An aspect of harmony in music of Johann Sebastian Bach

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"··· *tiefen Wissenschaft und Ausübung der Harmonie* ···"

Introduction

In 1752 F. W. Marpurg wrote in his "Vorwort zum Neudruck der Kunst der Fuge (Berlin), that Johann Sebastian Bach had a "*tiefen Wissenschaft und Ausübung der Harmonie*" [" a deep knowledge and grasp of harmony"] not surpassed by any of his contemporaries, see (5.1). And I think, nowadays we agree with Marpurg, also after comparing with 250 years more of written music.- I will try to "understand", to approach Bach's harmony, by analyzing one aspect of it.

Composing music and performing this in the meantone temperament can lead to impure tones. This was widely recognized, already long before Bach wrote his music. In which way did composers and performers cope with these problems?

Balthasar van der Pol (1889 - 1959) put forward in 1942 an idea about harmony in music of Johann Sebastian Bach:

In compositions by Bach, going through all modulations in pure intervals,
the pitch of the tone in beginning and end agree,

see [13], page 1117, see (2.1). At first sight I had no idea whether this idea by Van der Pol should be reasonable, and I tried to analyze some works of Bach with this "working hypothesis" in mind. It was a pleasant surprise for me to see that this hypothesis turned out to be correct in several pieces of Bach which I tried to consider from this point of view.

Being more and more convinced that this was the right idea, I was surprised that I got a contradiction in analyzing BWV 244/62. I thought I made a miscalculation, and studied several times that beautiful harmony of this moving choral of the St Matthew Passion. Until I found out that my computation was right, and that in thís case the working hypothesis by Van der Pol did not apply. It was, at least in this piece of music of Bach, curious to see that the thrilling diversion in harmony underlings the words "allerbängsten" and "so reisz mich aus den Ängsten". It seems as if Bach "on purpose" diverts from this "rule" (considered by Van der Pol) in order to illustrate the agony and passion described by these words; in other lines of the same choral the harmony is "as it should be".

It seems therefore that we could analyze harmony in of Johann Sebastian Bach with an *adapted* "working-hypothesis" as tool, see (4.1).

Everything below I consider as a humble way to understand an aspect of the compositional process of J. S. Bach; everything below should be considered as an attempt to find Bach's motivation and ideas, but we should refrain from considering these as facts or conclusions, also see (5.2). Probably many more aspects should be considered.

Material contained in this note is taken from [12].

1 Harmonics

Before entering in a discussion of Van der Pol's idea, we start with some preliminaries. This section contains well-known material. It is merely included in order to clarify the discussion in the next section.

Already in classical Greek science it was known that a (basic) tone has overtones; "pure intervals" derived from notes in a harmonic series are gentle to our ears (instead of "pure interval", the term "just interval" can be used); they have been considered important for enjoying music (and, philosophically, for much more). Two different musical instruments produce tones which consist of different patterns of harmonics, and this is one way we can distinguish them. A vibrating string can use its full length for the basic tone, but admitting more nodes and loops it can produce its overtones. A player of a wind instrument like a flute, an oboe or a trumpet can produce a basic tone, but (without changing the length of the vibrating column of air) can also force the vibrating air into an overtone. Your whole life you are hearing harmonics of all sort. And, almost all tones you hear come with many overtones.

(1.1) The intervals obtained from tones in a harmonic series are "pure intervals"; the ratios of frequencies are simple fractions. The first intervals in the sequence of harmonics are (instead of "perfect fifth" the term "pure fifth" can be used):

$$1 \; - \text{octave} \; - \; 2 \; - \text{prefect fifth} \; - \; 3 \; - \text{perfect quart} \; - \; 4$$
$$4 \; - \; \text{major third} \; - 5 - \text{minor third} \; - 6 - \; \cdots$$

Hence we obtain the following *ratios of frequencies*:

interval	ratio	number of semi-tones
octave	2/1	12
perfect 5-th	3/2	7
perfect 4-th	4/3	5
major 3-rd	5/4	4
minor 3-rd	6/5	3
⋯ etc.	⋯ etc.	
major 6-th	5/3	9
minor 6-th	8/5	8
major 2-nd	9/8 of 10/9	2
minor 2-nd	16/15	1

Note that a full tone (a major second) can appear in various disguises. For example the interval D-E is the second step in the natural scale of C-major, and it is also the first step in the natural scale of D-major. In case of the first step in a scale a major second is related with the ratio 9/8; in case of the second step in a major scale a major second is related with the ratio 10/9; see (5.3). In an analogous way the intervals F - G - A - B in the natural scale of C-major should correspond to the fractions 9/8, 10/9, respectively 9/8.

$$\text{C - } \tfrac{9}{8} \text{ - D - } \tfrac{10}{9} \text{ - E - } \tfrac{16}{15} \text{ - F - } \tfrac{9}{8} \text{ - G - } \tfrac{10}{9} \text{ - A - } \tfrac{9}{8} \text{ - B - } \tfrac{16}{15} \text{ - C}$$

In analyzing harmony in this way a step of a full tone should be considered in its harmonic surroundings in order to see which of the cases applies. For more details, see [14], Chapter 9, especially pp. 172/173; see [8], pag. 274; for minor scales see (5.4).

(1.2) These simple considerations have dramatic consequences. We would like to enjoy music in which all intervals played are pure. We like to tune our instruments with only pure intervals. Is this possible? As we easily see, the answer is negative.

Example 1. Consider a sequence of 12 steps, each a perfect 5-th (we go through the "circle of fifths"). At the end we seem to end with a tone which is seven octaves higher than the tone we started: the number of semi-tones equals $12 \times 7 = 7 \times 12$. However:

$$(\frac{3}{2})^{12} \quad \neq \quad 2^7, \quad \text{in fact} \quad (\frac{3}{2})^{12} > 2^7.$$

Indeed: suppose $(3/2)^{12} \stackrel{?}{=} 2^7$; this would imply that 3^{12} and 2^{19} should be equal; this is not the case, as follows by the property that natural numbers admit unique factorization, or by a simple computation: $3^{12} = 531441 \neq 524288 = 2^{19}$.

Or, even more simple: four intervals of a minor 3-rd seem to give an octave, $4 \times 3 = 12$, but

$$(\frac{6}{5})^4 = \frac{1296}{625} > 2.$$

And three intervals of a major 3-rd seem to give an octave, $3 \times 4 = 12$, but

$$(\frac{5}{4})^3 = \frac{125}{64} < 2.$$

Consequence. *It is not possible to tune a key-board instrument in such a way that all intervals are pure.*
At least one of the perfect 5-th intervals has to be tuned flat. Especially minor thirds are "far away" from pure.

Example 2. Here is another example, which will be studied again below: a sequence of two perfect fourth intervals and a sequence of a major fifth and a minor third, $5 + 5 = 10 = 7 + 3$, seem to give the same result (and they do on a key-board instrument); however:

$$\frac{4}{3} \times \frac{4}{3} = \frac{16}{9} = \frac{80}{45} \quad \neq \quad \frac{81}{45} = \frac{9}{5} = \frac{3}{2} \times \frac{6}{5}.$$

This example we will see in the choral BWV 244/32.

Example 3. Below, and see [13], page 1117, we see that fifth + fifth = sixth + fourth, $7 + 7 = 9 + 5$, gives

$$\frac{3}{2} \times \frac{3}{2} = \frac{81}{36} \neq \frac{80}{36} = \frac{5}{3} \times \frac{4}{3}.$$

See (5.5).

In these examples we see that even staying within the tones of one scale, pure intonation is not possible. And we start understanding what can happen when we introduce sharps and flats, or in case we try to follow the complicated harmonies and modulations as in music of Bach.

Example 4. Suppose we have a sequence of modulations which is as follows:

b – e – A – D;   D – G – A – A – b;

we see that the first half gives $\frac{4}{3} \times \frac{2}{3} \times \frac{4}{3} = \frac{32}{27}$, and not $\frac{6}{5}$; the second halve gives $\frac{2}{3} \times \frac{9}{8} \times \frac{9}{8} = \frac{27}{32}$; each of the halves seems to obstruct pure tuning, but the total gives a beginning and ending tone which are equal. This example we encounter in the Polonaise 1067/6, bars 3-4, 5-6.

(1.3) We said "dramatic consequences", but as we will see soon, it also offered an extra means of expression in the meantone tuning used in Bach's time.

There are several ways out of these difficulties. We discuss the two of the ways of tuning, which played a role in the time, and in the life and work of Johann Sebastian Bach. Clearly impure octaves are not allowed (they sound terrible).

The meantone temperament. One option is to make a decision to tune a certain set of interval pure, and all remaining interval as pure a possible, (5.6). There are various ways of doing this, and the choice which to adopt, depended on what was acceptable for the baroque ear, and what was easy to carry out while tuning an instrument, see (5.7). This practice was used (in various forms) in Bach's time. See (5.8).

The equal temperament. Another option (the well-tempered system) is to divide an octave into 12 equal parts. This results in a tuning where all intervals (not equal to an octave) are impure (it is easy to prove that indeed all these intervals are impure). Here are some examples. In this way a semi-tone in the equal temperament corresponds with $\sqrt[12]{2} \approx 1.059463$, and all other intervals correspond with powers of $\sqrt[12]{2}$:

interval	pure ratio	well-tempered ratio
minor 2-nd	$16/15 \approx 1.07$	1.059463
major 2-nd	$9/8 \approx 1.13$ or $10/9 \approx 1.11$	1.122462
minor 3-rd	$6/5 = 1.2$	1.189207
major 3-rd	$5/4 = 1.25$	1.259921
perfect 4-th	$4/3 \approx 1.333$	1.334840
perfect 5-th	$3/2=1.5$	1.498307
major 6-th	$5/3 \approx 1.67$	1.681793

For more information, see [14], pp. 178/179, see [8], p. 332. It is not clear whether any organ in Bach's time was tuned in the equal temperament; also see [4], page 196, especially the advice Georg Andrea Sorge gave Silbermann in 1748. Carl Philipp Emanuel Bach requested a keyboard instrument to be tuned in the equal temperament (or "well-tuned" ?), see [3], Introduction to Part one, p. 37. It was only around 1850 that the equal temperament was widely accepted. One can question whether we should distinguish "equal tempereramen" versus "well-tuned", see (5.9).

These aspects of harmony and of tuning instruments have been an important fact of life for Bach, such as:

- In writing music for performances Bach had to consider available instruments (and e.g. in the beginning of his career he had to write in different tonalities, "Chorton" and "Kammerton" for different instruments, see [7], page 1021).
- Bach was involved in many inspections of new organs and their tuning.

- Bach tuned his own key-board instruments, for his own use, and for renting; contemporaries mention his great skill in this, see (5.10).

- Bach probably thought several times in his life about tuning; his deep interest in this aspect is an aspect of his compositional process; it is also shown in his compositions BWV 846-869, *Das Wohltemperierte Klavier*, and the second series of 24 preludes and fugues BWV 870-893. However we should distinguish the questions whether Bach pursued a practical or a scientific-mathematical approach, see (5.2).

- In Bach's time certain tonalities had certain implications: "Tonart und musikalischen Affekt", see [9], see [10]. It seems clear that Bach was well-aware of this, and that he used this on many occasions. The St Matthew Passion BWV 244 is full of changes of tonality, where Bach changes into the tonality adequate for the particular "Affekt". Also see [5].

From the paper [13], see page 17.

- Bach, who knew that certain intervals where not well-tuned in the meantone temperament (awful to hear for his contemporaries), used these when an extreme expression was necessary; we give one example: (5.11).

Conclusion. *Bach knew in very fine detail the precise properties of intervals, the effect of using a certain interval in a certain composition, and he uses aspects of intervals and tonalities as extra means of expression.*

Let us try to consider music by Bach as if we still have the ability to hear pure and impure intervals, to distinguish various tonalities by the way certain intervals sound in that tonality, and let us try to develop a sensitive mind for this aspect of music by Bach.

Remark. In an analysis I will use numbers and (in)equalities, and one can wonder what Bach did and would have done. I think that Bach needed no computations whatsoever for immediately grasping the pitch of a tone, the interplay of melody and harmony; it was part of his life and he did not need such computations, I think. But let me try to do the necessary computations when I try to follow his intuition and craftsmanship on these aspects.

2 A "working hypothesis" by Van der Pol

As an introduction to the idea by Van der Pol let us imagine an experiment. We have two rooms connected by a door; in one room there is an experienced singer, in the other a key-board instrument. We start with the door open, the vocalist takes up the pitch of a given starting tone, and the door is closed. The singer sings only in pure intervals, going up twice by a perfect 4-th, and going down by a minor 3-rd and a major 5-th (not an unreasonable tune; see Example 2; we will see that Bach used this). At the end the door is opened again, and the final tone of the singer is compared with the starting tone as given by the key-board instrument. Bach would not have been surprised to hear that *the ending tone of the singer is lower in pitch than the starting tone*; we show it by a computation:

$$\frac{4}{3} \times \frac{4}{3} \times \frac{5}{6} \times \frac{2}{3} = \frac{80}{81} \quad < \quad 1.$$

Balthasar van der Pol, see [13], page 1117, writes: *"It would be of interest to see whether classical compositions always end in the same true key as that in which they begin. In some investigations that I carried out myself in this connection, I found that in Bach's music there was always strict agreement between the keynotes at the beginning and end of his pieces"*; see (5.12).

(2.1) Working hypothesis (Van der Pol, 1942): *In compositions by Bach, going through all modulations in pure intervals, the pitch of the tone in beginning and end agree.*

I must admit that it sounded not very convincing when I saw this for the first time. Hence I considered the harmony in some music of Bach, both vocal and instrumental, see (5.13). Surprisingly I found that in these case the "working hypothesis" by Van der Pol was correct; it was wonderful to find, that after complicated modulations, characteristic for Bach's music, the end tone was pure with respect to the beginning tone. Also see (5.14).

3 "am allerbängsten"

In further analysis however I found some disagreement between modulations in harmony of certain works by Bach versus the working hypothesis (2.1).

Here is an analysis of BWV 244/32, where I consider the harmony each time from the beginning to the end of a line:

Choral 32	St Matthew Passion		
bar.beat		modulation	Pf
0.4 → 3.1	Mir hat die Welt trüglich gericht't	g → $B^\flat$	6/5
3.1 → 5.1	mit Lügen und mit falschem G'dicht,	$B^\flat$ → F	3/4
5.1 → 7.1	viel Netz' und heimlich Strikken.	F→ F	1
7.1 → 8.1	Herr, nimm mein wahr	F → c	3/4
8.1 → 9.1	in dieser G'fahr,	c → $E^\flat$	6/5
9.1 → 11.3	b'hüt mich vor falschen Tücken !	$E^\flat$ → $B^\flat$	3/2

Choral 32	St Matthew Passion	Pf	Bach
total choral	g → $B^\flat$ → F → c → $E^\flat$→ $B^\flat$:	$\frac{6}{5} \times \frac{81}{80}$	$\frac{6}{5} \times \frac{648}{635}$ see (5.15)

Here we use Pf (the factor by Van der Pol) for the ratio of frequencies. Writing 11.3, we mean bar 11 and beat 3.

Would a singer come to the correct end tone using pure intonation of intervals? Would we listen to this choral and end up with a satisfied and harmonious feeling? It might be that Bach has "listened" to this in the following way. The modulation $B^\flat$-F-c-$E^\flat$-$B^\flat$ from 3.1 to 11.3 consists of -5-5+3+7:

twice a fourth down, then a minor third and a perfect fith up,

not unusual; see Example 2. We see that

$$3.1 \to 11.3: \qquad \frac{3}{4} \times 1 \times \frac{3}{4} \times \frac{6}{5} \times \frac{3}{2} = \frac{81}{80}:$$

the modulations $B^\flat$-F-c-$E^\flat$-$B^\flat$ give a tone at the end which is too high in comparison with the beginning tone: already the choice of modulations from fermate to the next fermate Bach composes a writhing tone to end with. Singers who would use pure intervals come into conflict with the instruments. The working hypothesis fails in this case. Here is what I think:

Bach composes a "writhing harmony",
which suits the "Lügen" described in this choral !

See (5.15).

(3.1) Here is a second example. We can compare the two chorals BWV 244/54 "O Haupt voll Blut und Wunden" and BWV 244/62 "Wenn ich einmal soll scheiden", see (5.16). These are on the same melody, but harmonies are different.

In case of BWV 244/54 the working hypothesis is valid. Here Bach stays within pure harmony. And, indeed, we feel satisfied and in harmony when listening to this music.

Then we look at the same melody, but a different harmony in BWV 244/62:

Choral 62	St Matthew Passion			Pf used
		modulation	Pf	by Bach
0.4 → 2.3	Wenn ich einmal soll scheiden,	a → C	3/5	idem
2.3 → 4.1	so scheide nicht von mir !	C → a	5/3	idem
4.1 → 6.3	Wenn ich den Tod soll leiden,	a→ C	3/5	idem
6.3 → 8.1	so tritt du dann herfür !	C → a	5/3	idem
8.1 → 10.3	Wenn mir am allerbängsten	a → C	3/5	1215/2048
10.3 → 12.3	wird um das Herze sein,	C → A	5/3	idem
12.3 → 14.3	so reisz mich aus den Ängsten	A → G	9/10 or 8/9	640/729
14.3→ 16.3	kraft deiner Angst und Pein.	G → E	5/3	idem

At first we consider the modulations from fermate to fermate. In the first six lines modulation moves from a to C (one can argue that the whole choral is in C; it does not make much difference for our considerations). In the last two we move from A to E, and

$$\frac{8}{9} \times \frac{5}{3} \qquad \neq \qquad \frac{3}{2} \qquad = \qquad \frac{9}{10} \times \frac{5}{3}.$$

Considering the modulations in this way, we can apply the working hypothesis by Van der Pol, if the modulation A → G, "so reisz mich aus den Ängsten" appears with the factor Pf = 9/10.

However, a further analysis, shows that Bach uses the "correct" harmony in all lines except in the two which contain

"Wenn mir am allerbängsten" and "so reisz mich aus den Ängsten" see (5.17). In these two passages the harmony is frightfull, is going "wrong" (we feel this when listening, choir members experience this). And, then, in that beautiful last sentence "kraft deiner Angst und Pein", quietness and harmony comes back in a beautiful way; where the faithful comes to rest and belief, the music assures us. We see that the drama of someone afraid of and forseeing his pain of death is exactly at those places where there is a deviation from the working hypothesis, while all other parts are in full harmony, in agreement with the hypothesis.

Note that some of the aspects observed in this section are about harmony at one particular moment, and some considerations are about more global structure in the music of Bach. One can argue that the "local" effects can easily be explained by the relation between text and harmony at that instant; this aspect of "illustration" is widely present in music by Bach. However the "global" aspects seem to fall under a different pattern; this is precisely what we try to put forward.

4 The "working hypothesis" revisited

Therefore I think we could analyze works by Bach with the following idea in mind.

(4.1) Revised working hypothesis. *In most compositions by Bach, going through all modulations in pure intervals, the pitch of the tone in beginning and end agree; however if the text demands for a writhing effect, the music illustraties this by a "writhing harmony".*

(4.2) Discussion. There are several aspects which should be mentioned (and perhaps many more).

- Much more evidence seems necessary before we can make the way of looking at Bach's music, mentioned in this note reasonable.
- How does earlier music relate with this idea, e.g. does music by Monteverdi fit into these ideas? What about works by contempory composers of Bach? Suppose that Bach indeed had in mind something like our hypothesis (2.1), did other composers have the same "solution" for this problem created by aspects of temperaments? See (5.12).
- It might very well be that, although the problem of intonation created in the meantone temperament was widely recognized already long before Bach, it was Bach who thought of meeting the problems in the way as was formulated by Van der Pol , see (2.1), and then used this in his compositional process in the way we try to formulate in our (4.1).
- Also we should compare works by Bach written in different periods of his life. Do we see a difference from our point of view?

- I was surprised to see that also in non-vocal music by Bach the hypothesis seemed to apply. Is this indeed a general pattern in his music?
- Do we know of any documents where this "rule" in harmony, this way of meeting problems created by the temperament, is described?
- Do we see any evidence in autographs by Bach supporting this way of considering the harmony he used?
- Often we see that a certain aspect of Bach's music can be "explained" in several ways, that there are various arguments which seem to have an influence on his compositional process. Considerations above should not be taken in an absolute way, I think, but should be taken into consideration in combination with other arguments.
- In many instances we see that Bach "illlustrates" extreme feelings by dissonants in harmony and in melody. From this point of view certain considerations of Section 3 can also be explained. However I think that the idea of Van der Pol, and the diversion of this "working hypothesis" in harmony as formulated in (4.1), and as we observe in music by Bach gives a complementary approach, explaining more aspects of this phenomenon.

5 Notes

(5.1) F. W. Marpurg in his "Vorwort zum Neudruck der Kunst der Fuge (Berlin), Frühjahr 1752" writes:

"*Thut man aber einen Blick in seine Schriften: so könte man aus allen, was jemahls in der Musik vorgegangen und täglich vorgehet, den Beweiß hernehmen, daß ihn keiner in der tiefen Wissenschaft und Ausübung der Harmonie, ich will sagen, einer tiefsinnigen Durcharbeitung sonderbarer, sinnreicher, von der gemeinen Art entfernter und doch dabey natürlichen Gedanken übertreffen wird;* $\cdots$ " See BD III, page 15, lines +1∼+6.

(5.2) Considering possiblities which ideas Johann Sebastian Bach might have had about composing and performing music, again and again I have the idea that Bach had a very *practical* approach to these aspects. Did Bach use a "correct" tuning in a certain system (temperament), or did he just tried to find the one which was the best for the situation at hand? For example, see [4], page 85: "Bach tuned another set of pipes entirely by ear, and won the contest handily, for a singer found it easier to sing a chorale in $B^\flat$ minor in Bach's tuning than in Neidhardt's. I feel, we should be very careful in assigning Bach "scientific" qualities where practical aspects of composing and performing music seem also possible explanations for effects we find in his music. For example, see [4], page 13: "The recorded opposition to the equal temperament on the part of such men as Werckmeister and even Sebastian Bach was to the rigorous mathematical treatment implied by the name "gleichschwebend." Theirs was a practical approximation to equality, ... ". Or, see [4], page 87: "Unfortunately, the more mathematically minded writers ... extreme accuracy .. is the all-important thing ... This is why Sebastian Bach and many others did not care for the equal temperament. They were not oppossed to the equal tuning itself, and their own tuning results were undoubtly comparable to the best tuning accomplished today ... they needed Mersenne to tell them that the complicated table could well have half their digists chopped off before using, and that, after all, a person who tunes accurately by beats gets results that the ear cannot distinguish from the successive powers of the 12th root of 2."

(5.3) A major second related with the ratio 9/8 is sometimes called a "major tone", and if related with the ratio 10/9 a "minor tone", see [11], pp. 132/133.
Here is a partial list of harmonics of C_1:

1	C_1	2	C_2	4	C_3	8	C_4
		3	G_2	5	E_3	9	D_4
				6	G_3	10	E_4
						12	G_4
						15	B_4
						16	C_5

In this way we see the factors related to the intervals C-D and D-E in the C-major scale.

A *harmonic* not equal to the basic tone is called an *overtone*. In the numbering of harmonics the basic tone is the first harmonic; in the numbering of overtones the octave of the basic tone (the second harmonic) is the first overtone; the third harmonic is the second overtone, and so on. A mathematician says: the n-th overtone is the $(n+1)$-st harmonic for $n \geq 1$.

(5.4) For an ascendig minor scale we can take:

$$\text{a - }\tfrac{9}{8}\text{ - b - }\tfrac{16}{15}\text{ - c - }\tfrac{10}{9}\text{ - d - }\tfrac{9}{8}\text{ - e - }\tfrac{10}{9}\text{ - f}^\sharp\text{ - }\tfrac{9}{8}\text{ - g}^\sharp\text{ - }\tfrac{16}{15}\text{ - a,}\quad \longrightarrow\ ;$$

for a descending (Eolic) minor scale we can take:

$$\text{a - }\tfrac{9}{8}\text{ - b - }\tfrac{16}{15}\text{ - c - }\tfrac{10}{9}\text{ - d - }\tfrac{9}{8}\text{ - e - }\tfrac{16}{15}\text{ - f - }\tfrac{9}{8}\text{ - g - }\tfrac{10}{9}\text{ - a,}\quad \longleftarrow \text{ Eolic;}$$

see [8], pag. 274.

(5.5) In what follows we will often just say: "we go from G to C", without specifying whether we mean a 4-th up, of a 5-th down. In computations this means that everything is determined up to factors 2.

(5.6) Pietro Aaron (1490 - 1545), writing his "*Thoscanello de la musica*", 1523, was probably the first who standarized a meantone temperament.

(5.7) Kirnberger published a system of tuning ("Die Kunst des reinen Satzes", Berlin 1779). Some people believe that this was the system of tuning used by Johann Sebastian Bach. In this system, Kirnberger III, the four 5-th C-G-D-A-E equally share the interval C-E consisting of two octaves and a pure major 3-rd, hence eachof these 5-th intervals have ratio equal to $\sqrt[4]{5} \approx 1.4953488$. The intervals E - B - $F^\sharp$ and the intervals $C^\sharp$ - $G^\sharp$ - $E^\flat$ - $B^\flat$ -F - C are tuned pure; the remaining $F^\sharp$ - $C^\sharp$ makes up for the difference: $2^7/(3/2)^7 \times 5 \approx 1.4983082$.

(5.8) Some people use the word *temperament* for "A system, some or all of whose interval cannot be expressed in rational numbers", while *a tuning* is used for "A system all of whose intervals can be expressed in rational numbers", see [4], page xii; I will use the term *tuning* for the practical way of tuning an instrument, and I will use *temperament* for a system describing the way a tuning can be performed.

(5.9) In Bach's time certainly the two notions "gleichschwebende Temperatur" and "wohltemperirt" were both available. The title "Das Wohltemperirte Klavier" perhaps does not imply it was for a keyboard instrument tuned in the equal temperament, but perhaps it only meant that it was written for a keyboard, as Werckmeister called it, "well-tuned", meaning playable in all keys, see [4], page 12 ("The title of Bach's famous "48" meant simply that the clavier was playable in all keys"), pp. 195, 194. And - of course - tuning an organ was more definite than tuning and retuning a clavichord.

In Spitta on Bach's tuning, we read that "...That he evolved all this by his own study and reflection, and not from theoretical treatises..", see [11], page 169.

(5.10) I do not know whether Bach (and his contemporaires) had an own way of tuning keyboard instruments, or perhaps various systems depending on the instrument, and what should be played on it, see [4], page 191: "The performer would retune his entire harpsichord when changing from sharp to flat", see page 194: "Thus Telemann ... used ... with variable intonation"; it seems that Bach was capable of tuning an instrument within a quarter of an hour; for references, see [11], p. 169, see [4], p. 191.

(5.11) In the St Matthew Passion twice the choir sings "laß ihn kreuzigen", in 244/45b in a, ending in B, and in 244/50b in b, ending in $C^\sharp$; in whatever meantone temperament, the ending fifth $C^\sharp$ – $G^\sharp$ must have been shocking for a listener in the 18-th century. It might well be that this key was the "worst" in the tuning used, e.g. see [4], page 12: "Werckmeister and Neidhardt explained clearly that in their systems the key of C would be the best and $D^\flat$ the worst ...".

(5.12) The fact that modulations using pure invervals might lead to a different ending tone was clear long before Bach's time. Probably Giovanni Pierluigi da Palestrina (± 1525 - 1594) took this into consideration while composing; certainly Christiaan Huygens (1629 - 1695) was well-aware of this aspect: see [6], page 20, Huygens gave the example C-G-D-A-E-C, where $(3/2) \times (3/4) \times (3/2) \times (3/4) \times (4/5)$ gives 81/80; also see [11], p. 169. But it is not clear to me whether before Bach the "remedy" (2.1) was ever considered or used.

In [11], on page 94 we find: *"Mr. Huygens observed long ago, that no voice or perfect instrument can always proceed by perfect intervals, without erring from the pitch first assumed. But as this would offend the ear of the musician, he naturally avoids it by his memeory of the pitch, and by tempering the intervals of the intermediate sounds, so as to return to it again."*

In [6] on pp. 224-225 we read: "But free modulation ... which Huygens did not particularly like: *.. that wandering through alien keys [is anept] because one forgets completely the one in which one has started. This enervates the power and the beauty of the music, nor does copnstant modulation delight the listeners... That it is rather questionable whether modern musicians with their figured counterpoint, where different words are pronounced at the ame time, are on the right track.* Huygens' most basic notion of music is that its function is to 'please', and to 'delight'... "

In this light, "memeory of the pitch, and by tempering the intervals of the intermediate sounds" is understandable; but it is not applicable to most music of Bach.

(5.13) In the following case I considered the harmony used, and in these cases I found agreement with the working hypothesis (2.1) by Van der Pol:

- The choral of the motet "Der Geist hilft", BWV 226.
- The Opening-choral of the motet "Jesu meine Freude", BWV 227.
- Sonata in e for traverso and basso continuo, BWV 1034.
- The first movement of the sonata in E for traverso and basso continuo, BWV 1035/1.

- The Polonaise and Double of the second orchestral suite: 1067/6,6a.
- Several chorals of BWV 244.

Certainly you will remark that this "evidence" is rather meager. Indeed, much more research seems necessary.

In BWV 244 the chorals 3, 15, 17, 37, 44 and 54 satisfy the hypothesis (2.1). In the chorals 10 ("an Handen und an Füßen gebunden in der Höll"), 25 ("züchtiget"), 32 ("Lügen"), 40 ("Angst", "Todespein") and 62 ("allerbängsten", "Ängsten") the hypothesis (2.1) does not apply, the harmony writhes, and we need the revised hypothesis (4.1). The harmony in choral 10 is "strange" in the line containing "und an Füßen"; perhaps Bach is less dogmatic than we intend to say in (4.1) ?

(5.14) In carrying out the analysis I encountered some foundational problems. If separate voices in a composition follow different harmonies (not uncommon in Bach's music), how should we consider the working hypothesis in this aspect? Should we analyze the harmony very closely, i.e. from tone to the next, or should we consider the harmony in larger units of music? In some cases different approaches can give different answers. A fundamental and systematic choice for such aspects seems to be necessary before going on; in general I tried to see how much of the "musical memory" of a singer can grasp (not a very well-defined or absolute criterion), and that I took as basis for my computations.

(5.15) It could be that Bach in every part chooses the harmony in 244/32 in such a way that finally the end tone agrees with the beginning. However a further analysis shows that the "Lügen" in 3.1 → 5.1 moreover does not obtain the Pf equal to 3/4 but the factor 96/125. We conclude that Bach in his BWV 244/32 deviates from the working hypothesis even further than we found in an analysis from fermate to fermate. A harmonic analysis shows:

Choral 32	St Matthew Passion		
bar.beat	modulation	Pf	Bach
0.4 → 3.1	g → B$^\flat$	6/5	6/5
3.1 → 5.1	B$^\flat$ → F	3/4	96/125
5.1 → 7.1	F→ F	1	1
7.1 → 8.1	F → c	3/4	3/4
8.1 → 9.1	c → E$^\flat$	6/5	6/5
9.1 → 11.3	E$^\flat$ → B$^\flat$	3/2	3/2

The harmony in the line "Lügen" makes it "even worse"! The factor 96/125 in the second line is obtained by the modulations

$$\mathrm{B}^\flat \text{ - } \mathrm{E}^\flat \text{ - } \mathrm{f}^\sharp \text{ - b - d - F:} \quad \frac{2}{3}\times\frac{6}{5}\times\frac{4}{3}\times\frac{6}{5}\times\frac{6}{5}=\frac{96}{125}=\frac{384}{500}\neq\frac{375}{250}=\frac{3}{4}.$$

(5.16) Let us consider the chorals BWV 244/15-17-44-54-62, all on the same melody. We see that the tonalities are: E - E$^\flat$ - D - F - a. As Wolff remarks, see [16], p. 324, see [15], p. 301, the accidentals are: 4♯ - 3♭ - 2♯ - 1♭ - 0. Is there an "explanation" for this? Did Bach have a "reason" for composing it in this way?

I think, any statement about can be a (dangerous) speculation. E.g. we also can observe that all tonalities fit within the relation text-Affekt. For example,

244/15: "Erkenne mich" in E, and Mattheson writes about E: *"... es ist vor extrem Verliebte"*, see [10], p. 89,

244/17: "Ich will bei dir stehen" in E$^\flat$, Heinicken: *"einem schönen Ton"*, see [10], p. 99,

244/44: "Befiehl du deine Wege" in D, Mattheson: *"Anleitung zu delicate Sachen"* see [10], p. 81,

244/54: "O Haupt voll Blut und Wunden" in F, Mattheson: *"schönsten Sentiments der Welt"*, [10], pag. 84,

244/62: “Wenn ich einmal soll scheiden” in a, Mattheson: *“etwas klagend, ehrbar und gelassen”*, [10], pag. 105;
is this an explanation (or a partial argument)? Perhaps the role of certain tonalities should not be taken in this absolute way? See [5], pp. 86 - 89.

And, note that the last of these 5 chorals ends in E. Does it suggest that the cycle of these five chorals is closing like a loop? That Bach emphasises the parallel drawn between the suffering of Christ and the mortality of all people? I rather stop with such speculations!

Probably a new way of considering this aspect along lines as set out in [5] seems necessary.

(5.17) In 244/62, from 8.1 to 10.3 we see:

$$\mathrm{a} - \mathrm{F}^{\sharp} - \mathrm{e} - \mathrm{e}^{\flat} - \mathrm{d} - \mathrm{C},$$

$$\frac{5}{6} \times \frac{9}{10} \times \frac{15}{16} \times \frac{15}{16} \times \frac{9}{10} = \frac{1215}{2048} \approx 0.5932 < 0.6 = \frac{3}{5}.$$

In 244/62, from 12.3 to 14.1 we see:

$$\mathrm{A} - \mathrm{F}^{\sharp} - \mathrm{G} - \mathrm{b} - \mathrm{e} - \mathrm{G} - \mathrm{C}^{\sharp} \text{ - } \mathrm{D} - \mathrm{G}$$

$$\frac{5}{6} \times \frac{16}{15} \times \frac{5}{4} \times \frac{4}{3} \times \frac{5}{6} \times \frac{16}{15} \times \frac{2}{3} = \frac{640}{729} \approx 0.8779 < 0.889 = \frac{8}{9} < 0.9 = \frac{9}{10}.$$

One can also argue that in 244/62, 8.1 → 10.3 the harmony is: a – B – e – F^7 – b – C; in this case we obtain the ratio 3/5; but we hear that this, although in some sense correct, it is far from the drama written by Bach. In the same way we could take for 244/62, 12.3 → 14.3: A – D – G – C – G – A – D – C, and arrive at 8/9.

R.L. Marshall, The Compositional

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