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Quenching of self-excited vibrations

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Abstract. Stable normal-mode vibrations in engineering can be undesirable and one of the possibilities for quenching these is by embedding the oscillator in an autoparametric system by coupling to a damped oscillator. There exists the possibility of destabilizing the undesirable vibrations by a suitable tuning and choice of nonlinear coupling parameters. An additional feature is that, to make the quenching effective in the case of relaxation oscillations, one also has to deform the slow manifold by choosing appropriate coupling; this is illustrated for Rayleigh and van der Pol relaxation.

Key words: autoparametric, quenching, relaxation oscillations, slow manifold

1. Introduction

Quenching of undesirable vibrations by energy absorption or exchange of energy can take different forms. Apart from straightforward damping mechanisms, one can use autoparametric coupling which we shall discuss here. The method is illustrated by a number of explicit examples, in particular for coupling with the van der Pol and the Rayleigh equations. These equations, in their uncoupled state, have solutions which can be transformed into each other. In their coupled states, the equations again show similar behaviour with small quantitative differences.

A typical formulation for autonomous systems runs as follows. Consider a one-degree-of-freedom oscillator, for example the van der Pol oscillator, described by the (2-dimensional) system

$$\dot{x} = f(x),$$

where $f(x)$ is a smooth 2-dimensional vector field, and assume that the equation has a stable periodic solution. Suppose that this corresponds with undesirable behaviour, as is often the case for flow-induced vibrations. One can think of vibrations of overhead lines in a windfield or of a movable dam in an estuary. Can we introduce a kind of energy absorber or, mathematically speaking, can we couple the equation to another system such that this periodic solution arises as an unstable normal mode in the full system? This would be a radical type of quenching. In general, we will speak of "quenching of an oscillator" if the amplitude of the oscillator is reduced. Sometimes this can be achieved by the introduction of the system

$$\dot{x} = f(x) + g(x, y), \tag{1}$$
$$\dot{y} = h(x, y), \tag{2}$$

in which y is n-dimensional, g and h are smooth vector fields, with $h(x, 0) = 0$. In this case system (1–2) is called autoparametric. In most cases, we also assume $g(x, 0) = 0$, so that the original periodic solution corresponds with a normal mode of the coupled system. Sometimes

$g(x, y)$ includes a perturbation, resulting in a normal mode close to the unperturbed one. The important questions are *"what are the conditions for the frequencies of the coupled oscillators?"* and *"what are the requirements for the coupling terms g and h to achieve effective destabilization of the normal mode?"* (or, more modestly, *"reduction of the amplitude of the normal mode"*).

Suppose $\phi(t):\mathbb{R}\to\mathbb{R}^2$ is a stable T-periodic solution of the equation

$$\dot{x} = f(x) + g(x, 0).$$

We shall study the stability of this normal mode or, alternatively, amplitude reduction, in system (1–2).

A general characterization of autoparametric systems is given in [1, pp. 1–4], see also [2, Chapter 12] and [3]. The autonomous character of the system (1–2) that we have chosen here, is not essential. A number of important problems of quenching involve external or parametric excitation.

Two fundamental aspects play a part in the analysis of these problems. First, there is the necessity of correct *tuning* of the coupled oscillators to create the appropriate resonance conditions. Autoparametric vibrations, resulting in possible destabilization of the normal mode, occur only in a limited region of the tuning parameters. Second we have to choose suitable *nonlinear coupling terms*, the expressions $g(x, y)$ and $h(x, y)$, to realise this destabilization by the mechanism of resonance. In this paper we will emphasize the second problem.

Our examples will be concerned with quenching of self-excited oscillations where we have introduced a small parameter. The analysis depends on the nature of the interaction: weak or strong. In the case of weak interaction we can employ the usual normal-form techniques which take the form of averaging or multiple timing. Strong interaction takes place in a self-excited autoparametric system with a relaxation oscillator. It turns out that, to destabilize relaxation oscillations, one needs, apart from correct tuning, rather strong interactions of a special form. This is tied in with the necessity to perturb the slow manifold which characterizes to a large extent the relaxation oscillation. Some of the results in this paper are discussed in more detail in [4].

2. Weak coupling of self-excited oscillations

We start by making some observations on weak self-excitation and weak interaction. We shall express the size of the interaction by using the small, positive parameter ε. Consider self-excited vibrations in the form of flow-induced vibrations as illustrated in Figure 1. Such phenomena are difficult to model and quite often one simply represents such excitation by using a Rayleigh or van der Pol oscillator. We will consider these two cases.

2.1. Coupling of Rayleigh vibrations

Consider the case of flow-induced vibrations represented by the Rayleigh oscillator embedded in the autoparametric system

$$\ddot{x} + x = \varepsilon(1 - \dot{x}^2)\dot{x} + \varepsilon(c_1x^2 + c_2xy + c_3y^2),$$
$$\ddot{y} + \varepsilon\kappa\dot{y} + q^2y = \varepsilon y(d_1x + d_2y).$$

Putting $y = 0$, we obtain a normal-mode solution of the system, which is a periodic solution $\phi(t)$ of the equation

$$\ddot{x} + x = \varepsilon(1 - \dot{x}^2)\dot{x} + \varepsilon c_1x^2,$$

a modified Rayleigh oscillator.

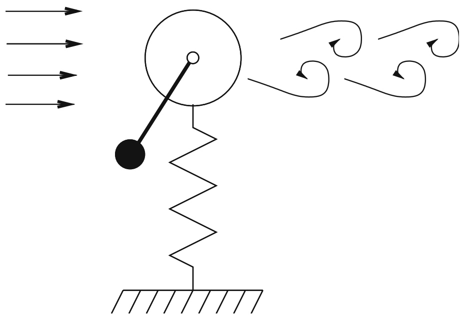

Figure 1. Example of an autoparametric system with flow-induced vibrations. The system consists of a single mass on a spring to which a pendulum is attached as an energy absorber. The flow excites the mass and the spring but not the pendulum.

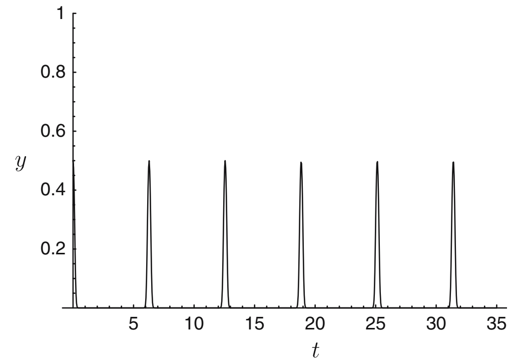

Figure 2. The solution $y(t)$ of the equation $\varepsilon\dot{y} = -xy(1-y), y(0) = 0{\cdot}5$ with $x(t) = \sin t$ which starts being attracted to the slow manifold $y = 0$. Although the stability between the slow manifolds $y = 0$ and $y = 1$ changes periodically, exponential sticking delays the departure from $y = 0$ and produces spike behaviour; $\varepsilon = 0{\cdot}01$.

The damping coefficient κ is positive, the frequency q and the coefficients c_i, d_i will be chosen suitably, *i.e.*, to provide optimal instability of the normal mode $\phi(t)$. The T-periodic solution $\phi(t)$ corresponds with self-excited vibrations and we linearize around this normal mode, putting $x = \phi(t) + u$, $y = y$, to find:

$$\ddot{u} + u = \varepsilon(1 - 3\dot{\phi}(t)^2)\dot{u} + \varepsilon(2c_1\phi(t)u + c_2\phi(t)y),$$
$$\ddot{y} + \varepsilon\kappa\dot{y} + q^2 y = \varepsilon d_1\phi(t)y.$$

The equation for y is Hill's equation with damping added, which can be reduced to Mathieu's equation by using that ε is small. It is well known that for the periodic solution of the modified Rayleigh oscillator we have the approximation

$$\phi(t) = 2\cos(t) + O(\varepsilon).$$

The estimate for the approximate solution is valid on the timescale $1/\varepsilon$, the estimate for the amplitude and period is valid for all time. Inserting the approximation into the equation for y yields

$$\ddot{y} + \varepsilon\kappa\dot{y} + (q^2 - 2\varepsilon d_1\cos(t))y = 0.$$

In parameter space, a relatively large instability domain arises on choosing $q = \frac{1}{2}$. The usual analysis (Poincaré-Lindstedt, averaging or harmonic balance) leads to the (known) requirement $|d_1| > \frac{1}{2}\kappa$ for instability of $y = 0$ (see for instance [5]).

We conclude that, on choosing $q = \frac{1}{2}, d_1 > \frac{1}{2}\kappa$, the solution $y = 0$ becomes unstable which destabilizes the normal mode in the y-direction. On choosing $c_2 \neq 0$, the solution $u = 0$ also becomes unstable which enforces the instability of the normal mode. The parameters c_1, c_3, d_2 play no part at this level of approximation.

Note that Abadi [6] studied this autoparametric system for $c_1 = c_2 = d_2 = 0$ with emphasis on the bifurcation phenomena in the case of an unstable normal mode.

2.2. Coupling of van der Pol vibrations

There is a strong relation between the solutions of the Rayleigh equations and the van der Pol equation, so we do not expect much change when replacing Rayleigh self-excitation by van der Pol self-excitation. Still, we have to check whether there are small differences. The system becomes

$$\ddot{x}+x=\varepsilon(1-x^2)\dot{x}+\varepsilon(c_1x^2+c_2xy+c_3y^2),$$
$$\ddot{y}+\varepsilon\kappa\dot{y}+q^2y=\varepsilon y(d_1x+d_2y).$$

Note, however, that there is a difference in the Lyapunov exponents of the normal modes in these cases. For the unperturbed Rayleigh oscillator the Lyapunov exponents are 0 and, with the approximation $\phi(t)=2\cos(t)+O(\varepsilon)$, $\lambda=-5\varepsilon+O(\varepsilon^2)$. For the unperturbed van der Pol oscillator we have 0 and $\lambda=-\varepsilon+O(\varepsilon^2)$, which implies slightly weaker attraction to overcome. On the other hand, the linearized equation for y does not change, leading to the same parameter conditions for instability of $y=0$ as in the case of the coupled Rayleigh oscillator.

3. Coupling with relaxation oscillations

Destabilization of relaxation oscillations is not so easy, and in these cases we will be content with quenching by reduction of the relaxation amplitude. For understanding and handling relaxation oscillations, geometric singular perturbation theory and slow manifolds are basic. We start with an introduction to this topic.

3.1. Intermezzo on slow manifolds

A different problem arises when we wish to quench a relaxation oscillation, *i.e.*, when we wish to reduce its amplitude(s). A relaxation oscillation is characterized by time-intervals with slow and fast motion; for the tuning we have to consider the timescale of the period of relaxation which is long with respect to a suitable small parameter. For instance, for the van der Pol equation,

$$\ddot{x}+x=\mu(1-x^2)\dot{x},\ \ \mu\gg 0,$$

we have the estimate

$$T_\mu=(3-2\log 2)\mu+O\left(\frac{1}{\mu^{\frac{1}{3}}}\right)\ \text{as}\ \mu\to\infty.$$

We will use results on relaxation oscillators which were summarised and extended by Grasman [7, Section 2.2]. We will discuss the nonlinearities by using results on slow manifolds in geometric singular perturbations; for introductions see [8], [9, Section 8.5] or the original papers by Fenichel [10–13].

Consider the autonomous system

$$\dot{x}=f(x,y)+\varepsilon\cdots,\quad x\in D\subset\mathbb{R}^n,$$
$$\varepsilon\dot{y}=g(x,y)+\varepsilon\cdots\quad y\in G\subset\mathbb{R}^m.$$

Note that the van der Pol equation with $\mu\gg 0$ would be represented here by the y-variable. In this context one often transforms $t\to\tau=t/\varepsilon$ so that

$$x'=\varepsilon f(x,y)+\varepsilon^2\cdots,\quad x\in D\subset\mathbb{R}^n,$$
$$y'=g(x,y)+\varepsilon\cdots\quad y\in G\subset\mathbb{R}^m$$

where the prime denotes differentiation with respect to τ.

In both these systems, with time variable t and time variable τ, we call y the fast variable, x the slow variable. The zero set of $g(x, y)$ is obtained from the equation $g(x, y) = 0$ and given by $\bar{y} = \psi(x)$, which in this autonomous case corresponds with a first-order approximation M_0 of the n-dimensional (slow) manifold M_ε in phase space. The flow on M_ε is to a first approximation described by $\dot{x} = f(x, \psi(x))$.

In geometric singular perturbation theory, for which Fenichel's results are basic, it is sufficient to assume that all real parts of the eigenvalues of the matrix

$$g_y(x, \psi(x)), x \in D,$$

are nonzero. Note that this requirement holds uniformly for $x \in D$ or in subsets of D. In this case the slow manifold M_ε is called *normally hyperbolic*. A manifold is called hyperbolic if the local linearization is structurally stable (real parts of eigenvalues all nonzero), it is normally hyperbolic if, in addition, the expansion or contraction near the manifold in the tranversal direction is larger than in the tangential direction (the slow drift along the slow manifold).

One might approach M_ε for instance by a stable branch, stay for some time near M_ε, and then leave again a neighbourhood of the slow manifold by an unstable branch. This produces solutions indicated as "pulse-like", "multi-bump solutions", etc. This type of exchanges of the flow near M_ε is what one often looks for in geometric singular perturbation theory.

3.1.1. *Existence of the slow manifold*

The question whether the slow manifold M_ε approximated by $\bar{y} = \psi(x)$ persists for $\varepsilon > 0$ was answered by Fenichel. The main result is as follows: If M_0 is a compact manifold which is normally hyperbolic, it persists for $\varepsilon > 0$, *i.e.*, there exists for sufficiently small, positive ε a smooth manifold M_ε close to M_0. Corresponding with the signs of the real parts of the eigenvalues, there exist stable and unstable manifolds of M_ε, smooth continuations of the corresponding manifolds of M_0, on which the flow is fast.

There are some differences between the cases that M_0 has a boundary or not. For details see [8, 14] and the original papers. It may surprise the reader that such subtleties have practical consequences as we shall briefly discuss.

3.1.2. *The compactness property*

Note that the assumption of compactness of D and G is essential for the existence and uniquess of the slow manifold. In many examples and applications M_0, the approximation of the slow manifold obtained from the fast equation, is not bounded. This can be remedied, admittedly in an artificial way, by applying a suitable cut-off of the vector field far away from the domain of interest. In this way, compact domains arise which coincide locally with D and G. However, this may cause some problems with the uniqueness of the slow manifold, as is shown in the following example.

Example. Consider the system

$$\dot{x} = 1, x(0) = x_0 > 0,$$
$$\varepsilon\dot{y} = -\frac{y}{x^2}\ , y(0) = y_0.$$

Putting $\varepsilon = 0$ produces $y = 0$, which corresponds with M_0; this is an (unbounded) part of the x-axis: $x \geq x_0$. We can obtain a compact domain for M_0 by putting $x_0 \leq x \leq L$ with L a positive constant, independent of ε. However, the limiting behaviour of the solutions depends on the initial condition. Integration of the equations yields

$$y(t) = y_0 \exp\left(\frac{1}{\varepsilon}\left(\frac{1}{x_0 + t} - \frac{1}{x_0}\right)\right).$$

As t increases, the solution for $y(t)$ tends to

$$y_0 \exp\left(-\frac{1}{\varepsilon x_0}\right).$$

So, after an initial fast transition, the solutions are all exponentially close to $y = 0$. There are, however, an infinite number of slow manifolds dependent on x_0, all tunnelling into this exponentially small neighbourhood of M_0 given by $y = 0$.

One might wonder about the practical use of exponential closeness, as such solutions cannot be distinguished numerically. The phenomenon is important and of practical use, when there is a change of stability, a bifurcation of the slow manifold. When a solution jumps off a slow manifold, exponentially close orbits may take some time to get away from the slow manifold, which became unstable. This interval of time is dependent on how close the solutions were to the stable part of the slow manifold. For explicit examples see [9, Chapter 8].

3.1.3. *Jumping phenomena*
Other interesting phenomena take place when the solutions jump repeatedly from one manifold to another, non-intersecting one. This may give rise to periodic solutions or oscillations with fast and slow motions. The classical example is the van der Pol relaxation equation; we present a more explicit case showing jumping phenomena.

Example. Consider the system

$$\dot{x} = f(x, y),$$
$$\varepsilon \dot{y} = -xy(1 - y).$$

We assume that the equation for x contains oscillatory solutions; as an illustration we replace it by an explicit function, $\sin t$. The slow manifolds are $y = 0$ and $y = 1$ with eigenvalues respectively $-x$ and x.

Choosing periodic behaviour of x, we find spike-like relaxation behaviour as shown in Figure 2.

3.2. Coupling with Rayleigh relaxation

We embed the Rayleigh oscillator in an autoparametric system of the form

$$\ddot{x} + x = \mu(1 - \tfrac{1}{3}\dot{x}^2)\dot{x} + \mu F(x, \dot{x}, y, \dot{y}),$$
$$\ddot{y} + \kappa \dot{y} + q^2 y = yG(x, \dot{x}, y, \dot{y}),$$

where κ is a positive damping coefficient. We assume that, if the $(y, \dot{y})$-vibration is absent, F vanishes to produce a pure van der Pol relaxation oscillation: $F(x, \dot{x}, 0, 0) = 0$. The functions F and G, and the remaining parameters have to be chosen to produce instability of the (periodic relaxation) normal mode, obtained by putting $y = 0$ in the case $\mu \gg 1$. To fix ideas, we shall later assume that F and G are polynomials in their arguments.

Differentiation of the first equation produces

$$\dddot{x} + \dot{x} = \mu(1 - \dot{x}^2)\ddot{x} + \mu \frac{dF}{dt}$$

and, putting $\dot{x} = w$,

$$\ddot{w} + w = \mu(1 - w^2)\dot{w} + \mu\frac{\mathrm{d}F}{\mathrm{d}t},$$

we obtain the perturbed van der Pol equation. To analyze the standard van der Pol equation, it is efficient to use Liénard's transformation. Putting $1/\mu = \varepsilon$, we generalize this transformation to obtain the following equivalent system

$$\begin{aligned}
\varepsilon\dot{w} &= z + w - \tfrac{1}{3}w^3 + F(x, w, y, \dot{y}),\\
\dot{z} &= -\varepsilon w,\\
\dot{x} &= w,\\
\ddot{y} + \kappa\dot{y} + q^2 y &= yG(x, w, y, \dot{y}).
\end{aligned}$$

To first approximation the slow manifold is given by

$$z = -w + \tfrac{1}{3}w^3 - F(x, w, y, \dot{y}).$$

The slow manifold is stable if

$$1 - w^2 + \frac{\partial F}{\partial w} < 0$$

and unstable when the expression is positive on the slow manifold. During the stable phase the oscillations will follow the stable manifold, while during the transition to an unstable phase the orbits leave the stable manifold in fast motion. This results in higher-dimensional relaxation oscillations. Choosing explicitly

$$F(x, \dot{x}, y, \dot{y}) = c_1\dot{x}y + c_2 x\dot{x}y + c_3\dot{x}y^2, \tag{3}$$

we find for the first approximation to the slow manifold

$$z = -w + \tfrac{1}{3}w^3 - c_1 wy - c_2 xwy - c_3 wy^2,$$

which is stable if

$$1 - w^2 + c_1 y + c_2 xy + c_3 y^2 < 0.$$

Quenching of the Rayleigh oscillator has been achieved if there is a reduction of the $(x, \dot{x})$ amplitudes when comparing pure and perturbed Rayleigh oscillations.

3.3. Coupling with van der Pol relaxation

For comparison we consider the van der Pol relaxation oscillator embedded in an autoparametric system of the form

$$\begin{aligned}
\ddot{x} + x &= \mu(1 - x^2)\dot{x} + F(x, \dot{x}, y, \dot{y}),\\
\ddot{y} + \kappa\dot{y} + q^2 y &= yG(x, \dot{x}, y, \dot{y}),
\end{aligned}$$

where κ is again a positive damping coefficient. We assume that if the $(y, \dot{y})$-vibration is absent, F vanishes to produce a pure van der Pol relaxation oscillation: $F(x, \dot{x}, 0, 0) = 0$. We follow the analysis of [4].

Again, the functions F and G, and the remaining parameters have to be chosen to produce instability of the (periodic relaxation) normal mode, obtained by putting $y = 0$ in the case $\mu \gg 1$. In the case $(y, \dot{y}) = (0, 0)$ we have pure van der Pol relaxation which is illustrated in Figure 3. To analyse the relaxation oscillation and to be more explicit, we assume that the interaction term F contains quadratic and cubic terms only and is again of the form (3). As an illustration we choose for the attached (y) oscillator $G = dx$.

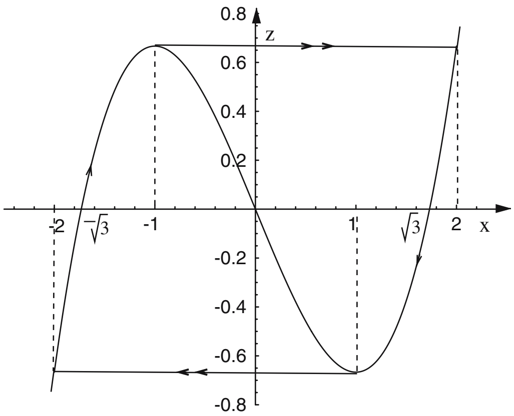

Figure 3. The phase plane of pure van der Pol relaxation. The slow manifold is approximated by a cubic curve, fast motion is indicated by double arrows.

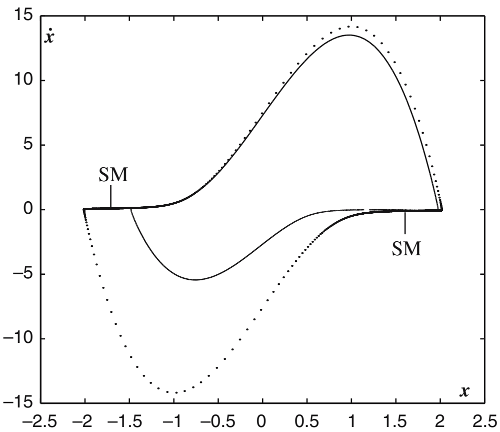

Figure 4. A periodic limit set of system (4–5) for $\mu = 10$, $c = -2{\cdot}2$, $d = 0{\cdot}03$, $\kappa = 0{\cdot}075$ with high starting values of the y-oscillation, projected on the x-$\dot{x}$ plane. The dotted orbit corresponds with the unperturbed van der Pol relaxation oscillation. In the perturbed state the slow manifolds are reduced and the limit cycle becomes asymmetric. *SM* is the stable part of the slow manifold.

We generalize the Liénard transformation $(x, \dot{x}) \to (x, z)$ for this problem to

$$\frac{1}{\mu}\dot{x} = z + x - \tfrac{1}{3}x^3 + c_1xy + \tfrac{1}{2}c_2x^2y + c_3xy^2,$$
$$\dot{z} = -\frac{1}{\mu}x - c_1x\dot{y} - \tfrac{1}{2}c_2x^2\dot{y} - 2c_3xy\dot{y}.$$

The slow manifold is given by

$$z = -x + \tfrac{1}{3}x^3 - c_1xy - \tfrac{1}{2}c_2x^2y - c_3xy^2.$$

It is stable if

$$1 - x^2 + c_1y + c_2xy + c_3y^2 < 0$$

and unstable if the expression is positive. The c_3-term is semi-definite, which is important as far as stability is concerned. So, we choose this term for our model of destabilisation of the relaxation oscillation. Replacing c_3 by c, we have the system

$$\ddot{x} + x = \mu(1 - x^2)\dot{x} + \mu c\dot{x}y^2, \tag{4}$$
$$\ddot{y} + \kappa\dot{y} + q^2y = dxy. \tag{5}$$

In generalised Liénard variables this becomes

$$\frac{1}{\mu}\dot{x} = z + x - \tfrac{1}{3}x^3 + cxy^2,$$
$$\dot{z} = -\frac{1}{\mu}x - 2cxy\dot{y},$$

with the equation for y added. The slow manifold is given by

$$z = -(1 + cy^2)x + \tfrac{1}{3}x^3,$$

which is unstable if $1 + cy^2 - x^2 > 0$. The slow manifold corresponds with a 3-dimensional cubic cylinder parallel to the $\dot{y}$-axis.

As a numerical illustration for the behaviour of the dynamics of system (4–5), we present a projection of the limit set on the $(x, \dot{x})$-plane (so the transients were left out) in Figure 4. For more numerical experiments see [4].

4. Discussion

Quenching of nonlinear oscillations is of considerable practical and mathematical interest. In the case of weak interaction between two oscillators within an autoparametric system, the theory is not complete, but extensive results are available (see the references). In the case of strong interactions, as is the case for coupled relaxation oscillators, the results are still very restricted. Geometric singular perturbation theory, involving fast-slow motion and slow manifolds, can be very helpful in these problems.

An important conclusion is that, to quench relaxation oscillations, apart from the usual tuning conditions, we have to choose the interaction such that strong deformation of the slow manifolds is possible. The conditions for stability/instability in the cases of Rayleigh and van der Pol relaxation look similar but are not identical.

When the normal-mode relaxation oscillation is destabilized, the dynamics of the coupled system is largely unexplored. It is clear, however, that periodic, quasiperiodic and chaotic limit sets are possible. More analysis and experiments are needed to understand these phenomena both qualitatively and quantitatively.

Interesting for exploration are the cases where the x-oscillator (1) that has to be quenched, has more dimensions than two and contains multifrequency oscillations. An extreme case arises if we replace Equation (1) by a wave equation. These problems are open for new research.

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