Abelian varieties isogenous to a Jacobian

Frans Oort

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Introduction

(0.1) Question *Given an abelian variety A; does there exist an algebraic curve C such that there is an isogeny between A and the Jacobian of C ?*

- If the dimension of A is at most three, such a curve exists; see (1.3).
- For any $g \geq 4$ there exists an abelian variety A of $\dim(A) = g$ over $\mathbb{C}$ such that there is no algebraic curve C which admits an isogeny $A \sim \mathrm{Jac}(A)$, see (3.1). One of the arguments which proves this fact (uncountability of the ground field) does not hold over a countable field.

Therefore:

- The question remains open over a countable field, see (3.4). One can expect that the answer to the question in general is negative for abelian varieties of dimension $g \geq 4$ over a given field.
- We offer a possible approach to this question via Newton polygons in positive characteristic, see (5.4).

1 Jacobians and the Torelli locus

(1.1) Jacobians. Let C be a complete curve over a field K. We write $J(C) = \mathrm{Pic}^0_{C/K}$.

In case C is irreducible and non-singular we know that $J(C)$ is an abelian variety. Moreover, $J(C)$ has a canonical polarization. This *principally polarized abelian variety* $\mathrm{Jac}(C) = (J(C), \Theta_C = \lambda)$ is called the *Jacobian* of C.

Suppose C is a geometrically connected, complete curve of genus at least 2 over a field K. We say that $J(C)$ is a *curve of compact type* if C is a stable curve such that:
- its geometrically irreducible components are non-singular, and
- its dual graph has homology equal to zero;

equivalently: C is stable, and for an algebraic closure $k \supset K$ the curve C_k is a tree of non-singular irreducible components;

equivalently (still $g \geq 2$): C is a stable curve, and $J(C)$ is an abelian variety.

For $g = 1$ we define "of compact type" as "irreducible + non-singular".

The terminology "a curve of compact type" is the same as "a good curve" (Mumford), or "a nice curve".

(1.2) The Torelli locus. By $C \mapsto \mathrm{Jac}(C) := (J(C), \Theta_C = \lambda)$ we obtain a morphism $j : \mathcal{M}_g \to \mathcal{A}_{g,1}$, from the moduli space of curves of genus g to the moduli space of principally polarized abelian varieties; this is called *the Torelli morphism.* The image

$$\mathcal{M}_g \twoheadrightarrow \mathcal{T}_g^0 \to \mathcal{A}_{g,1}$$

is called the *open Torelli locus.*

Let $\mathcal{M}_g^{\sim}$ be the moduli space of curves of compact type. The Torelli morphism can be extended to a morphism $\mathcal{M}_g^{\sim} \to \mathcal{A}_{g,1}$; its image

$$\mathcal{M}_g^{\sim} \twoheadrightarrow \mathcal{T}_g \to \mathcal{A}_{g,1}, \qquad \mathcal{T}_g = (\mathcal{T}_g^0)^{\mathrm{Zar}},$$

is the Zariski closure of $\mathcal{T}_g^0$; we say that $\mathcal{T}_g$ is the *closed Torelli locus.*

From now on let k be an algebraically closed field.

(1.3) *Suppose* $1 \leq g \leq 3$. *Then every abelian variety A of dimension g is isogenous with the Jacobian of a curve of compact type.*
Proof. In fact, a polarized abelian variety (A, λ) over an algebraically closed field is isogenous with a principally polarized abelian variety (B, μ). We know that there exists a curve C of compact type with $(B, \mu) \cong \mathrm{Jac}(C)$; for $g = 1$ this is clear; for $g = 2$ see A. Weil, [18], Satz 2; for $g = 3$ see F. Oort & K. Ueno, [14], Theorem 4. □

(1.4) Let $g \in \mathbb{Z}_{>0}$. We write $Y^{(cu)}(k, g)$ for the statement:

$Y^{(cu)}(k, g)$. *There exists an abelian variety A defined over k such that there does not exist a curve C of compact type of genus g defined over k and an isogeny $A \sim J(C)$* (here c stands for "of compact type", and u stands for "unpolarized").

2 Other formulations

(2.1) $Y^{(cp)}(k, g)$. *There exists a polarized abelian variety (A, λ) with* $\dim(A) = g$ *defined over k such that there does not exists a curve C of compact type (of genus g) defined over k and an isogeny* $(A, \lambda) \sim \mathrm{Jac}(C)$.

(2.2) $Y^{(iu)}(k, g)$. *There exists an abelian variety A with* $\dim(A) = g$ *defined over k such that there does not exists an irreducible curve C (of genus g) defined over k and an isogeny* $A \sim J(C)$.

(2.3) $Y^{(ip)}(k, g)$. *There exists a polarized abelian variety (A, λ) with* $\dim(A) = g$ *defined over k such that there does not exists an irreducible curve C (of genus g) defined over k and an isogeny* $(A, \lambda) \sim \mathrm{Jac}(C)$.

(2.4) Note that $Y^{(cu)} \Rightarrow Y^{(cp)} \Rightarrow Y^{(ip)}$ and $Y^{(cu)} \Rightarrow Y^{(iu)} \Rightarrow Y^{(ip)}$.

Given a point $[(A,\lambda)] = x \in \mathcal{A}_g$ in the moduli space of polarized abelian varieties we write $\mathcal{H}(x) \subset \mathcal{A}_g$ for the *Hecke orbit* of x; by definition $[(Y,\mu)] = y \in \mathcal{H}(x)$ if there exists an isogeny $A \sim B$ which maps λ to a rational multiple of μ.

(2.5) Here is a reformulation:

$$Y^{(ip)}(k,g) \quad \Longleftrightarrow \quad \mathcal{H}(\mathcal{T}_g^0)(k) \subsetneqq \mathcal{A}_g(k), \quad \text{where} \quad \mathcal{H}(\mathcal{T}_g^0) = \cup_{x \in \mathcal{T}_g^0} \ \mathcal{H}(x).$$

3 Over large fields

(3.1) *Suppose $g \geq 4$ and let k be an algebraically closed field which is uncountable, or a field such that* $\text{tr.deg.}_P(k) > 3g-3$ *(here P is the prime field of k). Then $Y^{(cu)}(k,g)$ holds.* For example if $k = \mathbb{C}$ we know that there is an abelian variety of dimension g not isogenous to a Jacobian variety of any curve of compact type.

(3.2) *We show that $Y^{(cp)}(k,g)$ holds for $k = \mathbb{C}$ and $g \geq 4$.*
Proof. In this case $\dim(\mathcal{M}_g \otimes \mathbb{C}) = 3g-3 < g(g+1)/2 = \dim(\mathcal{A}_g \otimes k)$. Hence $\mathcal{T}_g \otimes k$ is a proper subvariety of $\mathcal{A}_g \otimes k$. Write $\mathcal{H}(\mathcal{T}_g \otimes k)$ for the set of points corresponding with all polarized abelian varieties isogenous with a (polarized) Jacobian ($\mathcal{H}$ stands for "Hecke orbit"). We know that $\mathcal{H}(\mathcal{T}_g \otimes k)$ is a countable union of lower dimensional subvarieties. Hence $\mathcal{H}(\mathcal{T}_g(\mathbb{C})) \subsetneqq \mathcal{A}_g(\mathbb{C})$. □

An analogous fact can be proved in positive characteristic using the fact that Hecke correspondences are finite-to-finite on the ordinary locus, and that the non-ordinary locus is closed and has codimension one everywhere.

(3.3) A referee asked whether I could give an example illustrating (3.2). Let $V \subset \mathcal{A}_g \otimes \overline{\mathbb{Q}}$ be an irreducible subvariety with $\dim(V) > 3g-3$; for example choose V to be equal to an irreducible component of $\mathcal{A}_g \otimes \overline{\mathbb{Q}}$. Let η be its generic point, $K := \overline{\mathbb{Q}}(\eta)$, with algebraic closure $\overline{K} = L$; choose an embedding $L \hookrightarrow \mathbb{C}$. Over L we have a polarized abelian variety (A,λ) corresponding with $\eta \in \mathcal{A}_g(L)$. This abelian variety is not isogenous with the Jacobian of a curve. However, I do not know a "more explicit example".

The previous proof uses the fact that the transcendence degree of k is large. For a field like $\overline{\mathbb{Q}}$ this proof cannot be used. However we expect the following to be true.

(3.4) **Expectation** (N. Katz). *One can expect that $Y^{(cu)}(\overline{\mathbb{Q}},g)$ holds for every $g \geq 4$.* For a more general question see [8], 10.5.
One can expect that $Y^{(cu)}(\overline{\mathbb{F}_p},g)$ holds for every $g \geq 4$ and every prime number p.

Remark.

$$Y^{(cu)}(\overline{\mathbb{F}_p},g) \quad \Rightarrow Y^{(cu)}(\overline{\mathbb{Q}},g); \quad \text{see the proof of (5.1).}$$

4 Newton polygons and the p-rank

In order to present an approach to (3.4) we recall some notions.

(4.1) Manin and Dieudonné proved that isogeny classes of p-divisible groups over an algebraically closed field are classified by their Newton polygons, see [7], page 35. A symmetric Newton polygon (for height $h = 2g$) is a polygon in $\mathbb{Q} \times \mathbb{Q}$:

- starting at $(0, 0)$, ending at $(2g, g)$,
- lower convex,
- having breakpoints in $\mathbb{Z} \times \mathbb{Z}$, and
- a slope $\lambda \in \mathbb{Q}$, $0 \leq \lambda \leq 1$, appears with the same multiplicity as $1 - \lambda$.

See [12], 15.5.

An abelian variety A in positive characteristic determines a Newton polygon $\mathcal{N}(A)$ by taking the Newton polgon of $X = A[p^\infty]$. This defines a symmetric Newton polygon.

The finite set of symmetric Newton polygons belonging to $h = 2g$ is partially ordered by saying that $\xi' \prec \xi$ if no point of ξ' is below ξ, colloquially: if ξ' is "above" ξ.

(4.2) Let ξ be a symmetric Newton polygon. We write

$$W_\xi = \{[(B, \mu)] \mid \mathcal{N}(B) \prec \xi\} \subset \mathcal{A}_{g,1}; \quad W_\xi^0 = \{[(B, \mu)] \mid \mathcal{N}(B) = \xi\} \subset \mathcal{A}_{g,1}.$$

Grothendieck proved that under specialization Newton polygons go up and Grothendieck and Katz showed that the locus $W_\xi \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is *closed*, see [6]. The locus $W_\xi^0 \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is *locally closed* and $\overline{W_\xi^0} = W_\xi$.

One can also define W_ξ for $\mathcal{A}_g \otimes \mathbb{F}_p$; these loci are closed; we will focus on the principally polarized case.

These loci W_ξ are now reasonably well understood in the principally polarized case. The codimension of W_ξ in $\mathcal{A} := \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is precisely the length of the longest chain from ξ to the lowest Newton polygon ρ (g slopes equal to 0, and g slopes equal to 1: the "ordinary case"), see [11] and [13]. In particular the highest Newton polygon σ (all slopes equal to 1/2: the "supersingular case") corresponds to a closed subset of dimension $[g^2/4]$ (conjectured by T.Oda & F. Oort; proved by K.-Z. Li & F. Oort, [1], and reproved in [13]). In particular: *the longest chain of symmetric Newton polygons is equal to* $g(g+1)/2 - [g^2/4]$.

(4.3) For an abelian variety A over an algebraically closed field $k \supset \mathbb{F}_p$ we define the p-rank $f(A)$ of A by:

$$A(k)[p] \quad \cong \quad (\mathbb{Z}/p)^{f(A)}.$$

Here $G[p]$ for an abelian group G denotes the group of p-torsion points. It is easy to see that all values $0 \leq f \leq g$ do appear on $\mathcal{A}_g \otimes \mathbb{F}_p$.

(4.4) **Intersection with the Torelli locus.** We study intersection $W_\xi \cap (\mathcal{T}_g^0 \otimes \mathbb{F}_p)$ and the intersection $W_\xi \cap (\mathcal{T}_g \otimes \mathbb{F}_p)$. In low dimensional cases, and in some particular cases the dimension of these intersections is well-understood. However, in general these intersections are difficult to study. Some explicit cases show that in general the dimension of an irreducible component of $W_\xi \cap (\mathcal{T}_g^0 \otimes \mathbb{F}_p)$ need not be equal to $\dim(W_\xi) + \dim(\mathcal{M}_g) - \dim(\mathcal{A})$.

(4.5) The p-rank. We write

$$V_f = \{[(A,\lambda)] \in \mathcal{A}_g \otimes \mathbb{F}_p \mid f(A) \leq f\}.$$

This is called a p-rank stratum. We know:

$$\dim(V_f) \quad = \quad g(g+1)/2 - (g-f).$$

For principally polarized abelian varieties this was proved by Koblitz; the general case can be found in [9].

(4.6) Remark. Every $0 \leq f \leq g$ there exists a Newton polygon ξ such that $V_f = W_\xi$. In other words, the Newton polygon stratification refines the p-rank stratification.

We see in [2] that the dimension of every component of $V_f \cap (\mathcal{M}_g \otimes \mathbb{F}_p)$ equals $3g-3-(g-f)$ (for $g>1$).

5 From positive characteristic to characteristic zero.

In this section we formulate a question; a positive answer to this would imply that $Y^{(\cdot\cdot)}$ holds over $\overline{\mathbb{Q}}$. Here is the argument showing this last statement:

(5.1) *Suppose given g and a Newton polygon ξ* (of height $2g$).

$$W_\xi^0 \cap (\mathcal{T}_g \otimes \mathbb{F}_p) = \emptyset \quad \Rightarrow \quad Y^{(cu)}(\overline{\mathbb{Q}}, g).$$

Proof. The condition $W_\xi^0 \cap (\mathcal{T}_g \otimes \mathbb{F}_p) = \emptyset$ means that ξ does not appear on $\mathcal{M}_g^\sim$; hence there is an abelian variety A_0 with $\mathcal{N}(A_0) = \xi$ over $\overline{\mathbb{F}_p}$, which is not isogenous with the Jacobian of a curve of compact type over $\overline{\mathbb{F}_p}$. Choose an abelian variety A over $\overline{\mathbb{Q}}$ which has good reduction at p, and whose reduction is isomorphic with A_0 (this is possible by P. Norman & F. Oort, see [9]). We claim that the abelian variety A satisfies the condition $Y^{(cu)}(\overline{\mathbb{Q}}, g)$: a curve C of compact type over $\overline{\mathbb{Q}}$ with $A \sim_{\overline{\mathbb{Q}}} J(C)$ would have a Jacobian $J(C)$ with good reduction $J(C)_0 \sim A_0$; this shows that C has compact type reduction, and that $J(C_0) = J(C)_0 \sim_{\overline{\mathbb{F}_p}} A_0$; this is a contradiction; this proves the implication. □

(5.2) Note that for $g>1$ we have:

$$g(g+1)/2 - [g^2/4] < 3g-3 \quad \Longleftrightarrow g \leq 8;$$

$$g(g+1)/2 - [g^2/4] > 3g-3 \quad \Longleftrightarrow g \geq 9.$$

(5.3) Expectation. *Let $g = 11$, and let ξ be the Newton polygon with slopes $5/11$ and $6/11$. We expect:*

$$W_\xi^0 \cap (\mathcal{T}_{11} \otimes \mathbb{F}_p) \quad \stackrel{?}{=} \quad \emptyset,$$

i.e. we think that this Newton polygon should not appear on the moduli space of curves of compact type of genus equal to 11.

More generally one could consider $g \gg 0$, and ξ given by slopes i/g and $(g-i)/g$ such that $\mathrm{ggd}(i,g) = 1$, and such that the codimension of W_ξ in $\mathcal{A}$ is larger than $3g-3$:

(5.4) Expectation. *Suppose given g and a Newton polygon ξ (of height $2g$). Suppose:*

- *the longest chain connecting ξ with ρ is larger than $3g-3$;*
- *the Newton polygon has "large denominators".*

Then we expect that

$$W_{\xi}^{0} \quad \cap \quad (\mathcal{T}_g \otimes \mathbb{F}_p) \stackrel{?}{=} \emptyset.$$

Note that the first condition implies ($g = 8$ and $\xi = \sigma$) or $g > 8$. We say that the Newton polygon has *large denominators* if all slopes written as rational numbers with coprime nominator and denominator have a large denominator, for example at least eleven.

If a Newton polygon ξ does not apear on $\mathcal{T}_g$ one could expect that the set of slopes of ξ is not a subset of slopes of a Newton polygon appearing on any $\mathcal{T}_h$ with $h \geq g$.

(5.5) We do not have a complete list for which values of g and of ξ we have $W_{\xi}^{0} \cap \mathcal{T}_g \neq \emptyset$, not even for relatively small values of g.

One could also study which Newton polygons show up on the open Torelli locus $\mathcal{T}_g^0 \otimes \mathbb{F}_p$.

(5.6) Probably there is a genus g and a symmetric Newton polygon ξ for that genus such that ξ does show up on $\mathcal{T}_g \otimes \mathbb{F}_p$ and such that $Y^{(nu)}(\overline{\mathbb{Q}}, g)$ is true. In other words: it might be that our proposed attempt via (5.4) can confirm (3.4) for large g, but not for all $g \geq 4$.

(5.7) Conjecture. *Let $g', g'' \in \mathbb{Z}_{>0}$; let ξ', respectively ξ'' be a symmetric Newton polygon appearing on $\mathcal{T}_{g'}^0 \otimes \mathbb{F}_p$, respectively on $\mathcal{T}_{g''}^0 \otimes \mathbb{F}_p$; write $g = g' + g''$. Let ξ be the Newton polygon obtained by taking all slopes with their multiplicities appearing in ξ' and in ξ''. We conjecture that in this case ξ appears on $\mathcal{T}_g^0$.*

(5.8) We give some references.

In [3] the authors show that for every $g \in \mathbb{Z}_{>0}$ there exist an irreducible curve of genus g in characterstic 2 which is supersingular. One can expect that for every positive g and every prime number p there exists an irreducible curve of genus g in characterstic p which is supersingular; this would follow if (5.7) is true. For quite a number of values of g and p existence of a supersingular curve has been verified, see [15], Th. 5.1.1.

Next one can ask which Newton polygons show up on the hyperelliptic locus H_g. Here is a case where that dimension is known: *for $g = 3$ every component of $W_{\sigma} \cap (H_3 \otimes \mathbb{F}_p)$ has the expected dimension* $5 + 2 - 6 = 1$, see [10]; however already in this "easy case" the proof is quite non-trivial.

In [16] the authors show that for a hyperelliptic curve in characteristic two of genus $g = 2^n - 1$ and 2-rank equal to zero, the smallest slope equals $1/(n+1)$. In [17] we see which smallest slopes are possible on the intersection of the hyperelliptic locus with $V_{g,0}$ for $g < 10$.

In [4] it is shown that components of the intersection of $V_{g,f}$ with the hyperelliptic locus all have dimension equal to $g - 1 + f$ (i.e. codimension $g - f$ in the hyperelliptic locus). In [5] we find the question whether this intersection is transversal at every point.

Instead of the Newton polygon stratification one can consider another stratification, such as the the "Ekedahl-Oort stratification" or the "stratification by a-number". In [15], Chapter 2, especially Th. 2.4.1, also see the remark at the end of that chapter, we see that for an a-number m and a prime number p for every g with $g > pm + (m+1)p(p-1)/2$ the a-number m does not appear on $\mathcal{T}_g^0 \otimes \mathbb{F}_p$. As the a-number is not an isogeny invariant, this fact does not contribute directly to the validity of a statement like (5.4); however it does show that intersecting a stratification on $\mathcal{A}_g \otimes \mathbb{F}_p$ with the Torelli locus presents difficult and equally interesting problems.

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Frans Oort
Mathematisch Instituut
Budapestlaan 6
NL - 3584 CD TA Utrecht
The Netherlands
email: oort@math.uu.nl

Postbus 80010
NL - 3508 TA Utrecht
The Netherlands