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ON THE QUANTIFIER-FREE FRAGMENT OF
'LOGIC OF EFFECTIVE DEFINITIONS'

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A b s t r a c t. In [2] Jerzy Tiuryn has introduced Logic of Effective Definitions (LED) in which properties of effective definitional schemes are expressed. With respect to the quantifier-free part of it, he proved that each open formula is equivalent to one in a special conjunctive normal form. We prove that there is no finite bound to the number of conjuncts required for these normal forms.

K e y w o r d s: algorithmic logic, computability, algebraic structures, functional effective definitional schemes.

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1. Introduction; Basic Definitions

Let Σ be a finite signature for (possibly many-sorted) algebraic structures. There are several formalisms to describe classes of computable functions over structures of signature Σ . To mention some: relational and functional effective definitional scheme in the sense of Friedman [1], and fap, fap C, fap S and fap CS computable functions in the sense of Tucker [5]. Of these we consider only functional effective definitional schemes (feds) or the equivalent fap CS programs. Both these formalisms are universal in the sense that they provide a program for each partial function which is partially computable from an intuitive point of view.

In the sequel we shall use the terminology of Friedman's feds.

Definition: Let F_{Σ} be the set of all open first order formulae over signature Σ (which may contain the equality symbol $=$) and P_{Σ} the set of all finite Σ -polynomials. Furthermore let ω stand for the set of natural numbers.

Then we define a functional effective definitional scheme (feds) as a recursive function:

$$s : \omega \rightarrow (F_{\Sigma} \times P_{\Sigma}).$$

Let F be a partial computable function, defined over an algebraic structure \mathcal{A}_{Σ} with signature Σ . (suppose F is k -ary)

Then we can always express $F(x_1, \dots, x_k)$ by means of an infinite conditional program ϕ_F of the form: (v is output variable)

if $\delta_0(\vec{x})$ then $v := t_0(\vec{x})$
 else if $\delta_1(\vec{x})$ then $v := t_1(\vec{x})$
 else if $\delta_2(\vec{x})$ then $v := t_2(\vec{x})$
 else

where $\delta_i(\vec{x}) \in F_{\Sigma}$ and $t_i(\vec{x}) \in P_{\Sigma}$ and $\vec{x} = (x_1, \dots, x_k)$. So we can associate

F with a feds f with

$f(n) = (\delta_n(\vec{x}), t_n(\vec{x}))$, where the $\delta_i(\vec{x})$ and $t_i(\vec{x})$ are given by the way

ϕ_F is constructed.

(More about the power of feds can be found in [3].)

Note that if $F(a_1, \dots, a_k)$ is defined, the corresponding f (and ϕ_F)

must contain a $\delta_i(\vec{x})$ such that $\mathcal{U}_\Sigma \models \delta_i(\vec{a})$.

So if we define the effectively enumerable sequence $S_F(\vec{x})$ of open

Σ -formulae as $\{S_F^n(\vec{x}) \mid n \in \omega\} = \{\bigwedge_{1 \leq i \leq n-1} (\neg \delta_i(\vec{x})) \wedge \delta_n(\vec{x}) \mid n \in \omega\}$, this

S_F satisfies the condition

$\mathcal{U}_\Sigma \models S_F^n(a_1, \dots, a_k)$ iff program ϕ_F (computing F) with input a_1, \dots, a_k

terminates after the n-th step in the computation.

We shall call such a sequence S_F an effective definitional termination scheme (edts) for F.

(Note that this notion of edts is relative to the feds-formalism !)

Now we can express the termination of F by means of its corresponding

S_F :

$F(\vec{a})$ terminates iff for some $n \in \omega$

$$\mathcal{U}_\Sigma \models S_F^n(a_1, \dots, a_k).$$

In this paper we shall study what we call QF-formulae (quantifier-free formulae involving edts) defined by

1) $S \downarrow$ (more exactly $S(\vec{a}) \downarrow$ or $S \downarrow(\vec{a})$) is a QF-formula for all edts

$$S = \{\bigwedge_{1 \leq i \leq n-1} (\neg \delta_i(\vec{a})) \wedge \delta_n(\vec{a}) \mid n \in \omega\} \text{ with } \delta_i \in F_\Sigma.$$

2) If φ and ψ are QF-formulae, then $\varphi \wedge \psi$, $\varphi \vee \psi$ and $\neg \varphi$ are also QF-formulae.

The semantics of QF-formulae is given by:

1) $\mathcal{U}_\Sigma \models S(a_1, \dots, a_k) \downarrow$ if for some $n \in \omega$ $\mathcal{U}_\Sigma \models S^n(a_1, \dots, a_k)$.

2) For φ an arbitrary QF-formula, $\mathcal{U}_\Sigma \models \varphi(a_1, \dots, a_k)$ is defined out of

$\mathcal{U}_\Sigma \models S(a_1, \dots, a_k) \downarrow$ with induction over the construction of φ .

3) $\mathcal{U}_\Sigma \models \varphi$ iff $\mathcal{U}_\Sigma \models \varphi(a_1, \dots, a_k)$ for every $a_1, \dots, a_k \in \mathcal{U}_\Sigma$.

4) $\models \varphi$ iff for all \mathcal{U}_Σ with signature Σ : $\mathcal{U}_\Sigma \models \varphi$.

So, speaking informally, $\mathcal{U}_\Sigma \models S_F(a_1, \dots, a_k)^\dagger$ means that the program Φ_F which computes F over \mathcal{U}_Σ , terminates for inputs a_1, \dots, a_k . Furthermore we shall use

$S(a_1, \dots, a_k)^\dagger$ as abbreviation for the QF-formula
 $\neg(S(a_1, \dots, a_k)^\dagger)$.

Remark: Note that a different definition of $\models \varphi$ is possible too: one can focus attention to a certain class K of structures and define $\models \varphi$ by: for all $\mathcal{U} \in K$: $\mathcal{U} \models \varphi$. Some interesting problems arise if one takes $K = \{(\omega; S, P, 0)\}$: the class K only consists of the structure of natural numbers with successor, predecessor and zero. In this paper, however, we will consider the class of all possible Σ -structures.

The relation to Tiuryn's logic of effective definitions (LED, see [2]) is the following. The Quantifier-free part of LED consists of formulae built from atomic formulae of the form $s \dot{=} t$ (denoting: s and t both terminate and give the same output).

The s and t in $s = t$ are feds, and not our edts.

In [2] it is shown that each open LED formula is equivalent to one containing atomic subformulae of the form $s \dot{=} s$ only.

The edts are in one-one correspondence with formulae of the form $s \dot{=} s$ in the sense that given a feds s , there is a uniform method to find an edts R with $\models s \dot{=} s \leftrightarrow R^\dagger$ and given an edts R , a feds s with $\models s \dot{=} s \leftrightarrow R^\dagger$.

We conclude that each open LED formula is equivalent to an open formula containing atomic subformulae of the form T^\dagger (with T an edts) only.

For notational convenience sake we shall now introduce the following definitions:

Definition: The set AF of atomic forms = $\{A, A_1, A_2, \dots, A_{11}, A_{12}, \dots, A_{21}, \dots, B, B_1, B_2, \dots, C, C_1, C_2, \dots, D, D_1, D_2, \dots, E, E_1, E_2, \dots, F, F_1, F_2, \dots\}$.

The set PF of (propositional) forms is defined inductively as follows:

i) elements of AF are forms.

ii) if L and M are forms, then also $\neg L, L \wedge M, L \vee M$ are forms.

Note that from a propositional form L a QF-formula ϕ_L is obtained by substitution as follows:

substitute for every atomic form a in L a QF-formula of shape $T+$ with T and edts (of course, for all occurrences of a in L the same T must be used!).

Observe: i) all QF-formulae ϕ are obtained via some substitution from some propositional form.

ii) this form need not be unique (for instance $S+ \wedge \neg(S+)$ is obtained from both $A_1 \wedge \neg B_7$ and $A_1 \wedge \neg A_1$ etc.).

We say that ϕ has form L if ϕ can be obtained from propositional form L by substitution.

Notation: We shall use the letters L, M, P, Q to denote propositional forms, the letters R_i, S_i, T_i to denote edts and the Greek letters ϕ, χ, ψ, θ to denote QF-formulae.

Moreover, in this paper we will make use of the following symbols as abbreviations of expressions in the meta-language:

\triangleq for: is defined as,

\forall for: for every ... such that,

\exists for: there exists ... such that,

\nexists for: there is no ... such that,

s.t. for: such that

Definition: Let P and Q be two propositional forms. We say $P \leq_\Sigma Q$, P is Σ -reducible to Q, if for every QF-formula $\phi_P \in L_\Sigma$ with form P there exists a QF-formula

$\phi_Q \in L_\Sigma$ with form Q such that $\vdash \phi_P \leftrightarrow \phi_Q$.

We say $P \equiv_\Sigma Q$, if $P \leq_\Sigma Q$ and $Q \leq_\Sigma P$.

We shall omit the subscript Σ , when no confusion arises.

2. Some elementary observations.

Clearly, the following holds:

Lemma

- (i) \leq is transitive, i.e. if $L \leq M$ and $M \leq P$ (for certain forms L, M and P), then $L \leq P$.
- (ii) \leq is reflexive, i.e. for any form P : $P \leq P$.
- (iii) \equiv is an equivalence relation, i.e. \equiv is reflexive, symmetric and transitive.

Now we observe the following.

Proposition 0: Elements of PF can be seen as propositions in which the elements of AF are the propositional atoms. It then holds that if L and M denote two equivalent propositions in this sense, then also $L \equiv M$.

Example: $A \wedge (B \vee C)$ equiv. $(A \wedge B) \vee (A \wedge C)$, so $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.

Note that the converse is not true, for instance $A \equiv B$ and not $(A \text{ equiv. } B)$.

Next we notice that \leq induces a partial order (denoted by \leq as well) on PF/\equiv , the set of equivalence classes of PF, as follows:

Let $p, q \in PF/\equiv$; $p \leq q$ if for some $P \in p, Q \in q$: $P \leq Q$.

Proposition 1: (Tiuryn [2]).

- i). $A \vee B \equiv C$
- ii). $A \wedge B \equiv C$
- iii). $\neg A \vee \neg B \equiv \neg C$
- iv). $\neg A \wedge \neg B \equiv \neg C$

Proof: i) For all edts S, T , if $\phi_{A \vee B} = S \uparrow \vee T \uparrow$, then $\psi_C = R \uparrow$ with R such that $R^{2n} = S^n$ and $R^{2n+1} = T^n$, satisfies $\models \phi_{A \vee B} \leftrightarrow \psi_C$, because $(\exists n S^n) \vee (\exists m T^m) \leftrightarrow (\exists k R^k)$.

So $A \vee B \leq C$.

And conversely, for all edts R , if $\psi_C = R \uparrow$, then $\psi_{A \vee B} = R \uparrow \vee S_1 \uparrow$ with S_1 such

that $\models S_1 \uparrow$, satisfies $\models \phi_{A \vee B} \leftrightarrow \psi_C$, because

$(\exists n R^n) \vee (\exists m S_1^m) \leftrightarrow (\exists n R^n)$, so also $C \leq A \vee B$.

ii) For all edts S, T , if $\phi_{A \wedge B} = S \uparrow \wedge T \uparrow$, then $\psi_C = R \uparrow$ with $R^n = \bigvee_{i,j \leq n} (S^i \wedge T^j)$, satisfies $\models \phi_{A \wedge B} \leftrightarrow \psi_C$, because $(\exists n S^n) \wedge (\exists m T^m) \leftrightarrow (\exists k R^k)$.

So $A \wedge B \leq C$. And conversely, for all edts R , if $\psi_C = R \uparrow$, then $\phi_{A \wedge B} = R \uparrow \wedge T_1 \uparrow$ with T_1 such that $\models T_1 \uparrow$, satisfies $\models \phi_{A \wedge B} \leftrightarrow \psi_C$, because $(\exists n R^n) \wedge (\exists m T_1^m) \leftrightarrow (\exists n R^n)$, so also $C \leq A \wedge B$.

iii)

Let $\phi_{A \vee B}$ be an arbitrary QF-formula with form $'A \vee B'$, suppose

$\phi_{A \vee B} = 'S \uparrow \vee T \uparrow'$ for certain edts S and T . Note that

$\models 'S \uparrow \vee T \uparrow' \leftrightarrow '(S \uparrow \wedge T \uparrow)'$.

As $S \uparrow \wedge T \uparrow$ is a QF-formula $\phi_{A \wedge B}$ with form $A \wedge B$ we may conclude

(by ii) that there exists a QF-formula ψ_C with form C s.t.

$\models \phi_{A \wedge B} \leftrightarrow \psi_C$. Suppose $\psi_C = R \uparrow$ for some edts R . Hence

$\models \phi_{A \vee B} \leftrightarrow '(R \uparrow)'$. So as there exists a QF-formula ψ_C with form

$'C$ s.t. $\models \phi_{A \vee B} \leftrightarrow \psi_C$ (take $\psi_C = '(R \uparrow)'$), we have $'A \vee B' \leq 'C'$.

Conversely, for every QF-formula ψ_C with form C (suppose

$\psi_C = '(S \uparrow)'$) there exists a QF-formula $\phi_{A \vee B}$ s.t. $\models \psi_C \leftrightarrow \phi_{A \vee B}$.

(Take for instance $\phi_{A \vee B} = '(S \uparrow) \vee '(S \uparrow)'$).

So $'C \leq 'A \vee B'$ holds too. Therefore $'A \vee B' = 'C'$.

Proposition 2 (Tiuryn [2]).

For each propositional form P there is a propositional form $Q = \bigwedge_{1 \leq i \leq k} (A_i \vee \neg B_i)$

(for a certain k) such that $P \leq Q$.

Proof: Let P^1 be a conjunctive normal form of P , then by proposition 0: $P \equiv P^1$.

Furthermore $P^1 \leq \bigwedge_{1 \leq j \leq k} (A_{j1} \vee \dots \vee A_{jl_j} \vee \neg B_{j1} \vee \dots \vee \neg B_{jm_j})$

for certain k , l_j ($1 \leq j \leq k$) and m_j ($1 \leq j \leq k$),

and by proposition 1:

$A_{j1} \vee \dots \vee A_{jl_j} \leq A_j$ and

$\neg B_{j1} \vee \dots \vee \neg B_{jm_j} \leq \neg B_j$, so that

it is now not difficult to prove:

$$P \equiv P^1 \leq \bigwedge_{1 \leq i \leq k} (A_i \vee \neg B_i) = Q.$$

(Note that we have used the fact that the A_i and B_j are different atomic forms!)

Notation: define M^k as the form $\bigwedge_{1 \leq i \leq k} (A_i \vee \neg B_i)$; now proposition 2 says that \forall forms $P \exists k$ such that $P \leq M^k$.

Proposition 3:

$$\forall k: M^k \leq M^{k+1}.$$

Proof: For all edts S_i, T_i , if

$$\phi_{M^k} = \bigwedge_{1 \leq i \leq k} (S_i \uparrow \vee T_i \uparrow), \text{ then}$$

$$\psi_{M^{k+1}} = \bigwedge_{1 \leq i \leq k+1} (S_i \uparrow \vee T_i \uparrow) \text{ with}$$

edts S_{k+1} s.t. $\models S_{k+1} \uparrow$ and T_{k+1} an arbitrary edts, satisfies

$$\models \phi_{M^k} \leftrightarrow \psi_{M^{k+1}}. \quad \square$$

3. \equiv can divide PF into infinitely many equivalence classes.

We state now the main result of our paper:

Main Theorem:

There exists a signature Σ such that \equiv_{Σ} has infinitely many equivalence classes on PF.

To prove this theorem we will first prove the following proposition.

Proposition 4:

There exists a signature Σ such that there is no natural number $k \geq 1$ such that for all propositional forms P : $P \leq M^k$.

Proof:

Take $\mathcal{U} = \langle \omega, 3^{\omega}; 0, S, Ap \rangle$, in which ω is the set of natural numbers, 3^{ω} is the set of functions $\omega \rightarrow \{0, 1, 2\}$, S names the successor function on ω and Ap names the application function $ap: 3^{\omega} \times \omega \rightarrow \{0, 1, 2\}$ defined by $ap(\alpha, n) = \alpha(n)$.

Definition: If $\alpha \in 3^{\omega}$ and $\sigma = \eta_0 \wedge \dots \wedge \eta_{m-1}$ is a string of m symbols $\eta_i \in \{0, 1, 2\}$ ($m \geq 1$), then $\sigma * \alpha$ denotes the function $\beta \in 3^{\omega}$ given by

$$\beta(n) = \begin{cases} \eta_n & \text{if } 0 \leq n \leq m-1 \\ \alpha(n-m) & \text{if } n \geq m, \end{cases}$$

and $\bar{\alpha}(n)$ denotes the string $\alpha(0) \wedge \dots \wedge \alpha(n)$. (\wedge stands for concatenation.)

For a symbol η η^m will denote $\underbrace{\eta \wedge \dots \wedge \eta}_{m \text{ times}}$.

Furthermore, if σ is a string, we shall use the notations $|\sigma|$ for the length of σ , $\{0, 1, 2\}^*$ for the set of strings (built with 0, 1 and 2)

with finite length, and $\{0,1,2\}^m$ for the subset of $\{0,1,2\}^*$ that contains only all strings $\in \{0,1,2\}^*$ of length m .

Unless it is mentioned explicitly otherwise, a variable containing α (such as α, α^* , etc.) will range over $1*3^\omega$ and a variable containing β will range over $2*3^\omega$. (For notational convenience we shall often omit these ranges.)

Now we shall first prove the following:

Lemma: Let $\alpha \in 1*3^\omega$, $\beta \in 2*3^\omega$, and $H(\gamma)$ (with $\gamma \in 3^\omega$) the edts corresponding to the program:

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n: = 0; while  $\neg$ (Ap( $\gamma$ , n) = 0)
      do      n: = S(n)  od
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If $\phi_{L^1}(\alpha, \beta) \models H(\alpha) + \wedge H(\beta) +$, then $\exists \psi_{M^1}(\alpha, \beta)$ with form M^1 s.t. $\mathcal{A}_L \models \phi_{L^1} \leftrightarrow \psi_{M^1}$.

Proof:

Suppose $\exists \psi_{M^1}(\alpha, \beta)$ with form M^1 s.t. $\mathcal{A} \models \phi_{L^1} \leftrightarrow \psi_{M^1}$ (I).

Suppose this $\psi_{M^1}(\alpha, \beta) = S(\alpha, \beta) + \vee T(\alpha, \beta) +$

Take $\alpha^* = \lambda x \cdot 1$. Then: $\forall \beta (\in 2*3^\omega) \mathcal{A} \models \neg(H(\alpha^*) + \wedge H(\beta) +)$, so by (I):

$\forall \beta : \mathcal{A} \models \neg(\psi_{M^1}(\alpha^*, \beta))$, that is to say $\forall \beta : \mathcal{A} \models \neg(S(\alpha^*, \beta) + \vee T(\alpha^*, \beta) +)$,

: $\forall \beta : \mathcal{A} \models \neg(S(\alpha^*, \beta) +) \wedge \neg T(\alpha^*, \beta) +$,

: $\forall \beta : \mathcal{A} \models \neg(T(\alpha^*, \beta) +)$,

: $\forall \beta : \mathcal{A} \models T(\alpha^*, \beta) +$.

Now let us consider the edts $T(\alpha, \beta)$. Say $T(\alpha, \beta) = \{ \bigwedge_{0 \leq i \leq n-1} \neg \epsilon_i(\alpha, \beta) \wedge \epsilon_n(\alpha, \beta) \mid n \in \omega \}$ for certain $\epsilon_i(\alpha, \beta) \in F_{\{0, S, Ap\}}$.

(Note that $F_{\{0, S, Ap\}}$ only consists of finite conjunctions of (negations of) quantifier-free equations as no other relations are available.)

According to the definition of an edts these $\epsilon_i(\alpha, \beta)$ may contain equations $\delta_1 = \delta_2$ with $\text{val}_{\mathcal{A}} \delta_i \in 3^\omega$ (i.e. equations on the second domain of \mathcal{A}).

An equation of this kind, however, that contains only α or β as a free variable, can be one of the following only:

$$\alpha = \alpha, \beta = \beta, \alpha = \beta \text{ or } \beta = \alpha.$$

For every $\alpha \in 1 \cdot 3^{\omega}$ and $\beta \in 2 \cdot 3^{\omega}$ these four formulae are either always true or always false, independently of the choice of α and β . With respect to all other open first order formulae $\in F_{\Sigma}$ occurring in the edts $T(\alpha, \beta)$, we may say that as they all consist of a finite number of symbols, they contain a merely finite number of occurrences of Ap . Moreover, we know that the program ϕ_T associated with the edts T terminates for input α^* and all $\beta \in 2 \cdot 3^{\omega}$, which implies that only a finite number of ϵ_i are being evaluated during the execution of ϕ_T . Hence we may conclude that the values of α^* and β in only a finite number of arguments are used in executing ϕ_T .

We can therefore say that:

$\forall \beta \exists n_{\beta}$ s.t. for any $(\alpha^*)^1$ and β^1 with

$$\overline{(\alpha^*)^1}(n_{\beta}) = \overline{\alpha^*}(n_{\beta}) \text{ and}$$

$$\overline{\beta^1}(n_{\beta}) = \overline{\beta}(n_{\beta})$$

holds: $\mathcal{U} \models T((\alpha^*)^1, \beta^1)_{+}$.

For the present purposes we formulate König's Lemma as follows:

König's Lemma:

Let ρ be a predicate on $3^{\omega} \times \omega$ such that

- (i) $\rho(\beta, n) \rightarrow \rho(\beta, n+1)$,
- (ii) $\rho(\beta, n) \wedge \bar{\beta}(n) = \bar{\beta}'(n) \rightarrow \rho(\beta', n)$.

Then it holds that

$$\forall \beta \exists n \rho(\beta, n) \rightarrow \exists n \forall \beta \rho(\beta, n).$$

□

Now take

$$\rho(\beta, n) \triangleq [\forall (\alpha^*)^1 \forall \beta^1 \{ \overline{((\alpha^*)^1)}(n) = \overline{\alpha^*}(n) \wedge \overline{\beta^1}(n) = \overline{\beta}(n) \} \rightarrow \\ \rightarrow \mathcal{U} \models T((\alpha^*)^1, \beta^1) \}]$$

(Clearly, this ρ satisfies the conditions (i) and (ii) of König's Lemma.)

By König's Lemma there is an ℓ s.t. for any

$$\begin{aligned} & (\alpha^{*1}), \beta^1 \text{ with} \\ & \overline{(\alpha^*)^1}(\ell) = \overline{\alpha^*}(\ell) \text{ and} \\ & \overline{\beta^1}(\ell) = \overline{\beta}(\ell) \end{aligned}$$

$$\text{holds: } \mathcal{U} \models T((\alpha^*)^1, \beta^1) \uparrow.$$

(II).

Now we get for \mathcal{U} , observing that: $H(\alpha) \uparrow$ iff $\exists n \alpha(n) = 0$,

$$\begin{aligned} \forall \alpha, \beta : \phi_{L^1}(\alpha, \beta) & \leftrightarrow H(\alpha) \uparrow \wedge H(\beta) \uparrow \leftrightarrow \\ H(1^\ell * \alpha) \uparrow \wedge H(\beta) \uparrow & \leftrightarrow \phi_{L^1}(1^\ell * \alpha, \beta) \leftrightarrow \\ \psi_{M^1}(1^\ell * \alpha, \beta) & \leftrightarrow S(1^\ell * \alpha, \beta) \uparrow \vee T(1^\ell * \alpha, \beta) \uparrow \\ \text{(II)} & \\ \leftrightarrow S(1^\ell * \alpha, \beta) \uparrow & \leftrightarrow S'(\alpha, \beta) \uparrow \end{aligned}$$

$$\text{with } S'(\alpha, \beta) \triangleq S(1^\ell * \alpha, \beta).$$

So now $\exists S'$ s.t. $\forall \alpha \in 1 * 3^\omega, \beta \in 2 * 3^\omega$

$$\mathcal{U} \models \phi_{L^1}(\alpha, \beta) \leftrightarrow S'(\alpha, \beta) \uparrow.$$

Thus: $\exists S'$ s.t. $\forall \alpha \in 1 * 3^\omega, \beta \in 2 * 3^\omega$

$$\mathcal{U} \models H(\alpha) \uparrow \wedge H(\beta) \uparrow \leftrightarrow S'(\alpha, \beta) \uparrow.$$

Take $\alpha^{**} = 1 * (\lambda x. 0)$, then $\forall \alpha : \mathcal{U} \models H(\alpha^{**}) \uparrow$,

$$\text{so } \mathcal{U} \models H(\beta) \uparrow \leftrightarrow S'(\alpha^{**}, \beta) \uparrow :$$

$\exists S''$ such that $\forall \beta \in 2 \cdot 3^\omega$:

$$\mathcal{U} \models (H(\beta) \uparrow \leftrightarrow S''(\beta) \uparrow).$$

This provides us with the opportunity to decide

(P)... whether or not an arbitrary recursive function $\xi \in 3^\omega$ is zero in some argument $n \in \omega$:

we look for a zero in ξ (starting with checking if $\xi(0) = 0$, followed by checking if $\xi(1) = 0$, etc.) and in parallel with this we execute the program $\phi_{S''}$ corresponding to $S''(\beta)$, with input $2 \cdot \xi$.

(If $\phi_{S''}(2 \cdot \xi)$ terminates we know that $2 \cdot \xi$ (and also ξ) does not have a zero.)

But deciding (P) can easily be proved equivalent to deciding the Halting Problem for partial recursive functions on ω , which is known to be undecidable ([2]).

So supposition (I) is not true. □

Now we resume the proof of proposition 4:

Suppose that $\exists k \geq 1$ s.t. \forall prop. form $P: P \leq M^k$.

Let $L^N \triangleq (E_1 \wedge \neg F_1) \vee \dots \vee (E_N \wedge \neg F_N)$. Then $\forall N: L^N \leq M^k$, that is to say

$\forall N: \forall \phi_{L^N}$ with form $L^N \exists \psi_{M^k}$ with form M^k s.t. $\models \phi_{L^N} \leftrightarrow \psi_{M^k}$, so that certainly

$$\mathcal{U} = \langle \omega, 3^\omega, 0, S, \text{Ap} \rangle \models \phi_{L^N} \leftrightarrow \psi_{M^k} \cdot (1).$$

Let $\alpha_1, \dots, \alpha_N \in 1 * 3^\omega$, $\beta_1, \dots, \beta_N \in 2 * 3^\omega$.

Define $\vec{\alpha} = (\alpha_1, \dots, \alpha_N)$ and $\vec{\beta} = (\beta_1, \dots, \beta_N)$.

A formula $\phi = \phi(\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N)$ can now be written as $\phi(\vec{\alpha}, \vec{\beta})$.

We shall prove for $N > 1$:

if for $\phi_{L^N}(\vec{\alpha}, \vec{\beta}) = \bigvee_{1 \leq i \leq N} (H(\alpha_i) \uparrow \wedge H(\beta_i) \uparrow)$

$\exists \psi_{M^k}(\vec{\alpha}, \vec{\beta})$ with form M^k (with $k > 1$) s.t. $\mathcal{U} \models \phi_{L^N} \leftrightarrow \psi_{M^k}$,

then for $\phi_{L^{N-1}}(\vec{\alpha}, \vec{\beta}) = \bigvee_{1 \leq i \leq N-1} (H(\alpha_i) \uparrow \wedge H(\beta_i) \uparrow)$

$\exists \psi_{M^{k-1}}(\vec{\alpha}, \vec{\beta})$ with form M^{k-1} s.t. $\mathcal{U} \models \phi_{L^{N-1}} \leftrightarrow \psi_{M^{k-1}}$.

If we have proved this, then with the aid of (1) we can conclude that

$\forall N \geq 1$: if $\phi_{L^N}(\vec{\alpha}, \vec{\beta}) = \bigvee_{1 \leq i \leq N} (H(\alpha_i) \dot{+} \wedge H(\beta_i) \dot{+})$, then $\exists \psi_{M^1}(\vec{\alpha}, \vec{\beta})$ with form M^1 s.t. $\mathcal{U} \models \phi_{L^N} \leftrightarrow \psi_{M^1}$, and this contradicts the lemma just proved.

So suppose $N > 1$ and for $\phi_{L^N}(\vec{\alpha}, \vec{\beta}) = \bigvee_{1 \leq i \leq N} (H(\alpha_i) \dot{+} \wedge H(\beta_i) \dot{+}) \exists \psi_{M^k}(\vec{\alpha}, \vec{\beta})$ with form M^k for a certain $k > 1$ s.t. $\mathcal{U} \models \phi_{L^N} \leftrightarrow \psi_{M^k}$. (2).

Suppose this $\psi_{M^k}(\vec{\alpha}, \vec{\beta}) = \bigwedge_{1 \leq i \leq k} (S_i(\vec{\alpha}, \vec{\beta}) \dot{+} \vee T_i(\vec{\alpha}, \vec{\beta}) \dot{+})$

Define $\vec{\alpha}^* \triangleq \vec{\alpha}_1^*, \dots, \alpha_N^*$ with $\alpha_i^* = \lambda x \cdot 1$ ($1 \leq i \leq n$).

Now: $(\forall i) \nexists x \alpha_i(x) = 0$

so $(\forall i) \neg(H(\alpha_i^*) \dot{+})$, and thus $\forall \beta_1, \dots, \beta_N \in 2 * 3^\omega$, $\forall i \in \{1, \dots, n\}$

$$\neg(H(\alpha_i^*) \dot{+} \wedge H(\beta_i) \dot{+}),$$

so $\forall \vec{\beta}: \neg \phi_{L^N}(\vec{\alpha}^*, \vec{\beta})$, so that by (2):

$$\forall \vec{\beta}: \neg \psi_{M^k}(\vec{\alpha}^*, \vec{\beta}), \text{ that is to say,}$$

$$\forall \vec{\beta} \neg \bigwedge_{1 \leq i \leq k} (S_i(\alpha^*, \beta) \dot{+} \vee T_i(\vec{\alpha}^*, \vec{\beta}) \dot{+}),$$

$$\text{so } \forall \vec{\beta} \exists_{i \leq k} \mathcal{U} \models \neg(S_i(\vec{\alpha}^*, \vec{\beta}) \dot{+} \vee T_i(\vec{\alpha}^*, \vec{\beta}) \dot{+})$$

$$\forall \vec{\beta} \exists_{i \leq k} \mathcal{U} \models (S_i(\vec{\alpha}^*, \vec{\beta}) \dot{+} \wedge T_i(\vec{\alpha}^*, \vec{\beta}) \dot{+}) \text{ which implies that}$$

$$\forall \vec{\beta} \exists_{i \leq k} \mathcal{U} \models T_i(\vec{\alpha}^*, \vec{\beta}) \dot{+}.$$

By an argument analogous to the one in the proof of proposition 4 we can say that

$$\forall \vec{\beta} \exists_{i \leq k} \text{ s.t. } \exists n_{\vec{\beta}}^i \text{ s.t.}$$

$$\text{for any } (\vec{\alpha}^*)^1 = ((\alpha_1^*)^1, \dots, (\alpha_N^*)^1),$$

$$\vec{\beta}^1 = (\beta_1, \dots, \beta_N) \text{ with}$$

$$(\alpha_j^*)^1(n_{\vec{\beta}}^i) = \alpha_j^*(n_{\vec{\beta}}^i) \quad (1 \leq j \leq N)$$

$$\text{and } \beta_j^1(n_{\vec{\beta}}^i) = \beta_j(n_{\vec{\beta}}^i) \quad (1 \leq j \leq N)$$

$$\text{holds: } \mathcal{U} \models T_i((\vec{\alpha}^*)^1, \vec{\beta}^1) \dot{+}.$$

By a version of König's Lemma which is generalised to predicates

ρ on $(3^\omega)^N \times \omega$ there exists an m such that $\forall \vec{\beta} \exists_{i \leq k} \text{ s.t.}$

$$\text{for any } (\vec{\alpha}^*)^1 = ((\alpha_1^*)^1, \dots, (\alpha_N^*)^1),$$

$$\vec{\beta}^1 = (\beta_1, \dots, \beta_N) \text{ with}$$

$$(\alpha_j^*)^1(m) = \alpha_j^*(m) \quad (1 \leq j \leq N)$$

$$\text{and } \beta_j^1(m) = \beta_j(m) \quad (1 \leq j \leq N)$$

$$\text{holds: } \mathcal{U} \models T_i((\vec{\alpha}^*)^1, \vec{\beta}^1) \dot{+}. \quad (3).$$

Definition: If $\alpha \in 3^\omega$ and $\sigma \in \{0,1,2\}^*$, then $\sigma \sqsubseteq \alpha$ iff $\bar{\alpha}(\ell) = \sigma$ with $\ell = |\sigma|$.

Let $\vec{\sigma} = \sigma_1, \dots, \sigma_N$ be a list of strings $\in \{0,1,2\}^m$ and Γ the set of all these lists $\vec{\sigma}$.

Notice that $|\Gamma| = (3^m)^N < \infty$.

For every $\vec{\sigma} \in \Gamma$ we now define a recursive predicate $R^{\vec{\sigma}}$ on $(2 \cdot 3^\omega)^N$ by $R^{\vec{\sigma}}(\vec{\beta}) = R^{\vec{\sigma}}(\beta_1, \dots, \beta_N) \leftrightarrow$

$$(\sigma_1 \sqsubseteq \beta_1 \wedge \dots \wedge \sigma_N \sqsubseteq \beta_N).$$

Then: all $R^{\vec{\sigma}} = \{\vec{\beta} \mid R^{\vec{\sigma}}(\vec{\beta})\}$ are disjoint and $\bigcup_{\vec{\sigma} \in \Gamma} R^{\vec{\sigma}} = (2 \cdot 3^\omega)^N$. (4).

Now it follows from (3):

$$\forall \vec{\sigma} \in \Gamma \exists i_{\vec{\sigma}} \leq k \text{ s.t. } \forall \vec{\beta} \in R^{\vec{\sigma}} [T_{i_{\vec{\sigma}}}(\vec{\alpha}^*, \vec{\beta})] \quad (5).$$

Define $\vec{l}^m * \alpha \triangleq (l^m * \alpha_1, \dots, l^m * \alpha_N)$.

Note that $\phi_{L^N}(\vec{\alpha}, \vec{\beta}) \leftrightarrow \phi_{L^N}(\vec{l}^m * \alpha, \vec{\beta})$ (because $\phi_{L^N}(\vec{l}^m * \alpha, \vec{\beta}) \leftrightarrow$

$$\bigvee_{1 \leq i \leq N} (H(l^m * \alpha_i) \vdash \wedge H(\beta_i) \vdash) \xleftrightarrow{(0)} \bigvee_{1 \leq i \leq N} (H(\alpha_i) \vdash \wedge H(\beta_i) \vdash) \leftrightarrow \phi_{L^N}(\vec{\alpha}, \vec{\beta}).$$

$$\text{So } \phi_{L^N}(\vec{\alpha}, \vec{\beta}) \leftrightarrow \phi_{L^N}(\vec{l}^m * \alpha, \vec{\beta}) \xleftrightarrow{(1)} \psi_{M^k}(\vec{l}^m * \alpha, \vec{\beta}) \leftrightarrow$$

$$\bigwedge_{1 \leq i \leq k} (S_i(l^m * \alpha, \vec{\beta}) \vdash \vee T_i(l^m * \alpha, \vec{\beta}) \vdash) \leftrightarrow \bigwedge_{1 \leq i \leq k} (S_i \vdash \vee T_i)(\vec{l}^m * \alpha, \vec{\beta}) \xleftrightarrow{(4)}$$

$$\bigvee_{\vec{\sigma} \in \Gamma} [R^{\vec{\sigma}}(\vec{\beta}) \wedge \bigwedge_{1 \leq i \leq k} (S_i \vdash \vee T_i)(\vec{l}^m * \alpha, \vec{\beta})] \leftrightarrow \bigvee_{\vec{\sigma} \in \Gamma} [R^{\vec{\sigma}}(\vec{\beta}) \wedge \bigwedge_{\substack{1 \leq i \leq k \\ i \neq i_{\vec{\sigma}}}} (S_i \vdash \vee T_i)(\vec{l}^m * \alpha, \vec{\beta})]$$

$$(S_{i_{\vec{\sigma}}}(\vec{l}^m * \alpha, \vec{\beta}) \vdash \vee T_{i_{\vec{\sigma}}}(\vec{l}^m * \alpha, \vec{\beta}) \vdash) \xleftrightarrow{(5)}$$

$$\bigvee_{\vec{\sigma} \in \Gamma} [R^{\vec{\sigma}}(\vec{\beta}) \wedge \bigwedge_{\substack{1 \leq i \leq k \\ i \neq i_{\vec{\sigma}}}} (S_i \vdash \vee T_i)(\vec{l}^m * \alpha, \vec{\beta}) \wedge S_{i_{\vec{\sigma}}}(\vec{l}^m * \alpha, \vec{\beta}) \vdash].$$

Name the last formula $\theta(\vec{\alpha}, \vec{\beta})$.

$R^{\vec{\sigma}}$ recursive, so $\models R^{\vec{\sigma}}(\vec{\beta}) \leftrightarrow R_1^{\vec{\sigma}}(\vec{\alpha}, \vec{\beta}) \vdash$ for a certain edts $R_1^{\vec{\sigma}}$ and

$\models \neg R^{\vec{\sigma}}(\vec{\beta}) \leftrightarrow R_2^{\vec{\sigma}}(\vec{\alpha}, \vec{\beta}) \vdash$ for a certain edts $R_2^{\vec{\sigma}}$, (whereby must be noticed:

$$\mathcal{U} \models R_1^{\vec{\sigma}}(\vec{\alpha}, \vec{\beta}) \vdash \leftrightarrow R_2^{\vec{\sigma}}(\vec{\alpha}, \vec{\beta}) \vdash).$$

Now $\theta(\vec{\alpha}, \vec{\beta}) \leftrightarrow \bigvee_{\vec{\sigma} \in \Gamma} [R^{\vec{\sigma}}(\vec{\beta}) \wedge \bigwedge_{\substack{1 \leq i \leq k \\ i \neq i_{\vec{\sigma}}}} (S_i'' \vdash \vee T_i'' \vdash)(\vec{\alpha}, \vec{\beta}) \wedge S_{i_{\vec{\sigma}}}''(\vec{\alpha}, \vec{\beta}) \vdash]$ for

certain edts $S_i'', T_i'' \leftrightarrow \bigvee_{\vec{\sigma} \in \Gamma} [\bigwedge_{\substack{1 \leq i \leq k \\ i \neq i_{\vec{\sigma}}}} ((S_i'' \vdash \wedge R_1^{\vec{\sigma}} \vdash) \vee (T_i'' \vdash \wedge R_1^{\vec{\sigma}} \vdash))(\vec{\alpha}, \vec{\beta})$

$$\wedge (S_{i \rightarrow \sigma}'' \vdash \wedge K_1 \vdash) (\vec{\alpha}, \vec{\beta}) \leftrightarrow \forall_{\sigma \in \Gamma} [\bigwedge_{\substack{1 \leq i \leq k \\ i \neq i \rightarrow \sigma}} ((S_i'' \vdash \wedge R_1 \vdash) \vee \neg (T_i'' \vdash \vee K_2 \vdash)) (\vec{\alpha}, \vec{\beta})$$

$$\wedge (S_{i \rightarrow \sigma}'' \vdash \wedge R_1 \vdash) (\vec{\alpha}, \vec{\beta})] \leftrightarrow \forall_{\sigma \in \Gamma} [\bigwedge_{\substack{1 \leq i \leq k \\ i \neq i \rightarrow \sigma}} (S_i \vdash \vee T_i \vdash) (\vec{\alpha}, \vec{\beta}) \wedge S_{i \rightarrow \sigma} \vdash (\vec{\alpha}, \vec{\beta}) \vdash]$$

for certain $S_i \vdash, T_i \vdash$ with $\forall i, j: \neg (S_i \vdash \wedge S_j \vdash), \neg (T_i \vdash \wedge T_j \vdash)$, and $\neg (S_i \vdash \wedge T_j \vdash)$ for all σ, σ' with $\sigma \neq \sigma'$

$$\leftrightarrow \bigwedge_{1 \leq i \leq k-1} (S_i' \vdash \vee T_i' \vdash) (\vec{\alpha}, \vec{\beta}) \wedge S_k' \vdash (\vec{\alpha}, \vec{\beta}) \vdash \text{ for certain } S_i', T_i'. \quad (*)$$

To prove the last equivalence we take the following into consideration:

Consider the first two terms of

$$\begin{aligned} & \bigwedge_{\sigma \in \Gamma} \{ \vec{\sigma}^1, \vec{\sigma}^2, \dots, \vec{\sigma}^{(3^m N)} \} [(\bigwedge_{\substack{1 \leq i \leq k \\ i \neq i \rightarrow \sigma}} (S_i \vdash \vee T_i \vdash) \wedge S_{i \rightarrow \sigma} \vdash) (\vec{\alpha}, \vec{\beta})] : \\ & \left(\bigwedge_{\substack{1 \leq i \leq k \\ i \neq i \rightarrow 1}} (S_i \vdash \vee T_i \vdash) \wedge S_{i \rightarrow 1} \vdash \right) (\vec{\alpha}, \vec{\beta}) \vee \\ & \left(\bigwedge_{\substack{1 \leq i \leq k \\ i \neq i \rightarrow 2}} (S_i \vdash \vee T_i \vdash) \wedge S_{i \rightarrow 2} \vdash \right) (\vec{\alpha}, \vec{\beta}). \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \left(\bigwedge_{1 \leq i \leq k-1} (S_i^* \vdash \vee T_i^* \vdash) \wedge S_k^* \vdash \right) (\vec{\alpha}, \vec{\beta}) \vee \\ & \left(\bigwedge_{1 \leq i \leq k-1} (S_i^{**} \vdash \vee T_i^{**} \vdash) \wedge S_k^{**} \vdash \right) (\vec{\alpha}, \vec{\beta}) \end{aligned}$$

for certain edts $S_i^*, T_i^*, S_i^{**}, T_i^{**}$ with $\forall i, j: \neg (S_i^* \wedge S_j^{**}), \neg (T_i^* \wedge T_j^{**}), \neg (S_i^* \wedge T_j^{**})$ and $\neg (T_i^* \wedge S_j^{**})$.

This again is to be written as:

$$(\bigwedge_{1 \leq i \leq k} \chi_i) \vee (\bigwedge_{1 \leq j \leq k} \chi_j'), \text{ for formulae } \chi_i, \chi_j' \text{ with}$$

$$\chi_i = \begin{cases} (S_i^* \vee T_i^*) (\vec{\alpha}, \vec{\beta}) & (1 \leq i < k) \\ S_k^* (\vec{\alpha}, \vec{\beta}) \vdash & (i = k) \end{cases} \quad \text{and}$$

$$\chi_i' = \begin{cases} (S_i^{**} \vee T_i^{**}) (\vec{\alpha}, \vec{\beta}) & (1 \leq i < k) \\ S_k^{**} (\vec{\alpha}, \vec{\beta}) \vdash & (i = k) \end{cases}$$

(Note that $\neg (\chi_i \wedge \chi_j') (\forall i, j)$. (6).)

And this formula is equivalent to $\bigwedge_{1 \leq i \leq k} (\chi_i \vee \chi_i')$.

(Proof: Suppose $(\bigwedge \chi_i) \vee (\bigwedge \chi_i')$, then either $\bigwedge \chi_i$ or $\bigwedge \chi_i'$,

suppose $\bigwedge \chi_i$ (if $\bigwedge \chi_i'$: analogous), then $\chi_1 \wedge \dots \wedge \chi_k$, so $(\chi_1 \vee \chi_1') \wedge \dots \wedge (\chi_k \vee \chi_k')$

thus $\bigwedge (\chi_i \vee \chi_i')$.

Conversely, suppose $\bigwedge (\chi_i \vee \chi_i')$, then $\chi_i \vee \chi_i'$;

Suppose χ_i , then by (6): $\neg \chi_j' (\forall j)$ and so $\bigwedge_i \chi_i$, because $\bigwedge (\chi_i \vee \chi_i')$.

Now suppose χ_i' , then $\neg \chi_j (\forall j)$ and so $\chi_i' (\forall j)$: $\bigwedge_j \chi_j'$.

Thus: always $(\bigwedge_i \chi_i) \vee (\bigwedge_j \chi_j')$.

Now: for $i < k$:

$$\chi_i \vee \chi_i' \leftrightarrow ((S_i^{*+} \vee T_i^{*+}) \vee (S_i^{**+} \vee T_i^{**+})) (\vec{\alpha}, \vec{\beta}) \leftrightarrow$$

$$((S_i^{*+} \vee S_i^{**+}) \vee (T_i^{*+} \vee T_i^{**+})) (\vec{\alpha}, \vec{\beta}) \leftrightarrow$$

$$(S_i^{*+} \vee \neg(T_i^{*+} \wedge T_i^{**+})) (\vec{\alpha}, \vec{\beta}) \leftrightarrow$$

$$(S_i^{*+} \vee \neg(T_i^{*+})) (\vec{\alpha}, \vec{\beta}) \leftrightarrow (S_i^{*+} \vee T_i^{*+}) (\vec{\alpha}, \vec{\beta}) \text{ and}$$

$$\chi_k \vee \chi_k' \leftrightarrow (S_k^{*+} \vee S_k^{**+}) (\vec{\alpha}, \vec{\beta}) \leftrightarrow S_k' (\vec{\alpha}, \vec{\beta}) + \text{ for certain } S_i', T_i'.$$

$$\text{So: } \bigwedge_{1 \leq i \leq k} (\chi_i \vee \chi_i') \leftrightarrow \bigwedge_{1 \leq i \leq k-1} (S_i^{*+} \vee T_i^{*+}) (\vec{\alpha}, \vec{\beta}) \wedge S_k' (\vec{\alpha}, \vec{\beta}) +, \text{ that is to say,}$$

the union of the first two terms of the disjunction $\bigvee_{\vec{\sigma} \in \Gamma} []$ is to be written in the same form as one term of the disjunction.

By pursuing this with the next terms we get the result (*)

So far we have proved that for our special $\phi_{L^N} (\vec{\alpha}, \vec{\beta})$ there exists a formula

θ_{p^k} with form

$$p^k \triangleq (C_1 \vee \neg D_1) \wedge \dots \wedge (C_{k-1} \vee \neg D_{k-1}) \wedge C_k \text{ such that } \mathcal{U} \models \phi_{L^N} \leftrightarrow \theta_{p^k} :$$

$$\phi_{L^N} (\vec{\alpha}, \vec{\beta}) \leftrightarrow \theta_{p^k} (\vec{\alpha}, \vec{\beta}) = \bigwedge_{1 \leq i \leq k-1} (S_i^{*+} \vee T_i^{*+}) (\vec{\alpha}, \vec{\beta}) \wedge S_k' (\vec{\alpha}, \vec{\beta}) +.$$

$$\text{Now take } \alpha^{**} = (\alpha_1, \dots, \alpha_{N-1}, \alpha_N^{**})$$

$$\text{with } \alpha_N^{**}(x) = 1 * (\lambda x. 0)$$

$$\text{and } \beta^{**} = (\beta_1, \dots, \beta_{N-1}, \beta_N^{**})$$

$$\text{with } \beta_N^{**} = \lambda x. 2.$$

Then: $\mathcal{U} \models H(\alpha_N^{**})^\dagger \wedge H(\beta_N^{**})^\dagger$,

so $\forall \alpha_1, \dots, \alpha_{N-1}, \beta_1, \dots, \beta_{N-1} : \mathcal{U} \models \phi_{L_N}^{\rightarrow}(\alpha^{**}, \beta^{**})$

thus $\forall \alpha_1, \dots, \alpha_{N-1}, \beta_1, \dots, \beta_{N-1} : \mathcal{U} \models \theta_{p_k}(\overrightarrow{\alpha^{**}}, \overrightarrow{\beta^{**}})$.

And this implies: $\forall \alpha_1, \dots, \beta_{N-1} : \mathcal{U} \models S_k'(\overrightarrow{\alpha^{**}}, \overrightarrow{\beta^{**}})^\dagger$, what is to say

$$\begin{aligned} \forall \alpha_1, \dots, \beta_{N-1} \exists n_{\alpha_1}, \dots, n_{\beta_{N-1}} \text{ s.t. for any } (\overrightarrow{\alpha^{**}})^1 &= (\alpha_1^1, \dots, \alpha_{N-1}^1, (\alpha_N^{**})^1), \\ (\overrightarrow{\beta^{**}})^1 &= (\beta_1^1, \dots, \beta_{N-1}^1, (\beta_N^{**})^1) \\ \text{with } \overline{\alpha_j^1}(n_{\alpha_1}, \dots, \beta_{N-1}) &= \overline{\alpha_j^1}(n_{\alpha_1}, \dots, \beta_{N-1}) \quad (1 \leq j < N) \\ (\overrightarrow{\alpha^{**}})^1(n_{\alpha_1}, \dots, \beta_{N-1}) &= \overline{\alpha_N^{**}}(n_{\alpha_1}, \dots, \beta_{N-1}), \\ \text{and } \overline{\beta_j^1}(n_{\alpha_1}, \dots, \beta_{N-1}) &= \overline{\beta_j^1}(n_{\alpha_1}, \dots, \beta_{N-1}) \quad (1 \leq j < N) \\ (\overrightarrow{\beta^{**}})^1(n_{\alpha_1}, \dots, \beta_{N-1}) &= \overline{\beta_N^{**}}(n_{\alpha_1}, \dots, \beta_{N-1}) \end{aligned}$$

holds: $\mathcal{U} \models S_k'((\overrightarrow{\alpha^{**}})^1, (\overrightarrow{\beta^{**}})^1)^\dagger$.

So by König's Lemma $\exists m'$ s.t. $\forall \alpha_1, \dots, \beta_{N-1}$ holds that

$$\begin{aligned} \text{for any } (\overrightarrow{\alpha^{**}})^1 &= (\alpha_1^1, \dots, \alpha_{N-1}^1, (\alpha_N^{**})^1), \\ (\overrightarrow{\beta^{**}})^1 &= (\beta_1^1, \dots, \beta_{N-1}^1, (\beta_N^{**})^1) \end{aligned}$$

$$\text{with } \overline{\alpha_j^1}(m') = \overline{\alpha_j^1}(m') \quad (1 \leq j < N),$$

$$\overline{(\alpha_N^{**})^1}(m') = \overline{\alpha_N^{**}}(m'),$$

$$\text{and } \overline{\beta_j^1}(m') = \overline{\beta_j^1}(m') \quad (1 \leq j < N),$$

$$\overline{(\beta_N^{**})^1}(m') = \overline{\beta_N^{**}}(m') :$$

$$\mathcal{U} \models S_k'((\overrightarrow{\alpha^{**}})^1, (\overrightarrow{\beta^{**}})^1)^\dagger. \quad (7).$$

$$\begin{aligned} \text{So: } \phi_{L_N}(\alpha_1, \dots, \alpha_{N-1}, \alpha_N^{**}, \beta_1, \dots, \beta_{N-1}, \beta_N) &\leftrightarrow \\ \phi_{L_N}(\alpha_1, \dots, \alpha_{N-1}, \alpha_N^{**}, \beta_1, \dots, \beta_{N-1}, 2^{m'} * \beta_N) &\leftrightarrow \\ \theta_{p_k}(\alpha_1, \dots, \alpha_{N-1}, \alpha_N^{**}, \beta_1, \dots, \beta_{N-1}, 2^{m'} * \beta_N) &\leftrightarrow \end{aligned}$$

$$\bigwedge_{1 \leq i \leq k-1} (S_i' \vdash \vee T_i' \uparrow) (\alpha_1, \dots, \alpha_{N-1}, \alpha_N^{**}, \beta_1, \dots, \beta_{N-1}, 2^{m'} * \beta_N) \wedge$$

$$S_k'(\alpha_1, \dots, \alpha_{N-1}, \alpha_N^{**}, \beta_1, \dots, \beta_{N-1}, 2^{m'} * \beta_N) \uparrow \quad (7) \quad \leftrightarrow$$

$$\bigwedge_{1 \leq i \leq k-1} (S_i' \vdash \vee T_i' \uparrow) (\alpha_1, \dots, \alpha_{N-1}, \alpha_N^{**}, \beta_1, \dots, \beta_{N-1}, 2^{m'} * \beta_N)$$

$$\text{Let } \phi_{L, N-1}(\alpha_1, \dots, \alpha_{N-1}, \beta_1, \dots, \beta_{N-1}) = \bigvee_{1 \leq i \leq N-1} (H(\alpha_i) \uparrow \wedge H(\beta_i) \uparrow)$$

$$\text{Then } \forall \alpha_1, \dots, \alpha_{N-1}, \beta_1, \dots, \beta_{N-1}: \phi_{L, N-1}(\alpha_1, \dots, \alpha_{N-1}, \beta_1, \dots, \beta_{N-1}) \leftrightarrow$$

$$\bigvee_{1 \leq i \leq N-1} (H(\alpha_i) \uparrow \wedge H(\beta_i) \uparrow) \vee (H(\alpha_N^{**}) \uparrow \wedge H(\beta_N^{***}) \uparrow)$$

$$\text{with } \beta_N^{***} = 2^*(\lambda x.0) \text{ (because } \mathcal{U} \models (H(\alpha_N^{**}) \uparrow) \text{ and } \mathcal{U} \not\models (H(\beta_N^{***}) \uparrow))$$

$$\leftrightarrow \phi_{L, N}(\alpha_1, \dots, \alpha_{N-1}, \alpha_N^{**}, \beta_1, \dots, \beta_{N-1}, \beta_N^{***})$$

$$\leftrightarrow \bigwedge_{1 \leq i \leq k-1} (S_i' \vdash \vee T_i' \uparrow) (\alpha_1, \dots, \alpha_{N-1}, \alpha_N^{**}, \beta_1, \dots, \beta_{N-1}, 2^{m'} * \beta_N^{***})$$

$$\leftrightarrow \bigwedge_{1 \leq i \leq k-1} (\tilde{S}_i \vdash \vee \tilde{T}_i \uparrow) (\alpha_1, \dots, \alpha_{N-1}, \beta_1, \dots, \beta_{N-1})$$

$$\text{for some edts } \tilde{S}_i, \tilde{T}_i.$$

Thus we conclude that for $N > 1$:

$$\text{if for } \phi_{L, N}(\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N) = \bigvee_{1 \leq i \leq N} (H(\alpha_i) \uparrow \wedge H(\beta_i) \uparrow)$$

$$\exists \psi_{M^k}(\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N) \text{ with form } M^k \text{ (} k > 1 \text{) s.t. } \mathcal{U} \models \phi_{L, N} \leftrightarrow \psi_{M^k},$$

$$\text{then for } \phi_{L, N-1}(\alpha_1, \dots, \alpha_{N-1}, \beta_1, \dots, \beta_{N-1}) = \bigvee_{1 \leq i \leq N-1} (H(\alpha_i) \uparrow \wedge H(\beta_i) \uparrow)$$

$$\exists \psi_{M^{k-1}}(\alpha_1, \dots, \alpha_{N-1}, \beta_1, \dots, \beta_{N-1}) \text{ with form } M^{k-1} \text{ s.t. } \mathcal{U} \models \phi_{L, N-1} \leftrightarrow \psi_{M^{k-1}}.$$

and this leads to a contradiction as we saw earlier.

Q.E.D.

Corollary 1. For the signature Σ of proposition 4:

PF with order \leq has no maximal element.

Proof: Suppose there is a maximal element M , that is to say: \forall prop. forms

$$P : P \leq M, \text{ then by proposition 2: } \exists k_0 \text{ s.t. } M \leq M^{k_0},$$

$$\text{so } \forall P : P \leq M^{k_0}, \text{ so } \exists k \text{ s.t. } \forall P : P \leq M^k.$$

Contradiction.

At last we are now able to prove our Main Theorem, which we shall repeat first:

Theorem: For some $\Sigma : \equiv$ has infinitely many equivalence classes on PF.

Proof: Take the signature Σ of proposition 4, and suppose that there are only finitely many equivalence classes on PF.

Then there are only finitely many M^k which are not mutually equivalent, suppose that these are $M^{k_1}, \dots, M^{k_\ell}$, with $k_1 < \dots < k_\ell$.

By proposition 2: \forall forms $P \exists k_P$ s.t. $P \leq M^{k_P}$. But $M^{k_P} \leq M^{k_\ell}$ for all P by proposition 3; so $\forall P : P \leq M^{k_\ell}$.

Thus $\exists k$ s.t. $\forall P : P \leq M^k$, and this contradicts Proposition 4.

Q.E.D.

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