Newton Polygon strata in the moduli space of abelian varieties.

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Introduction

We consider p-divisible groups (also called Barsotti-Tate groups) in characteristic p, abelian varieties, their deformations, and we draw some conclusions.

For a p-divisible group (in characteristic p) we can define its Newton polygon. This is invariant under isogeny. For an abelian variety the Newton Polygon of its p-divisible group is "symmetric". We are interested in the strata defined by Newton Polygons in local deformation spaces, or in the moduli space of polarized abelian varieties.

In deformation theory of a p-divisible group it is difficult to keep track of behavior of its Newton Polygon. Isogeny correspondences between components of the moduli space of polarized abelian varieties in characteristic zero are finite-to-finite; however in general such correspondences blow up an down in characteristic p. Hence an isogeny invariant seems difficult to follow in local deformation theory. That is the true origin of fascinating aspects of the problem we are considering.

Grothendieck showed that Newton polygons go up under specialization, see [5], page 149, see [10], Th. 2.3.1 on page 143; we obtain Newton Polygon strata as closed subsets in the deformation space of a p-divisible group or in the moduli space of polarized abelian varieties in positive characteristic.

In 1970 Grothendieck conjectured the converse. In [5], the appendix, we find a letter of Grothendieck to Barsotti, and on page 150 we read: "$\cdots$ *The wishful conjecture I have in mind now is the following: the necessary conditions* $\cdots$ *that G' be a specialization of G are also sufficient. In other words, starting with a BT group $G_0 = G'$, taking its formal modular deformation* $\cdots$ *we want to know if every sequence of rational numbers satisfying* $\cdots$ *these numbers occur as the sequence of slopes of a fiber of G as some point of S.*"

In this paper we show that this conjecture for p-divisible groups by Grothendieck indeed is true, see (2.1). We show that the analogon for *principally* (quasi-)polarized formal groups is true, see (3.1). We show that the analogon is true for *principally* polarized abelian varieties, see (3.2); this is a particular case of the theorems we prove about Newton Polygon strata. As a result we can give precise information on these strata in Section 3 (their dimension can be computed from combinatorial data; we show that generically on such a stratum the a-number equals one) in case of *principally* polarized abelian varieties.

We study deformations keeping track of information about the Newton Polygon. The proof relies on two rather different aspects of deformation theory of p-divisible groups.

- On the one hand we have studied deformations of simple p-divisible groups, *keeping the Newton Polygon constant.* We use methods and results derived from "Purity" as obtained in [8]. This works fine for simple groups. However the use of "catalogues" for non-isoclinic groups does not seem to give what we want; it is even not clear that nice catalogues exist in general.

- On the other hand we studied deformation theory with $a(G_0) = 1$ as closed fiber; here we study deformations where *the Newton Polygon jumps*. This works by using a non-commutative version of the theorem of Cayley-Hamilton from linear algebra, see [18]. Note however that the Cayley-Hamilton approach breaks down essentially for $a \geq 2$.

In this paper a combination of the two methods give what we want. Hence, in spirit, the proof of a straight statement is not uniform. We have not been able to unify these in one straightforward method. We wonder what Grothendieck would have substituted for our proof.

Here is a survey of results on Newton Polygon strata inside $\mathcal{A} := \mathcal{A}_{g,1} \otimes \mathbb{F}_p$.

- *For every symmetric Newton Polygon ξ the locus $\mathcal{W}_\xi(\mathcal{A}) = W_\xi \subset \mathcal{A}$ is closed* (Grothendieck-Katz).
- *For every ξ and every irreducible component $W \subset W_\xi$ its dimension can be computed easily:* $\dim(W) = \mathrm{sdim}(\xi)$*;* for the notation of this combinatorial invariant, see (4.1).
- *Generically on W the Newton Polygon equals ξ and generically the a-number is at most one,* see (4.2).
- *The supersingular locus $\mathcal{S}_{g,1}$ has "many components"* (for $p >> 0$; Deuring-Eichler for $g = 1$, Katsura-Oort for $g \leq 3$, Li-Oort for all g), but we expect:
- *for every $\beta \neq \sigma$ the locus W_β is geometrically irreducible* (?), see (5.1).

We remark that in case of quasi-polarized p-divisible groups, or polarized abelian varieties, *without supposing that the polarization is principal*, there are counterexamples to the existence of deformations with given closed fiber, and expected Newton Polygon for the generic fiber.

The interesting phenomenon that Hecke correspondences (finite-to-finite in characteristic zero) cause blowing up and down between various strata in positive characteristic is the origin of these interesting facts. It seems a miracle that for the principally polarized case we obtain such coherent, aesthetically beautiful theorems.

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1 Some definitions, notations, and results we are going to use

Throughout the paper we fix a prime number p. We apply notions as defined and used in [18], and in [8]. For a p-divisible group G, or an abelian variety X, over a field of positive characteristic we use its Newton Polygon, abbreviated by NP, denoted by $\mathcal{N}(G)$, respectively $\mathcal{N}(X)$. For dimension d and height $h = d + c$ of G (respectively dimension $g = d = c$ of X) this is a lower convex polygon in $\mathbb{R} \times \mathbb{R}$ starting at $(0, 0)$ ending at (h, c) with integral break points, such that every slope is non-negative and at most equal to one. *We write $\beta \prec \gamma$ if* *every point of* γ *is on or below* β (the locus defined by γ contains the one defined by β). For further details we refer to [18].

(1.1) A theorem by Grothendieck and Katz, see [9], 2.3.2, says that for any family $\mathcal{G} \to S$ of p-divisible groups (in characteristic p) and for any Newton Polygon γ there is a unique closed set $W \subset S$ containing all points s at which the fiber has a Newton Polygon equal to or lying above γ:

$$s \in W \quad \Leftrightarrow \quad \mathcal{N}(\mathcal{G}_s) \prec \gamma.$$

This set will be denoted by

$$W_\gamma =: \mathcal{W}(\mathcal{G} \to S) \subset S.$$

In case of symmetric Newton Polygons we write

$$\mathcal{W}_\gamma(\mathcal{A}_g \otimes \mathbb{F}_p) = W_\gamma$$

for the Newton Polygon stratum given in the moduli space of polarized abelian varieties in characteristic p. We will study this mainly inside $\mathcal{A} := \mathcal{A}_{g,1} \otimes \mathbb{F}_p$, the moduli space of *principally* polarized abelian varieties in characteristic p.

(1.2) In "Purity" we obtained, see [8], Th. 4.1:
If in a family of p-divisible groups (say, over an irreducible scheme) the Newton Polygon jumps, then it already jumps in codimension one.

(1.3) For a commutative group scheme G over a field K (in characteristic p), we write $a(G)$ for the dimension of the L-vector space $\mathrm{Hom}(\alpha_p, G_L)$, where $L \supset K$ is any perfect field containing K.

Note that there exist examples in which

$$\dim_K(\mathrm{Hom}(\alpha_p, G)) < \dim_L(\mathrm{Hom}(\alpha_p, G_L)).$$

However, if $a(G) = 1$, then $\dim_K(\mathrm{Hom}(\alpha_p, G)) = 1 = \dim_L(\mathrm{Hom}(\alpha_p, G_L))$.

Note that $\mathrm{Hom}(\alpha_p, G) \neq 0$ iff the local-local part of G is non-trivial, i.e. iff G is not ordinary. Hence if we write $a(G) \leq 1$ we intend to say: either G is ordinary, or $a(G) = 1$.

(1.4) We recall one of the results obtained by the method of "Purity" as described in [8], Corollary 5.12:
Let G_0 *be an absolutely simple p-divisible group over a field* K *of characteristic p. Then there exists a deformation* G_η *with* $\mathcal{N}(G_\eta) = \mathcal{N}(G_0)$ *and* $a(G_\eta) \leq 1$.

(1.5) We recall one of the results obtained by the method of "Cayley-Hamilton" as described in [18], Theorem 3.2; from that we see, using the notation as in (2.10):
Let G_0 *be a p-divisible group over an algebraically closed field* $k \supset \mathbb{F}_p$ *with* $a(G_0) \leq 1$. *In* $\mathcal{D} := \mathrm{Def}(X_0)$ *there exists a coordinate system* $\{t_j \mid j \in \diamondsuit(\rho)\}$ *and an isomorphism* $\mathcal{D} \cong \mathrm{Spf}(k[[t_j \mid j \in \diamondsuit(\rho)]])$ *such that for any* $\gamma \succ \mathcal{N}(X_0)$ *we have*

$$\mathcal{W}_\gamma(\mathcal{D}) = \mathrm{Spf}(R_\gamma), \quad \text{with} \quad R_\gamma := k[[t_j \mid j \in \diamondsuit(\gamma)]] = k[[t_j \mid j \in \diamondsuit(\rho)]]/(t_j \mid j \notin \diamondsuit(\gamma)).$$

Corollary. *Let* G_0 *be a p-divisible group over a field* K *with* $a(G_0) \leq 1$. *In* $\mathrm{Def}(G_0)$ *every Newton Polygon* $\gamma \succ \mathcal{N}(G_0)$ *is realized.*

In fact, in $\mathrm{Def}(G_0)$ the NP-strata are closed subsets, and we have seen that the finite union of strata belonging to all NP strictly above γ is properly contained in the stratum given by γ.

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For symmetric Newton Polygons we have analogous statements with

$$\mathrm{Def}(G_0, \lambda_0) \cong \mathrm{Spf}(k[[t_j \mid j \in \triangle(\rho)]]),$$

using the notation as in (4.1).

(1.6) Displays. Given a Dieudonné module M of a p-divisible group, and a W-base for the W-free module, the map $F : M \to M$ is given by a matrix, called a display. Mumford showed that deformations of certain p-divisible groups can be given by writing out a display over a more general base ring. What we need is contained in [19], [20]; also see [14], [15]. Below we construct deformations of local-local p-divisible groups. We shall write out the display, and use several times (without further mention) that this defines a deformation, see [19], Chapter 3, in particular his Corollary 3.16. Deformations of polarized formal p-divisible groups can be described with the help of displays, see [15], Section 1.

Notations and results as described in [18] and in [8] will be used below.

We write K for an arbitrary field in characteristic p, and k for an algebraically closed field.

2 Deformations of p-divisible groups

(2.1) Theorem (conjectured by Grothendieck, Montreal 1970). *Let K be a field of characteristic p, and let G_0 be a p-divisible group over K. We write $\mathcal{N}(\mathcal{G}_0) =: \beta$ for its Newton Polygon. Suppose given a Newton Polygon γ "below" β, i.e. $\beta \prec \gamma$. Then there exists a deformation G_η of G_0 such that $\mathcal{N}(\mathcal{G}_\eta) = \gamma$.*

Deformations of p-divisible groups. For p-divisible groups there exists a solution to the universal deformation problem in equi-characteristic: given G_0 over a field K, there exists a formal p-divisible group $\mathcal{G} \to \mathrm{Spf}(A)$ which is universal for this problem, and $A \cong K[[t_1, \cdots, t_{dc}]]$. As finite group schemes are "algebraizable", the same holds for certain limits, and this results in a "universal family" denoted by $\mathcal{G} \to \mathrm{Spec}(A)$, see [7], 2.4.4. We use the passage from formal p-divisible groups over $\mathrm{Spf}(A)$ to p-divisible groups over $\mathrm{Spec}(A)$ without further comments.

We say that $\mathcal{H}_\eta$ is a *deformation* of G_0, if there exists a complete local domain B of characteristic p with residue class field K and fields of quotients $Q(B)$,

$$Q(B) \supset B \to K \quad \text{and a } p\text{-divisible group} \quad \mathcal{H} \to \mathrm{Spec}(B)$$

with special fiber $\mathcal{H} \otimes K \cong G_0$ and generic fiber $\mathcal{H} \otimes Q(B) = \mathcal{H}_\eta$.

Reduction of the problem. Inside $\mathcal{D} := \mathrm{Spec}(A)$, the base of the universal deformation space of G_0, for a given Newton Polygon γ there is a closed $V_\gamma \subset \mathcal{D}$ carrying all p-divisible groups with Newton Polygon equal or "above" γ, see [10], 2.3.2. In order to prove Theorem (2.1) we have to show that there is an irreducible component of V_γ with generic point parameterizing a p-divisible group having Newton Polygon exactly equal to γ. Hence it suffices to show (2.1) in case $K = k$, an algebraically closed field, and in case G_0 is a local-local p-divisible group. From now on we make these assumptions, and we write h for the height of G_0, and d for its dimension, and $h = d + c$.

(2.2) *Filtered groups.* Consider a p-divisible group G_0 over an algebraically closed field k. Suppose its Newton Polygon $\beta = \mathcal{N}(G_0)$ has $m+1$ points with integral coordinates

$$\#(\beta \cap (\mathbb{Z} \times \mathbb{Z})) = m + 1;$$

then there exist simple groups Z_i, with $1 \leq i \leq m$, corresponding with the slopes between integral points in γ (we arrange these slopes in some order), and an isogeny $\sum Z_i \to G_0$. We define a filtration

$$0 = G_0^{(0)} \subset \cdots \subset G_0^{(i)} \subset \cdots \subset G_0^{(m)} = G_0$$

by taking the image

$$\sum_{i \leq j} Z_i \quad \longrightarrow \quad G_0^{(j)} \subset G_0.$$

Such a filtration has the following property:

- The successive quotients $G_0^{(i)}/G_0^{(i-1)}$, with $0 < i \leq m$ are *simple* p-divisible groups.

This is called a *maximal filtration* of G_0. We will consider deformations of filtered groups.

Remark. We could have taken the filtration for example such that the successive quotients have non-decreasing slopes; however we do not need such a condition for our construction below.

(2.3) *A base adapted to a filtration.* Suppose given a maximal filtration of G_0 over k as above. We write d_i for the dimension of $G_0^{(i)}$ and c_i for the dimension of its dual. Let $M_0 = \mathbb{D}(G_0)$ be the covariant Dieudonné module. This has a filtration by $\mathbb{D}(G_0^{(i)}) =: M_0^{(i)} \subset M_0^{(m)} = M_0$. We write $W = W_\infty(k)$. We say that $\{x_1, \cdots, x_d;\ y_1, \cdots, y_c\} \subset M_0$ is a *base adapted to the filtration* if it is a W-base for M_0 with $y_j \in VM_0$, and moreover $x_j \in M_0^{(i)}$ iff $j \leq d_i$ and $y_j \in M_0^{(i)}$ iff $j \leq c_i$.
Claim: *Such a base exists.* In fact, note that if $H \subset G$ is a sub-p-divisible group, then $\mathbb{D}(H) \subset \mathbb{D}(G)$ is a submodule such that the quotient has no W-torsion; hence it is a W-direct summand. Using this, we conclude the existence of a W-base of the given form.

(2.4) **Lemma.** *There exists a deformation $\{\mathcal{G}^{(i)} \mid 0 \leq i \leq m\}$ of filtered p-divisible groups such that every sub-quotient*

$$\mathcal{Y}^{(i)} := \mathcal{G}^{(i)}/\mathcal{G}^{(i-1)}$$

is an absolutely simple p-divisible group of constant slope, with $a(\mathcal{Y}_\eta^{(i)}) \leq 1$ for all $0 < i \leq m$.
Proof. We show that this is a direct consequence of [8], Corollary (5.12). It suffices to prove the lemma in case G_0 is local-local. We describe deformations of p-divisible groups (in equal characteristics), and in particular of filtered p-divisible groups with the help of displays. Consider $M_0 = \mathbb{D}(G_0)$ and a basis adapted to the filtration as above. We consider the display

of M_0, i.e. the matrix of the σ-linear map $F: M_0 \to M_0$ on this W-basis, can be given in block form

$$\begin{pmatrix} A & pB \\ c & pD \end{pmatrix},$$

and its display is

$$\begin{pmatrix} A & B \\ c & D \end{pmatrix} = \begin{pmatrix} (A_{a,b}) & (B_{a,f}) \\ (C_{e,b}) & (D_{e,f}) \end{pmatrix}, \quad 1 \leq \ a, \ b, \ e, \ f \ \leq m.$$

Here, for fixed $a = b = e = f = i$ the four "diagonal" blocks give the display of the induced display on $Y_0^{(i)} = G_0^{(i)}/G_0^{(i-1)}$. As this is the display of a filtered group we have $A_{a,b} = 0$ for $a > b, \quad \cdots \quad , D_{e,f} = 0$ for $e > f$.

All formal deformations of G_0 are given by a $d \times c$ matrix T, with $T_{i,j} = (t_{i,j}, 0, \cdots)$, the Teichmüller lifts of the variables $t_{i,j}$. According to the mumbers d_i, and c_i this matrix is written in blockform $T = (\tau_{a,b} \mid 1 \leq a, b \leq m)$, and the universal deformation given by T is the display

$$\begin{pmatrix} A + TC & B + TD \\ C & D \end{pmatrix}.$$

For every simple factor $Y_0^{(i)}$ there is a matrix $\tau_{i,i}$ giving a deformation of $Y_0^{(i)}$ to $Y_\tau^{(i)}$ with generic fiber having the same Newton Polygon as $Y_0^{(i)}$ and a-number equal to one: this is precisely [8], Corollary (5.12). Using the matrix T in block form with these blocks as diagonal elements, and $\tau_{i,j} = 0$ for all $i \neq j$ we achieve a deformation of filtered groups of the filtered G_0 which satisfies the requirements of the lemma. This finishes the proof. •

(2.5) Remark. In case we are in the polarized case, the base will be in slightly different from the choice in the proof of (2.4); however it looked unnecessary to take already here precautions needed in that case.

(2.6) We recall the notion of a display *in normal form*; in this paper we need only the "modulo p version", which is slightly different from the one considered in [18], 2.1. Let Y be a local-local p-divisible group of dimension u and codimenison v over a field. Suppose Y is given by a display. Here, a W-basis for the display is said to be in *normal form* if we have a basis $\{x_1, \cdots, x_u; \ y_1, \cdots y_v\}$ with $F(x_i) = x_{i+1}$ for $i < u$ and $F(x_u) = y_1$ (i.e. the left hand upper block has entries equal to one just below the diagonal, rest zero, and the left hand lower block has one entry one in the right hand upper corner and rest zero). [Note that such a base can exist only if $a(Y) = 1$.]

Suppose a filtration $\{H_0^{(i)} \mid 0 \leq i \leq m\}$ of a local-local G_0 considered as above moreover has the property that $a(H_0^{(i)}/H_0^{(i-1)}) = 1$. We say a base adapted to the filtration is in *normal form adapted to the filtration* if the base can be induced on each of the subfactors $\mathbb{D}(H_0^{(i)}/H_0^{(i-1)})$ and if on each of these factors it is in normal form.

Claim. *A display associated with a p-divisible group over a field K with a filtration with the extra condition $a(H_0^{(i)}/H_0^{(i-1)}) = 1$ for all i, allows the choice of a base in normal form adapted to the filtration.* Indeed, for $N = \mathbb{D}(Y)$ as above, with $a(Y) = 1$, there exists a vector $x = x_1 \in N$ with $F^{u-1}x \notin VN + pN$; this follows by methods of p-linear algebra; note that the rank of the map induced by F on N/VN equals $d - 1$; note that this is the same of the rank of the matrix on any base; then $F^u x \notin pN$; hence for this module N a base in normal form exists. We can lift these, chosen for each of the relative factors, to a base in normal form for $\mathbb{D}(G_0)$.

(2.7) Lemma. *Suppose $\{H_0^{(i)} \mid 0 \leq i \leq m\}$ is a filtered local-local p-divisible group over a field K, with $a(H_0^{(i)}/H_0^{(i-1)}) = 1$. Then there exists a deformation $\{\mathcal{H}^{(i)} \mid 0 \leq i \leq m\}$ of filtered p-divisible groups such that $\mathcal{H}_\eta^{(i)}/\mathcal{H}_\eta^{(i-1)}$ and $H_0^{(i)}/H_0^{(i-1)}$ have the same slopes, and such that $a(\mathcal{H}_\eta^{(m)}) \leq 1$.*

Proof. It suffices to prove this lemma in case we work over an algebraically closed field k. We use a base $\{x_1, \cdots, x_d;\ y_1, \cdots, y_c\}$ for $M_0 = \mathbb{D}(G_0)$ in normal form adapted to the filtration; its existence was showed above. Next we choose the deformation matrix T. As above, this is in block form $T = (\ \tau_{i,j} \mid 1 \leq\ i,\ j \leq m)$. We choose variables $s_1, \cdots, s_{m-1}$, and their Teichmüller lifts $S_i = (s_i, 0, \cdots) \in W(k[[s_j \mid 1 \leq j \leq m-1]])$; we define: $\tau_{i,j} = 0$ if $i+1 \neq j$; we define $\tau_{i,i+1}$, with $1 \leq i < m$ as the matrix consisting of zero elements, except in the left hand upper corner, and there the element equals S_i; i.e. $T_{d_{i-1}+1, c_i+1} = S_i$, $\ 1 \leq i < m$, and all other elements in the matrix T are equal to zero (we write $d_0 = 0$). This matrix T defines a display; hence it defines a p-divisible group $\mathcal{H}_S \to \mathrm{Spec}(k[[s_1, \cdots, s_{m-1}]])$. By construction the deformation respects the filtration; moreover it leaves the successive quotients undeformed; hence the Newton Polygon of its generic fiber equals $\beta := \mathcal{N}(G_0)$. Next we compute the Hasse-Witt matrix of the deformed group. This is given by the matrix $(A+TC)$ mod p. In A we have diagonal blocks with zeros everywhere, except on the subdiagonal in the blocks, where the elements are one (this is because we have chosen the base such that on partial quotients the matrix is in normal form). The contribution of the deformation is changing (by the variables s_i) the right hand upper corner of the blocks in $(A+TC)$ mod p immediately above the diagonal. Note that the element $(A+TC)_{c_i+1, d_{i+1}-1}$ is non-zero modulo p, for $1 \leq i \leq m-1$. We claim: *the rank of the Hasse-Witt matrix equals* $d-1$, i.e. $a(\mathcal{H}_\eta) = 1$; in fact, in the Hasse-Witt matrix of the display, delete the d_1-column and the (c_m+1)-th row; then permute columns and rows in such a way that the elements 1 which were just below the diagonal in the same order come on the diagonal, and then such that the elements $S_1, \cdots, S_{m-1}$ in this order are on the diagonal ; in the new matrix (modulo p), we have non-zero elements on the diagonal, and zeros below the diagonal; hence its determinant is non-zero; hence the original matrix has rank equal to $d-1$, which proves the claim. This finishes the proof. •

(2.8) Proposition. *Suppose G_0 is a p-divisible group over a field K. There is a deformation G_η of G_0 such that $a(G_\eta) \leq 1$ and $\mathcal{N}(G_0) = \mathcal{N}(G_\eta)$.*

Proof. We show that this follows, using (2.4) and (2.7). Indeed, as we have seen above it suffices to start with a local-local p-divisible group G_0. By (2.4) there is a deformation to a filtered group H with successive quotients having each $a = 1$. We choose an irreducible component of this: over $S_1 = \mathrm{Spec}(A_1) \subset \mathrm{Spec}(A)$, with $R_1 = A/I_1$, there is a filtered group with successive quotients having all generically $a = 1$, and having the same Newton polygons as in the special fiber. Write $L_1 = Q(A_1)$ for its field of fractions; the generic fiber $\mathcal{G} \otimes_A L_1$ is a filtered group, with the same Newton polygon β, with successive quotients having all a-number equal to one. We consider the universal deformation space of the filtered group $\mathcal{G} \otimes L_1$. By (2.4) there is a deformation with constant Newton Polygon having a-number equal to one in the generic fiber; this implies there exists an integral complete local domain $R_2 \to L_1$, with $Q(R_2) = L_2$ and a deformation $\mathcal{H} \to R_2$ of $\mathcal{G} \otimes L_1$ with $a(\mathcal{H}_\eta) = 1$ and $\mathcal{N}(H_\eta) = \beta$. Consider the ring Γ_1

$$\begin{array}{ccc} R_1 & \hookrightarrow & L_1 \\ \uparrow & & \uparrow \\ \Gamma_1 & \hookrightarrow & R_2 \end{array}$$

of all elements in R_2 mapping into R_1. The display defining $\mathcal{H} \to R_2$ is a lift of the display of $\mathcal{G} \otimes R_1$, hence we conclude that it is defined by a matrix with elements in Γ_1. This proves (2.8). •

(2.9) Proof of (2.1). We show that this follows, using (2.8) and [18]. Indeed, by (2.8) we have a deformation of G_0, keeping the Newton Polygon constant, and achieving $a(G_\eta) \leq 1$ for the generic fiber over $L = K(\eta)$. Using the corollary in (1.5) we conclude that G_η over L admits a deformation to a fiber having Newton Polygon equal to $\gamma \succ \mathcal{N}(G_\eta) = \mathcal{N}(G_0)$. As in the previous proof we conclude that this shows that in $\mathrm{Def}(G_0)$ the Newton Polygon γ is realized. This finishes the proof of (2.1). •

(2.10) We use the following notation: we fix integers $h \geq d \geq 0$, and we write $c := h - d$. We consider Newton Polygons ending at (h, c). For a point $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ we write $(x, y) \prec \gamma$ for the property "*the point (x, y) is on or above the Newton Polygon γ*". For a Newton Polygon β we write:

$$\diamondsuit(\beta) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < c, \quad y < x, \quad (x, y) \prec \beta\},$$

and we define

$$\dim(\beta) := \#(\diamondsuit(\beta)).$$

Note that for the "ordinary" Newton Polygon $\rho := d\cdot(1, 0) + c\cdot(0, 1)$ the set of points $\diamondsuit = \diamondsuit(\rho)$ is a parallelogram; this explains our notation. Note that $\#(\diamondsuit(\rho)) = d\cdot c$.

(2.11) Theorem. *Suppose given a p-divisible group G_0 over a field K. Let γ be a Newton Polygon with $\gamma \succ \mathcal{N}(G_0) =: \beta$. Consider the closed formal subset $\mathcal{W}_\gamma(\mathcal{D}(G_0)) =: V_\gamma \subset \mathcal{D}(G_0)$. The dimension of every component of V_γ equals $\dim(\gamma) = \#(\diamondsuit(\gamma))$; generically on such a closed set the Newton Polygon is γ; on V_γ the a-number generically is at most one.* (In fact on V_γ the a-number generically is equal to one iff $\gamma \neq \rho := d\cdot(1, 0) + c\cdot(0, 1)$.)

Proof. It suffices to prove the theorem in case G_0 is defined over an algebraically closed field k. In (2.8) we have seen that every p-divisible group can be deformed to one with the same Newton Polygon, and having a-number at most one; as in the proof of (2.8) "transitivity of methods" shows that every irreducible component of V_β has at its generic point these properties. In [18] we find a description of the deformation theory of p-divisible groups with $a \leq 1$, see (1.5). Hence we know that deforming p-divisible groups with $a = 1$ every Newton Polygon below β can be achieved. In [18] the dimension of the locus $V_\gamma(a = 1) \subset \mathrm{Def}(G_0)$ is computed to be purely equal to $\diamondsuit(\gamma)$. We see that the theorem follows from (2.8) and [18]. •

3 Deformations of principally quasi-polarized formal groups and of principally polarized abelian varieties

(3.1) Theorem (the principally polarized variant of the conjecture by Grothendieck, Montreal 1970). *Let K be a field of characteristic p, and let (G_0, λ_0) be a* principally *quasi-polarized p-divisible group over K. We write $\mathcal{N}(G_0) = \beta$ for its Newton Polygon. Suppose given a symmetric Newton Polygon γ "below" β, i.e. $\beta \prec \gamma$. Then there exists a deformation (G_η, λ) of (G_0, λ_0) such that $\mathcal{N}(\mathcal{G}_\eta) = \gamma$.*

(3.2) Corollary. *Let K be a field of characteristic p, and let (X_0, λ_0) be a* principally *polarized abelian variety over K. We write $\mathcal{N}(X_0) = \beta$ for its Newton Polygon. Suppose given a symmetric Newton Polygon γ "below" β, i.e. $\beta \prec \gamma$. Then there exists a deformation $(\mathcal{X}_\eta, \lambda)$ of (G_0, λ_0) such that $\mathcal{N}(\mathcal{X}_\eta) = \gamma$.*

Proof. The existence of a formal polarized abelian scheme as wanted follows from (3.1) using the Serre-Tate theorem on the equivalence between formal deformations of polarized abelian schemes and the corresponding quasi-polarized p-divisible groups, see [9], Th. 1.2.1. By the Chow-Grothendieck algebraization method for polarized formal schemes ("formal GAGA"), see [4], III^1.5.4, it follows that we obtain an actual abelian scheme. Its Newton Polygon can be read off from its p-divisible group. Hence we see that (3.2) follows from (3.1). •

In order to give a proof of (3.1) at first we make the reduction to the case that (G_0, λ_0) is defined over an algebraically closed field, and that G_0 is of local-local type (for polarized formal groups this is the same as saying that it is a local p-divisible group): if we assume the theorem proved for this special case; then it follows in general. From now on we G_0 is a local p-divisible group with a principal quasi-polarization over an algebraically closed field; we write h for the height of G_0, and d for its dimension; then $h = 2d$, and G_0 is a local-local p-divisible group.

(3.3) Analogous to the definitions as in (2.2) we consider a filtration related to (G_0, λ), a *principally quasi-polarized filtered formal group* over an algebraically closed field.

Suppose the number of points with integral coordinates on the Newton Polygon $\beta = \mathcal{N}(G)$ is $m + 1$:

$$\#(\beta \cap (\mathbb{Z} \times \mathbb{Z})) = m + 1.$$

We write s for the number of $G_{1,1}$-factors in the isogeny type of G_0; i.e. the number of slopes equal to $1/2$ is $2s$; we speak of the *even* case, respectively the *odd* case, in case s is even, respectively odd; we write $s = 2t$, respectively $s = 2t + 1$. We write $m = 2n + s$. A *maximal symplectic filtration* of a *principally* quasi-polarized p-divisible group (G_0, λ) is a filtration

$$0 = G_0^{(0)} \subset \cdots \subset G_0^{(i)} \subset \cdots \subset G_0^{(m)} = G_0$$

- such that $G_0^{(i)}/G_0^{(i-1)}$ is simple for every $0 < i \leq m$, such that
- moreover $\lambda_0 : G_0 \to (G_0)^t$ induces an isomorphism on the subquotients

$$G_0^{(i)}/G_0^{(i-1)} \to (G_0^{(m-i+1)}/G_0^{(m-i)})^t$$

for every $0 < i \leq (m+1)/2$.

Let us explain the last condition. Note that an exact sequence of p-divisible groups

$$0 \to X \longrightarrow Y \longrightarrow Z = Y/X \to 0$$

yields an exact sequence

$$0 \to Z^t \longrightarrow Y^t \longrightarrow X^t \to 0.$$

The last condition in the definition of a (maximal) symplectic filtration says that for $0 < i \leq (m+1)/2$ we have a natural commutative diagram:

$$\begin{array}{ccccccc}
G_0 & \hookleftarrow & G_0^{(m-i+1)} & \longrightarrow & \left(G_0^{(m-i+1)}/G_0^{(i-1)}\right) & \hookleftarrow & \left(G_0^{(i)}/G_0^{(i-1)}\right) \\
\lambda_0 \downarrow\cong & & & & \downarrow\cong & & \\
 & & & & & & \downarrow\cong \\
G_0^t & \longrightarrow & \left(G_0^{(m-i+1)}\right)^t & \longleftarrow & \left(G_0^{(m-i+1)}/G_0^{(i-1)}\right)^t & & \\
 & & & & \Vert\cong & & \\
 & & & & G_0^{(m-i+1)}/G_0^{(i-1)} & \to & \left(G_0^{(m-i+1)}/G_0^{(m-i)}\right).
\end{array}$$

Such diagrams follow if the composite map $G_0^{(i-1)} \to G_0 \to (G_0)^t \to (G_0^{(m-i+1)})^t$ is zero for every $0 < i \leq (m+1)/2$.

Note that in the "odd case", i.e. m is odd, there is a "middle" step in the filtration, and it is required to be self dual; in case m is even, there is no middle step, and we need only consider $i \leq m/2$.

The conditions necessary for a filtration to be maximal and symplectic can also be expressed on the Dieudonné module $\mathbb{D}(G_0) = M_0$. One has to assume that $N_0^{(i)} \subset M_0$ has a torsion-free quotient, in order to obtain $G_0^{(i)} \subset G_0$. The property of being symplectic can be expressed with the help of the bilinear, non-degenerate, skew pairing on M_0 induced by the principal quasi-polarization.

In this way we see: if $X \subset G_0$ is a p-divisible subgroup, and λ_0 is a principal quasi-polarization, such that $(X \to G_0 \to (G_0)^t \to X^t) = 0$, then there exists a unique $X^\perp \subset G_0$ such that λ_0 induces a principal quasi-polarization on $X^\perp/X$.

(3.4) We recall a lemma which was proved in [11], 6.1. Over $\mathbb{F}_{p^2}$ we consider two series of quasi-polarized supersingular p-divisible groups.

$$\mathbf{I}_r \qquad (S, \tau_r), \quad r \in \mathbb{Z}_{\geq 0} \quad S = G_{1,1}, \quad \deg(\tau_r) = p^{2r},$$

$$\mathbf{II}_r \qquad (T, \nu_r), \quad r \in \mathbb{Z}_{\geq 0} \quad T = (G_{1,1})^2, \quad \deg(\nu_r) = p^{2r}.$$

These are defined as follows. On the Dieudonné modules we take in the first case:

$$M = W{\cdot}x \oplus W{\cdot}Fx, \quad Fx = Vx, \quad \epsilon \in W - pW, \quad \epsilon^\sigma = -\epsilon, \quad < x, Fx >= p^r{\cdot}\epsilon.$$

In the second case:

$$M = W{\cdot}x \oplus W{\cdot}Fx \oplus W{\cdot}y \oplus W{\cdot}Fy, \quad Fx = Vx, \quad Fy = Vy,$$

and

$$< x, y >= p^r, \quad < x, Fx >= 0 =< y, Fy >=< x, Fy >=< y, Fx > .$$

With these notations: *For every supersingular quasi-polarized p-divisible group* (H,ς) *over an algebraically closed field k there is an isomorphism:*

$$(H,\varsigma) \quad \cong \quad \bigoplus_j \ (H_j,\varsigma_j),$$

where each of the summands (H_j,ς_j) *is of type* **(I)** *or of type* **(II)***, with some degree for the quasi-polarization.*

Remark. Suppose (S,τ_r) is of type $(\mathbf{I}_r)$ and u is an integer, with $u \geq r$. Then there is an isogeny $\psi : S \to S$ such that $(S,\psi^*\tau_r) \cong (S,\tau_u)$.

(3.5) Lemma. *Given a principally quasi-polarized p-divisible group* (G_0,λ_0) *over an algebraically closed field k, there exists a maximal symplectic filtration* $(\{G_0^{(i)} \mid 0 \leq i \leq m\},\lambda_0)$.

Proof. A maximal symplectic filtration which will be constructed in the proof will depend on choices: the subgroups appearing in the first half of the filtration and their order will depend on the choice which factors in the isogeny type are chosen to appear on which place. We shall prove the lemma for local-local p-divisible groups (and, if necessary, one can insert later the local-étale factors and their duals).

The lemma will be proved by induction on the height of G_0. So we assume that for all principally polarized p-divisible groups of smaller height the lemma has been established.

In the induction step we write (Y,λ) for the pair (G_0,λ) in the lemma. We choose a direct sum of simple p-divisible groups H, an isogeny $\varphi : H \to Y$ and we construct $\varsigma := \varphi^*(\lambda)$, arriving at a quasi-polarized p-divisible group (H,ς). We choose this isogeny φ in such a way that on all supersingular factors of type **(I)** the degree of the induced quasi-polarization is equal to some fixed number p^{2r}; by the remark above this choice can be made.

We perform the induction step by choosing choosing a simple direct summand $D \overset{\leftarrow}{\hookrightarrow} H$ such that the induced map

$$(\varsigma|_D \colon D \to H \overset{\varsigma}{\longrightarrow} H^t \to D^t) = 0.$$

After we make such a choice, we define $(X \subset Y) = \mathrm{Im}(\varphi) = (\varphi(X) = D \subset H))$. We conclude that

$$(X \hookrightarrow Y \overset{\lambda}{\longrightarrow} Y^t \to X^t) = 0;$$

we see that in this case λ induces a principal quasi-polarization on $X^{\perp}/X$, and we carry on by induction.

As for the choice of $D \subset Y$ as above there are in general four basically different choices possible.

3) In case Y admits a simple factor of slope not equal to $1/2$, we can choose for $D \subset H$ such a factor. In this case $\mathrm{Hom}(D,D^t) = 0$, and we are done.

2) In case (H,ς) contains a supersingular direct summand T of Type **(II)**, we choose $D \subset H$ such that $\mathbb{D}(D) = W{\cdot}x \oplus W{\cdot}Fx \subset \mathbb{D}(T) \subset \mathbb{D}(Y)$, using the notation explained above. This is a totally isotropic subspace, and a choice $X \subset Y$ as desired follows.

1) Suppose (H,ς) contains at least two direct summands of Type **(I)** (by construction with quasi-polarizations of the same degree)

$$W{\cdot}e \oplus W{\cdot}Fe \oplus W{\cdot}f \oplus W{\cdot}Ff = N \subset \mathbb{D}(H),$$

with $< e, Fe >= p^r{\cdot}\epsilon =< f, Ff >$, etc. We choose $\xi \in W(k)$ with $\xi \cdot \xi^{\sigma} = -1$; note that $\xi^{\sigma^2} = \xi$; in this case $z := e + \xi{\cdot}f$ generates a Dieudonné submodule $N' = W{\cdot}z \oplus W{\cdot}Fz \subset N$

which is isotropic for the form induced by ς; it is the Dieudonné module $N' = \mathbb{D}(D)$ of a p-divisible group $D \subset H$ of height 2, and its image $X \subset Y$ satisfies the condition $(X \hookrightarrow Y \to Y^t \to X^t) = 0$. In this case a choice for the induction step can be made.
0) Suppose that (H, ς) satisfies none of the possibilities in the previously considered three cases. Then either $Y = 0 = H$, and we are done. Or Y is supersingular of height 2; in this case it is of Type (I_0), and it is filtered by $0 \subset Y$, which is of the form wanted in the lemma. Hence induction finishes the proof of (3.5). •

(3.6) Definition. *A symplectic base adapted to a symplectic filtration.* Suppose given a principally quasi-polarized p-divisible group (G_0, λ) over an algebraically closed field k. We write $M := \mathbb{D}(G_0)$. Suppose $\{G_0^{(i)} \mid 0 \leq i \leq m\}$ is a symplectic filtration as in (3.3). Suppose $Y^{(i)} := G_0^{(i)}/G_0^{(i-1)}$ has dimension d_i and codimension c_i. Note that in this case $d_i = c_{m-i+1}$. Let us write $D_i := d_1 + \cdots + d_i$, i.e. this is the dimension of $G_0^{(i)}$.

We say that $\{x_1, \cdots, x_d;\ y_1, \cdots, y_d\}$ is a symplectic base adapted to this symplectic filtration if

$$x_1, \cdots, x_{D_i}, y_{D_{m-i}+1}, \cdots, y_d \in M_i := \mathbb{D}(G_0^{(i)})$$

is a W-basis for $M := \mathbb{D}(G_0)$ (note this definition, in the symplectic case, is slightly different from (2.3)), and if it is symplectic i.e. $< x_i, y_i >= 1, \quad 0 < i \leq h$, and all other products between base vectors are zero. We say moreover this base adapted to a filtration is *in normal form* if it is in normal form modulo p, in the sense of (2.6), i.e. normal modulo p on all simple subfactors.

(3.7) Lemma. *Suppose given a principally polarized p-divisible group (G, λ) over an algebraically closed field k, with a maximal symplectic filtration $\{X_i \mid 0 \leq i \leq m\}$. We write $\mathbb{D}(G) = M$, and $\{N_i = \mathbb{D}(X_i) \mid 0 \leq i \leq m\}$ for its filtered Dieudonné module.*
(a) *There exists symplectic base adapted to this symplectic filtration.*
(b) *If moreover $a(X_i/X_{i-1}) = 1$, for all $0 < i \leq m$, there exists a normal symplectic base adapted to this symplectic filtration.*
Proof. In case the number m of steps in the filtration is even we change slightly the notation; in that case we introduce an extra trivial step in the middle. In case m is odd, we do not change the filtration. In both cases we obtain $m = 2n + 2t + 1$. We achieve a filtration

$$0 = N_0 \subset N_1 \subset \cdots \subset N_n \subset \cdots \subset N_{n+t} \subset N_{n+t+1} \subset \cdots \subset N_{n+2t+1} \subset \cdots \subset N_m.$$

On this module there is an alternating non-degenerate bilinear form, denoted by $< -, - >$; we have $N_i^\perp = N_{m-i}$. We will choose the base on M inductively by considering $P_j = N_{n+t+j+1}/N_{n+t-j}$. Note that the paring on M induces a pairing with the same properties on each P_j, $\ 0 \leq j \leq n+t$. Induction starts at $j = 0$: if $P_0 = 0$ (the case the original filtration had an even number of steps) we are done; if $P_0 \neq 0$, then $X_{n+t+j+1}/X_{n+t-j} \cong G_{1,1}$, the module P_0 has rank 2, and the choice of a symplectic base in normal form is easy: $P_0 = W{\cdot}e \oplus W{\cdot}Fe$.

Induction hypothesis: On P_j for $0 \leq j < n+t$ there exits a symplectic base adapted to the filtration, in normal form in case **(b)**. Induction step: We are going to construct the same for P_{j+1}.

In order to simplify, we choose a different notation, which will only be used in the induction step. We fix j as before, and write $P := P_j$, and $Q := P_{j+1}$. We assume we have already a basis $\{B_1'', \cdots, B_d''; \ b_1'', \cdots, b_d''\}$ in the required form for P. We write $N = N_{n+t-j}/N_{n+t-j-1} \subset Q = N_{n+t+j+2}/N_{n+t-j-1}$, and $N^\perp \subset Q$; note that $N^\perp = N_{n+t+j+1}$,

$$0 \subset N \subset N^\perp \subset Q \quad \text{and} \quad N^\perp/N = P$$

. We choose a basis $\{A_1, \cdots, A_u; \ a_1, \cdots, a_v\}$ for $N \subset Q$, and a basis $\{C_1'', \cdots, C_v''; \ c_1'', \cdots, c_u''\}$ for $Q/N^\perp$; these are chosen such that the duality $N \xrightarrow{\sim} (Q/N^\perp)^D$ agrees with the choices made, and in case **(b)**, such that these bases are in normal form; these choices follow in case **(a)** from the structure of the problem: compare the various possibilities in the proof of (3.5); in case **(b)** we consider the finite length Dieudonné modules $U := N/p{\cdot}N$ and $W := (Q/N^\perp)/p{\cdot}(Q/N^\perp)$; these are in perfect duality; both are generated by one element, as Dieudonné modules, in case **(b)**; a generator $A_1^{(0)}$ for U not in $FU + VU$ has the property that $F^u(A_1^{(0)})$ generates V^vU; using this we choose dual bases for U and W in normal form; then we lift these to the required bases for N and $N^D = Q/N^\perp$. By these preparations we have chosen bases:

$$\text{for} \quad N: \quad \{A_1, \cdots, A_u; \ a_1, \cdots, a_v\}, \quad N \subset Q,$$

$$\text{for} \quad P: \quad \{B_1'', \cdots, B_d''; \ b_1'', \cdots, b_d''\}, \quad P = N^\perp/N,$$

$$\text{for} \quad Q/N^\perp: \quad \{C_1'', \cdots, C_v''; \ c_1'', \cdots, c_u''\},$$

which satisfy the conditions for being symplectic on P and on $N \oplus (Q/N^\perp)$, and being normal in **(b)** as explained:

$$\begin{array}{ccccc} A_i, a_j \in N & \subset & N^\perp & \subset & Q \\ & & \downarrow & & \downarrow \\ & & B_i'', b_j'' \in N^\perp/N & \subset & Q/N \\ & & & & \downarrow \\ & & & & C_i'', c_j'' \in \ Q/N^\perp. \end{array}$$

Note that Q is a successive extension with three quotients as given above. In order to perform the induction step, we lift the given bases for the three (sub)quotients to a W-base

$$\{A_1, \cdots, A_u; \ B_1', \cdots, B_d'; \ C_1', \cdots, C_v'; \ c_1', \cdots, c_u'; \ b_1, \cdots, b_d; \ a_1, \cdots, a_v\}$$

for Q, using $Q \to Q/N \to Q/N^\perp$ and $N^\perp \to N^\perp/N$, such that this base respects the filtration, and such that the last $u+d+v$ base vectors are in VQ. We are going to show it can be changed into a symplectic base adapted to the filtration (which will be normal by the choices already made in case of **(b)**). Note that the following inner products already satisfy the conditions in the condition of being symplectic: $< A, - >$, $< a, - >$, $< B', B' >$, $< B', b' >$; by this we mean that $< A_i, z >= 0$ for all vectors in the given base, unless $z = c_i'$, in which case it equals one, etc. We are going to change some of the base vectors, such that the base still is adapted to the filtration, that it becomes symplectic (and the condition of being normal will remain unchanged).

Compute: $< B'_i, C'_j >= \beta_{i,j}$, and $< B'_i, c'_j >= \gamma_{i,j}$; write $B_i := B'_i - \sum_j (\beta_{i,j} \cdot a_j + \gamma_{i,j} \cdot A_j)$. After this step all products of the form $< B_i, - >$ satisfy the conditions required.

Compute: $< C'_i, C'_j >= \xi_{i,j}$, and $< C'_i, c'_j >= \delta_{i,j} + \varepsilon_{i,j}$ (here $\delta_{i,j}$ is the Kronecker delta) and $< C'_i, b_j >= \zeta_{i,j}$; write $C_i := C'_i - \sum_j (\xi_{i,j} \cdot a_j + \varepsilon_{i,j} \cdot A_j) - \sum_j \zeta_{i,j} \cdot B_j$. Compute $< c'_i, c'_j >= p \cdot \eta_{i,j}$ and $< c_i.b_j >= p \cdot \theta_{i,j}$; write $c_i = c'_i - \sum_j \eta_{i,j} \cdot F \cdot VA_j - \sum_j \theta_{i,j} \cdot F \cdot VB'_j$. In this way we obtain a base for Q which satisfies the requirements: it is symplectic, it is adapted to the filtration in the sense of (3.3), and in case **(b)** it is in normal form. This ends the induction step. Performing all steps we find a base for M satisfying the conditions of the lemma. •

(3.8) Lemma. *Start with data and the notation as in* (3.3). *There exists a deformation* $(\{\mathcal{G}^{(i)} \mid 0 \leq i \leq m\}, \lambda)$ *of quasi-polarized filtered p-divisible groups such that all slopes of*

$$Y^{(i)} := \mathcal{G}^{(i)} / \mathcal{G}^{(i-1)}$$

are constant, and $a(Y_\eta^{(i)}) \leq 1$ *for all* $0 < i \leq m$.

Proof. We adapt the proof of (2.4) with small changes. It suffices to prove the lemma in case we have a local-local p-divisible group. We choose a symplectic base $\{x_1, \cdots, x_h;\ y_1, \cdots, y_h\}$ for $\mathbb{D}(G_0)$ adapted to the filtration as constructed in the previous lemma. The display matrix is written in block form and the blocks $A_{i,i}, B_{i,m-i}, C_{m-i,i}, D_{m-i,m-i}$ describe the display of $N_0 = \mathbb{D}(G_0^{(i)}/G_0^{(i+1)})$; here the choice of the base is slightly different form the one made in (2.4), as we have to adapt to the polarized case. By [8], Corollary (5.15) we can choose $\tau_{i,m-i}$ (for $0 < i \leq (m+1)/2$) which deforms $G_0^{(i)}/G_0^{(i-1)}$ keeping the Newton polygon constant, and achieving $a = 1$. In case $G_0^{(i)}/G_0^{(i-1)}$ is supersingular of height 2, we already have $a = 1$, and we choose $\tau_{i,m-i} = 0$ We complete the matrix T in a symmetric way; this means that the block $\tau_{m-i,i}$ is obtained from $\tau_{i,m-i}$ (for $1 \leq i \leq n$) by symmetry in the main diagonal of T. All block matrices $T_{i,j}$ outside the anti-diagonal are required to be zero. Hence T is defined, and the deformation $\mathcal{G}_\tau$ thus obtained satisfies the requirements of the lemma. This proves Lemma (3.8). •

(3.9) Lemma. *Suppose* $(\{H_0^{(i)} \mid 0 \leq i \leq m\}, \lambda)$ *is a local principally quasi-polarized filtered p-divisible local group over an algebraically closed field k, with all* $a(H_0^{(i)}/H_0^{(i-1)}) = 1$. *Then there exists a deformation* $\{\mathcal{H}^{(i)} \mid 0 \leq i \leq m\}$ *of filtered p-divisible groups with the same slopes, such that* $a(\mathcal{H}_\eta^{(m)}) = 1$.

Proof. We adapt the proof of (2.7) with small changes. We reduce to the local-local case. For the case of the Dieudonné module of principally quasi-polarized formal p-divisible groups we choose a symplectic base in normal form adapted to the symplectic filtration. We study deformations displays on this base. We choose variables $s_1, s_2, \cdots, s_{m-1}$, with $s_j = s_{m-j}$ and their Teichmüller lifts $S_i \in W(k[[s_i \mid 1 \leq i \leq m-1]]/(\cdots s_j = s_{m-j} \cdots))$; we define the block matrix $\tau_{i,m-i}$ with S_i, $1 \leq i < m$, in the left hand upper corner; we define $\tau_{i,j} = 0$ if $j \neq m - i$ (i.e. equal to zero if not in the upper-antidiagonal). The display matrix in block form $T = (\tau_{i,j} \mid 1 \leq i, j \leq m)$ defines a deformation $\mathcal{H}_s$ which satisfies the requirements of (3.9). •

(3.10) Corollary. *Let* (G_0, λ_0) *be a principally quasi-polarized p-divisible group over a field K, and let* ξ *be a symmetric Newton polygon,* $\xi \succ \mathcal{N}(G_0)$. *There is a deformation* (G_η, λ_η) *of* (G_0, λ_0) *such that* $\xi = \mathcal{N}(G_\eta)$ *and* $a(G_\eta) \leq 1$.

Proof of (3.1) and of (3.10). From (3.8) and (3.9) we conclude that a principally quasi-polarized p-divisible group (G_0, λ_0) defined over a field K can be deformed to a principally quasi-polarized p-divisible group G_η with the same Newton Polygon and with $a(G_\eta) = 1$: use the universal deformation space of (G_0, λ_0). For this (over some field) it follows from [18] that it can be deformed to the situation where the generic fiber has a given Newton Polygon, see (1.5). Hence using the same methods as in the proof of (2.8) we see that Theorem (3.1) and Corollary (3.10) follow from (3.8), (3.9) and [18]. •

(3.11) Corollary. *Let (X_0, λ_0) be a principally polarized abelian variety over a field K and let ξ be a symmetric Newton polygon, $\xi \succ \mathcal{N}(G_0)$. There is a deformation (G_η, λ_η) of (G_0, λ_0) such that $\xi = \mathcal{N}(G_\eta)$ and $a(G_\eta) \leq 1$.*
Proof. This follows from (3.10) and [18], using Serre-Tate and Chow-Grothendieck. In fact, the p-divisible group $X_0[p^\infty]$ plus principal polarization can be deformed as in (3.10). The rest follows as in the proof of (3.2). •

(3.12) Here is an easy example which might explain the esssence of what we are doing. Consider a supersingular elliptic curve E, with its unique principal polarization, and let $X_0 = E \times E$ be equiped with the "diagonal" principal polarization λ_0. Study deformations which stay inside the supersingular locus. In this case this is still not so difficult. The display of the universal deformation over Spf$(k[[s, t, u]])$ gives a Hasse-Witt matrix equal to

$$H = \begin{pmatrix} s & t \\ t & u \end{pmatrix}.$$

For abelian surfaces being supersingular amounts to having p-rank equal to zero. Hence the supersingular locus here is given by $H \cdot H^{(p)} = 0$. One easily solves the four equations thus obtained. Choose $\zeta \in W$ with $\zeta^\sigma{\cdot}\zeta = -1$, and consider all deformations given by $s = \zeta{\cdot}t$ and $u = \zeta^{-1}{\cdot}t$ over Spf$(k[[t]])$. Clearly there are $p + 1$ of such choices; these are exactly the components of the local deformation space of the supersingular locus in the deformation space of principally polarized abelian surfaces. We can also argue as follows (and obtain the same result). We see that (X_0, λ_0) admits exactly $p+1$ maximal symplectic filtrations; each of these defines a regular supersingular deformation space (keeping the filtration); in this way we find back the $p + 1$ branches of the local supersingular moduli space at (X_0, λ_0).

(3.13) Remark. Here is another example. Suppose (G_0, λ) is a principally quasi-polarized p-divisible group, such that $G_0 = H'_0 \oplus H''_0$, say with $a(H'_0) = 1 = a(H''_0)$, where H' is a simple formal group with slope s, with $0 < s < 1/2$. We can choose a deformation of the filtered group $0 \subset H'_0 \subset G_0$, and arrive at a filtered p-divisible group $0 \subset \mathcal{H}' \subset \mathcal{X}$ with a principal quasi-polarization. However we can also arrive at $0 \subset \mathcal{H}'' \subset \mathcal{Y}$. This last case gives an example of a filtered p-divisible group over a base, where the steps in the filtration have *decreasing* slopes. It is easy to show that the p-divisible groups $\mathcal{X}$ and $\mathcal{Y}$ (over some common base) are not isomorphic. This phenomenon deserves further study, especially in case of more isogeny factors.

In case of a p-divisible group over a complete local base such that the Newton Polygon is constant we find in [10], Th. 2.4.2 the existence of a filtration by isoclinic groups of increasing slopes if certain conditions on Hodge polygons are satisfied. We see that such a filtration in general does not exist.

(3.14) We work this out in a special case. Suppose given $H := G_{2,1} \oplus G_{1,2}$ and a quasi-polarization μ on it with $\mathrm{Ker}(\mu) = G_{2,1}[V] \oplus G_{1,2}[F] = (\alpha_p)^2$. We construct $(\mathcal{G}, \lambda) \to \mathbb{P}^1$ in the usual way of "taking the quotient by a variable α_p", i.e. above the point $(x, y) \in \mathbb{P}^1$ the fiber $(\mathcal{G}, \lambda)_{(x,y)} = H/(x, y)(\alpha_p)$, and the polarization μ descends to a principal polarization on $\mathcal{G}_{(x,y)}$. Around the point $0 = (0, 1)$ we have a filtration $(\mathcal{G}, \lambda)$, and every fiber of $\mathcal{G}'$ has Newton Polygon equal to $(2, 1)$; on can show that the factor group does not come from a subgroup (and here the filtration is by increasing slopes). Around the point $\infty = (1, 0)$ we have a filtration $\mathcal{G}'' \subset \mathcal{G}$; here the filtration is by decreasing slopes and around the point $\infty \in \mathbb{P}^1$ no isoclinic filtration with increasing slopes is possible.

Conversely, both local situations can be reconstructed by deformation of filtered groups.

(3.15) We have seen deformations obtained as deformations of filtered groups. However, in general an isoclinic, non-constant $\mathcal{G} \to B$ over an integral base need not be derived from a deformation of a filtered group. A component of the Newton Polygon stratum locally at a pont need not be given by a deformation of filtered groups.

In case $X_0[p^\infty] \cong G_{2,1} \oplus G_{1,2}$ the local deformation space of (X_0, λ) keeping the Newton Polygon constant has two irreducible branches of dimension equal to 2. One can be given by a filtered deformation (the one with the increasing slopes), the other does contain a one-dimensional filtered deformation (the one with the decreasing slopes). Here we see an example that we can move to the interior of a branch, but not reach the full branch by filtered deformations. Here is another example:

Consider an irreducible component $W \subset \mathcal{S}_{3,1,n}$ of the supersingular locus (with some level structure given by an integer $n \geq 3$ not divisible by p). This has a very singular point $x \in W$ obtained by any flag type quotient $(E^3 \to E^3/(\alpha_p)^2 \to X_0) = (F : E^3 \to E^3/E^3[F])$; this is the pont described in [11], (9.4.16) on page 59. The universal family over W, locally at x does not come from the deformation of any maximal symplectic filtration of X_0; in fact, no subspace on which the $a = 1$ case on W is realized can be obtained by a filtered deformation.

4 Newton Polygon strata

We recall some notations. We fix $g \in \mathbb{Z}_{>0}$, and we write $\mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ for the moduli space of *principally* polarized abelian varieties in characteristic p.

For a symmetric Newton Polygon β belonging to g we write $W_\beta \subset \mathcal{A}$ for the subset of all $[(X, \lambda)] \in \mathcal{A}$ with $\mathcal{N}(X) \prec \beta$, which is a closed subset by [10], Th. 2.3.1 on page 143.

For an irreducible subset $W \subset \mathcal{A}$ we write $\mathcal{N}(-, W \subset \mathcal{A})$ for the Newton Polygon of the generic fiber, and $a(-, W \subset \mathcal{A})$ for the a-number of the generic fiber above W.

(4.1) We fix an integer g. For every *symmetric* Newton Polygon ξ of height $2g$ we define:

$$\triangle(\xi) = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid y < x \leq g, \quad (x, y) \prec \xi\},$$

and we write

$$\mathrm{sdim}(\xi) := \#(\triangle(\xi)).$$

For the ordinary symmetric Newton Polygon $\rho = g{\cdot}((1, 0) + (0, 1))$ indeed $\triangle = \triangle(\rho)$ is a triangle; this explains our notation. But you can rightfully complain that the "triangle" $\triangle(\beta)$ in general is not a triangle.

(4.2) Theorem, see [17]. *Suppose given p, and g as as above. For every Newton Polygon β we have:*

(a) *For every irreducible component W of $W_\beta := \mathcal{W}_\beta(\mathcal{A}) \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ we have*

$$\mathcal{N}(-, W \subset \mathcal{A}) = \beta \quad \textit{and} \quad a(-, W \subset \mathcal{A}) \leq 1,$$

i.e. generically on W the Newton Polygon of $\mathcal{X} \to W$ equals β, and generically the a-number is at most one.

(b) *The dimension of every irreducible component W of W_β equals* $\mathrm{sdim}(\beta) = \#(\triangle(\beta))$.

This follows form (3.10) and [18]; compare with (2.9) and with (1.5) •

(4.3) Remark. In particular a supersingular principally polarized abelian variety can be deformed (keeping polarization and Newton Polygon) to one with $a = 1$. Hence we have a new proof for [11], Theorem (4.9). In particular we obtain the dimension of the supersingular locus, and the number of components of the supersingular locus, expressed as a class number. The difference in methods between [11] and this paper is as follows. In [11] *polarized flag type quotients*, and several variants were considered; with the correct definitions (especially once having found the notion of "rigid" quotients) the fact that every component of the supersingular locus generically has $a = 1$ results; also the dimension of the supersingular locus can be computed by methods of flag type quotients. In the proof in this paper we avoid such considerations, but instead we consider deformations of *filtered p-divisible groups*; it works for all Newton Polygons, and it shows that $a \leq 1$ generically on Newton Polygon strata; for those the method of Cayley Hamilton works, as in [18]; it gives a computation for the dimension of the various strata. Of course, in both cases we obtain the same answer for the dimension of the supersingular locus: $\dim(\mathcal{S}_{g,1}) = [g^2/4]$, as conjectured by Tadao Oda and the present author, see [16], pp. 615/616. By the method of flag type quotients we obtain a global description of every component of the supersingular locus, as to be found in [11].

(4.4) Remark. We have studied deformations of *principally* polarized abelian varieties. However if we would consider deformations of abelian varieties with a non-separable polarization things are not as uniform as above. In case of polarized p-divisible groups, or polarized abelian varieties, *without supposing the polarization is principal*, there are counterexamples to the existence of deformations with given closed fiber, and expected Newton Polygon for the generic fiber, see [18], Remark 6.8: there exist supersingular polarized abelian varieties of dimension $g = 3$, i.e. its Newton Polygon equals $3{\cdot}(1,1)$, which cannot be deformed to a polarized abelian variety with Newton Polygon equal to $(2,1) + (1,2)$. In the non-principally polarized case dimensions of strata may be different from what we compute in the principally polarized case. In [11], Section 12 some examples have been worked out. The phenomenon that Hecke correspondences blow up and down is present for Newton Polygons which allow $a \geq 2$. Also see (5.8).

5 Some questions and conjectures

(5.1) For every Newton Polygon β (and every g and every p) we obtain $W_\beta \subset \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$. For $\beta = \sigma$, the supersingular Newton Polygon, this locus has "many" components (for $p >> 0$;

in fact this number is a class number, asymptotically going to ∞ with $p \to \infty$).
Conjecture. *Given p, g, and $\beta \neq \sigma$ we conjecture that the locus W_β is* geometrically irreducible.

(5.2) We consider complete subvarieties of moduli spaces. It is known that for any field K, and any *complete* subvariety $W \subset \mathcal{A}_g \otimes K$, the dimension of W is at most $(g(g+1)/2) - g$. We wonder is this maximum ever achieved? If yes, in which cases?
Conjecture. *Let $g \geq 3$. Suppose W is a complete subvariety $W \subset \mathcal{A} = \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ of dimension equal to* $(g(g+1)/2) - g$ (the maximal possible dimension for complete subvarieties of this; see [3]). *We expect that under these conditions W is equal to the locus V_0 of principally polarized abelian varieties with p-rank equal to zero.* (This locus is complete and has the right dimension.)

If this is true, probably we have a proof for:

(5.3) Conjecture. *Let $g \geq 3$. Let $W \subset \mathcal{A}_g \otimes \mathbb{C}$ be a* complete *subvariety. We expect that under these conditions:*

$$\dim(W) < (g(g+1)/2) - g.$$

Hecke orbits are dense in $\mathcal{A}_g \otimes \mathbb{C}$. Chai proved the same for Hecke orbits of *ordinary* polarized abelian varieties in positive characteristic, see [1]. In his case only ℓ-power isogenies need to be considered for one prime $\ell \neq p$.

(5.4) Conjecture. *Fix a polarized abelian variety $[(X, \lambda)] = x \in \mathcal{A} \otimes \mathbb{F}_p$. Choose any prime number $\ell \neq p$. Consider the Hecke orbit of x admitting isogenies of degree equal to a product of a power of ℓ and of a succession of isogenies with kernel α_p. We conjecture that this Hecke orbit is everywhere dense in the Newton Polygon stratum W_β with $\beta := \mathcal{N}(X)$.*
This will be studied in [2].

(5.5) As in the case of simple p-divisible groups, for any p-divisible group G there is a catalogue for all groups isogenous with G. Probably there exists an irreducible one (constructed, using the methods of Section 2). It could be useful to study that case.

(5.6) Conjecture; Foliations. We expect that the following facts to be true. For every Newton polygon β there should exist integers $i(\beta)$, $c(\beta), \in \mathbb{Z}_{\geq 0}$, and for every point $[(X, \lambda)] = x \in \mathcal{A} = \mathcal{A} \otimes \mathbb{F}_p$ with $\mathcal{N}(X) = \beta$ there should exist a closed subset $x \in \mathcal{I}(x) = \mathcal{I}_\beta(x) \subset W_\beta \subset \mathcal{A}$, and a locally closed subset $x \in \mathcal{C}(x) = \mathcal{C}_\beta(x) \subset W_\beta \subset \mathcal{A}$ such that:

- $\dim(\mathcal{I}(x)) = i(\beta)$ and $\dim(\mathcal{C}(x)) = c(\beta)$.
- For every geometric point $[(Z, \zeta)] = z \in \mathcal{C}(x)(k)$ there is an isomorphism $(Z[p^\infty], \zeta) \cong (X[p^\infty], \lambda)$. All irreducible components of the locally closed set $\mathcal{C}(x)$ contain x, and it is the maximal closed set with this and the property just mentioned.
- For every geometric point $[(Y, \mu)] = y \in \mathcal{I}(x)$ there is a Hecke-correspondence using only iterates of α_p-isogenies relating $[(X, \lambda)]$ and $[(Y, \mu)]$. All irreducible components of the

closed set $\mathcal{I}(x)$ contain x, and it is the maximal closed set with this and the property just mentioned.

- The dimensions are complementary: $i(\beta) + c(\beta) = \mathrm{sdim}(\beta)$, and locally at x their intersection is zero dimensional.
- If moreover $a(X) \leq 1$, the (locally) closed sets $\mathcal{I}(x)$ and $\mathcal{C}(x)$ are regular at $x \in \mathcal{A}$, intersect transversally at x, and together their tangent spaces span the tangent space of $x \in W_\beta$.
- Examples:
for the supersingular locus we have $i(\sigma) = \mathrm{sdim}(\sigma) = [g^2/4]$ and $c(\sigma) = 0$;
for the ordinary locus we have $i(\rho) = 0$, and $c(\rho) = \mathrm{sdim}(\sigma) = (g(g+1))/2$;
for the case the p-rank equals one, i.e. $\beta = g\cdot(1,0) + (1,1) + g\cdot(0,1)$ we have $i(\beta) = 0$, and $c(\beta) = \mathrm{sdim}(\sigma) = ((g(g+1))/2) - 1$.
We have: p-rank $f(\beta) < g-1$ iff $i(\beta) > 0$.
We have: $\beta \neq \sigma$ iff $c(\beta) > 0$.
- There is an easy combinatorial argument by which the numbers $i(\beta)$ and $c(\beta)$ can be read off from the Newton Polygon diagram of β.

(5.7) In general $G[p]$ does not determine a p-divisible group G. But in some cases it does. Let β be a symmetric Newton Polygon. For a pair of relatively prime integers (m,n) we have defined in [8], Section 5 a p-divisible group $H_{m,n}$; it is characterized by: $H_{m,n} \sim G_{m,n}$, and for an algebraically closed field $k \supset \mathbb{F}_p$, the ring $\mathrm{End}(H_{m,n} \otimes k)$ is a maximal order in $\mathrm{End}^0(G_{m,n} \otimes k)$. We define H_β to be the direct sum of all $H_{m,n}$ ranging over all slopes of β. We expect:
Conjecture. *Suppose (G,λ) is a principally quasi-polarized p divisible group over an algebraically closed field k, such that $G[p] \cong H_\beta[p]$; then (?) we should conclude $G \cong H_\beta$.*

Note that in the special cases $\beta = \rho$ (the ordinary case), and $\beta = \sigma$ (supersingular) this conjecture is true; the conjectural statement above seems the natural generalization of this. Special cases have been proved.

(5.8) **Conjecture** (Newton Polygon strata, the non-principally polarized case). Let ξ be a symmetric Newton Polygon and consider all possible polarized abelian varieties, where the polarization need not be principal. This gives a stratum $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$. Let $f = f(\xi)$ be the p-rank of ξ, i.e. this Newton polygon has exactly f slopes equal to zero. We expect: *under these conditions, there is an irreducible component*

$$W \subset \mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p) \quad \text{with} \quad \dim(W) = ((g(g+1)/2) - (g-f),$$

i.e. we expect that there is a component of every Newton Polygon stratum which is a whole component of its p-rank stratum.

If this is the case, we see that there are "many" pairs of polarized abelian variety (X,λ) and a Newton Polygon $\gamma \succ \mathcal{N}(X)$ such that there exist no deformation of (X,λ) to a polarized abelian variety with Newton Polygon equal to γ, namely consider $\beta \prec \gamma$ with $\beta \neq \gamma$ and $f(\beta) = f(\gamma)$.

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