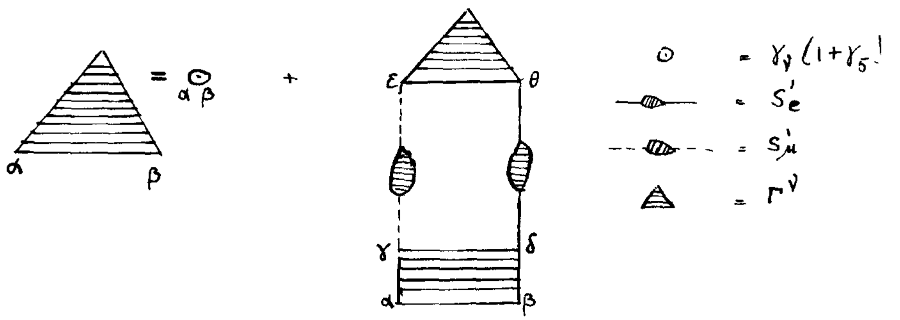


Fig. 2.

The square black box denotes the sum of all those diagrams with one electron going out at β and one muon going in at α (with four vectors p^e and p^μ) and one electron going in at δ and one muon going out at γ (four vectors now $p^e + s$ and $p^\mu + s$) that cannot become disconnected by cutting one electron and one muon line. In fig. 3 we give some of the pertinent and non-pertinent diagrams.

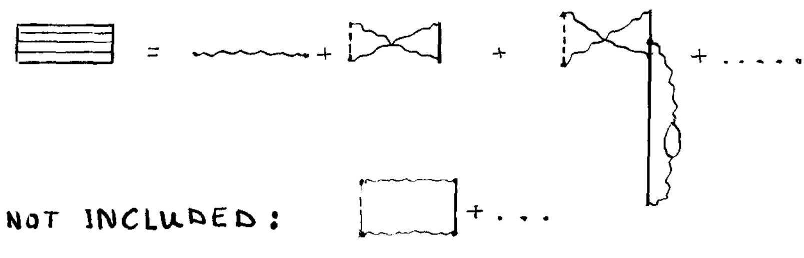


Fig. 3.

In formula:

$$\Gamma^{\nu}_{\beta\alpha}(p^{e}, p^{\mu}) = [\gamma_{\nu}(1+\gamma_5)]_{\beta\alpha} + \int d_4 s\, [S'_{e}(p^{e}+s)]_{\delta\theta}\, \Gamma^{\nu}_{\theta\epsilon}(p^{e}+s, p^{\mu}+s)$$

$$\times [S'_{\mu}(p^{\mu}+s)]_{\epsilon\gamma}\, K_{\beta\delta\gamma\alpha}(s, p^{e}, p^{\mu}) \qquad (4a)$$

or schematically:

$$\Gamma^{\nu} = \gamma_{\nu}(1+\gamma_5) + \int S'_{\mu} K S'_{e} \Gamma^{\nu} . \qquad (5)$$

A simple example is given in fig. 4. This diagram does not contain divergences for fixed s, and its contribution I to the S-matrix can be written as

$$I = [\bar{\Psi}_{e}(0)]_{\beta} \Gamma^{\nu}_{\beta\alpha}(\text{spec}) [\psi_{\mu}(0)]_{\alpha} [\bar{\Psi}_{\nu_{\mu}}(0)\gamma_{\nu}(1+\gamma_5)\psi_{\nu_{e}}(0)] \frac{g}{\sqrt{2}}\, \delta_4(p^{\mu} - p^{e} - Q)$$

with Q is the sum of the neutrino four-vectors and where $\Gamma^{\nu}(\text{spec})$ is a term in Γ^{ν} and can be written as in (4a):

$$\Gamma^{\nu}_{\beta\alpha}(\text{spec}) = \int d_4 s\, e^4 \cdot \frac{1}{(2\pi)^4} \left[\int d_4 t\, \gamma_{\mu} \frac{-1}{(\not{p}^{e}+\not{t}) - i m_{e}} \gamma_{\sigma} \right]_{\beta\delta}$$

$$\times \left[\frac{-1}{(\not{p}^{e}+\not{s}) - i m_{e}} \right]_{\delta\theta} [\gamma_{\nu}(1+\gamma_5)]_{\theta\epsilon} \left[\frac{-1}{\not{p}_{\mu}+\not{s} - i m_{e}} \right]_{\epsilon\gamma}$$

$$\times \left[\gamma_{\mu} \frac{-1}{(\not{p}_{\mu}+\not{s}-\not{t}) - i m_{\mu}} \cdot \gamma_{\sigma} \frac{-i}{(t-s)^2} \frac{-i}{t^2} \right]_{\gamma\alpha} \qquad (4b)$$

The first and last term in (4b) in square brackets together form the (here finite) contribution to K, see sect. 5 and eq. (4b).

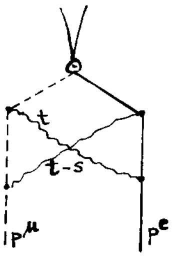


Fig. 4.

In sect. 5 it is shown that

$$K = [Z_2(e)Z_2(\mu)]^{-1}\,\overline{K} \tag{6}$$

with finite $\overline{K}$. By finite we always mean independent of the cut-off introduced by means of the method of regulators [5]. We then find the following integral equation for

$$\overline{\Gamma^\nu} \equiv [Z_2(e)Z_2(\mu)]^{\frac{1}{2}}\,\Gamma^\nu\,,$$

$$\overline{\Gamma^\nu} = [Z_2(e)Z_2(\mu)]^{\frac{1}{2}}\,\gamma_\nu(1+\gamma_5) + \int \overline{S}_\mu\,\overline{K}\,\overline{S}_e\,\overline{\Gamma^\nu}\,, \tag{7}$$

where $S'_\mu = Z_2(\mu)\overline{S}_\mu$ and $S'_e = Z_2(e)\,\overline{S}_e$.
(Even in order e^2, $Z_2(e) \neq Z_2(\mu)$). We will show in sect. 4 that in each order:

$$\text{(i)}\;\; [Z_2(\mu)Z_2(e)]^{\frac{1}{2}} - Z_2(e) \text{ is finite}\,, \tag{8a}$$

$$\text{(ii)}\;\; \int \overline{S}_e\,\overline{K}_e\,\overline{S}_e\,\overline{\Gamma^\nu_e} - \int \overline{S}_\mu\,\overline{K}\,\overline{S}_e\,\overline{\Gamma^\nu} \text{ is finite} \tag{8b}$$

($\overline{K}_e$ equals $\overline{K}$ but the ingoing lepton is an electron instead of a muon and $\overline{\Gamma^\nu_e}$ equals $\overline{\Gamma^\nu}$ in the same way).

Then, all that remains to be shown is the finiteness of QW:

$$QW = Z_2(E)\gamma_\nu(1+\gamma_5) + \int \overline{S}_e\overline{K}_e\overline{S}_e\,\overline{\Gamma^\nu_e} \tag{7a}$$

If $\gamma_\nu(1+\gamma_5)$ were replaced by γ_ν in (7a) this would be pure electrodynamics and the finiteness of this expression has been proved [4]. The divergent part in the integral with $\gamma_\nu(1+\gamma_5)$ replaced by $\gamma_\nu\gamma_5$ is however the same as that belonging to the pure quantum electrodynamical case. For, the effect of bringing γ_5 to the right of the integrand in (7a) is a replacement of m_e by $-m_e$ in one free electron-line only. And the divergent part of this expression is the same as that of the same expression with $+m_e$ (and still γ_5 at the end of the integrand). This comes about because the highest possible divergence of the graph is logarithmic, and differentiation to m_e renders it finite (for details see sect. 4). So the divergent parts in (7a) cancel, and all we have to do, it to prove the relations (8a) and (8b).

4. A RELATION BETWEEN Z-FACTORS

In this section we prove the relations (8a) and (8b). Equivalent to (8a) is: $Z_2(\mu) - Z_2(e)$ is finite or, since $Z_1 = Z_2$ it is to be shown that $Z_1(\mu) - Z_1(e)$ is finite. We will show this by proving that $\Delta Z(m) = Z_1(m) - Z_1(e)$ is differentiable. Then, since $\Delta Z(m_e) = 0$, we will have proved (8a). By $Z_1(m)$ we mean $Z_1(\mu)$ with only in the incoming and outgoing muon line the mass m_μ

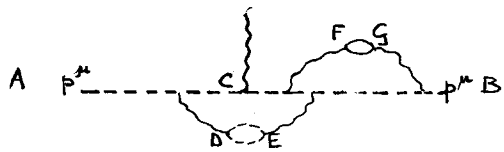


Fig. 5.

by m. In fig. 5 we illustrate this: in AC and BC we have a mass m, in DE however a mass m_μ and in FG a mass m_e. (Minus the (logarithmically) divergent part of this graph contributes to Z_1.) In the graphs that contribute to $Z_1(m)$, as for example does fig. 5, we have only an overall (logarithmic) divergence, all other (inner) divergences being removed by mass and charge renormalization. The mass m occurs only in the fermion propagators: $(t + im)/t^2 + m^2 - i\epsilon)$, and differentiating the denominator to m increases the convergence of the integrand (making it finite) whereas the factor im in the nominator never contributes to the divergence, as is seen by power-counting. So, $Z_1(\mu) - Z_1(\mathrm{e})$ is finite.

Relation (8b) is proved in the same way. Now we only replace the mass m_μ of the incoming muon by m (see fig. 6).

The expressions in (8b) are only related to the part of the graph CDEFG. Replacing in CD m_μ by m_e means replacing $\overline{K}$ by $\overline{K}_e$, and replacing m_μ by m means replacing $\overline{K}$ by $\overline{K}_m$. In the same way in DM we replace $\overline{S}_\mu$ by $\overline{S}_m$, and changing the mass in ME means going to a new vertex function $\overline{\Gamma_m}$. But at K for example we do not change anything.

We now proceed by induction. Suppose (8b) is valid to order $(2n-2)$. Then, by (8a) and (8b), $\overline{\Gamma_m}$ is finite to order $(2n-2)$. This means that in (8b) we only have a logarithmic over-all divergence. Consider as an example fig. 6: MEN is finite, its divergence having disappeared by going over from

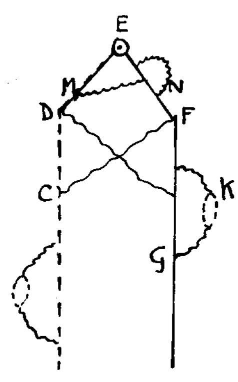


Fig. 6.

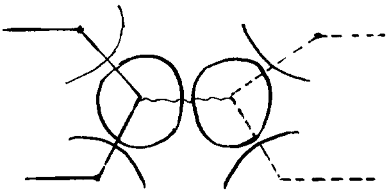


Fig. 7.

Γ_m to $\overline{\Gamma}_m$. The overall divergence is the final integration, as indicated by $\int d_4 s$ in formula (4a). But now differentiation to m makes (8b) finite to order $(2n)$, thus completing the proof by induction. Bremsstrahlung never makes K or S' more divergent and the proof goes on in this case unchanged.

5. PHOTON, ELECTRON AND MUON GIVES AGAIN QUANTUM-ELECTRODYNAMICS

In the preceding sections we have extensively used electrodynamical results, although we deal with two kinds of fermions. But these results go on. For example, both for muon and electron self-energy terms a Ward identity holds, and charge-renormalization at the electron as well as the muon vertex holds, as is illustrated in fig. 7: we split the divergent parts of propagators in two parts. Then, at each vertex the bare charge e_0 is replaced by e. This means that the kernel $\overline{K}$ is finite (after mass and charge renormalizations of both electron and muon). A complete proof would start with the coupled integral equations for quantum electrodynamics, but these are only trivially modificated, so the proof of renormalization is in our case the same. Our kernel K describes μ-$\bar{e}$ instead of μ-e scattering, so a combinatorial correction like in Compton scattering need not be included when taking the limit $m_\mu \rightarrow m_e$.

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