

A UNITARY MODEL FOR INDEPENDENT CLUSTER PRODUCTION

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The Lippman-Schwinger equation is generalized so as to include the description of multiparticle production. Unitarity is thereby automatically satisfied. Inelastic diffraction is found to be a general consequence of non-decreasing cross sections and a certain smoothness of the potential in going from the inelastic to the elastic case.

1. Introduction

It is well established that unitarity plays a crucial role in high-energy hadron-hadron scattering. It is therefore desirable, even if one constructs only phenomenological models to incorporate unitarity from the outset. In this context it then becomes important to find the simplest scheme which meets this requirement. In this spirit we propose an explicitly covariant Lippman-Schwinger (LS) equation [1–3] generalized for many-particle production. Some of the advantages of such an equation are:

- (i) the equation is linear but the solution nevertheless obeys the non-linear unitarity equation;
- (ii) the interaction is described by a potential on which no other restrictions are *a priori* imposed than that it be Hermitian and covariant;
- (iii) the equation can actually be solved at high energy under some simplifying, but not unrealistic, assumptions on the potential. As we will show in this paper, the formalism is particularly well suited for studying the effect of the uncorrelated production of cluster or mesons on the leading particle spectrum. In fact, under quite general conditions such as non-decreasing cross-sections and smoothness of the potential in going from the inelastic to the elastic case, it can be proved that there should be inelastic diffraction at high energy.

In sect. 2 we give a description of the LS formalism generalized to many particle production. In sect. 3 we apply this to nucleon-nucleon scattering for the special case that the only correlations between the initial and final state are by way of the nucleons under the above assumption on the smoothness of the potential. In sect. 4

the longitudinal momentum distribution of the final nucleons and produced particles are then calculated resulting, for the nucleon case, in a singularity in the extreme forward direction. It turns out that both distributions can be expressed in terms of the Koba-Nielsen-Olesen (KNO) [4] multiplicity distribution function which can be calculated explicitly in this model and is found to be in excellent agreement with experiment [5].

2. The Lippmann-Schwinger equation for many-particle systems

In this section we will develop a formalism which appears to be the simplest scheme to guarantee unitarity and some basic analytic properties of the scattering amplitude. Let us indicate our many-particle states by $\alpha, \beta, \gamma, \dots$, where, for example, $\gamma = (k_1, \dots, k_n)$. Only the four-momenta are written, but other quantum numbers such as spin and isospin could easily be included; however, such complications will be ignored here.

The requirement of unitarity of the S -matrix implies that the scattering amplitude $M_{\beta\alpha}$ describing the transition $\alpha \rightarrow \beta$ satisfies the condition

$$M_{\beta\alpha} - M_{\alpha\beta}^* = 2\pi i \int_{\gamma} M_{\gamma\beta}^* M_{\gamma\alpha} \delta^4(P_{\gamma} - P_{\alpha}) \quad \text{for all} \quad P_{\alpha} = P_{\beta}, \quad (2.1)$$

where P_{α} is the total four-momentum of the state α and \int_{γ} implies the summation over all intermediate states γ , which will be defined later. We now regard $M_{\beta\alpha}$ as the limit of a function $M_{\beta\alpha}(P)$ for $P \rightarrow P_{\alpha}$, such that $P^2 \rightarrow P_{\alpha}^2 + i\epsilon$, and we postulate the following Lippmann-Schwinger equation for $M_{\beta\alpha}(P)$:

$$M_{\beta\alpha}(P) = V_{\beta\alpha}(P) + \int_{\gamma} V_{\beta\gamma}(P) L_{\gamma}(P) M_{\gamma\alpha}(P), \quad (2.2)$$

where $V_{\beta\alpha}(P)$ is a Hermitian interaction potential $[V_{\beta\alpha}(P)]^* = V_{\alpha\beta}(P)$, and where as stated above,

$$M_{\beta\alpha} \equiv \lim_{\substack{P \rightarrow P_{\alpha} \\ P^2 \rightarrow P_{\alpha}^2 + i\epsilon}} M_{\beta\alpha}(P). \quad (2.3)$$

To facilitate the notation we now write eq. (2.2) in operator form:

$$M(P) = V(P) + V(P)L(P)M(P) \quad (2.4)$$

with $V(P) = V^{\dagger}(P)$ and $L(P)$ diagonal.

To see what condition unitarity puts upon $L(P)$ we observe that (we sometimes omit the index P):

$$M - M^\dagger = VLM - M^\dagger L^\dagger V = (M^\dagger - M^\dagger L^\dagger V)LM - M^\dagger L^\dagger (M - VLM) = M^\dagger (L - L^\dagger)M,$$

which leads, by eq. (2.1), to the condition

$$\text{Im } L_\gamma(P) = \pi \delta^4(P_\gamma - P). \quad (2.5)$$

The choice of $\text{Re } L_\gamma(P)$ is in fact arbitrary, as a different choice merely implies a change in the potential V as we can see by observing that eq. (2.4) is equivalent to the following set of two equations [1, 2]:

$$M = W + W(L - L')M \quad \text{with} \quad W = W^\dagger; \quad L, L' \text{ diagonal}; \\ L' = L'^\dagger, \quad (2.6)$$

$$W = V + VL'W. \quad (2.7)$$

So, we can always add a Hermitian operator L' to L , corresponding to a different $\text{Re } L_\gamma(P)$, and this results in a different Hermitian potential W which is the solution of eq. (2.7).

So, we are free to make a choice for $\text{Re } L_\gamma(P)$ without changing the content of the formalism at all. We will of course use this freedom to satisfy some basic analytic properties in $s = P^2$ of $M_{\beta\alpha}(P)$, like branch points at every energy where a new channel opens up, in the hope that we can then give a good description of many-particle processes with a comparatively simple potential.

To this end we require $L_\gamma(P)$ to satisfy a dispersion relation in $s = P^2$:

$$L_\gamma(P) = \frac{1}{\pi} \int_0^\infty \frac{ds'}{s' - s} \text{Im } L_\gamma(P') \quad \text{with} \quad s' = P'^2; \quad s = P^2, \quad (2.8)$$

which if s approaches the real axis from above is equivalent to

$$L_\gamma(P) = \int_0^\infty \frac{2x \, dx}{x^2 - 1 - i\epsilon} \delta^4(P_\gamma - xP). \quad (2.9)$$

In order to see the implications of the choice (2.8) we perform the integration over s' in eq. (2.8), which results in:

$$L_\gamma(P) = \int_0^\infty \frac{ds'}{s' - s} \delta^4(P_\gamma - P') = \frac{2P_\gamma^{-2}}{P_\gamma^2 - s} (1 - \mathbf{v}_\gamma^2)(1 - \mathbf{v}^2) \delta^3(\mathbf{v}_\gamma - \mathbf{v}), \quad (2.10)$$

where \mathbf{v} is the total three-velocity of the intermediate state γ [$\mathbf{v}_\gamma = \mathbf{P}_\gamma/P_{0\gamma}$; $\mathbf{v} = (\mathbf{P}/P_0)$]. It should also be noted that the expression $(1 - \mathbf{v}_\gamma^2)(1 - \mathbf{v}^2) \delta^3(\mathbf{v}_\gamma - \mathbf{v})$ is an explicitly covariant quantity so that $L_\gamma(P)$ is Lorentz invariant. So we see

that there is conservation of total three-velocity in the intermediate states which contribute to eq. (2.2); s , the square of the total energy can be off shell. As in the total c.m.s., conservation of total three-velocity is equivalent to conservation of total three-momentum, it is clear that eq. (2.10) is nothing but the covariant generalization of conservation of total three-momentum in the conventional LS formalism (in this connection it is interesting to note that in the non-relativistic limit eq. (2.2) with $L_\gamma(P)$ as in eq. (2.8) reduces to the conventional LS equation).

Concerning the choice for $L_\gamma(P)$ in eq. (2.8) we remark that we can actually introduce a cut-off in the dispersion integral in eq. (2.9) by the transformation;

$$V \rightarrow \exp\left(\frac{\delta(P-P_\alpha) \cdot P}{s}\right) \exp\left(\frac{\delta(P-P_\beta) \cdot P}{s}\right) V$$

and similarly for M , which leaves V and M unchanged in the physical region ($P = P_\alpha = P_\beta$), but results in the same equation i.e. eq. (2.2), but with

$$L_\gamma(P) = \int_0^\infty \frac{2x \, dx}{x^2 - 1 - i\epsilon} e^{2\delta(1-x)} \delta^4(P_\gamma - xP),$$

instead of eq. (2.9).

To conclude: We have to solve eq. (2.2) with $L_\gamma(P)$ as in eq. (2.9) for some reasonable potential V , taking $P = P_\alpha = P_\beta$, where $P_\alpha(P_\beta)$ is the total four-momentum of the initial (final) state. In the next section we will give an example of how to treat eq. (2.2) in the case of multiparticle production from the scattering of two nucleons.

3. An application to multiparticle production

For multiparticle production the question arises, How does one choose the potential $V_{\beta\alpha}$ when both β and α contain an arbitrary number of particles? In making such a choice we must strike a balance between what is realistic and what can still be calculated. Let the initial state with two nucleons and n mesons (or clusters) be $\alpha = (p_1, p_2, k_1, \dots, k_n)$ and let the final state be $\beta = (p'_1, p'_2, k'_1, \dots, k'_m)$, then we take:

$$V_{\beta\alpha}(P) = [(p_1 + p_2)^2 (p'_1 + p'_2)^2]^{1/2} W(p'_1 p'_2 | p_1 p_2 | P) \\ \times f_m^*(P, p'_1, p'_2, k'_1, \dots, k'_m) f_n(P, p_1, p_2, k_1, \dots, k_n), \quad (3.1)$$

where for $n = 0$ $f_0(P, p_1, p_2) = 1$ by definition.

We see that $V_{\beta\alpha}$ is chosen such that the only correlations between initial and final state arise from the nucleon-nucleon potential W and the dependence on the total four-momentum P . In particular, as the function $f_m(P, p'_1, p'_2, k'_1, \dots, k'_m)$ does

not depend on p_1 and p_2 separately, the only correlation between the outgoing direction of the mesons and the incoming direction is by way of the correlation between initial and final nucleons. In this way, as we will see later, the phases of the functions f_n do not enter into the equations at all, which is a rather satisfactory feature as we have next to no guide of how to construct such a phase [6, 7].

Now we turn to a fundamental question concerning our approach, i.e. What can we learn from such an LS equation which cannot be learned from unitarity alone? Why introduce such an equation and with it the concept of a potential? The answer is that the requirement of simple properties for the potential implies important consequences for the scattering and production amplitudes. We require, in particular, that the potential V in eq. (3.1) should vary smoothly when one goes from the inelastic case to the elastic one; we will specify this condition later, but loosely speaking it means that there are no drastic changes in the potential V if we change the number of produced particles (especially to $n = 0$) or if we go from $p_1 + p_2 \neq p'_1 + p'_2$ to $p_1 + p_2 = p'_1 + p'_2$. Inserting the potential of eq. (3.1) into our LS equation [eq. (2.2)] reduces the complexity considerably.

Defining analogously to eq. (3.1):

$$M_{\beta\alpha}(P) = [(p_1 + p_2)^2 (p'_1 + p'_2)^2]^{1/2} T(p'_1 p'_2 | p_1 p_2 | P) f_m^*(P, p'_1, p'_2, k'_1, \dots, k'_m) f_n(P, p_1, p_2, k_1, \dots, k_n) \quad (3.2)$$

Eq. (2.2) reduces to a similar equation, but now for the two-nucleon amplitude T only:

$$T(p'_1 p'_2 | p_1 p_2 | P) = W(p'_1 p'_2 | p_1 p_2 | P) + \int \frac{d^3 q_1}{2q_{10}} \frac{d^3 q_2}{2q_{20}} W(p'_1 p'_2 | q_1 q_2 | P) C(q_1 q_2 | P) T(q_1 q_2 | p_1 p_2 | P), \quad (3.3)$$

where $C(q_1 q_2 | P)$ has no longer the simple form $L_\gamma(P)$ of eq. (2.9) but now contains the sum over all possible intermediate states with two fixed nucleon momenta $q_1 q_2$:

$$C(q_1 q_2 | P) = C_{\text{el}}(q_1 q_2 | P) + C_{\text{inel}}(q_1 q_2 | P), \quad (3.4)$$

$$\text{Im } C_{\text{el}}(q_1 q_2 | P) = \pi (q_1 + q_2)^2 \delta^4(q_1 + q_2 - P), \quad (3.5)$$

$$\text{Im } C_{\text{inel}}(q_1 q_2 | P) = \pi (q_1 + q_2)^2 \sum_{n=1}^{\infty} \int \prod_{i=1}^n \frac{d^3 k_i}{k_{i0}} |f_n(P, q_1, q_2, k_1, \dots, k_n)|^2 \delta^4(q_1 + q_2 + \sum_{l=1}^n k_l - P), \quad (3.6)$$

and where $\text{Re } C$ is obtained from (see eq. (2.8)):

$$C(q_1 q_2 | P) = \frac{1}{\pi} \int_0^\infty dx \frac{2x}{x^2 - 1 - i\epsilon} \operatorname{Im} C(q_1 q_2 | xP), \quad (3.7)$$

where the variable xP should only be inserted into the δ^4 function in eq. (3.6).

We indeed observe, by looking at eq. (3.6), that the phase of the functions f_n does not play any role at all. For further use we mention some other formulae. Full elastic and inelastic unitarity is automatically guaranteed and expressed by

$$T(p'_1 p'_2 | p_1 p_2 | P) - T^*(p_1 p_2 | p'_1 p'_2 | P) \quad (3.8)$$

$$= 2i \int \frac{d^3 q_1}{2q_{10}} \frac{d^3 q_2}{2q_{20}} T^*(q_1 q_2 | p'_1 p'_2 | P) \operatorname{Im} C(q_1 q_2 | P) T(q_1 q_2 | p_1 p_2 | P).$$

In the case of elastic two-nucleon scattering (that is, $P = p_1 + p_2 = p'_1 + p'_2$) in the forward direction ($t = 0$ or $p'_1 = p_1$), eq. (3.8) reduces to the optical theorem, the normalization being such that

$$\sigma_{\text{tot}}(s) = (2\pi)^3 \left(\frac{s}{s - 4m^2} \right)^{1/2} \operatorname{Im} T(p_1 p_2 | p_1 p_2 | p_1 + p_2). \quad (3.9)$$

Another quantity of interest is the two-nucleon momentum distribution function:

$$\frac{d\sigma}{(d^3 p'_1 / 2p'_{10})(d^3 p'_2 / 2p'_{20})} = (2\pi)^3 |T(p'_1 p'_2 | p_1 p_2 | P)|^2 \operatorname{Im} C(p'_1 p'_2 | P). \quad (3.10)$$

For elastic scattering we take instead of $\operatorname{Im} C$ only $\operatorname{Im} C_{\text{el}}$ in eq. (3.10), and for inelastic scattering only $\operatorname{Im} C_{\text{inel}}$. If there are only two nucleons in the *initial* state, as we will assume henceforth, we have

$$P = p_1 + p_2.$$

Now we will deal with the condition of smoothness on the potential V . The physical idea behind this is very simple: we assume that on the level of the potential nothing drastic happens if we go from the case where we produce n particles to the case where we produce $n - 1$ particles; and this is supposed to be true for all n up to $n = 0$, i.e. elastic scattering. The singular behaviour we finally find in the actual distributions (the peak near $x = 1$ in the leading particle distribution) is solely due to conservation of four-momentum, as we will show.

As V in eq. (3.1) is written in the form of a product, we have to specify the condition for the two-nucleon potential W and the function f separately. We require, in particular:

- (i) For the two-nucleon potential $W(p'_1 p'_2 | p_1 p_2 | p_1 + p_2)$: if we define

$$x_1 = \frac{2p'_{10}}{\sqrt{s}}, \quad x_2 = \frac{2p'_{20}}{\sqrt{s}} \quad \text{and} \quad s = (p_1 + p_2)^2, \quad (3.11)$$

in the total c.m.s., then W is such that we can interchange the limits $x_1, x_2 \rightarrow 1$ and $s \rightarrow \infty$.

(ii) For the functions $|f_n(P, p'_1, p'_2, k_1, \dots, k_n)|^2$: The functions

$$h_n(P, p'_1, p'_2) \equiv \int \prod_{i=1}^n \frac{d^3 k_i}{2k_{i0}} |f_n(P, p'_1, p'_2, k_1, \dots, k_n)|^2 \quad (3.12)$$

(and $h_0 \equiv 1$) exist, have the smoothness property (3.11) and are such that, written in terms of x_1, x_2 , the ratio h_n/h_{n+1} varies at most as a power of $\ln s$ for all $n \geq 0$ and s large.

Now we return to our master equation (3.3) for the two-nucleon amplitude T . As the potential W in eq. (3.3) is supposed to change smoothly from the inelastic to the elastic case, the same will in general be true for amplitude T . This follows from the appearance of T in the left-hand side of eq. (3.3). So, by looking at eq. (3.10), it is clear that any possible singular behaviour of the inelastic two-nucleon distribution at the edge of phase space, such as single and double inelastic diffraction, necessarily has to come from the factor $\text{Im } C_{\text{inel}}$ in eq. (3.10). Generally speaking, as we will show, such singular behaviour should indeed be expected as long as we demand that the potential is such that

$$0 < C_0 \leq \sigma_{\text{el}}, \sigma_{\text{inel}} \leq C_1 (\ln s)^2 \quad \text{as} \quad s \rightarrow \infty. \quad (3.13)$$

Before we do this, however, we will first discuss which kinematical situation of the final nucleons will contribute dominantly to our equation, and so to the scattering. First we notice that, as according to eq. (3.13) the cross sections are assumed not go to 0 at high energy, higher partial waves will increasingly contribute, and as a result the scattering will be dominantly along the incoming direction in the total c.m.s. or, in other words, the potential W should damp high transverse momentum transfers to the final nucleons. As a result, the final nucleons will tend to come out in opposite hemispheres, along the incoming axis. So if we define the incoming direction as the direction $\mathbf{p}_1 - \mathbf{p}_2$ in the total c.m.s. and the outgoing direction as $\mathbf{p}'_1 - \mathbf{p}'_2$ in the c.m.s. of the *final* nucleons, we can state that the angle θ between these two directions in any of the two systems will be small. Looking at the two-nucleon distribution in eq. (3.10) we see that this information is solely contained in the factor $|T|^2$ as the factor $\text{Im } C_{\text{inel}}(p'_1 p'_2 | p_1 + p_2)$ is independent of this angle as it does not depend on p_1, p_2 separately.

4. Momentum and multiplicity distributions

In order to demonstrate that $\text{Im } C_{\text{inel}}$ indeed develops a singularity at the edge of phase space, we will now calculate $\text{Im } C_{\text{inel}}$ in the independent emission model (IEM), in which we will adjust the weights of the different channels so as to fulfil the smoothness condition (3.12) and such that $\sigma_{\text{el}}/\sigma_{\text{inel}}$ will only weakly depend on s as in eq. (3.13) (by “depending weakly on s ” we always mean “varies at most as a power of $\ln s$ for large s ”). We choose the IEM, as we can then easily calculate all quantities involved. This choice is in fact no limitation as we are only interested in the “global” quantity $\text{Im } C_{\text{inel}}$. We could in fact have taken any short-range (in the *exclusive* sense, i.e. for a fixed number of produced particles) correlation model, as long as we adjust the weights as to fulfil our smoothness condition.

So we choose in the c.m.s. of the final nucleons $p'_1 p'_2$:

$$|f_n(P, p'_1, p'_2, k_1, \dots, k_n)|^2 = g_n(s) \prod_{i=1}^n \frac{\beta}{\pi} e^{-\beta k_{\perp i}} e^{-\alpha(2k_i \cdot P/s)}, \quad (4.1)$$

where $g_0(s) = 1$, $s = P^2 = (p_1 + p_2)^2$, and the symbol \perp stands for “transverse to the outgoing direction $\mathbf{p}'_1 - \mathbf{p}'_2$ ”. The factor $e^{-\alpha(2k_i \cdot P/s)}$ has been included because otherwise the function h_n in (3.12) would not exist and the dispersion integral in eq. (3.7) would diverge owing to the fact that the virtual mesons in the intermediate state would be able to carry away an arbitrary amount of energy (see also the discussion at the end of sect. 1). In appendix A it is now shown that in the dominant kinematical region, where \mathbf{p}'_1 and \mathbf{p}'_2 are in opposite hemispheres in the total c.m.s. we can write

$$\begin{aligned} \text{Im } C_{\text{inel}}(p'_1 p'_2 | p_1 + p_2) &= 2\beta e^{-2\alpha P \cdot Q/s} (p'_1 + p'_2)^2 \\ &\times \left[g_1(s) \delta(Q^2 - \mu^2) + (Q^2)^{-1} \int_0^\infty \frac{d\rho \rho}{[\Gamma(\rho + 1)]^2} \phi(\rho, s) \left(\frac{Q^2}{s}\right)^\rho \right], \end{aligned} \quad (4.2)$$

where $Q = p_1 + p_2 - p'_1 - p'_2$, so Q^2 is the missing mass squared to the nucleons; $s = (p_1 + p_2)^2 = P^2$ is the total energy squared, and where $g_n(s)$ is written in the form:

$$g_n(s) = \frac{1}{\Gamma(n)} \int_0^\infty d\rho \rho^{n-1} \left(\frac{s}{\bar{\mu}^2}\right)^{-\rho} \phi(\rho, s) \quad \text{and} \quad g_0(s) = 1.$$

$\bar{\mu}^2 \equiv e^{2\gamma} k_\perp^2 + \mu^2$ is the average of the square of the transversal mass ($\gamma = 0.5772$ is Euler's constant). We observe that in eq. (4.2) we still have the as yet unknown function $\phi(\rho, s)$ (corresponding to $g_n(s)$) which has to be chosen such that $\sigma_{\text{el}}/\sigma_{\text{inel}}$ depends only weakly on s (see eq. (3.13)). This last condition tells us that $\phi(\rho, s)$ can only weakly depend on s as we can see as follows.

As σ_{el} and σ_{inel} should have approximately the same energy dependence and as

they are the integrated distributions in eq. (3.10) for the elastic case and the inelastic case, respectively, it is clear that as the factor $|T|^2$ in eq. (3.10) is smooth and damping in the transverse directions, the integrals over longitudinal phase space of the $\text{Im } C_{\text{inel}}$ and $\text{Im } C_{\text{el}}$ should have approximately the same energy dependence. Another way of saying the same thing is that in eq. (3.3) the elastic and inelastic intermediate states have to contribute both at high energy, i.e. C_{el} and C_{inel} ($C = C_{\text{el}} + C_{\text{inel}}$ in eq. (3.3)) have to have approximately the same energy dependence when integrated over the dominant region of phase space, which is the longitudinal direction. As $\text{Im } C_{\text{el}} = \pi(p'_1 + p'_2)^2 \delta^4(p_1 + p_2 - p'_1 - p'_2)$ is s -independent when integrated over the longitudinal directions, it follows that $\text{Im } C_{\text{inel}}$ has to depend weakly on s if integrated over the longitudinal directions, which by inspection of eq. (4.2) means that $\phi(\rho, s)$ can only weakly depend on s , or in other words: $\text{Im } C_{\text{inel}}$ in eq. (4.2) approximately scales in the Feynman sense. So now we have the following conditions:

(a) $\phi(\rho, s)$ depends weakly on s ,

$$(b) g_n(s) \equiv \frac{1}{\Gamma(n)} \int_0^\infty d\rho \rho^{n-1} \left(\frac{s}{\mu^2}\right)^{-\rho} \phi(\rho, s) \quad \text{with} \quad g_0(s) = 1 \quad (4.3)$$

is such that $g_n(s)/g_{n+1}(s)$ depends only weakly on s .

The last condition arises from eq. (3.12) as $h_n(s) \simeq g_n(s) (\ln s)^n$. Let us now first see what happens in the standard treatment of the IEM [8–10]. There one chooses $g_n(s) = (1/n!) \lambda^n$ independent of s (the factor $1/n!$ arises from Bose statistics) so that we obtain $\phi(\rho, s) = (s/\mu^2)^\rho \theta(\lambda - \rho)$ (which in the integrand of eq. (4.2) would act like $(\ln s)^{-1} (s/\mu^2)^\lambda \delta(\lambda - \rho)$). Clearly $\phi(\rho, s)$ is strongly s -dependent and thus violates condition (a) in eq. (4.3) leading to vanishing cross sections at high energy. In order to avoid this, one then introduces a factor $s^{-\lambda}$ in $|T_{\text{inel}}|^2$ in eq. (3.10) but *not* in $|T_{\text{el}}|^2$, resulting in a highly non-smooth amplitude T , and so a highly non-smooth W ; alternatively, which is completely equivalent to the above procedure, one defines $g_n(s) = s^{-\lambda} (1/n!) \lambda^n$ for $n \geq 1$ but $g_0(s) = 1$, which clearly violates condition (b) in eq. (4.3). So we see that, if we accept our smoothness condition on the potential, we have to discard the standard IEM (for *identical* mesons at least) as the sole mechanism of production, as it would lead to asymptotically vanishing cross sections. Therefore we try the more general ansatz:

$$g_n(s) = a_n \lambda^n(s) \quad \text{with} \quad g_0(s) = 1 \quad \text{and so} \quad a_0 = 1. \quad (4.4)$$

It is now shown in appendix B that such an ansatz in combination with the conditions in eq. (4.3) leads to the following conditions on $\lambda(s)$ and the a_n :

$$\lambda(s) \simeq \left(\ln \frac{s}{s_0} \right)^{-1}. \quad (4.5)$$

This result was also obtained by Białas and Kotanski [10] in their unitarization

scheme for the IEM, where they used non-diffractive uncorrelated production as an input, while in our approach it arises in a self-consistent way.

In appendix B it is also shown that the coefficients a_n in eq. (4.4) are such that the series

$$\sum a_n x^n \text{ has a radius of convergence } R = 1, \quad (4.6)$$

and that as a consequence $\phi(\rho, s)$ will act in eq. (4.2) as

$$\phi(\rho, s) \underset{s \rightarrow \infty}{\underset{c > 0}{\simeq}} c \left(\frac{s_0}{\bar{\mu}^2} \right)^\rho \left(\rho \ln \frac{s}{s_0} \right)^r \quad \text{for some } r. \quad (4.7)$$

Actually the coefficient c in eq. (4.7) could still depend very weakly on $\rho \ln(s/s_0)$, i.e. like $\ln(\rho \ln(s/s_0))$ or weaker, but this we ignore as it would only change some very minor details which are unobservable anyhow, such as $\ln \ln s$ terms in the cross sections. Looking at property (4.6) we observe again that coefficients a_n of the type $1/n!$, as in the standard IEM, are not allowed; it follows that Bose statistics for the secondary particles suppress the cross sections too much. Interpreting this result we see that, as the factor $1/n!$ in the standard IEM arose from Bose statistics and as by eq. (4.6) such a strong n -dependent factor is not allowed, it looks as if Bose statistics are not allowed to have much influence. In this spirit we are then led to the following speculation. Suppose that one actually produces clusters with a variable mass so that two clusters of a different mass are non-identical, then, as there would be so many quantum states available we would expect Bose statistics to play virtually no role, and we would in fact have some modified Boltzmann statistics, implying coefficients a_n which would only weakly depend on n , in accordance with property (4.6). We should mention that if we introduce variable masses, nothing in the previous formulae would change except that $\bar{\mu}^2$ in eq. (4.3) should also have to be averaged over the cluster mass spectrum. It should also be noted that nothing in the above statements prevents us from adding a component $g'_n = (1/n!) \lambda_0^n s^{-\lambda_0}$ to the $g_n(s) = a_n \lambda^n(s)$, as prescribed by (4.5) and (4.6), normalizing to $g_0(s) = 1$ of course, resulting in a two-component model; in the following we will omit such a component. The point we have made above is that there has to exist a component of the form (4.5) and (4.6). Later we will see that this is what is commonly called the diffractive component (in the sense that all $c_n^{\text{diffractive}}$ have the same energy dependence for $s \rightarrow \infty$ and n fixed).

As a check on the consistency of our approach we will now ask under which condition the principal value part of the dispersion integral in eq. (3.7), describing the contribution of the virtual intermediate states, exists, i.e. When will the expression $\text{Im } C_{\text{inel}}(q_1 q_2 | xP)$ converge for $x \rightarrow \infty$? As in the intermediate state the total energy is not conserved, the δ^4 function in eq. (3.6) will no longer provide any convergence and as $\sum a_n x^n$ has a convergence radius of 1 we obtain the condition (in fact otherwise the integrand in eq. (3.7) would diverge exponentially)

$$\lambda(s) \int \frac{d^3k}{k_0} \frac{\beta}{\pi} e^{-\beta k_1^2} e^{-\alpha 2k \cdot P/s} \leq 1, \quad (4.8)$$

which leads us to

$$\lambda(s) \leq \left(\ln \frac{s}{\alpha^2 \bar{\mu}^2} \right)^{-1}. \quad (4.9)$$

This is entirely compatible with eq. (4.5) as long as

$$s_0 \leq \alpha^2 \bar{\mu}^2 \quad (\alpha, \bar{\mu} \text{ as in eq. (4.2)}). \quad (4.10)$$

We now return to the calculation of $\text{Im } C_{\text{inel}}$ in eq. (4.2). Having obtained the expression (4.7) for $\phi(\rho, s)$ we can now write down $\text{Im } C_{\text{inel}}$ explicitly at high energy:

$$\begin{aligned} \text{Im } C_{\text{inel}}(p'_1 p'_2 | p_1 + p_2) &= 2\beta e^{-2\alpha P \cdot Q/s} (p'_1 + p'_2)^2 \\ &\times \left[a_1 \left(\ln \frac{s}{s_0} \right)^{-1} \delta(Q^2 - \mu^2) + c \left(\ln \frac{s}{s_0} \right)^r (Q^2)^{-1} \int_0^\infty \frac{d\rho \rho^{r+1}}{[\Gamma(\rho+1)]^2} \left(\frac{s_0}{\bar{\mu}^2} \frac{Q^2}{s} \right)^\rho \right], \end{aligned} \quad (4.11)$$

where

$$Q = p_1 + p_2 - p'_2 - p'_2 \text{ and } P = p_1 + p_2.$$

In order to see what happens in the diffractive limit, i.e., when Q^2 the square of the missing mass to the nucleons, is small compared to s , we take the limit $(Q^2/s)_{\downarrow 0}$ in eq. (4.11) and find

$$\text{Im } C_{\text{inel}}(p'_1 p'_2 | p_1 + p_2) \underset{Q^2/s \downarrow 0}{\simeq} 2\beta c \left(\ln \frac{s}{s_0} \right)^r \left(\frac{Q^2}{s} \right)^{-1} \left| \ln \left(\frac{Q^2}{s} \right) \right|^{-(r+2)}. \quad (4.12)$$

So we observe that there appears a singularity at $(Q^2/s)_{\downarrow 0}$, resulting in single and double diffraction of the final nucleons, as by eq. (3.10) the two-nucleon momentum distribution is proportional to $\text{Im } C_{\text{inel}}$ and as the factor $|T|^2$ in eq. (3.10) can never become 0 for $(Q^2/s)_{\downarrow 0}$ because the smoothness property of T tells us that $\text{Im } T = [1/(2\pi)^3] \sigma_{\text{tot}} > 0$ in that limit.

Now in order to calculate the complete two-nucleon distribution in the longitudinal direction we would have to solve eq. (3.3) for some potential W . This we can actually do analytically if W in the total c.m.s. is of the form

$$W(p'_1 p'_2 | p_1 p_2 | P) = \tilde{W}(\mathbf{p}_{1\perp} - \mathbf{p}'_{1\perp}, \mathbf{p}_{2\perp} - \mathbf{p}'_{2\perp}, s) e^{-\gamma(p_1 + p_2 + p'_1 + p'_2) \cdot P/s}, \quad (4.13)$$

where $s = P^2$.

So we can solve the nucleon-nucleon amplitude (elastic and inelastic) if the poten-

tial depends essentially on the total energy and the difference in the transverse momenta. It is now easy to see from eq. (3.3), as $C(q_1 q_2 | P)$ only acts in the longitudinal direction at high energy that, in that case, the amplitude T will be of the same form (4.13), even with the same γ (compare this with the discussion at the end of sect. 1). We will not give the solution here but will refer to a forthcoming paper dealing with the behaviour of the two-nucleon amplitude in the transverse direction and so in impact parameter space. Assuming then that W and thus T is of the form (4.13), we obtain by integrating eq. (3.10) over the transverse momenta of the final nucleons, choosing $\gamma = \alpha$ for a convenience and dropping the term for the production of exactly one particle

$$\frac{d\sigma_{\text{inel}}}{dx_1 dx_2} \simeq h(s) \left(\ln \frac{s}{s_0} \right)^r \int_0^\infty \frac{d\rho \rho^{r+1}}{[\Gamma(\rho+1)]^2} \left(\frac{s_0}{\mu^2} \right)^\rho (1-x_1)^{\rho-1} (1-x_2)^{\rho-1}, \quad (4.14)$$

where $x_1 = 2p'_{10}/\sqrt{s}$, $x_2 = 2p'_{20}/\sqrt{s}$, and with the additional restriction that the final nucleons are in opposite hemispheres as otherwise the factor $(p'_1 + p'_2)^2/s \approx O(s^{-1}) \approx 0$ in eq. (4.11). The function $h(s)$ is determined by the solution of eq. (3.3) but turns out to be such that, for any value of r , the cross sections are always $\leq \text{const} \times \pi b_0^2(s)$, where $b_0(s)$ is the range of the potential W in impact parameter space. So in this sense the Froissart bound is obeyed. If we demand that all the cross sections have the same energy dependence asymptotically, then we have to choose r such that

$$b_0^2(s) \sim \begin{cases} (\ln s)^r & \text{for } r > 0 \\ \ln \ln s & \text{for } r = 0 \\ \text{constant} & \text{for } r < 0 \end{cases}$$

and as a result all cross sections will behave like $\text{const.} \times \pi b_0^2(s)$ and so the function $h(s)$ in eq. (4.14) will be a constant. Taking this to be the case, one obtains for the nucleon momentum distribution function from eq. (4.14)

$$\frac{d\sigma_{\text{inel}}}{dx} \simeq \text{const.} \left(\ln \frac{s}{s_0} \right)^r \int_0^\infty \frac{d\rho \rho^r}{[\Gamma(\rho+1)]^2} \left(\frac{s_0}{\mu^2} \right)^\rho (1-x)^{\rho-1}. \quad (4.15)$$

We observe that this distribution is flat for small x , but becomes singular near $x = 1$, which is qualitatively in agreement with experiment. The behaviour near $x = 1$ is given by

$$\frac{d\sigma_{\text{inel}}}{dx} \underset{x \uparrow 1}{\simeq} \text{const.} \left(\ln \frac{s}{s_0} \right)^r (1-x)^{-1} |\ln(1-x)|^{-(r+1)}. \quad (4.16)$$

We will now consider the multiplicity distribution in the so-called KNO limit [4], i.e. $n \rightarrow \infty$, $\ln s \rightarrow \infty$, $n/\ln s$ fixed. In appendix C it is shown that we obtain

$$\sigma_n \simeq \text{const.} \frac{n^{r-1}}{[\Gamma(\frac{n}{\ln s} + 1)]^2} \left(\frac{s_0}{\bar{\mu}^2}\right)^{n/\ln s} \quad (4.17)$$

Observe that the parameter ρ which we have been using in the foregoing text is, in this limit, nothing but $\rho = n/\ln s$. Eq. (4.17) then gives us for the average multiplicity

$$\langle n \rangle = \rho_0 \ln s \quad \text{with} \quad \rho_0 = \int_0^\infty \frac{d\rho \rho^r}{[\Gamma(\rho+1)]^2} \left(\frac{s_0}{\bar{\mu}^2}\right)^\rho \bigg/ \int_0^\infty \frac{d\rho \rho^{r-1}}{[\Gamma(\rho+1)]^2} \left(\frac{s_0}{\bar{\mu}^2}\right)^\rho \quad (4.18)$$

(this is for $r > 0$, for $r = 0$ $\langle n \rangle \sim \ln s / \ln \ln s$, while for $r < 0$ it is constant as the lower limit of the ρ integration is really $(\ln s)^{-1}$).

For the KNO scaling function $[\psi = \langle n \rangle (\sigma_n / \sigma_{\text{inel}})]$ we obtain from eq. (4.16)

$$\psi(z) = A \frac{(\rho_0 z)^{r-1}}{[\Gamma(\rho_0 z + 1)]^2} \left(\frac{s_0}{\bar{\mu}^2}\right)^{\rho_0 z} \quad \text{with} \quad z = \frac{n}{\langle n \rangle}, \quad (4.19)$$

which, owing to the conventional normalizations $\int \psi(z) dz = \int z \psi(z) dz = 2$, leaves us with only two constants, e.g. ρ_0 and r , where $0 < r \leq 2$ as $\sigma \leq (\ln s)^2$. Taking $\rho_0 = 1$ and $r = 2$ we then obtain an excellent fit to the data, for which we refer to ref. [5]. The fact that $\rho_0 = 1$, giving the averaged multiplicity of the produced particles as $\langle n \rangle \simeq \ln s$, by eq. (4.18), implies that we are really producing clusters which each decay into roughly 2–3 particles [11]. The value $r = 2$ would imply that the cross sections actually go like $(\ln s)^2$ asymptotically. Another item of interest is that, as we can see by comparing eq. (4.15) with eq. (4.19), we have a direct connection between the KNO scaling function and the leading particle spectrum:

$$\frac{1}{\sigma_{\text{inel}}} \frac{d\sigma_{\text{inel}}}{dx} = \frac{1}{\rho_0} \int_0^\infty d\rho \rho \psi\left(\frac{\rho}{\rho_0}\right) (1-x)^{\rho-1}, \quad (4.20)$$

which is rather general, and has been extensively discussed by Benecke et al. [12]. To conclude we will write down the inclusive one-cluster momentum distribution as a function of $x = 2k_0/\sqrt{s}$ in the total c.m.s.:

$$\frac{1}{\sigma_{\text{inel}}} \frac{d\sigma_{\text{inel}}}{dx} = \frac{1}{x \rho_0} \int_0^\infty d\rho \rho \psi\left(\frac{\rho}{\rho_0}\right) (1-x)^\rho. \quad (4.21)$$

Again we see that the distribution can be simply expressed in terms of the KNO function in eq. (4.19). We also observe the resulting relation between the leading particle and cluster spectrum:

$$\left(\frac{d\sigma}{dx}\right)_{\text{leading particles}} = \frac{x}{1-x} \left(\frac{d\sigma}{dx}\right)_{\text{clusters}} \quad (4.22)$$

5. Summary

It has been shown, in the context of a single-component model for independent cluster production, that the observed momentum distribution of the final particles as well as the multiplicity distribution may be regarded as consequences of a certain smoothness property. This smoothness property, loosely speaking, says that the scattering of the leading particles in the particle production case connects smoothly to the elastic scattering.

One of us (de Groot) is greatly indebted to Professor L. Van Hove for continuous interest and helpful criticism during the later stages of this work.

Appendix A.

We define the generating function

$$Z_y(p'_1 p'_2 | P) = \pi(p'_1 + p'_2)^2 \sum_{n=2}^{\infty} y^n g_n(s) \int \prod_{i=1}^n \frac{d^3 k_i}{k_{i0}} \frac{\beta}{\pi} e^{-\beta k_{i1}^2} \delta^4(p'_1 + p'_2 + \sum_{l=1}^n k_l - P). \quad (\text{A.1})$$

We have not included the term for $n = 1$, as it is written down explicitly in eq. (4.2). To obtain $\text{Im } C_{\text{inel}}$ in eq. (4.2) we have (the factor $e^{-2\alpha(P \cdot Q/s)}$ arises directly from the factors $e^{-\alpha(2k_i \cdot P/s)}$ in eq. (4.1)):

$$\text{Im } C_{\text{inel}} = 2\beta e^{-2\alpha P \cdot Q/s} (p'_1 + p'_2)^2 \delta(Q^2 - \mu^2) + e^{-2\alpha P \cdot Q/s} Z_1(p'_1 p'_2 | P), \quad (\text{A.2})$$

where

$$Q = P - p'_1 - p'_2 \quad (\text{A.3})$$

is the total meson four-momentum.

Although we need Z_y here only for $y = 1$, we take y arbitrary for future use. As we have

$$g_n(s) \equiv \frac{1}{\Gamma(n)} \int_0^{\infty} d\rho \rho^{n-1} \left(\frac{s}{\mu^2}\right)^{-\rho} \phi(\rho, s), \quad (\text{A.4})$$

we can write (A.1) as:

$$\begin{aligned}
Z_y(p'_1 p'_2 | P) &= \pi(p'_1 + p'_2)^2 \int_0^\infty \frac{d\rho}{\rho} \left(\frac{s}{\bar{\mu}^2}\right)^{-\rho} \phi(\rho, s) \sum_{n=2}^\infty \frac{(y\rho)^n}{(n-1)!} \int \prod_{i=1}^n \frac{d^3 k_i}{k_{i0}} \frac{\beta}{\pi} e^{-\beta k_{i0}^2} \delta^4\left(Q - \sum_{l=1}^n k_l\right) \\
&= \pi(p'_1 + p'_2)^2 \int_0^\infty \frac{d\rho}{\rho} \left(\frac{s}{\bar{\mu}^2}\right)^{-\rho} \phi(\rho, s) F_{y\rho}(Q),
\end{aligned}$$

where $F_z(Q)$ has been calculated in ref. [9]:

$$\begin{aligned}
F_z(Q) &= \sum_{n=2}^\infty \frac{z^n}{(n-1)!} \int \prod_{i=1}^n \frac{d^3 k_i}{k_{i0}} \frac{\beta}{\pi} e^{-\beta k_{i0}^2} \delta^4\left(Q - \sum_{l=1}^n k_l\right) \\
&\simeq \frac{2\beta}{\pi} (Q^2)^{-1} \frac{1}{[\Gamma(z)]^2} \left(\frac{Q \cdot Q}{\bar{\mu}^2}\right)^z.
\end{aligned}$$

So we can write

$$Z_y(p'_1 p'_2 | P) \simeq 2\beta \frac{(p'_1 + p'_2)^2}{Q^2} \int_0^\infty \frac{d\rho}{\rho} \left(\frac{s}{\bar{\mu}^2}\right)^{-\rho} \frac{\phi(\rho, s)}{[\Gamma(y\rho)]^2} \left(\frac{Q \cdot Q}{\bar{\mu}^2}\right)^{y\rho}, \quad (\text{A.5})$$

which for $y = 1$ gives us eq. (4.2) if we include the term for $n = 1$ as given by (A.2).

Appendix B

In the following we mean by “ $\phi(\rho, s)$ depends weakly on s ” that $\phi(\rho, s)$ for all $\rho > 0$ varies at most as a power of $\ln s$ for large s . Let $g_n(s)$ be of the form $a_n \lambda^n(s)$ with $a_0 = 1$, then by the definition of $\phi(\rho, s)$ in (4.2) we have:

$$a_n \lambda^n(s) = \frac{1}{\Gamma(n)} \int_0^\infty d\rho \rho^{n-1} \left(\frac{s}{\bar{\mu}^2}\right)^{-\rho} \phi(\rho, s) \quad \text{with} \quad \lambda(s) > 0. \quad (\text{B.1})$$

As a_n is independent of s it follows that ϕ is of the form

$$\phi(\rho, s) = \left(\frac{s}{\bar{\mu}^2}\right)^\rho h\left[\frac{\rho}{\lambda(s)}\right]. \quad (\text{B.2})$$

As by property (4.3) (a), $\phi(\rho, s)$ depends only weakly on s , it is clear that, in order to compensate the strong s -dependence of the factor $(s/\bar{\mu}^2)^\rho$ for all $\rho > 0$, the function $h[\rho/\lambda(s)]$ should asymptotically mainly depend exponentially on its argument:

$$h\left[\frac{\rho}{\lambda(s)}\right] = e^{-c_0 \rho / \lambda(s)} \tilde{\phi}\left[\frac{\rho}{\lambda(s)}\right] \quad \text{for some} \quad c_0 \in (0, \infty), \quad (\text{B.3})$$

where (i) $(s/\bar{\mu}^2)^\rho e^{-c_0[\rho/\lambda(s)]}$ depends weakly on s and (ii) $\tilde{\phi}[\rho/\lambda(s)]$ depends weakly on s .

The first property gives us:

$$\lim_{s \rightarrow \infty} \lambda(s) \ln \frac{s}{\bar{\mu}^2} = c_0 \quad \text{and so} \quad \lambda(s) \underset{s \rightarrow \infty}{\simeq} c_0 \left(\ln \frac{s}{s_0} \right)^{-1},$$

where as $c_0 \in (0, \infty)$ [$c_0 > 0$ as $\lambda(s) > 0$], we can, without loss of generality, take $c_0 = 1$, absorbing the factor c_0^n in (B.1) into the a_n . So:

$$\lambda(s) \underset{s \rightarrow \infty}{\simeq} \left(\ln \frac{s}{s_0} \right)^{-1}, \quad (\text{B.4})$$

while the second property combined with (B.2)–(B.4) tells us that ϕ is asymptotically of the form:

$$\phi(\rho, s) = \left(\frac{s}{\bar{\mu}^2} \right)^\rho e^{-\rho/\lambda(s)} \tilde{\phi} \left(\frac{\rho}{\lambda(s)} \right) \underset{s \rightarrow \infty}{\simeq} \left(\frac{s_0}{\bar{\mu}^2} \right)^\rho \tilde{\phi} \left(\rho \ln \frac{s}{s_0} \right), \quad (\text{B.5})$$

where $\tilde{\phi}[\rho \ln(s/s_0)]$ depends only weakly on s and so $\tilde{\phi}(y)$ varies at most as a power of y for large y or, in other words, there exists an r such that for each $\epsilon > 0$ one can find a y_0 such that if $y > y_0$:

$$c y^{-\epsilon} \leq y^{-r} \tilde{\phi}(y) \leq c y^\epsilon. \quad (\text{B.6})$$

In order to see that the series $\sum a_n x^n$ has a radius of convergence $R = 1$ we note that

$$a_n = \frac{1}{\Gamma(n)} \int_0^\infty dy y^{n-1} e^{-y} \tilde{\phi}(y),$$

and thus

$$\sum_{n=0}^\infty a_n x^n = 1 + x \int_0^\infty dy e^{(x-1)y} \tilde{\phi}(y), \quad (\text{B.7})$$

and, as by property (B.6) $\phi(y)$ cannot depend exponentially on y for large y , we see that the integral can only exist for $x \leq 1$.

Appendix C

For $n \geq 2$ we can write with $x_1 = 2p'_{10}/\sqrt{s}$, $x_2 = 2p'_{20}/\sqrt{s}$:

$$\sigma_n = \int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2} \frac{d\sigma_n}{\frac{dx_1}{x_1} \frac{dx_2}{x_2}} \sim \frac{1}{n!} \frac{d^n}{dy^n} \bigg/ \int_{y=0}^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2} Z_y(p'_1 p'_2 | p_1 + p_2),$$

as $(p'_1 + p'_2)^2 \simeq x_1 x_2 s$ and $Q^2 = (p_1 + p_2 - p'_1 - p'_2)^2 = s(1 - x_1)(1 - x_2)$ we find with Z_y from appendix A:

$$\begin{aligned} \sigma_n(s) &\sim \frac{1}{n!} \frac{d^n}{dy^n} \bigg/ \int_{y=0}^1 dx_1 \int_0^1 dx_2 \frac{d\rho}{\rho} \left(\frac{s}{\mu^2}\right)^{(y-1)\rho} \frac{\phi(\rho, s)}{[\Gamma(y\rho)]^2} (1-x_1)^{y\rho-1} (1-x_2)^{y\rho-1} \\ &= \frac{1}{n!} \frac{d^n}{dy^n} \bigg/ \int_0^\infty \frac{d\rho}{\rho} \left(\frac{s}{\mu^2}\right)^{(y-1)\rho} \frac{\phi(\rho, s)}{[\Gamma(y\rho+1)]^2}. \end{aligned} \quad (C.1)$$

As we can prove that under suitable conditions on the analyticity of a general function $f(y, \rho)$ we have

$$\lim_{\substack{n \rightarrow \infty \\ \ln s \rightarrow \infty \\ n/\ln s = x > 0 \text{ fixed}}} \frac{\ln s}{n!} \frac{d^n}{dy^n} \bigg/ \int_0^\infty d\rho f(y, \rho) s^{(y-1)\rho} = f(1, x), \quad (C.2)$$

we find that as $\phi(\rho, s)$ only depends weakly on s :

$$\sigma_n(s) \underset{\substack{n \rightarrow \infty \\ \ln s \rightarrow \infty \\ n/\ln s \text{ fixed}}}{\sim} \frac{1}{n} \frac{\phi(n/\ln s, s)}{[\Gamma(n/\ln s + 1)]^2}, \quad (C.3)$$

and as by eq. (4.7) $\phi(\rho, s) \underset{s \rightarrow \infty}{\simeq} (s_0/\mu^2)^\rho (\rho \ln s)^r$ we obtain in this limit ($n \rightarrow \infty$, $\ln s \rightarrow \infty$, $n/\ln s > 0$ fixed):

$$\sigma_n(s) \sim \frac{n^{r-1}}{[\Gamma(n/\ln s + 1)]^2} \left(\frac{s_0}{\mu^2}\right)^{n/\ln s} \quad (C.4)$$

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