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REGULAR EXTENSIONS OF ITERATIVE ALGEBRAS
AND METRIC INTERPRETATIONS

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A b s t r a c t. An algebra is said to be iterative if every nontrivial finite system of fixed-point equations has unique solution.

The paper discusses possibilities of finding topological structure for a given iterative algebra so that the unique solution of every system S can be approximated by a sequence of elements generated from S .

K e y w o r d s: fixed-point semantics, iterative algebras, approximations of unique fixed points.

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Introduction

Iterative algebraic theories have been introduced by C.C. Elgot in [3] as an alternative way to define semantics of programming languages by means of fixed points. An algebraic counterpart of this notion is the notion of an iterative algebra introduced by one of us in [5].

Relationships between these two notions are discussed there. The basic assumption behind the notion of an iterative algebra is the existence and uniqueness of solutions of finite nontrivial systems of fixed-point algebraic operations. The main disadvantage of this approach is that one has no way, in general, to obtain (to generate) these unique solutions.

In [5] has been discussed a possibility of defining a partial order on an extension of a given iterative algebra so that the unique solution is obtainable as a least upper bound of a countable chain of "finite" approximations. Such extension satisfying a certain universality condition is called a regular extension of a given iterative algebra. In fact it is a regular algebra in the sense of [8] .

The main result of [5] states that every iterative algebra admits a regular extension, which is unique up to an isomorphism. The proof of this result given in [5] is constructive in the sense that it gives a canonical regular extension for each iterative algebra. But on the other hand, the proof, being constructive, is very long and technical - it involves quite detailed combinatorial study of infinite regular trees.

Here we provide a shorter but nonconstructive proof of existence of regular extensions for an arbitrary iterative algebra. Then this proof is used to answer one of the open problems stated in [5] - we show that there exists an iterative algebra \underline{A} such that its regular extension is not iterative, hence by the uniqueness of regular extensions, every regular extension of \underline{A} is not an iterative algebra.

The main disadvantage of unique fixed-point approach, in our opinion, is a lack of methods to generate the unique solutions of fixed-point equations, or at least to generate "approximants" of these solutions. One way to repair this situation is to find a partial order on a certain extension of a given iterative algebra \underline{A} , so that a unique solution of a fixed-point system in \underline{A} corresponds to the least solution of the same system in the extension. Moreover we require that the latter solution is obtainable as a least upper bound of a chain of ω iterations of this system. This condition means that we expect the extension to be a regular algebra (cf. [5], [8]).

This idea has been discussed in detail in [5] - it turned out that sometimes it is impossible, in the sense that some elements of \underline{A} must be collapsed by every partial order which works.

In the second part of this paper we are studying another possible way of getting approximations of unique solutions. This is done by employing a distance function. The idea of using metric spaces to the semantics of programming languages is not new - there is a number of papers investigating this approach [1], [1], [6]. It follows from Banach fixed-point theorem that if we find a complete metric space structure on a given iterative algebra \underline{A} such that all basic operations become contractive then the unique fixed point of a vector $\vec{P} = (P_1, \dots, P_n)$ of nontrivial n -ary polynomials in \underline{A} is a topological limit of the sequence $\{\vec{P}^{\rightarrow k}(\vec{a}) : k < \omega\}$, which is independent of the choice of $\vec{a} \in A^n$.

We show that there is an iterative algebra such that for no distance function all basic operations are contractive, and for no distance function the above sequence is converging to the unique fixed point of \vec{P} .

Our main positive result of this section is that for every countable iterative algebra \underline{A} over a monadic signature there is a distance function d such that $\langle A, d \rangle$ is weakly contractive, i.e. for every basic n -ary operation f , and for all $\vec{a}, \vec{b} \in A^n$, if $\vec{a} \neq \vec{b}$ then $d(f(\vec{a}), f(\vec{b})) < \max\{d(a_i, b_i) : i \leq n\}$. We also show that the above result does not hold for uncountable structures. Our positive result implies that for every finite iterative algebra over a monadic signature there is a distance function which makes that algebra into a complete metric space with contractive operations.

1. Preliminary definitions and results.

1.1. Let Σ be a signature and let $n \in \omega$. By $T_\Sigma(n)$ we denote the set of all terms over Σ with variables among $\{x_1, \dots, x_n\}$. Let $k \in \omega$, a vector $\vec{P} \in (T_\Sigma(n))^k$ is said to be an ideal vector of polynomial symbols if none of its components P_i ($i \leq k$) is a single variable from $\{x_1, \dots, x_n\}$. If $\vec{P} \in T_\Sigma(n)$ and \underline{A} is a Σ -algebra then $P_{\underline{A}} : A^n \rightarrow A$ is a polynomial determined in \underline{A} by \vec{P} .

1.2. A Σ -algebra \underline{A} is said to be iterative (cf.[5]) if for every $n, k \in \omega$, for every ideal vector $\vec{P} \in T_\Sigma(n+k)^k$ of polynomial symbols the following hold:

1.2.1. For every $\vec{a} \in A^k$ the equation

$$\vec{x} = \vec{P}_{\underline{A}}(\vec{x}, \vec{a})$$

has unique solution in A^n . We denote this solution by $(\vec{P}_{\underline{A}})^+(\vec{a})$.

1.2.2. There is $\vec{a} \in A^k$ such that

$$a_0 \neq ((\vec{p}_A)^+(\vec{a}))_0,$$

i.e. the first component of \vec{a} and that of $(\vec{p}_A)^+(\vec{a})$ are different.

It turns out that iterative algebras and iterative algebraic theories (cf.[2]) correspond each to other in a natural way described in [5].

1.3. Let Σ be a signature and let X be a set disjoint with Σ . Denote by $\bar{R}_\Sigma(X)$ the set of all finite and/or infinite Σ -trees with variables among X (for a formal definition see [8]) which have only a finite number of different subtrees. $\bar{R}_\Sigma(X)$ gets a structure of a Σ -algebra in a usual way (cf.[8]).

1.3.1. Theorem ([5])

For every iterative Σ -algebra A and for every function $f: X \rightarrow A$ there is a unique extension $f: \bar{R}_\Sigma(X) \rightarrow A$ of the mapping f to a Σ -homomorphism.

1.3.2. For $n \in \omega$, let $X_n = \{x_1, \dots, x_n\}$. For every $t \in \bar{R}_\Sigma(X_n)$ and for every iterative Σ -algebra A we define $t_A: A^n \rightarrow A$ by $t_A(\vec{a}) = (\vec{a})(t)$, where $\vec{a} \in A^n$, $(\vec{a}): X_n \rightarrow A$ is a function $(\vec{a})(x_i) = a_i$ (for $1 \leq i \leq n$), and $(\vec{a}): (\bar{R}_\Sigma(X_n) \rightarrow A)$ is the unique extension of (\vec{a}) to a Σ -homomorphism. Functions of the form t_A , where $t \in \bar{R}_\Sigma(X_n)$ are called iterative polynomials in A .

The next result is implicitly contained in [5], Prop. 3.4.

1.3.3. Theorem ([5])

For every $n \in \omega$, and for every $t \in \bar{R}_\Sigma(X_n) - X_n$ there exist $k \in \omega$, and an ideal vector $\vec{p} \in T_{\Sigma}(k+n)^k$ such that in every iterative Σ -algebra A , t_A is equal to the first component of $(\vec{p}_A)^+$.

And conversely, for every ideal vector $\vec{p} \in T_{\Sigma}(k+n)^k$ there exist $t_1, \dots, t_k \in \bar{R}_\Sigma(X_n) - X_n$, such that in every iterative Σ -algebra A , $(\vec{p}_A)^+ = (t_{1A}, \dots, t_{kA})$.

Since Σ -homomorphisms preserve unique fixed points, we obtain the following.

1.3.4. Corollary

If $h: A \rightarrow B$ is a Σ -homomorphism between iterative Σ -algebras A, B , then for every $n \in \omega$, $t \in \bar{R}_\Sigma(X_n)$, $\vec{a} \in A^n$,

$$h(t_A(\vec{a})) = t_B(h(\vec{a})).$$

1.4. Let A be a Σ -algebra. A triple $\langle A, \leq, 1 \rangle$ is said to be an ordered algebra if \leq is a partial order in A such that all Σ -operations are \leq -monotone, and $1 \in A$ is the least element in that order.

Let $\langle A, \leq, 1 \rangle$ be an ordered Σ -algebra, let $n, k \in \omega$, $\vec{p} \in T_{\Sigma}(n+k)^k$, and let $\vec{a} \in A^k$. Define a map $f: A^n \rightarrow A^n$ by $f(\vec{x}) = \vec{p}_A(\vec{x}, \vec{a})$. Let $L_{\vec{p}, \vec{a}} = \{f^i(1, \dots, 1) : i < \omega\}$.

A subset $X \subseteq A$ is said to be an iteration in $\langle \underline{A}, \leq, \perp \rangle$ if there exist $n, k \in \omega$, $\vec{p} \in T(n+k)^n$, $\vec{a} \in A^k$ such that the projection onto the first coordinate $\pi_1^n(l_{\vec{p}, \vec{a}}) = X$.

An ordered Σ -algebra $\langle A, \leq, \perp \rangle$ is said to be a regular algebra (cf. [8]) if for every $n, k \in \omega$, $\vec{p} \in T_\Sigma(n+k)^k$, $\vec{a} \in A^k$,

1.4.1. The set $L_{\vec{p}, \vec{a}}^\perp$ has least upper bound in A^n with respect to \leq defined componentwise.

1.4.2. $\vec{p}_A (\sup L_{\vec{p}, \vec{a}}^\perp, \vec{a}) = \sup L_{\vec{p}, \vec{a}}^\perp$.

We denote $\sup L_{\vec{p}, \vec{a}}^\perp$ by $(\vec{p}_A)^\nabla(\vec{a})$.

1.5. Let $\langle \underline{A}, \leq_{\underline{A}}, \perp_{\underline{A}} \rangle$, $\langle \underline{B}, \leq_{\underline{B}}, \perp_{\underline{B}} \rangle$ be regular Σ -algebras. A function

$h : A \rightarrow B$ is said to be a regular homomorphism if for every $n, k \in \omega$, $\vec{p} \in T_\Sigma(n+k)^k$, and for every $\vec{a} \in A^k$,

$$h((\vec{p}_A)^\nabla(\vec{a})) = (\vec{p}_B)^\nabla(h(\vec{a})).$$

In the above formula application of h to vectors is defined componentwise in the usual way.

1.6. Let a set X be disjoint from Σ .

Denote by $R_\Sigma(X)$ the Σ -algebra $\bar{R}_\Sigma(X \cup \{\perp\})$, where \perp is an element belonging to neither Σ nor to X .

Define a syntactic order \leq on trees from $R_\Sigma(X)$ as follows: for

$t_1, t_2 \in R_\Sigma(X)$, $t_1 \leq t_2$ if t_2 can be obtained from t_1 by replacing some occurrences of \perp in t_1 by some elements of $R_\Sigma(X)$.

1.6.1. Theorem ([8])

For every regular Σ -algebra $\langle \underline{A}, \leq_{\underline{A}}, \perp_{\underline{A}} \rangle$ and for every function $f : X \rightarrow A$ there is exactly one extension of f to a regular homomorphism $\bar{f} : R_\Sigma(X) \rightarrow \underline{A}$.

1.7 Let $n \in \omega$, and let $X_n = \{x_1, \dots, x_n\}$ be the set of individual variables. Let $t \in R_\Sigma(X_n)$ and let $\langle \underline{A}, \leq_{\underline{A}}, \perp_{\underline{A}} \rangle$ be a regular Σ -algebra. We define

$t_{\underline{A}} : A^n \rightarrow A$, by $t_{\underline{A}}(\vec{a}) = \vec{a}(t)$, where $\vec{a} \in A^n$, $(\vec{a}) : X_n \rightarrow A$ is a function $(\vec{a})(x_i) = a_i$, and $(\vec{a}) : R_\Sigma(X_n) \rightarrow \underline{A}$ is the unique extension of (\vec{a}) to a

regular homomorphism. Functions of the form $t_{\underline{A}}$, where $t \in R_\Sigma(X_n)$ are called $(n\text{-ary})$ regular polynomials in $\langle \underline{A}, \leq_{\underline{A}}, \perp_{\underline{A}} \rangle$.

1.7.1. Theorem ([8], [9])

Let $\langle \underline{A}, \leq_{\underline{A}}, \perp_{\underline{A}} \rangle$, $\langle \underline{B}, \leq_{\underline{B}}, \perp_{\underline{B}} \rangle$ be regular Σ -algebra. Let $h : A \rightarrow B$ be a function. The following conditions are equivalent:

- (i) h is a regular homomorphism of $\langle \underline{A}, \leq_{\underline{A}}, \perp_{\underline{A}} \rangle$ into $\langle \underline{B}, \leq_{\underline{B}}, \perp_{\underline{B}} \rangle$.
- (ii) h is a Σ -homomorphism, $h(\perp_{\underline{A}}) = \perp_{\underline{B}}$, if E is an iteration in $\langle \underline{A}, \leq_{\underline{A}}, \perp_{\underline{A}} \rangle$ then $\sup_{\leq_{\underline{B}}} h(E)$ exists and $\sup_{\leq_{\underline{B}}} h(E) = h(\sup_{\leq_{\underline{A}}} E)$.

(iii) for every $n \in \omega$, for every $t \in R_{\Sigma}(X_n)$, and for every $a \in A^n$,

$$h(t_{\underline{A}}(\vec{a})) = t_{\underline{B}}(h(\vec{a})).$$

Remark: The original definition of a regular homomorphism was given in [8] as condition (ii) of the above theorem. Equivalence of (ii) and (iii) is proved in [8], Prop. 3.15. Equivalence of (i) and (iii) follows from the following normal form theorem proved in [9], Thm. 3.1.

1.7.2. Theorem ([9])

For every $k \in \omega$, and for every $t \in R_{\Sigma}(X_k)$, there exist $n \in \omega$ and a vector $\vec{p} \in T_{\Sigma}(n+k)^n$ such that for every regular Σ -algebra $\langle \underline{A}, \leq_{\underline{A}}, \perp_{\underline{A}} \rangle$, $t_{\underline{A}}$ is equal to the first component of $(\vec{p}_{\underline{A}})^{\vee}$.

And conversely, for every $\vec{p} \in T_{\Sigma}(n+k)^n$ there exist $t_1, \dots, t_n \in R_{\Sigma}(X_k)$ such that for every regular Σ -algebra $\langle \underline{A}, \leq_{\underline{A}}, \perp_{\underline{A}} \rangle$, $(\vec{p}_{\underline{A}})^{\vee} = (t_{1\underline{A}}, \dots, t_{n\underline{A}})$.

2. Regular extensions of iterative algebras

2.1. An iterative Σ -algebra \underline{A} is said to admit a regular extension (cf. [5])

if there is a regular Σ -algebra $\underline{A}_R = \langle A_R, \leq, \perp \rangle$ and a map $\varphi_{\underline{A}} : A \rightarrow A_R$ subject to the following conditions:

2.1.1. for every $n, k \in \omega$, for an arbitrary ideal vector $\vec{p} \in T_{\Sigma}(n+k)^n$, and for every $\vec{a} \in A^k$,

$$\varphi_{\underline{A}}((\vec{p}_{\underline{A}})^{\vee}(\vec{a})) = (\vec{p}_{\underline{A}_R})^{\vee}(\varphi_{\underline{A}}(\vec{a})).$$

2.1.2. for every regular Σ -algebra $\langle \underline{B}, \leq, \perp \rangle$ and for every function $f : A \rightarrow B$ satisfying

$$f((\vec{p}_{\underline{A}})^{\vee}(\vec{a})) = (\vec{p}_{\underline{B}})^{\vee}(f(\vec{a}))$$

for all $n, k \in \omega$, ideal $\vec{p} \in T(n+k)^n$, and $\vec{a} \in A^k$; there is exactly one regular homomorphism $f^* : \underline{A}_R \rightarrow \underline{B}$ with $f^* \varphi_{\underline{A}} = f$.

The pair $(\varphi_{\underline{A}}, \underline{A}_R)$ is called a regular extension of \underline{A} .

2.2. An iterative algebra \underline{A} is said to admit a faithful regular extension if there is a regular extension $(\varphi_{\underline{A}}, \underline{A}_R)$ with $\varphi_{\underline{A}}$ being injective.

It is easy to prove that every iterative algebra admits at most one (up to isomorphism) regular extension, i.e. if $(\varphi_{\underline{A}}, \underline{A}_R)$ and $(\psi_{\underline{A}}, \underline{A}'_R)$ are regular extensions of the same iterative algebra \underline{A} , then there is a regular isomorphism $\xi : \underline{A}_R \rightarrow \underline{A}'_R$ with $\xi \varphi_{\underline{A}} = \psi_{\underline{A}}$. In [5] it is proved that there is an iterative algebra which does not admit a faithful regular extension. However, the main result of [5] states that

2.3 Theorem ([5])

Every iterative algebra admits a regular extension.

The aim of this section is to give a shorter proof of 2.3. First we introduce some useful notions.

2.4. Let \underline{A} be an iterative Σ -algebra. Let $i_A^* : \bar{R}_\Sigma(A) \rightarrow \underline{A}$ be the unique extension of $\text{id} : A \rightarrow A$ ($\text{id}(x)=x$) to a Σ -homomorphism, (cf.1.3.1).

Define a binary relation $e_{\underline{A}} \subseteq R_\Sigma(A)^2$ by $e_{\underline{A}} = \{(t, t') \in R_\Sigma(A)^2 : t, t' \in \bar{R}_\Sigma(A) \text{ and } i_A^*(t) = i_A^*(t')\}$.

A quasi-order r (i.e. a reflexive and transitive binary relation) in $R_\Sigma(A)$ is said to be compatible if

2.4.1. $\leq \subseteq r$, where \leq is the synthactic partial order in $R_\Sigma(A)$, cf.1.6.

2.4.2. for every $n \in \omega$, for every n -ary operation symbol f in Σ and for all

$\vec{t}, \vec{t}' \in R_\Sigma(A)$, if for all $1 \leq i \leq n$, $(t_i, t'_i) \in r$, then $(f(\vec{t}), f(\vec{t}')) \in r$.

If r is a compatible quasi-order in $R_\Sigma(A)$ then the relation $\sim_r = \{(t, t') \in R_\Sigma(A) : (t, t') \in r \text{ and } (t', t) \in r\}$ is a congruence in $R_\Sigma(A)$ and $\langle R_\Sigma(A)/\sim_r, \subseteq_r, |\cdot| \rangle$ is an ordered Σ -algebra, where $R_\Sigma(A)/\sim_r$ is the quotient structure, $|\cdot|$ is the equivalence class determined by \sim_r , and for $t, t' \in R_\Sigma(A)$, $|t| \subseteq_r |t'|$ iff $(t, t') \in r$.

2.5. Let \bigcup_A be the set of all compatible quasi-orders r in $R_\Sigma(A)$ which satisfy the following two conditions:

2.5.1. $e_{\underline{A}} \subseteq r$.

2.5.2. For every iteration E in $\langle R_\Sigma(A), \leq, \cdot \rangle$ (\leq is the synthactical order) and for every $t \in R_\Sigma(A)$, if for all $x \in E$, $(x, t) \in r$, then $(\sup^{\leq} E, t) \in r$, where $\sup^{\leq} E$ is the least upper bound of E in $\langle R_\Sigma(A), \leq, \cdot \rangle$.

The following result follows immediately from our definitions.

2.6. Proposition

(i) For a compatible quasi-order r in $R_\Sigma(A)$, $r \in \bigcup_A$ iff $\langle R_\Sigma(A)/\sim_r, \subseteq_r, |\cdot| \rangle$ is a regular algebra and $e_{\underline{A}} \subseteq r$.

(ii) \bigcup_A is closed under arbitrary intersections.

Let $r_{\underline{A}} = \bigcap \bigcup_A$. To avoid complex notations we denote $\sim_{r_{\underline{A}}}$ by $\sim_{\underline{A}}$, and $\subseteq_{\underline{A}}$ by $\subseteq_{\underline{A}}$.

Let $\underline{A}_R = \langle R_\Sigma(A)/\sim_{\underline{A}}, \subseteq_{\underline{A}}, |\cdot| \rangle$.

2.7. Theorem

The pair $\langle \underline{A}_R, \varphi_{\underline{A}} \rangle$, where $\varphi_{\underline{A}} : A \rightarrow R_\Sigma(A)/\sim_{\underline{A}}$ is defined by $\varphi_{\underline{A}}(a) = |a|$ for $a \in A$, is a regular extension of \underline{A} .

Proof. It follows from 2.4 and 2.5 that if E is an iteration in $\langle R_\Sigma(A), \leq, \perp \rangle$ where \leq is the syntactical order then $|E| = \{|t| : t \in E\}$ is an iteration in \underline{A}_R , where $|t|$ is the equivalence class of $\sim_{\underline{A}}$ determined by t . Moreover, $\sup^E |E| = |\sup^{\leq E}|$. Therefore, since $| \cdot | : R_\Sigma(A) \rightarrow R_\Sigma(A)/\sim_{\underline{A}}$ is a Σ -homomorphism preserving bottom elements, it follows from Theorem 1.7.1. that $| \cdot |$ is a regular homomorphism. Below in the proof we apply mappings to vectors, extending them componentwise.

Let $n, k \in \omega$ and let $p \in \Gamma_\Sigma(n+k)^k$ be an ideal vector of polynomial symbols. Then for every $\vec{a} \in A^k$,

$$i_{\underline{A}}^*((\vec{p}_{R_\Sigma(A)})^+(\vec{a})) = (\vec{p}_{\underline{A}}^*)^+(i_{\underline{A}}(\vec{a})) = (\text{by (1.3.4)}) i_{\underline{A}}^*((\vec{p}_{\underline{A}})^+(\vec{a})). \quad (\text{since } i_{\underline{A}}^* \upharpoonright A = \text{id.})$$

Therefore by 2.5.1 we obtain

$$(2.7.1) \quad |(\vec{p}_{R_\Sigma(A)})^+(\vec{a})| = |(\vec{p}_{\underline{A}})^+(\vec{a})|, \text{ for every } n, k \in \omega, \text{ ideal vector}$$

$$\vec{p} \in T_\Sigma(n+k)^k, \vec{a} \in A^k.$$

Since $R_\Sigma(A)$ is both iterative and regular,

$$(2.7.2) \quad (\vec{p}_{R_\Sigma(A)})^+(\vec{a}) = (\vec{p}_{R_\Sigma(A)})^\nabla(\vec{a}), \text{ for every } n, k \in \omega, \text{ ideal vector}$$

$$P \in T_\Sigma(n+k)^k, a \in A^k. \text{ Let } n, k \in \omega, \text{ let } P \in T_\Sigma(n+k)^k \text{ and } a \in A^k.$$

Then we have

$$\begin{aligned} \varphi_{\underline{A}}((\vec{p}_{\underline{A}})^+(\vec{a})) &= \varphi_{\underline{A}}((\vec{p}_{R_\Sigma(A)})^+(\vec{a})) \quad (\text{by (2.7.1)}) \\ &= \varphi_{\underline{A}}((\vec{p}_{R_\Sigma(A)})^\nabla(\vec{a})) \quad (\text{by (2.7.2)}) \\ &= (\vec{p}_{\underline{A}})^\nabla(\varphi_{\underline{A}}(\vec{a})). \quad (\text{since } | \cdot | \text{ is a regular homomorphism}). \end{aligned}$$

Thus $(\varphi_{\underline{A}}, \underline{A}_R)$ satisfies 2.1.1. Now let $\langle \underline{B}, \leq_B, \perp \rangle$ be a regular Σ -algebra and let $f : A \rightarrow B$ be a function satisfying:

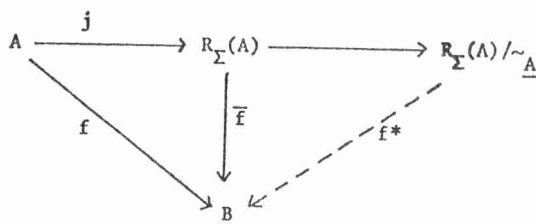
$$(2.7.3) \quad \text{for every } n, k \in \omega, \text{ ideal vector } \vec{p} \in T_\Sigma(n+k)^k, \text{ and for every } \vec{a} \in A^k, \\ f((\vec{p}_{\underline{A}})^+(\vec{a})) = (\vec{p}_{\underline{B}})^\nabla(f(\vec{a})).$$

We will prove that there exists exactly one regular homomorphism

$$f^* : \underline{A}_R \rightarrow \underline{B} \text{ such that } f^* \varphi_{\underline{A}} = f.$$

Since $\{|a| : a \in A\}$ generates \underline{A}_R (i.e. \underline{A}_R is the least regular subalgebra of \underline{A}_R containing $\{|a| : a \in A\}$), so there is at most one f^* satisfying required property.

Consider the following diagram:



where j is the identity embedding, and \bar{f} is the unique extension of f to a regular homomorphism. It follows from [9], Prop. 5.2. that f^* exists so that the above diagram commutes if

(2.7.4) for all $t, t' \in R_{\Sigma}(A)$, if $t \sim_{\underline{A}} t'$ then $\bar{f}(t) = \bar{f}(t')$.

In order to prove (2.7.4) let us consider the quasi-order $r = \{(t, t') \in R_{\Sigma}(A)^2 : \bar{f}(t) \leq_{\underline{B}} \bar{f}(t')\}$.

It follows from [9], Prop. 5.1 that $\langle R_{\Sigma}(A)/\sim_r, \subseteq_r, [1] \rangle$ is a regular algebra. Let us observe that $r \in \bigcup_{\underline{A}} \underline{A}$ implies (2.7.4). To prove the former, by Proposition 2.6 it remains to show -

(2.7.5) $e_{\underline{A}} \subseteq r$.

Let $t \in \bar{R}_{\Sigma}(A)$. Since t has only a finite number of different subtrees, there is $m \in \omega$ such that only m different elements of A occur in t . Call these elements a_1, \dots, a_m .

Let ξ be a tree which is obtained from t by replacing every a_i in t by x_i for $i \leq m$. Obviously $\xi \in R_{\Sigma}(X_m)$ and $t = \xi_{\bar{R}_{\Sigma}(A)}(\vec{a})$, where $\vec{a} = (a_1, \dots, a_m)$.

Now we have

$$\begin{aligned}
 f(i_{\underline{A}}^*(t)) &= f(i_{\underline{A}}^*(\xi_{\bar{R}_{\Sigma}(A)}(\vec{a}))) \\
 &= f(\xi_{\underline{A}}(\vec{a})) && \text{(by (1.3.4))} \\
 &= \xi_{\underline{B}}(f(\vec{a})) && \text{(by (1.3.3), (1.7.2), (2.7.3))} \\
 &= \xi_{\underline{B}}(\bar{f}(\vec{a})) \\
 &= \bar{f}(\xi_{R_{\Sigma}(A)}(\vec{a})) && \text{(by (1.7.1))} \\
 &= \bar{f}(t). && \text{(by (2.7.2.), (1.7.2), (1.3.3))}
 \end{aligned}$$

Thus we have proved that \bar{f} restricted to $\bar{R}_{\Sigma}(A)$ coincides with $fi_{\underline{A}}^*$, and therefore condition (2.7.5) holds. This completes the proof of the theorem.

The next result answers one of the problems left in [5].

2.8 Theorem

There exists an iterative algebra \underline{A} such that the algebra \underline{A}_R is not iterative.

Before we start proving this theorem we present an idea of the proof.

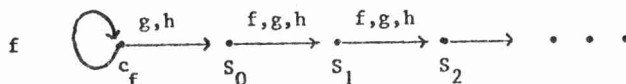
First we present an iterative algebra $\underline{A} = \langle A, f, g, h \rangle$, where operations f, g, h are unary. Then we prove that there exist $a, b \in A$ such that a and b are collapsed in (every) regular extension, $f(a) = b$ in \underline{A} , and if $s \in A$ is the unique fixed point of f then s and a are not collapsed in (every) regular extension. Thus $|a|$ and $|s|$ are two different fixed points of f in \underline{A}_R .

2.8.1 (The construction of \underline{A})

$\underline{A} = \langle A, f, g, h \rangle$, where f, g, h are unary functions in A , and A is the union of countably many sets C_w , where w is a nonempty finite word (i.e. a sequence) over $\{f, g, h\}$ such that for no $u \in \{f, g, h\}^*$ (by $\{f, g, h\}^*$ we denote the set of all words over $\{f, g, h\}$) and no $n \in \omega$, $w = u^n$; such words will be called noncyclic.

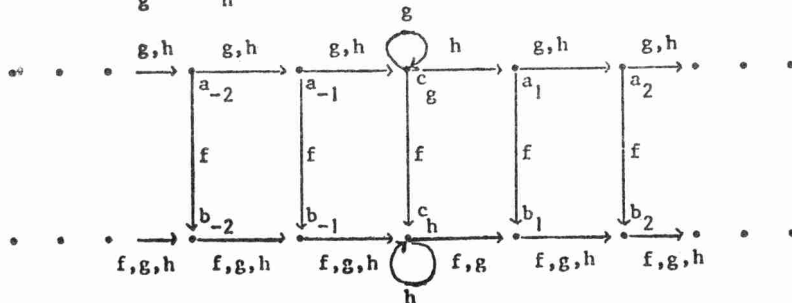
Since the fixed-point equations in \underline{A} are of the form $x = w(x)$, where $w \in \{f, g, h\}^*$, it is enough to consider only equations $x = w(x)$ with w being noncyclic. It is intended that the equation $x = w(x)$ has its unique solution in C_w , for every noncyclic $w \in \{f, g, h\}^*$. We define C_w together with operations defined on it, in a form of a graph with the convention that there is an arrow from the node a to the node b , which is labelled by f (resp. g or h) if $f(a) = b$ (resp. $g(a) = b$ or $h(a) = b$)

The set C_f is defined as follows:



It is obvious that the only fixed-point equation solvable in C_f is of the form $x = f^n(x)$, for some $n > 0$.

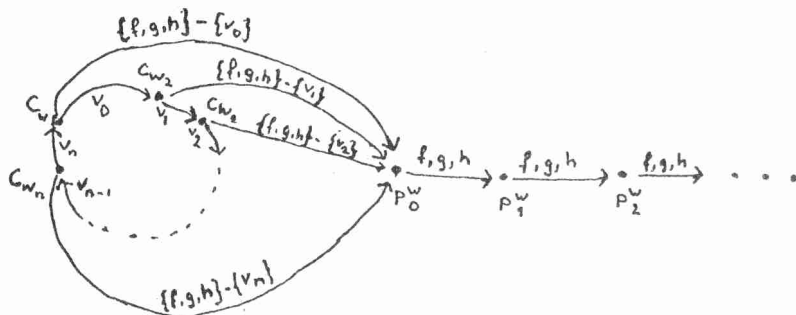
Sets C_g and C_h are defined in the following way:



where $C_g = \{c_g\} \cup \{a_n : n \geq 1\} \cup \{a_{-n} : n \geq 1\}$, and $C_h =$

$C_h = \{c_h\} \cup \{b_n : n \geq 1\} \cup \{b_{-n} : n \geq 1\}$. It is obvious that the only fixed-point equation solvable in C_g (in C_h resp.) is of the form $x = g^n(x)$ ($x = h^n(x)$) for some $n > 0$.

Let $w \in \{f, g, h\}^*$ be a noncyclic word of greater than one, let $w = v_0 v_1 \dots v_n$ with $v_i \in \{f, g, h\}$ for $i \leq n$. Let $w_i = v_i, v_{i+1}, \dots, v_n, v_0, \dots, v_{i-1}$ for $i \leq n$. The set C_w is defined as follows:



So $C_w = \{c_{w_0}, \dots, c_{w_n}\} \cup \{p_i^w : i < \omega\}$. If $u \in \{f, g, h\}^*$ is a cyclic permutation of a noncyclic word w then we put $D_u = C_w$ with operations defined the same. Moreover we require that if $u, w \in \{f, g, h\}^*$ are noncyclic words none of which is a cyclic permutation of another then $C_u \cap D_w = \emptyset$.

Clearly $w_i(c_{w_i}) = c_{w_i}$ for $i \leq u$. Moreover it is easy to see that if $r(x) = x$ is solvable in C_w , with $r \in \{f, g, h\}^*$ then there exist $i \leq n$ (n is the length of w) and $k \in \omega$ such that $r = (w_i)^k$.

The above remarks show that $A = \bigcup \{C_w \mid w \in \{f, g, h\}^*, w \text{ is nonempty and noncyclic}\}$ is an iterative algebra.

2.8.2 $(C_g \sim_A C_n)$

To see this let us observe:

$$C_g = (g_A)^+ \sim_A \sup\{g^n(1) : n < \omega\} \leq \sup\{g^n(b_{-n}) : n < \omega\} \sim_A \sim_A \sup\{C_h : n < \omega\} = C_h.$$

Similarly we have:

$$C_n = (h_A)^+ \sim_A \sup\{h^n(1) : n < \omega\} \leq \sup\{h^n(a_{-n}) : n < \omega\} \sim_A \sim_A \sup\{C_g : n < \omega\} = C_g.$$

The above sup's are taken in $\langle R_\Sigma(A), \leq, \perp \rangle$ where $\Sigma_1 = \{f, g, h\}$
 $\Sigma_u = \emptyset$, $fwu \neq 1$; \leq is the syntactical partial order in $R_\Sigma(A)$.

2.8.3 $(C_f \not\leq_A C_g)$

To prove this it is enough to find a quasi-order $r \in \bigcup_A$ such that
 $(c_g, c_f) \notin r$.

First we introduce necessary notations. Let Σ be a signature such that
 $\Sigma_1 = \{f, g, h\}$, $\Sigma_n = \emptyset$ for $n \neq 1$. Let $f^\omega \in R_\Sigma(A)$ be the infinite tree satisfying
 $f_{R_\Sigma(A)}(f^\omega) = f^\omega$. Let $B = \{f^\omega\} \cup \{f^n(C_p) : n < \omega\}$ (here $f^n(C_f)$ is a term
 from $R_\Sigma(A)$), and let $D = R_\Sigma(A) \setminus (B \cup \{w(1) : w \in \{f, g, h\}^*\})$.

Let r be the least binary relation in $R_\Sigma(A)$ satisfying the following conditions:

- (1) If $t_1 \in R_\Sigma(A)$, and $t_2 \in D$ then $(t_1, t_2) \in r$.
- (2) If $t_1, t_2 \in B$ then $(t_1, t_2) \in r$.
- (3) If $t \in B$ then for every $w \in \{f, g, h\}^*$, $(w(1), t) \in r$
 iff there is $n < \omega$ with $w = f^n$.

- (4) For all $w, u \in \{f, g, h\}^*$, $(w(1), u(1)) \in r$ iff
 w is a prefix of u .

It is easy to check that r is a compatible quasi-order in $R_\Sigma(A)$, and that
 $e_A \subseteq r$. Obviously, $(c_g, c_f) \notin r$. To complete the proof of Theorem 2.8 it is
 enough to show that r satisfies 2.5.2.

Let E be an iteration in $\langle R_\Sigma(A), \leq, \perp \rangle$ and let $t \in R_\Sigma(A)$ be such that
 for all $x \in E$, $(x, t) \in r$. If E is finite then obviously $(\sup^{\leq} E, t) \in r$, so let
 us assume that E is infinite. Thus $E \subseteq \{w(1) : w \in \{f, g, h\}^*\}$ and by (1) the
 case $t \in D$ is trivial. However, if $t \in B$ then by (3) $\sup^{\leq} E = f^\omega$, and by (2)
 $(f^\omega, t) \in r$. As there are not other possible cases, r satisfies condition 2.5.2.

3. Metric interpretations.

3.1. Let A be a Σ -algebra and let d be a metric on A . The pair $\langle A, d \rangle$ is
 said to be a contractive algebra if for every $n < \omega$ and for every n -ary
 Σ -operation f , there exists a constant $0 \leq C_f < 1$ such that for all $\vec{a}, \vec{b} \in A^n$,

$$d(f_{\underline{A}}(\vec{a}), f_{\underline{A}}(\vec{b})) \leq C_f \cdot \max\{d(a_i, b_i) : 1 \leq i \leq n\}.$$

3.2. Let $\langle \underline{A}, d \rangle$ be a pair where \underline{A} is a Σ -algebra and d is a metric on \underline{A} .

$\langle \underline{A}, d \rangle$ is said to be an algebra with convergence if all Σ -operations in \underline{A} are continuous with respect to d , and for every $n, k \in \omega$, and for an ideal vector $\vec{p} \in T_{\Sigma}(n+k)^n$, for every $\vec{a} \in \underline{A}^k, \vec{b} \in \underline{A}^n$ the sequence $\{\vec{p}_{\underline{A}}^{(m)}(\vec{b}, \vec{a}) : m < \omega\}$ has a limit in \underline{A}^n which is independent of \vec{b} . Here the elements $\vec{p}_{\underline{A}}^{(m)}(\vec{b}, \vec{a})$ are defined inductively:

$$\vec{p}_{\underline{A}}^{(0)}(\vec{b}, \vec{a}) = \vec{b};$$

$$\text{for } m \geq 0, \vec{p}_{\underline{A}}^{(m+1)}(\vec{b}, \vec{a}) = \vec{p}_{\underline{A}}(\vec{p}_{\underline{A}}^{(m)}(\vec{b}, \vec{a}), \vec{a}).$$

The next result is a slight modification of Banach fixed-point theorem.

3.3. (Banach fixed-point theorem)

If $\langle \underline{A}, d \rangle$ is a contractive algebra and d is a complete metric then \underline{A} is an iterative algebra and $\langle \underline{A}, d \rangle$ is an algebra with convergence. Moreover, for every $n, k \in \omega$, for every ideal vector $\vec{p} \in T_{\Sigma}(n+k)^n$, and for arbitrary $\vec{a} \in \underline{A}^k, \vec{b} \in \underline{A}^n$, $\lim_{m \rightarrow \infty} \vec{p}_{\underline{A}}^{(m)}(\vec{b}, \vec{a}) = (\vec{p}_{\underline{A}})^+(\vec{a})$ and $(\vec{p}_{\underline{A}})^+$ is a vector of contractive mappings.

It is natural to ask if for a given iterative algebra \underline{A} there is a metric d such that $\langle \underline{A}, d \rangle$ is a contractive algebra, or such that this pair is an algebra with convergence. The next result has been proved in [2] for algebraic theories - the proof for algebras remains the same and thus will be omitted.

3.4. Proposition ([2])

If \underline{A} is an iterative algebra and $\langle \underline{A}, d \rangle$ is a contractive algebra, then $\langle \underline{A}, d \rangle$ is an algebra with convergence.

As we shall see, in general an iterative algebra \underline{A} may not possess a metric d such that $\langle \underline{A}, d \rangle$ is an algebra with convergence. This gives rise to the following definition.

3.5. Let \underline{A} be Σ -algebra and d a metric on \underline{A} . The pair $\langle \underline{A}, d \rangle$ is said to be a weakly contractive algebra if for every $n \in \omega$ and for every n -ary Σ -operation f , for all $\vec{a}, \vec{b} \in \underline{A}^n$, if $\vec{a} \neq \vec{b}$ then $d(f(\vec{a}), f(\vec{b})) < \max\{d(a_i, b_i) : 1 \leq i \leq n\}$.

Obviously, if \underline{A} is finite and $\langle \underline{A}, d \rangle$ is weakly contractive then it is a contractive algebra.

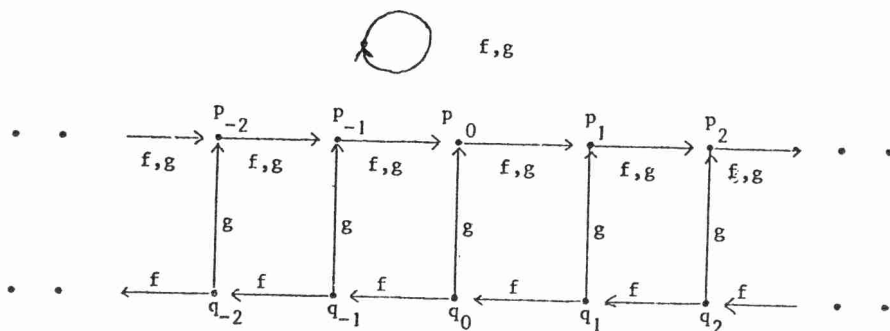
3.6. Theorem

There exists a countable iterative algebra \underline{A} such that:

(1) there is a metric d_1 such that $\langle \underline{A}, d_1 \rangle$ is an algebra with convergence;

- (2) there is a metric d_2 such that $\langle \underline{A}, d_2 \rangle$ is weakly contractive;
 (3) for no metric d , $\langle \underline{A}, d \rangle$ is weakly contractive with convergence. Therefore
 for no metric d , $\langle \underline{A}, d \rangle$ is contractive.

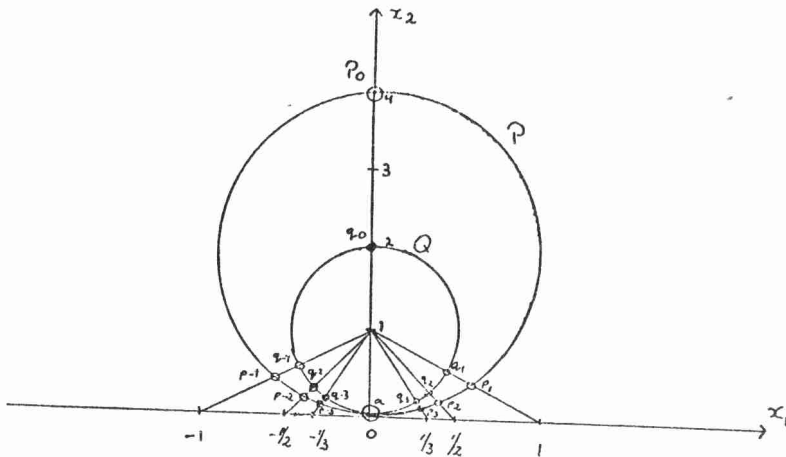
Proof. Let $\underline{A} = \langle A, f, g \rangle$, where f and g are unary functions and \underline{A} is defined by the following graph:



So $A = \{a\} \cup \{p_k : k \in \mathbb{Z}\} \cup \{q_k : k \in \mathbb{Z}\}$, where \mathbb{Z} stands for the set of all integers, and operations are defined above. Note that \underline{A} is an iterative algebra as all ideal fixed-point equations have a as the unique solution, since the second part of the picture contains no loops.

To prove (1) we embed A into real plane \mathbb{R}^2 and take metric on A induced by the distance function in \mathbb{R}^2 : $(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

We embed A into \mathbb{R}^2 in the following way. Let Q be the circle defined by the equation $x_1^2 + (x_2 - 1)^2 = 1$, and let P be the circle $x_1^2 + (x_2 - 2)^2 = 4$. For every new $0 < n < \omega$ let L_n be the straight line connecting points $(-\frac{1}{n}, 0)$ and $(0, 1)$ and let R_n be the straight line connecting $(\frac{1}{n}, 0)$ and $(0, 1)$. Now we are ready to describe an embedding φ of A into \mathbb{R}^2 . Let $\varphi(a) = (0, 0)$, $\varphi(q_0) = (0, 2)$, $\varphi(p_0) = (0, 4)$. For every $0 < n < \omega$, let $\varphi(p_n)$ (respectively $\varphi(q_n)$) be the point $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$ such that $x_1 > 0$, and $\vec{x} \in P \cap R_n$ ($\vec{x} \in Q \cap R_n$, respectively). Similarly, for every $0 < n < \omega$, let $\varphi(p_{-n})$ ($\varphi(q_{-n})$, respectively) be the point $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$ such that $x_1 < 0$, and $\vec{x} \in P \cap L_n$ ($\vec{x} \in Q \cap L_n$, respectively). The following picture illustrates the situation.



It is easy to see that the metric d in A which is induced by the distance function S in R^2 makes $\langle \underline{A}, d \rangle$ into an algebra with convergence.

Property (2) follows from a general result (Theorem 3.8 in this paper) which depends only on the cardinality of A .

To prove (3) suppose d is a metric on A such that $\langle A, d \rangle$ is a weakly contractive algebra with convergence.

Let $n < \omega$, then $g_{\underline{A}}^{n+1}(q_{-n}) = p_0$ and $g_{\underline{A}}^{n+1}(q_{-n+1}) = p_1$. Therefore, since $\langle \underline{A}, d \rangle$ is weakly contractive we have

$$(3.6.1) \text{ for every } n < \omega, d(p_0, p_1) < d(q_{-n}, q_{-n+1}).$$

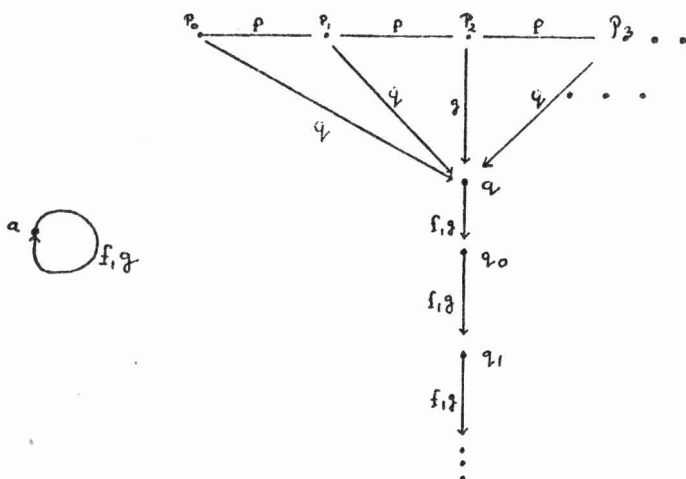
As $\langle \underline{A}, d \rangle$ is an algebra with convergence, $\lim_{n \rightarrow \infty} q_{-n} = \lim_{n \rightarrow \infty} f_{\underline{A}}^n(q_0) = a$. Thus $d(q_{-n}, q_{-n+1}) \rightarrow 0$ and by (3.6.1) $d(p_0, p_1) = 0$. Obtained contradiction proves (3) and completes the proof of 3.6.

Remark. (1) and (3) of the above theorem confirm the hypothesis stated in [2] that the notion of "contractiveness" of a given structure is essentially stronger than that of being a structure with convergence.

3.7. Proposition

There exist a countable iterative algebra \underline{A} such that for no metric d , $\langle \underline{A}, d \rangle$ is an algebra with convergence.

Proof. Let $\underline{A} = \langle A, f, g \rangle$ be the following algebra presented in the picture below.



Observe that \underline{A} is really an iterative algebra as the part defined on the set $A \setminus \{a\}$ contains no loop.

Suppose d is a metric in A which makes $\langle \underline{A}, d \rangle$ into an algebra with convergence. Then $\lim_n p_n = \lim_n f_{\underline{A}}^n(p_0) = (f_{\underline{A}})^+ = a$.
Therefore

$$a = g_{\underline{A}}(a) = g_{\underline{A}}(\lim_n p_n) = \lim_n g_{\underline{A}}(p_n) = q.$$

Obtained contradiction completes the proof of 3.7.

3.8. Theorem

Let \underline{A} be a monodic iterative Σ -algebra, i.e. Σ contains only constants and unary operation symbols. If A is countable then there exists a metric d on A such that $\langle \underline{A}, d \rangle$ is weakly contractive.

Proof. Let \underline{A} be a unary iterative Σ -algebra with A being countable. Let $D = \{(a, b) \in A^2 : a \neq b\}$. Define a binary relation $r \subseteq D^2$ as follows: for $(a_1, a_2), (b_1, b_2) \in D$, $((a_1, a_2), (b_1, b_2)) \in r$ iff there exists an ideal polynomial symbol $p \in T_{\Sigma}(1)$ such that $a_1 = p_{\underline{A}}(b_1)$ and $a_2 = p_{\underline{A}}(b_2)$.

It is easy to see that r is transitive.

We show that

(3.8.1) for every $(a, b) \in D$, $((a, b), (a, b)) \notin r$.

Indeed, if $((a, b), (a, b)) \in r$ for a certain $(a, b) \in D$, then for some ideal polynomial symbol $p \in T_{\Sigma}(1)$, $a = p_{\underline{A}}(a)$ and $b = p_{\underline{A}}(b)$, so $a = b$ and we get contradiction with $(a, b) \in D$.

From the above remarks it follows that $r \cup \{(a,b), (a,b) : (a,b) \in D\}$ is a partial order in D . It follows from Zorn's lemma that every partial order can be extended to a linear order (i.e. to such a partial order where any two elements of the carrier are comparable). It is well known that every countable linear order can be embedded into the poset of reals $\langle R, \leq \rangle$ where \leq is a natural partial order in reals. Therefore every countable linear order can be embedded into the open interval $(\frac{1}{2}, 1)$ with order induced from reals.

Let $\varphi : D \rightarrow (\frac{1}{2}, 1)$ be such an embedding.

Define $d : A^2 \rightarrow R$ as follows

$$d(a,b) = \begin{cases} 0, & \text{if } a = b \\ \min\{\varphi(a,b), \varphi(b,a)\}, & \text{if } a \neq b \end{cases}$$

It is a routine matter to check that d is metric with required properties.

The next result shows that 3.8 does not hold for uncountable structures.

3.9. Theorem

There exists an iterative monadic algebra $\underline{A} = \langle A, \{f_\alpha : \alpha < \omega_1\} \rangle$ with $\text{card}(A) = \mathfrak{c}/2_1$ such that for no metric d on A , $\langle \underline{A}, d \rangle$ is weakly contractive.

Proof. Let $A = \{a\} \cup \{\alpha : \alpha < \omega_1\}$. For every $\alpha < \omega_1$, $f_{\alpha A} : A \rightarrow A$ is defined as follows:

$f_{\alpha A}(a) = a$ and for $\beta < \omega_1$,

$$f_{\alpha A}(\beta) = \begin{cases} \beta+1 & \text{if } \alpha \leq \beta \\ \alpha & \text{if } \beta < \alpha \end{cases}$$

It is easy to see that each $f_{\alpha A}$ restricted to $\{\alpha : \alpha < \omega_1\}$ is strictly increasing, so \underline{A} is an iterative algebra.

Suppose d is a metric on A which makes $\langle \underline{A}, d \rangle$ into a weakly contractive algebra. Define a sequence of reals $\{r_\alpha : \alpha < \omega_1\}$, where $r_\alpha = d(a, \alpha)$. If $\alpha < \beta < \omega_1$ then $r_\beta = d(a, \beta) = d(f_{\beta A}(a), f_{\beta A}(\alpha)) < d(a, \alpha) = r_\alpha$.

But this is an evident contradiction as there is no sequence of reals $\{r_\alpha : \alpha < \omega_1\}$ satisfying the above inequality for all $\alpha < \beta < \omega_1$.

From the above result one easily derives the following.

3.10. Corollary

There exists an iterative algebra $\underline{A} = \langle A, f \rangle$ with f a binary operation, such that $\text{card}(A) = \mathfrak{c}/2_1$ and for no metric d on A , $\langle \underline{A}, d \rangle$ is weakly contractive.

Without success we tried to answer the following questions.

Problem 1

Does Theorem 3.8 hold without any assumptions on Σ ?

As a corollary of Theorem 3.8 we obtain

3.11. Corollary

Every finite monadic iterative algebra admits a complete metric d such that $\langle A, d \rangle$ is a contractive algebra.

Problem 2

Does Corollary 3.11 hold for polyadic algebras?

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